POINCARÉ DUALITY PAIRS AND THE INVARIANT $E_*(G, W, M)$

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In this work, through the theory of relative homology of groups due to Dicks and Dunwoody, denoted by $H_*(G, W, M)$ (where G is a group, W is a G-set and M is a $\mathbb{Z}_2G$-module), we present some calculations of the invariant $E_*(G, W, M)$ (defined via the cokernel of the corestriction map in homology), for particular modules (for example $M = Hom(\mathbb{Z}_2W, \mathbb{Z}_2)$), when (G, W) satisfies certain duality conditions. In the following we present the principal results of this work. We begin with the definition of PD^n-pairs and its topological interpretation due to Dicks and Dunwoody ([3]).

Definition 1. *([3]) A pair (G, W) is a* ***Poincaré duality pair of dimension*** *n, or simply a PD^n- par, if there exists natural isomorphisms*

$$H^k(G; M) \simeq H_{n-k}(G, W; M)$$

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for all $\mathbb{Z}_2G$-modules M and all $k \in \mathbb{Z}$.

Theorem 1. *([3]) Let X be a compact n-manifold, $\widetilde{X}$ its universal covering space, and suppose that $\widetilde{X}$ and the components of the boundary $\partial\widetilde{X}$ are all contractible; let $G = \pi_1(X)$ and let W be the G-set of components of $\partial\tilde{X}$. Then (G, W) is a Poincaré duality pair of dimension n.* □

The next theorem provides, through the topological interpretation for PD^n-pairs given by Theorem 1, a description of the set of orbit representatives in W and the family of isotropy subgroups.

Theorem 2. *([4]) Let X be a compact n-manifold, which is also a CW-complex, with boundary $\partial X = \bigcup_{i \in I} X_i$, where X_i, $i \in I$, are the components of ∂X. Consider $\widetilde{X}$ the universal covering of X and suppose that $\widetilde{X}$ and the components of the boundary $\partial\widetilde{X}$ are all contractible. Let $G = \pi_1(X)$ be and W the G-set of components of $\partial\widetilde{X}$. Then,*
(i) $E = \{\widetilde{X_i} \mid i \in I\}$ is a set of orbit representatives in W, where, for each $i \in I$, $\widetilde{X_i}$ is a copy of the universal covering of X_i.
(ii) For each $\widetilde{X_i} \in E$, we have $G_{\widetilde{X_i}} = \pi_1(X_i)$.
(iii) (G, W) is a PD^n-pair. □

The next result characterize the types of PD^n-pairs with the notation of Dicks and Dunwoody.

Theorem 3. *([4]) Let (G, W) be a PD^n-pair. Then, only one of the statements is true:*
(i) W consists of exactly two elements and the G-action in W is transitive.
(ii) W consists of exactly two elements and the G-action in W is trivial.
(iii) $[G : G_w] = \infty$, for all $w \in W$, and W is infinite. □

Now, through the invariant $E_*(G,W,M)$, for particular modules, we present some results about certain homological finiteness conditions, i.e., when G is a PD^n-grupo, $n > 1$, or (G,W) is a PD^n-pair. Firstly we present the definition of $E_*(G,W,M)$.

Definition 2. *Let (G,W) be a pair where G is a group and W is a G-set. Consider E a set of orbit representatives for the G-action in W, such that $[G : G_w] = \infty$ for all $w \in E$ and let M be a $\mathbb{Z}_2G$-module. We define*

$$E_*(G,W,M) = 1 + dim\ coker\ cor_W^G$$

where $cor_W^G : \bigoplus_{w \in E} H_1(G_w; M) \longrightarrow H_1(G;M)$ is the corestriction homomorphism induced in homology by the inclusion maps $G_w \overset{i}{\hookrightarrow} G$.
For the sake of simplicity, for $M = \overline{\mathbb{Z}_2G} = Hom(\mathbb{Z}_2G, \mathbb{Z}_2)$, we denote $E_*(G,W,\overline{\mathbb{Z}_2G})$ by $\overline{E_*}(G,W)$ and, for $M = \overline{\mathbb{Z}_2W} := \bigoplus_{w\in E} Hom(\mathbb{Z}_2(G/G_w), \mathbb{Z}_2)$, we denote $E_*(G,W,\overline{\mathbb{Z}_2W})$ by $E_*(G,W)$.

Remark 1. *The definition of $E_*(G,W,M)$ is independent of the choice of the set of orbit representatives E.*

Theorem 4. *Let G be a group, W a G-set and E a set of orbit representatives for W. Suppose that $[G : G_w] = \infty$, $\forall w \in E$.*
(i) If (G,W) is a PD^n-pair with $n > 1$ then $\overline{E_*}(G,W) = 1$.
(ii) If G is a PD^n-group, $n > 1$, then $\overline{E_*}(G,W) = 1$. □

Theorem 5. *Let (G,W) be a Poincaré duality pair of dimension n. Consider E a set of orbit representatives for the G-orbits of W and suppose $[G : G_w] = \infty$ for all $w \in E$. Then $E_*(G,W) = 1$.* □

Example 1. *Let Y be a compact surface with boundary. This surfaces are obtained by removing a finite number of open disks from the closed compact surfaces (sphere, torus, projective plan, Klein bottle, etc.). Let $\tilde{Y}$ be the universal recovering of Y. In all cases $\tilde{Y}$ is contractible. Moreover, the boundary components of $\partial\tilde{Y}$ are also contractible, except in the case of the sphere minus a open disk, which the boundary is a circle. Let $G = \pi_1(Y)$ be and W the set of the components of the boundary $\partial\tilde{Y}$. I follow from Theorem 1 that (G,W) is a PD^2-pair (except in the case of the sphere minus a open disk) and, by Theorem 5, $E_*(G,W) = 1$.*

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