Scale-free percolation

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Many real-world networks, such as WWW, social, financial, neural, or biological networks, exhibit a number of fairly general patterns:

- the length of a smallest path between two vertices is small w.r.t. the system size (*small world*),
- the degrees of vertices exhibit a power law (*scale-free network*),
- vertices that are geographically close are likely to be connected (*geometric clustering*),
- vertices with high degree are likely to be connected even if far away from each other (*hierarchies*).

It is a challenge to find good mathematical network models that are rich enough to capture these properties but sufficiently simple to be amenable to a rigorous analysis.

Scale-free percolation, as introduced by Deijfen, van der Hofstad, and Hooghiemstra (2013), is an excellent candidate that meets all of the above criteria. It denotes a percolation model on $\mathbb{Z}^d$, where two lattice points x and y are connected by an edge with probability

$$p_{x,y} = 1 - \exp\left\{-\lambda \frac{W_x W_y}{|x-y|^\alpha}\right\} \tag{1}$$

where $\lambda > 0$ is a percolation parameter, W_x and W_y are i.i.d. edge weights with power law distribution

$$P(W_x > w) \approx w^{-(\tau-1)}, \qquad w > 0,$$

and $\alpha > 0$ denotes the exponent for the long-range connections. It arises as a marriage of the (non-spatial) Norros-Reittu model and the (non scale-free) long-range percolation.

After an introduction to the model and a discussion of the various parameters, we focus on the regime where the expected degrees are infinite. The main result is a criterion for transience vs. recurrence and the emergence of a specific hierarchical network as subgraph of the infinite cluster.