Stable Minimal Hypersurfaces Along the Ricci Flow

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Introduction

1 Introduction and Main Results

Let (M^3, g_0) be a smooth closed orientable Riemannian 3-manifold. Consider the Ricci flow $(M, g(t))$, $t \in [0, T)$, with initial metric $g(0) = g_0$. Suppose that the Ricci curvature of the metric g_0 is positive. Then, there exists no orientable compact stable minimal surface in (M^3, g_0). By the maximum principle, proved by Hamilton [8], the Ricci curvature keeps positive along the Ricci flow and then no orientable compact stable minimal surface appears along the Ricci flow. Now, assume the initial metric g_0 has no orientable compact stable minimal surface instead assuming the Ricci curvature is positive. Fernando Marques and Andre Neves proved in [5] that, in the case when the scalar curvature R_{g_0} is positive, if (M^3, g_0) has no orientable compact stable minimal surface then there exists a $0 < \varepsilon(g_0) \leq T$ such that there is no orientable compact stable minimal surface in $(M^3, g(t))$, for all $t \in [0, \varepsilon(g_0))$. An interesting problem is to prove that, in this case of positive scalar curvature, $\varepsilon(g_0) = T$. One of the interests in to solve this problem is to study the type of singularities can appears along the Ricci flow when no stable closed minimal surfaces there exit in the initial metric. Note that, in high dimension, if the Ricci flow of the initial metric is positive, we can not guaranty the positivity of the Ricci curvature along the Ricci flow. Then, a very interesting question is to obtain conditions to guaranty that stable minimal hypersurfaces won't appear along the Ricci flow if the initial metric does not have stable minimal hypersurfaces. In this work, we prove that if the inicial metric is spherically symmetric with no closed minimal hypersurface then no closed minimal hypersurface appear along the Ricci flow. As a consequence of that result, we obtain that the Ricci flow develops a singularity of type II or shrinks to a round point. In fact, we may assume that the singularity is of type I, since if the singularity is of type II nothing we have to do. Note that, in the spherically symmetric case, to say that $g_0 = ds^2 + \psi(s)^2 g_{st}$ has no compact stable minimal surface is equivalent to say that there exists one, and only one, $s_0 \in (-1, 1)$ such that $\psi'(s_0) = 0$. In this case, s_0 is a maximum point, since $\psi : [-1, 1] \to \mathbb{R}$ has a maximum point and this maximum point can not be neither -1 nor 1. Lets call this point a pole. Hence, we can first take a rescaling limit around the pole P at the maximal time T to get an ancient κ-solution and then take a backward limit around the pole P again to get a non-flat gradient shrinking soliton. If the shrinking soliton is compact, then we know that it is the round sphere $\mathbb{S}^3$. This implies that the original solution shrinks to a round point as the time tends to the maximal time T. While if the shrinking soliton is noncompact, then we know that it is $\mathbb{S}^2 \times \mathbb{R}$. But since the limit is taking around the pole and the metric is rotationally symmetric, it can not be $\mathbb{S}^2 \times \mathbb{R}$.

2 Important Examples

2.1 Example 1: Simply Connected Lie Groups

The first important step here is to show examples of metrics g_0 on $\mathbb{S}^3$ for which the Ricci curvature is not positive and there is no orientable compact stable minimal surfaces in $(\mathbb{S}^3, g_0)$. Hence, let $\mathbb{G}$ be a 3-dimensional Lie group with a left invariant metric g_0. Suppose that $\mathbb{G}$ is diffeomorphic to the 3-sphere $\mathbb{S}^3$. Suppose that $(\mathbb{G}, g_0)$ has positive scalar curvature. Then, there is no compact stable minimal surface in $(\mathbb{G}, g(t))$, for all $t \in [0, T)$, where $g(t)$ is the Ricci flow with $g(t) = g_0$. In fact, suppose that there exists $t \in [0, T)$ such that $(\mathbb{G}, g_t)$ has a compact stable minimal surface Σ. Since the scalar curvature is positive, Σ is a 2-sphere. Another hand, for every $t \in [0, T)$, $(\mathbb{G}, g(t))$ is a Lie group and $g(t)$ is a left invariant metric with positive scalar curvature. Since $\mathbb{G}$ is diffeomorphic to the 3-sphere, it follows by the Classification Theorem for H-spheres in simply connected 3-dimensional metric Lie group due to Meeks-Mira-Perez-Ros that every minimal sphere in $\mathbb{G}$ has index 1. This contradiction implies that $(\mathbb{G}, g(t))$ has no compact stable minimal surface in $(\mathbb{G}, g(t))$, for all $t \in [0, T)$.

3 Spherically Symmetric Case

In this section, we consider metrics on $\mathbb{S}^3$ given by

$$g_0 = \varphi(x)^2 dx^2 + \psi(x)^2 g_{st}$$

in which g_{st} is the standard metric of constant curvature 1 on $\mathbb{S}^2$. We have punctured the sphere $\mathbb{S}^3$ at its north and south poles, and identified the remaining manifold with $(-1,1) \times \mathbb{S}^2(1)$, with x the coordinate on $(-1,1)$ and $\mathbb{S}^2(1)$ the unit sphere. The Ricci tensor of the metric g_0 is given by

$$Ric_{g_0} = n\left\{-\frac{\psi_{xx}}{\psi} + \frac{\psi_x\varphi_x}{\varphi\psi}\right\} dx^2 + \left\{-\frac{\psi\psi_{xx}}{\varphi^2} + \frac{\psi\varphi_x\psi_x}{\varphi^3} - (n-1)\frac{\psi_x^2}{\varphi^2} + n - 1\right\} g_{st}\,.$$

Now, consider the Ricci flow $g(t)$ with initial metric g_0. Each such metric may be identified with functions $\varphi, \psi : (-1,1) \times [0,T) \to \mathbb{R}_+$ via

$$g(x,t) = \varphi(x,t)^2 (dx)^2 + \psi(x,t)^2 g_{st}$$

where g_{st} is the standard round unit-radius metric on $\mathbb{S}^2$. Smoothness at the poles requires that φ and ψ satisfy suitable boundary condition. Under Ricci flow, the quantities φ and ψ evolve by

$$\frac{\partial\varphi(x,t)}{\partial t} = n\left(\frac{\psi_{xx}}{\psi\varphi} - \frac{\psi_x\varphi_x}{\varphi^2\psi}\right),$$

$$\frac{\partial\psi(x,t)}{\partial t} = \frac{\psi_{xx}}{\varphi^2} - \frac{\varphi_x\psi_x}{\varphi^3\psi} + (n-1)\frac{\psi_x^2}{\varphi^2\psi} - \frac{n-1}{\psi},$$

respectively.

Putting $v = \frac{\psi_x}{\varphi}$ we can find a linear parabolic equation

$$\partial_t v = A(x,t)v_{xx} + B(x,t)v_x + C(x,t)v$$

for suitable coefficients A, B and C. Furthermore, at the extremes $x = \pm 1$ we have seen that $v \to \mp 1$. In order to simplify notation, considerer a more geometric quantity that is the distance s to the equator given by

$$s(x) = \int_0^x \varphi(\mu)d\mu\,.$$

Then,

$$\frac{\partial}{\partial s} = \frac{1}{\varphi(x)}\frac{\partial}{\partial x}$$

and

$$ds = \varphi(x)dx\,.$$

Using this notation the metric can be written as

$$g_0 = ds^2 + \psi(s)^2 g_{st}\,.$$

Note also that a slice $\{s\} \times \mathbb{S}^3$ is minimal if and only if $\psi'(s) = 0$. Also it is a stable minimal surface if, and only if, $\psi'(s) = 0$ and $\psi''(s) \geq 0$.

The main result of this section is the following.

Theorem

Let g_0 be a spherically symmetric metric on $\mathbb{S}^3$. Suppose that there is no compact stable minimal surface in $(\mathbb{S}^3, g_0)$. Then, there is no compact stable minimal surface in $(\mathbb{S}^3, g(t))$, for all $t \in [0,T)$, where $g(t)$ is the Ricci flow with $g(0) = g_0$.

Remark

Note that, in the theorem above, the condition on the dimension was significant just for the singularity analysis. Hence, we have proved the following result. Let g_0 be a spherically symmetric metric on $\mathbb{S}^n$, $n \geq 3$. Suppose that there is no stable minimal hypersurface in $(\mathbb{S}^n, g_0)$. Then, there is no stable minimal hypersurface in $(\mathbb{S}^n, g(t))$, for all $t \in [0, T)$, where $g(t)$ is the Ricci flow with $g(0) = g_0$.

For the general case, we have proved the following result.

Theorem

Let g_0 be a Riemannian metric on $\mathbb{S}^n$ with positive scalar curvature. Suppose that there is no stable minimal hypersurface in $(\mathbb{S}^n, g_0)$. Then, there is no non-degenerated stable minimal hypersurface in $(\mathbb{S}^n, g(t))$, for all $t \in [0, T)$, where $g(t)$ is the Ricci flow with $g(0) = g_0$.

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