ON SHAPES OF SYMPLECTIC MANIFOLDS AS SEEN BY HOLOMORPHIC CURVES

RENATO VIANNA

Abstract. We will expose the joint work in progress with Dmitry Tonkonog and Egor Shelukhin, that in particular give bounds to embeddings of product neighbourhoods $D \times L \subset T^*L \cong \mathbb{R}^n \times L$ for L a Lagrangian torus in a symplectic manifold X and D star-shape. A more detailed explanation of the results is described below.

We define the shape $Sh_L(X)$ of a Lagrangian submanifold of a symplectic manifold (X, ω) to be the subset of $H^1(L, \mathbb{R})$ formed by the classes which are a flux of a Lagrangian isotopy. We also define $Sh_L^\star(X) \subset H^1(L, \mathbb{R})$ to be classes α such that $t_0\alpha = \mathrm{Flux}(\{L_t\}_{t \in [0;t_0]})$ for some Lagrangian isotopy $\{L_t\}_{t\in[0;1]}$, $0 \leq t_0 \leq 1$. We show how to use J-holomorphic disks with boundary on L to get bounds on $Sh_L^\star(X)$ in general and on $Sh_L(U)$, $L \subset U \subset X$, provided that L is monotone and $U \hookrightarrow T^*L$ is what we call a Liouville neighbourhood. Along the way, we describe an invariant of a Lagrangian L under the action of $\mathrm{Symp}(X)$, relying on the work of Fukaya-Oh-Ohta-Ono, when dimension of X is 4 or 6. We intuitively think of this invariant as "low-area holomorphic disks with non-zero count". We expect to be able to generalise this notion to higher dimensions. We use these bounds and almost toric fibrations to describe $Sh_L(X)$ and $Sh_L^\star(X)$ in some examples, including $Sh_L^\star(\mathbb{C}^n)$, $Sh_L(\mathbb{C}^n)$ (n= 2,3), L a toric fibre, $Sh_L^\star(X)$ for X a del Pezzo surface and L a monotone fibre as described in the work of the last author. Computing $Sh_L^\star(X)$ completely determine the function $def_L : H^1(L, \mathbb{R}) \to (0, \infty]$ defined by Entov-Ganor-Membrez. Moreover, each time we are able to completely describe $Sh_L^\star(X)$, for L a torus, we get embeddings of product neighbourhoods $U_k = D_k \times L \subset T^*L \cong \mathbb{R}^n \times L$, with $D_k \mapsto Sh_L^\star(X)$ as $k \to \infty$. Finally, for L a torus fibre in a compact toric (and almost toric for $n = 2$) manifold X, we give exotic embeddings of a product neighbourhood $D \times L \subset T^*L \cong \mathbb{R}^n \times L$ into X, with $D \subset \mathbb{R}^n$ unbounded.