ON SHAPES OF SYMPLECTIC MANIFOLDS AS SEEN BY HOLOMORPHIC CURVES

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Abstract. We will expose the joint work in progress with Dmitry Tonkonog and Egor Shelukhin, that in particular give bounds to embeddings of product neighbourhoods $D \times L \subset T^*L \cong \mathbb{R}^n \times L$ for L a Lagrangian torus in a symplectic manifold X and D star-shape. A more detailed explanation of the results is described below.

We define the shape $Sh_L(X)$ of a Lagrangian submanifold of a symplectic manifold (X, ω) to be the subset of $H^1(L, \mathbb{R})$ formed by the classes which are a flux of a Lagrangian isotopy. We also define $Sh^\star_L(X) \subset H^1(L, \mathbb{R})$ to be classes α such that $t_0\alpha = \mathrm{Flux}(\{L_t\}_{t\in[0;t_0]})$ for some Lagrangian isotopy $\{L_t\}_{t\in[0;1]}$, $0 \leq t_0 \leq 1$. We show how to use J-holomorphic disks with boundary on L to get bounds on $Sh^\star_L(X)$ in general and on $Sh_L(U)$, $L \subset U \subset X$, provided that L is monotone and $U \hookrightarrow T^*L$ is what we call a Liouville neighbourhood. Along the way, we describe an invariant of a Lagrangian L under the action of $\mathrm{Symp}(X)$, relying on the work of Fukaya-Oh-Ohta-Ono. We intuitively think of this invariant as "low-area holomorphic disks with non-zero count". We use these bounds and almost toric fibrations to describe $Sh_L(X)$ and $Sh^\star_L(X)$ in several examples, including $Sh^\star_L(\mathbb{C}^n)$ $\forall n > 0$, $Sh_L(\mathbb{C}^m)$ (m= 2,3), L a toric fibre, $Sh^\star_L(X)$ for X a del Pezzo surface and L a monotone fibre as described in the work of the last author. Computing $Sh^\star_L(X)$ completely determine the function $def_L : H^1(L, \mathbb{R}) \to (0, \infty]$ defined by Entov-Ganor-Membrez. Moreover, each time we are able to completely describe $Sh^\star_L(X)$, for L a torus, we get embeddings of product neighbourhoods $U_k = D_k \times L \subset T^*L \cong \mathbb{R}^n \times L$, with $D_k \mapsto Sh^\star_L(X)$ as $k \to \infty$. Finally, for L a torus fibre in a compact toric (and almost toric for $n = 2$) manifold X, we give exotic embeddings of a product neighbourhood $D \times L \subset T^*L \cong \mathbb{R}^n \times L$ into X, with $D \subset \mathbb{R}^n$ unbounded.