Bernoulli property for certain hyperbolic billiards

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Classification: Dynamical systems and ergodic theory

1 Abstract

In this work, we show that hyperbolic billiards constructed originally by Bussolari-Lenci has the Bernoulli property. These billiards do not satisfy the standard Wojtkowski-Markarian-Donnay-Bunimovich technique for the hyperbolicity of focusing or mixed billiards in the plane. Our proof employs a locally ergodic theorem (LET). *Joint work with Roberto Markarian - Universidad de la Republica, Uruguay.*

2 Introduction

We are interested in the following domain Ω (see [1]): take a unit square and replace three of its sides with circular arcs of curvature $\mathcal{K}_d \in (-\sqrt{2}, 0)$ having their endpoints in the vertices of the square, so the arcs are convex relative to the interior of the square. Now perturb the fourth side into a focusing circular arc of curvature $\mathcal{K}_f \ll 1$ and add two strips as shown in Figure 1. The resulting billiard is hyperbolic by [1] and do not satisfy condition B2 in [3].

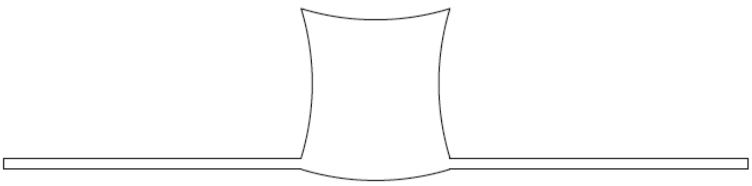

Figura 1: The main billiard table.

3 Generalities

Phase space: We identify $\mathcal{M}$ with the rectangle $[0, L] \times [-\pi/2, \pi/2]$, where L is the perimeter of Ω: each (q, u) is identified with the pair (s, α), where s is the arclength coordinate of q (relative to a fixed choice of the origin $s = 0$ and oriented counterclockwise) and α is the angle (oriented clockwise) between u and the inner normal to Γ in q. The Lebesgue measure on $\Omega \times S^1$ induces an $\mathcal{F}$-invariant measure μ on $\mathcal{M}$ which, in the above coordinates, is described by $d\mu(s, \alpha) = c \cos \alpha ds d\alpha$.

Singularity set: Let us indicate with $\partial\mathcal{M}$ the set of all pairs $(s, \alpha) \in \mathcal{M}$ where s corresponds to a vertex of Γ or $\alpha = \pm\pi/2$. Define $\mathcal{S}_1^+ := \mathcal{F}^{-1}\partial\mathcal{M}$ and $\mathcal{S}_1^- := -\mathcal{S}_1^+$. For every $k \geq 1$, define iteratively $\mathcal{S}_{k+1}^+ = \mathcal{S}_k^+ \cup \mathcal{F}^{-1}\mathcal{S}_k^+$ and $\mathcal{S}_k^- = -\mathcal{S}_k^+$. For technical reasons, we also define $\mathcal{R}_k^\pm = \partial\mathcal{M} \cup S_k^\pm$, called the *singularity sets* of $\mathcal{F}^k$ e $\mathcal{F}^{-k}$.

Under the above assumptions, $\mathcal{F}$ is a piecewise differentiable map with singularities, of type studied by Katok and Strelcyn in [5].

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4 Theoretical Background

Denoted by l the length of the strips. The geometric constants $l, \mathcal{K}_f$, $\mathcal{K}_d$ are chosen via the following procedure. One starts by fixing arbitrary values of $\mathcal{K}_d$. Then $\mathcal{K}_f$ is determined by a geometric condition that we presently describe, with the help of Fig. 2. For $s' \in \mathcal{M}^-$ and $s'' \in \mathcal{M}^+$, consider the straight line passing through s' e s'', and let $I(s', s'')$ be its intersection with the disc $D_{-2}(s')$. The curvature $\mathcal{K}_f$ must be so small that

$$\forall s' \in \mathcal{M}^-, \forall s'' \in \mathcal{M}^+,\ I(s', s'') \subset D_4(s''). \tag{1}$$

Finally, l is chosen such that $l \geq 1/\mathcal{K}_f$. For any $x \in \mathcal{M}$, define three cones in $T_x\mathcal{M}$ as in [1]:

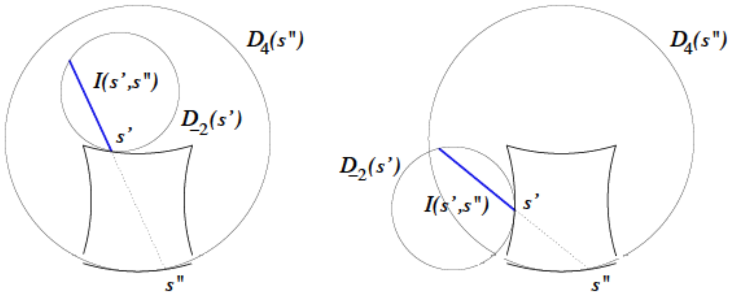

Figura 2: Condition 1 for two different choices of s' .

$$C_0(x) := \left\{ v \in T_x\mathcal{M} : -\frac{\cos\alpha}{|\mathcal{K}|} \leq f^+(v) \leq 0 \right\};$$

$$C_1(x) := \{ v \in T_x\mathcal{M} : -\infty < f^+(v) \leq 0 \};$$

$$C_2(x) := \left\{ v \in T_x\mathcal{M} : \frac{\cos\alpha}{2|\mathcal{K}|} \leq f^+(v) \leq \frac{\cos\alpha}{|\mathcal{K}|} \right\}.$$

For each $x = (s, \alpha)$, the choice $C(x) := C_i(x)$ will depend on s, s_{-1}, and what happens to the trajectoy between the collisions at s_{-1} and s.

Theorem 1 (Bussolari-Lenci, [1]). *The cone field C just defined is eventually strictly invariant relative to the map $\mathcal{F}$.*

By the Spectral Theorem (see [5]) for maps with singularities implies that $\mathcal{F}$ has at most countably many ergodic components of positive measure (the natural measure μ), with each ergodic component further decomposed into finitely many Bernoulli components cyclically permuted by $\mathcal{F}$, in which its restriction on the Bernoulli component is a K-automorphism. The map $\mathcal{F}$ restricted to the Bernoulli component is Bernoulli (see [2]). We prove that there exists a mensurable set $\mathcal{H} \subset \mathcal{M}$ of full measure such that for every $x \in \mathcal{H}$, there is a neighborhood of x in $\mathcal{M}$ contained up to a set of zero measure in a Bernoulli component of $\mathcal{F}$ (see Theorem A).

Following [6], to measure the expansion generated by the action of $D\mathcal{F}^k$ with $k > 0$ on vectors in (U, C) with respect to quadratic form Q, we define

$$\sigma_C(D_x\mathcal{F}^k) = \inf_{v \in intC(x)\setminus\{0\}} \sqrt{\frac{Q_{\mathcal{F}^k x}(D_x\mathcal{F}^k v)}{Q_x(v)}},$$

and

$$\sigma_C^*(D_x\mathcal{F}^k) = \inf_{v\in intC(x)\setminus\{0\}} \frac{\sqrt{Q_{\mathcal{F}^k x}(D_x\mathcal{F}^k v)}}{\|v\|}.$$

5 Main Results

Define the following sets

- $\mathcal{R}_\infty^\pm := \bigcup_{k\geq 1}\mathcal{R}_k^\pm$,
- $\mathcal{R} := \mathcal{R}_\infty^- \cap \mathcal{R}_\infty^+$,
- $\mathcal{N}^\pm := \{x \in \mathcal{M}\setminus\mathcal{R}_\infty^\pm : \exists k > 0 \text{ tal que } \mathcal{F}^{(\pm)n}x \in \mathcal{M}^0 \ \ \forall n \geq k\}$,
- $\mathcal{N} := \mathcal{N}^- \cap \mathcal{N}^+$,
- $\mathcal{N}' := (\mathcal{R}_\infty^- \cap \mathcal{N}^+) \cup (\mathcal{R}_\infty^+ \cap \mathcal{N}^-)$,
- $\mathcal{H} := \mathcal{M}\setminus(\mathcal{R}\cup\mathcal{N}\cup\mathcal{N}')$.

Lemma 1. *We have $\mu(\mathcal{H}) = 1$ and $m_-(\mathcal{S}_1^- \cap \mathcal{N}^+) = 0$, where m_- is the measure induced by the Riemann metric g on $\mathcal{S}_1^-$*

Now, we recall the relevant definitions for the application of the Local Ergodic Theorem (LET) proved in [4].

Definition 1. *A point $x \in \mathcal{M}\setminus\partial\mathcal{M}$ is called sufficient if there exist $l \in \mathbb{Z}$, $N \in \mathbb{N}$, an open set O and an invariant continuous cone field $(O\cup\mathcal{F}^{-N}O, K)$ such that*

- $x \notin \mathcal{R}_{|l|}^+ \cap \mathcal{R}_{|l|}^-$,
- *O is a neighborhood of $\mathcal{F}^{l+N}x$ and $O\cap\mathcal{R}_N^- = \emptyset$,*
- *$\sigma_K(D_y\mathcal{F}^N) > 3$ for all $y \in \mathcal{F}^{-N}O$.*

For the next results one must consider the following conditions:

- **L1-Regularity:** *$\mathcal{R}_k^-$ is a union of finitely many arcs of class C^2 that can only intersect at their boundaries for every $k > 0$.*
- **L2-Alignment:** *$T_y\mathcal{R}_k^- \subset C_i(y),\ i = 0, 1, 2,$ for every $y \in \mathcal{R}_k^- \cap \mathcal{F}^{-N}O$.*
- **L3-Sinai-Chernov-Ansatz:** *$\sigma_C(D_y\mathcal{F}^n) \to \infty$ when $n \to +\infty$ for almost everywhere $y \in \mathcal{R}_1^-$.*
- **L4- Non-Contraction:** *$\exists\ a > 0$ such that $\|D_y\mathcal{F}^m v\| \geq a\|v\|$ for every $n > 0, v \in C_i(y)$ and every $y \in E\setminus\mathcal{R}_m^+$.*

+ similar conditions with $\mathcal{R}_k^-, \mathcal{F}, C_i, \mathcal{F}^{-N}O$ replaced by $\mathcal{R}_k^+, \mathcal{F}^{-1}, C_i', O$

Theorem 2 (Del Magno-Markarian,[4])**.** *Suppose that L1 is satisfied, and that $x \in \mathcal{M}\setminus\partial\mathcal{M}$ is a sufficient point satisfying L2-L4. Then there exists a neighborhood of x contained up to a set of zero μ-measure in a Bernoulli component of $\mathcal{F}$.*

Theorem A 1. *Every point x of $\mathcal{H}$ has a neighborhood contained $(mod\ 0)$ in a Bernoulli component of $\mathcal{F}$.*

Demonstração. The wanted conclusion follows at once by applying Theorem 2 to points of $\mathcal{H}$. Since every point of $\mathcal{H}$ is sufficient, we need to show that for each of these points, Conditions L1-L4 of Theorem 2 are satisfied. □

Proposition 1. *Every point $x \in \mathcal{H}$ satisfies Conditions L1-L4.*

Corollary 1. *Every Bernoulli component of $\mathcal{F}$ is open $(mod\ 0)$.*

Demonstração. Let B be a Bernoulli component. Since $\mu(B) > 0$, we have $\mu(B \cap \mathcal{H}) > 0$. Let $x \in B \cap \mathcal{H}$, and let U be the neighborhood of x as in Theorem A. The set $V := \bigcup_{n\in\mathbb{Z}} \mathcal{F}^n U$ is open. Moreover, since V is invariant and contained $(mod\ 0)$ in B, it follows that $B = V\ (mod\ 0)$. □

Corollary 2. *The map $\mathcal{F} : \mathcal{H} \longrightarrow \mathcal{H}$ is Bernoulli.*

Demonstração. By Theorem A, every point of $\mathcal{H}$ has a neighborhood contained up to a set of zero measure in a Bernoulli component of $\mathcal{F}$. The same is true for every connected component of $\mathcal{H}$, and so for every $\mathcal{M}_i \cap \mathcal{H}$ such that $\mathcal{M}_i \subset \mathcal{M}^- \cup \mathcal{M}^+$. Since $\mu(\mathcal{H}) = 1$, we conclude that every set $\mathcal{M}_i \subset \mathcal{M}^- \cup \mathcal{M}^+$ is contained $(mod\ 0)$ in a single Bernoulli component of $\mathcal{F}$. Since Ω is connected, it follows that all sets $\mathcal{M}_i$ belongs to the same Bernoulli component, i.e, $\mathcal{F}$ is Bernoulli. □

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