Cosine family and second order differential equations

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1 Introduction

One of the tools used in the study of first order abstract differential equations defined in Banach spaces is the theory of semigroups of bounded linear operators. This approach begin with the work of Hille and Yosida in 1948 and since then, has motivated a large number of works.

In particular, this theory is also used in the study of second order abstract differential equations because we can transform a second order equation on a first order system. Moreover, the inspiration to define the concept of semigroups comes from ordinary equations, and we often say that the semigroups have the properties of the exponential function, present in the solution of many ODE's.

However, there is another approach for the abstract second order problem, based on the solution of second order ordinary equations, which envolves cosine and sine functions. In this cas, we work with two families of bounded linear operators: the cosine and sine families, whose properties resemble a lot the properties of cosine and sine functions of real numbers. The first results of this line of work appeared on the end of 1960s, with the works from de Da Prato [1], Giusti [4] and Fattorini [2, 3]. Furthermore, a compilation of this theory can be found in [8, 9, 10].

One of the questions is if there exists a relation between the cosine and sine families of operators and the second order homogeneous Cauchy problem, as happen in the case of semigroups theory and the first order Cauchy problem. The answer is affirmative and it is the objective of this work.

Thus, we will show that every second order abstract differential equation of the form $u''(t) = Au(t),\ t \in \mathbb{R}$ which is well-posed in a certain sense gives rise to a strongly continuous cosine family

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of bounded linear operators with infinitesimal generator A, and conversely, every strongly continuous cosine family of bounded linear operators with infinitesimal generator A may be associated with the well-posed second order differential equation $u''(t) = Au(t),\ t \in \mathbb{R}$.

This work is more detailed in [7].

2 Results

Let $A : D(A) \subset X \to X$ be a linear operator, not necessarily bounded, with $D(A)$ dense in X, and let $x, y \in X$. By a linear second order initial value problem in X, we mean

$$u''(t) = Au(t), \ \ t \in \mathbb{R}, \tag{1}$$

$$u(0) = x, \tag{2}$$

$$u'(0) = y. \tag{3}$$

Definition 1 *A function $u : \mathbb{R} \to X$ is a solution $u(t) = u(t; x; y)$ of the initial value problem (1)-(3) if:*

(i) $u \in C^2(\mathbb{R}; X)$;

(ii) $u(t) \in D(A)$ for all $t \in \mathbb{R}$;

(iii) u satisfies (1)-(3).

Notice that if (1) and (2) are to be satisfied, then x is necessarily in $D(A)$. Since A may be unbounded, and thus $D(A) \subsetneq X$, we cannot expect to obtain a solution to the initial value problem for every $x \in X$.

Definition 2 *We say that the initial value problem (1)-(3) is well-posed if:*

(i) (1)-(3) has a unique solution $u(\cdot, x, 0)$ for each $x \in D(A)$ and $y = 0$;

(ii) for every sequence $(x_n)_{n\in\mathbb{N}}$ in $D(A)$ such that $\lim_{n\to\infty} x_n = 0$, we have $\lim_{n\to\infty} u(t, x_n, 0) = 0$, uniformly for t in compact subsetf of $\mathbb{R}$.

Theorem 1 *Let the initial value problem (1)-(3) be well-posed, and for each $x \in D(A)$ and $t \in \mathbb{R}$, let $C(t)$ be the extension in $\mathcal{L}(X)$ of $x \mapsto u(t, x, 0)$ on $D(A)$. Then, $(C(t))_{t\in\mathbb{R}}$ is a strongly continuous cosine family of bounded linear operators in X.*

Theorem 2 *Let $(C(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine family with infinitesimal generator A and associated sine family $(S(t))_{t\in\mathbb{R}}$. Then the second order initial value problem $(1)-(3)$ is well-posed. Moreover, $x \in D(A)$ and $y \in E$, then the unique solution $u(\cdot, x, y)$ of $(1)-(3)$ is given by*

$$u(t, x, y) = C(t)x + S(t)y, \ t \in \mathbb{R}.$$

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