Some constructions and counterexamples on topological dynamics

Marisa dos Reis Cantarino (IME - USP)
Manuel Valentim de Pera Garcia (IME - USP)
Sônia Regina Leite Garcia (IME - USP)

1 Introduction

Let $(X, \mathcal{B}, T)$ denote a *dynamical system* (X is a set, $\mathcal{B}$ is a σ-algebra on X and $T : X \longrightarrow X$ is a $\mathcal{B}$-measurable function) and (X, T) denote a *topological dynamical system* (X is compact and $T : X \longrightarrow X$ is continuous). Let $M_T(X, \mathcal{B})$ be the set of *T-invariant measures* on $(X, \mathcal{B})$. We will use this terminology through this text.

A classical result on dynamical systems is the following:

Theorem 1.1 (Birkhoff - 1923) *Let $(X, \mathcal{B}, T)$ be a dynamical system and $\mu \in M_T(X, \mathcal{B})$. Then $\forall f \in L^1(X, \mathcal{B}, \mu)$,*

$$\tilde{f}(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f \circ T^j(x)$$

exists a. e. in X. Furthermore, $\tilde{f} \in L^1$, $\int_X \tilde{f} d\mu = \int_X f d\mu$ e $\tilde{f} \circ T = \tilde{f}$.

When the above average is computed over $\{q(n) : n \in \mathbb{N}\}$, where q is a polynomial with integer coefficients, we have:

Theorem 1.2 (Bourgain - 1989) *For any measure-preserving system $(X, \mathcal{B}, \mu, T)$, for any polynomial $q(t) \in \mathbb{Z}[t]$, and for any $f \in L^p(X, \mathcal{B}, \mu)$ with $p > 1$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f \circ T^{q(n)}(x)$$

exists μ-almost everywhere.

How do we translate Bourgain's result to topological dynamics? First, we consider that "almost everywhere" can be "translated" to *residual* (co-meager) in topological spaces, and we assume f to be continuous. Next, we need to ensure that the Birkhoff sums converge in order to ask if the sums of the polynomial iterations of T converge. Finally, we want the limit to be $\int_X f d\mu$. For this end, we take (X, T) to be *totally uniquely ergodic* (to see the reason for this hypothesis, see [6]).

This leads to the question:

Question 1.1 (Bergelson - 1996) *Let (X, T) be a totally uniquely ergodic topological dynamical system with $\mathcal{M}_T(X, \mathcal{B}) = \{\mu\}$, $p \in \mathbb{Z}[t]$ and $f \in C(X)$. Is it true that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f \circ T^{p(j)}(x) = \int_X f d\mu$$

on a residual set?

Ronnie Pavlov proved that the answer to this question is no if the degree of p is at least 2. In fact, his result is a more general one, and to prove it we require some constructions.

Theorem 1.3 (Pavlov - 2007) *For any increasing sequence $\{p_i\}$ of integers with upper Banach density zero, there exist a totally minimal, totally uniquely ergodic, and topologically mixing topological dynamical system (X, T) and $f \in C(X)$ such that there exists a residual set where*

$$\frac{1}{n} \sum_{i=0}^{n-1} f \circ T^{p_i}$$

does not converge.

2 Constructions

All the following constructions will use symbolic topological dynamical systems (X, σ) where $X = \overline{\mathcal{O}_\sigma^+(x)}$, $x \in \{0,1\}^{\mathbb{N}}$ and σ is the left-shift map.

2.1 Minimal

Our first construction will provide a minimal symbolic topological dynamical system. For this end, we define inductively th sequences $\{n_k\}$, $\{\omega_k\}$ and $\{A_k\}$ with the following rules:

(1) n_k is a positive integer;

(2) ω_k is a word on the alphabet $\{0,1\}$ of length n_k;

(3) A_k is a set of words on the alphabet $\{0,1\}$ of length n_k;

(4) $n_1 = 1, \omega_1 = 0, A_1 = \{0,1\}$;

(5) $n_{k+1} \geq n_k |A_k|$ and is a multiple of n_k;

(6) A_{k+1} is the set of words ω of length n_{k+1} that are concatenations of elements of A_k, each of them appearing at least once on ω;

(7) ω_{k+1} is a element of A_{k+1} which begins with w_k.

Now, we define x to be the limit of w_k: $x[m] = \omega_k[m]$ if $m \leq n_k$.

2.2 Totally minimal

Here to have (X, σ) totally minimal we set the rules (1) - (4) from the previous section and:

(5) $n_{k+1} \geq (k!)^2 n_k |A_k| + k! + n_k^2$;

(6) A_{k+1} is the set of words w' which are concatenations of words in A_k and the word 1 with the following properties: the word 1 does not appear at the beginning or end of w', only one 1 can be concatenated between two words of A_k and all $w \in A_k$ appears in w' at all positions indexed modulo $k!$;

(7) ω_{k+1} is a element of A_{k+1} which begins with w_k.

We define x to be the limit of w_k.

2.3 Totally minimal, totally uniquely ergodic, and topologically mixing

Finally, to have (X, σ) totally minimal, totally uniquely ergodic, and topologically mixing we set the rules (1) - (4) from section 2.1 and:

(5) Fix $\{d_k\}_{k\in\mathbb{N}}$ positive reals such that $\sum d_k < \infty$;

(6) $n_{k+1} = c_k (k+1)! |A_k| n_k + p$, where $c_k > n_k > \frac{1}{d_k}$ is integer, p is a prime which satisfies $n_k < p \leq 2n_k$;

(7) A_{k+1} is the set of words w' of length n_{k+1} according to the construction of section 2.2 and $\forall w \in A_k$ and $\forall 0 \leq i < k!$

$$fr^*_{i,k!}(w, w') \in \left[\frac{1-d_k}{k!|A_k|}, \frac{1+d_k}{k!|A_k|}\right],$$

where $fr^*_{i,k!}(w, w')$ is the frequency of w in w' at position $i(\text{mod k!})$. It tends to $\dfrac{1}{k!|A_k|}$ as $k \to \infty$;

(8) ω_{k+1} is a element of A_{k+1} which begins with w_k.

We define x to be the limit of w_k.

3 Generalizations

Theorem 1.1 was generalized to $\mathbb{Z}^d$-actions:

Theorem 3.1 (Wiener - 1939) *Let (X,T) be a $\mathbb{Z}^d$-action and $\mu \in \mathcal{M}_T(X,\mathcal{B})$. Then $\forall f \in L^1(X,\mathcal{B},\mu)$*

$$\tilde{f}(x) := \lim_{n\to\infty} \frac{1}{n^d} \sum_{g\in[0,n)^d} f \circ T^g(x)$$

exists a. e. in X. Furthermore $\tilde{f} \in L^1$, $\int_X \tilde{f} d\mu = \int_X f d\mu$ e $\tilde{f} \circ T = \tilde{f}$.

And Yuri Lima proved the following:

Theorem 3.2 (Lima - 2012) *Given $P \in \mathbb{Z}^d$ with upper Banach density zero, there exists a $\mathbb{Z}^d$-action (X,T) totally minimal and totally uniquely ergodic and $f \in C(X)$ such that there exists a residual set where*

$$\frac{1}{|P \cap (-n,n)^d|} \sum_{g\in P\cap(-n,n)^d} f \circ T^g(x)$$

fails to converge. Furthermore, $(X,\mathcal{B},\mu,T)$ can have arbitrarily large topological entropy.

To do so, he adapted constructions from [6] and [2] to $\mathbb{Z}^d$-actions.

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