ABSTRACT

A proof of Seidel's conjectures on the volume of ideal tetrahedra in hyperbolic 3-space

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It is well known that transcendental methods are typically involved in calculating even simple geometric invariants in hyperbolic geometry (say, distance, area, volume, etc.). This has been observed already by Gauss, who referred to volume-related problems in hyperbolic geometry as a 'jungle'. One way to deal with this kind of difficulty is to express a given geometric invariant as a monotonic function of algebraic expressions.

As a toy example, let us consider the projective model of hyperbolic n-space. We take an $(n+1)$-dimensional $\mathbb{R}$-linear space V equipped with a bilinear symmetric form of signature $--\cdots-+$; the hyperbolic n-space is nothing but the open n-ball of positive points

$$\mathbb{H}^n := \{p \in \mathbb{P}_{\mathbb{R}}V \mid \langle p,p\rangle > 0\}$$

inside the projective space. Hyperbolic n-space is endowed with the distance function

$$d(p,q) := \operatorname{arcCosh}\sqrt{\frac{\langle p,q\rangle\langle q,p\rangle}{\langle p,p\rangle\langle q,q\rangle}}$$

for $p,q \in \mathbb{H}^n$. Clearly, distance is a monotonic function of the *tance*

$$\operatorname{ta}(p,q) := \frac{\langle p,q\rangle^2}{\langle p,p\rangle\langle q,q\rangle}.$$

In practical applications, it is usually much simpler to consider the tance instead of the distance.

J. J. Seidel's conjectures concern applying an analogous idea to the case of the volume formula of an ideal simplex in $\mathbb{H}^n$. An ideal simplex is nothing but an $n+1$-tuple of points $(v_1,\dots,v_{n+1})$ in the ideal boundary (the *absolute*) of $\mathbb{H}^n$. We will actually call $(v_1,\dots,v_{n+1})$ a *labelled* ideal simplex because the vertices are specified in a particular order. Choosing representatives $v_i \in V$, $i = 1\dots,n+1$, we associate to an ideal simplex a *Gram matrix* $G := (g_{ij})$, $g_{ij} := \langle v_i, v_j\rangle$. Clearly, a Gram matrix of an ideal simplex depends on the choice of the representatives $v_i \in V$. Among all the Gram matrices of a given ideal simplex, there is a single

one,[1] *DSG*, that is *doubly stochastic* (a matrix is called doubly stochastic if all its entries are non-negative and the sum of entries in every row and every column equals 1). Seidel claims that:

Speculation 1. The determinant and the permanent[2] of *DSG* completely determine the volume of the associated ideal tetrahedron.

Speculation 4. The volume is a monotonic function of the determinant and of the permanent of *DSG*.

We prove that both conjectures are true in the case $n = 3$, i.e., in the case of ideal tetrahedra in 3-dimensional hyperbolic space.[3] In fact, we prove a stronger version of Speculation 1: the determinant and permanent of *DSG* determine not only the volume of the corresponding tetrahedron, they determine the tetrahedron itself (modulo isometries).

Roughly speaking, the proof goes as follows. We begin by describing a classifying (moduli) space of all labelled non-degenerate ideal tetrahedra in $\mathbb{H}^3$. This is accomplished by invoking a simple linear algebra fact: Let $(v_1,\dots,v_n)$ and $(w_1,\dots,w_n)$ be n-tuples in a finite dimensional linear space equipped with a symmetric bilinear form. Assume that the kernel of the form restricted to the subspaces $\mathbb{R}v_1+\dots+\mathbb{R}v_n$ and $\mathbb{R}w_1+\dots+\mathbb{R}w_n$ is null. Then the n-tuples differ by an element I of the orthogonal group

$$\mathrm{O}\,V := \big\{J\in \mathrm{GL}\,V \mid \langle Jv,Jw\rangle = \langle v,w\rangle \text{ for all } v,w\in V\big\},$$

$Iv_i = w_i$, if and only if their Gram matrices are the same. This fact plus an appropriate choice of coordinates allow us to describe the space of labelled ideal tetrahedron as the region bounded by an equilateral triangle Δ in the Euclidean plane. The altitudes of Δ divide the triangle into six smaller triangles — each is a copy of the space of (non-labelled) ideal tetrahedra.

The fact that we obtain six copies of the space of ideal tetrahedra is simple to explain. There is a natural action of the 4-symmetric group S_4 on the space of labelled ideal tetrahedra (permutation of vertices). But this action has a kernel which is isomorphic to Klein's four group H and we arrive at an effective $S_3 = S_4/H$ action. This S_3-action simply permutes the 'three' dihedral angles of an ideal tetrahedron (opposite dihedral angles of an ideal tetrahedron in hyperbolic 3-space are necessarily equal). At the level of Δ, the mentioned S_3-action corresponds to the reflections in the altitudes.

Once Δ is obtained, Speculation 1 can be proved as follows. It is easy to obtain a necessary and sufficient condition on a pair of real numbers α,β such that α and β are respectively

[1] Some degenerate ideal simplices do not admit a doubly stochastic Gram matrix (this is the case, say, for the ideal simplex $(v,v,\dots,v)$).

[2] The permanent of a matrix $G=(g_{ij})$ is defined by the expression $\operatorname{per}G := \sum_{\sigma\in S_{n+1}} g_{1\sigma(1)}g_{2\sigma(2)}\cdots g_{(n+1)\sigma(n+1)}$, where S_{n+1} stands for the symmetric $n+1$-group.

[3] We also prove Speculation 3 which concerns the minimum possible value of the permanent of some doubly stochastic matrices.

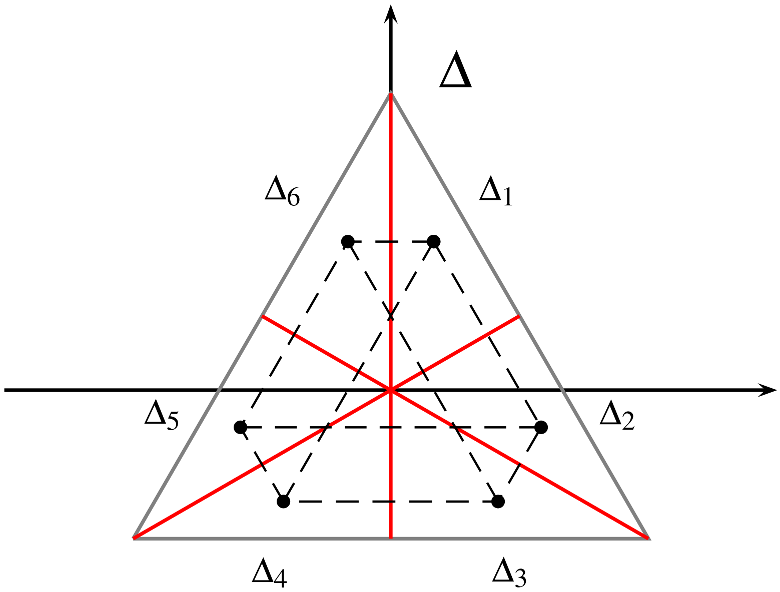

Figure 1 – The set Δ.

the determinant and permanent of the doubly stochastic matrix of at least one ideal tetrahedron (this is an application of the so-called Sylvester's criterion in linear algebra). Writing down the formulae stating that the determinant and permanent of the doubly stochastic matrix of an arbitrary ideal tetrahedron respectively equal α and β leads to solving (in the generic case) a polynomial of degree three. It turns out that, once a single solution of this polynomial is obtained, the other solutions are the obvious ones: they correspond to the Gram matrices of tetrahedra that differ only as *labelled* tetrahedra.[4] Figuring out the region Δ was a crucial step in the solution of Speculation 1 since the equations relating the determinant and permanent of a doubly stochastic matrix to coordinates in other parameterizations of the classifying space of tetrahedra tend to be quite involved (polynomials of higher degree).

Solving the above mentioned polynomial of degree three allows us to express the entries of the double stochastic matrix *DSG* of an ideal tetrahedron in terms of its determinant α and permanent β. In this way, one finds an explicit formula for the volume in terms of α and β. Indeed, one of the simplest ways to calculate the volume of an ideal hyperbolic tetrahedron was discovered by Milnor, who found an expression in terms of Lobachevsky's function л and the dihedral angles of the tetrahedron:

$$\mathrm{vol}\,(T) = л(\theta_1) + л(\theta_2) + л(\theta_3),$$

where

$$л(x) := -\int_0^x \log|2\sin t|\,\mathrm{d}t.$$

[4] It could be that Seidel did not state Speculation 1 in its full generality because he did not make a clear distinction between labelled and non-labelled tetrahedra. His Speculation 1 was possibly made after an observation that there existed tetrahedra with different Gram matrices but with the same volume. However, these tetrahedra are geometrically the same as non-labelled tetrahedra.

It is quite simple[5] to express the dihedral angles in terms of the entries of *DSG* and therefore we arrive at a volume formula that depends only on α and β. Differentiation plus a (non-straightforward) manipulation of inequalities leads to a proof of Speculation 4.

It should be mentioned that Speculation 1 was given a ‘counter-example’ in “Seidel’s problem on the volume of a non-euclidean tetrahedron”, (Abrosimov, 2010). Actually, the author of the mentioned article do not deal with the doubly stochastic Gram matrix of vertices of an ideal tetrahedron. Instead, they consider a normalized Gram matrix of the points which are *polar* to the faces of the tetrahedron.

Seidel’s speculations combine very well with some aspects of the elementary representation theory of the symmetric group which we recall below (our approach follows almost literally the book “Representation Theory” of Fulton-Harris).

Let $n \in \mathbb{N}$ be a natural number. A *partition* $\lambda = (\lambda_1, \dots, \lambda_k)$ of n is a choice of non-null natural numbers $\lambda_1, \dots, \lambda_k$ satisfying $n = \sum_{j=1}^{k} \lambda_j$ and $\lambda_i \geqslant \lambda_{i+1}$ for every i. The *Young diagram* of a partition λ is a union of squares organized as follows: the first horizontal line consists of λ_1 adjacent squares, the second horizontal line consists of λ_2 adjacent squares, etc.; the k-th horizontal line consists of λ_k adjacent squares. The squares in consecutive lines are drawn in such a way that the first square of the upper line is adjacent to the first square of the lower line. A *Young tableau* T_λ of a Young diagram is obtained by filling the squares of the diagram with the numbers $1, \dots, n$ (without repetition). The results we are interested in do not depend on a particular Young tableau associated to a Young diagram, so we assume that the diagram is always filled in some particular way (say, from 1 to n following lines). Below we illustrate the 5 Young tableau related to the partitions of the number 4.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

Figure 2 – Young tableau

Let T_λ be a Young tableau. We define two subgroups $L_\lambda, C_\lambda \leqslant S_n$ of the symmetric group as follows: the subgroup L_λ consists of all permutations that preserve the rows of the

[5] Curiously, the entries of *DSG* constitute lengths of the sides of an Euclidean triangle whose internal angles are exactly the dihedral angles of the corresponding ideal tetrahedron, see Subsection **??**.

Young tableau and the subgroup C_λ consists of all permutations that preserve the columns of the Young tableau.

In the group algebra $\mathbb{C}S_n$ we introduce the *Young projectors*

$$a_\lambda := \frac{1}{|L_\lambda|}\sum_{g\in L_\lambda} g, \quad b_\lambda := \frac{1}{|C_\lambda|}\sum_{g\in C_\lambda}(\operatorname{sgn} g)g,$$

where $\operatorname{sgn} g$ denotes the sign of the permutation $g \in S_n$. The *Schur symmetrizer* is defined to be the element $c_\lambda := a_\lambda b_\lambda \in \mathbb{C}S_n$. (The Schur symmetrizer is non-null because $L_\lambda \cap C_\lambda = 1$.)

The subspace $V_\lambda := (\mathbb{C}S_n)c_\lambda \leqslant \mathbb{C}S_n$ is an irreducible representation of S_n (and every irreducible representation of S_n is isomorphic to some V_λ).

We are now ready to introduce the Schur functors.

Given a partition λ of n, we denote $\mathbb{S}^\lambda V := V^{\otimes n}c_\lambda$, where the action of c_λ is by permutation of indices. In particular, when $\lambda = (n)$, the Schur functor $\mathbb{S}^\lambda$ is the n-th symmetric power functor. When $\lambda = (1,1,\dots,1)$, we have $\mathbb{S}^\lambda = \bigwedge^n$, that is, $\mathbb{S}^\lambda$ is n-th exterior power functor. For other partitions, $\mathbb{S}^\lambda$ 'interpolates' between these extremal cases.

The space $\mathbb{S}^\lambda V$ can be visualized as follows: their elements are (finite) linear combinations of *decomposable* terms of the type

$$v_1\dots v_n := (v_1\otimes\cdots\otimes v_n)c_\lambda.$$

For instance, in the exterior power case, $v_1\dots v_n = \sum_{g\in S_n}(\operatorname{sgn} g)v_{g1}\otimes\cdots\otimes v_{gn}$. In the symmetric power case, $v_1\dots v_n = \sum_{g\in S_n} v_{g1}\otimes\cdots\otimes v_{gn}$.

One can readily see that $\mathbb{S}^\lambda$ is indeed a functor from the category of finite dimensional linear spaces to itself. Indeed, we already know how $\mathbb{S}^\lambda$ behaves at the level of objects. At the level of morphisms, let $f : V \to W$ be a linear map. It suffices to define $\mathbb{S}^\lambda f : \mathbb{S}^\lambda V \to \mathbb{S}^\lambda W$ for decomposable elements: $(\mathbb{S}^\lambda f)(v_1\dots v_n) := fv_1\dots fv_n$. We have just arrived at the *Schur functors.*

It should be noted that, if V is endowed with a symmetric bilinear form $\langle -,-\rangle$, then $\mathbb{S}^\lambda V$ has a naturally induced symmetric bilinear form. This is well known, for example, in the case of the exterior power functor: the induced symmetric bilinear form on $\bigwedge^n V$ is given, at the level of decomposible elements, by

$$\langle v_1\wedge\cdots\wedge v_n, w_1\wedge\cdots\wedge w_n\rangle := \det a_{ij},$$

where $a_{ij} := \langle v_i, w_j\rangle$. In general, we have:

4.1. Definition — Theorem ("Schur functors in classic geometries", work in progress of Grossi) *Let λ be a partition of n, let V_λ be the associated representation of the symmetric group S_n, and let $\mathbb{S}^\lambda$ be the corresponding Schur functor. Then*

$$\langle v_1\dots v_n, w_1\dots w_n\rangle := \sum_{g\in S_n}\chi(g)\, a_{1g(1)}\cdot a_{2g(2)}\cdot\dots\cdot a_{ng(n)}$$

gives a induced symmetric bilinear form on $\mathbb{S}^{\lambda}V$, where $a_{ij} := \langle v_i, w_j \rangle$ and χ is the character of the representation.

The above formula, in the cases of the exterior and symmetric powers, corresponds respectively to the determinant and permanent of the matrix with entries a_{ij}. In "Group characters and algebra", Littlewood-Richardson call

$$\sum_{g \in S_n} \chi(g)\, a_{1g(1)} \cdot a_{2g(2)} \cdot \cdots \cdot a_{ng(n)}$$

the *immanant* of the matrix with entries a_{ij}.

There are deep connections between Schur functors and the geometry of $\mathbb{H}^n$. For instance, let $p_1, p_2, p_3 \in \mathbb{H}^3$ be pairwise distinct points. Then $p_1 \wedge p_2 \wedge p_3 \in \bigwedge^3 V$ corresponds (under the Hodge star operator $\star : \bigwedge^3 V \to \bigwedge^1 V = V$) to the polar point of the plane generated by p_1, p_2, p_3. In order to calculate the angle θ between the planes generated by, say, p_1, p_2, p_3 and q_1, q_2, q_3, one essentially has to calculate the tance between $p := p_1 \wedge p_2 \wedge p_3$ and $q := q_1 \wedge q_2 \wedge q_3$ in $\bigwedge^3 V$; explicitly, the mentioned angle is given by

$$\cos^2\theta = \frac{\langle p, q \rangle^2}{\langle p, p \rangle \langle q, q \rangle},$$

where the symmetric bilinear form in the previous formula is the one induced by the exterior power functor in $\bigwedge^3 V$.

Hence, it is no accident that the determinant and permanent (of a suitably normalized Gram matrix of ideal points) play an important role in volume problems. Since non-degenerate ideal tetrahedra in $\mathbb{H}^3$ form a 2-dimensional manifold, determinant and permanent suffice. In the general case of a tetrahedron whose vertices are all inside $\mathbb{H}^3$, the classifying space is 6-dimensional. There are, as we saw above, 5 Schur functors related to the representations of the symmetric group S_4. Therefore, we make the following

Speculation. *Let T be a tetrahedron whose vertices (v_1, v_2, v_3, v_4) are all in $\mathbb{H}^3$. Then T is determined, up to isometry, by the trace and all immanants of the doubly stochastic Gram matrix associated to the vertices. The volume of T is a monotonic function of the trace and of all immanants of this matrix.*