The discriminant in the relative asymptotic directions for a surface in $\mathbb{R}^5$

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Introduction. The second fundamental form α determines the shape operators associated to the family of normal vector fields on a surface S immersed in $I\!R^n, n \geq 3$, and hence their corresponding principal configurations. The study of this dynamics goes back to the works of Monge and Darboux , who described the behavior of the principal curvature lines in the neighborhood of umbilic points of analytic surfaces in euclidean 3-space. A complete treatment of the subject in terms of structural stability of the principal lines for surfaces of class $C^r, r \geq 4$ has been provided more recently (Gutierrez and Sotomayor [6], [8], Bruce and Fidal [4]). The generic behavior of principal configurations on surfaces in $I\!R^4$ has been studied along these lines by Ramirez Galarza and Sánchez Bringas in [10]. Besides the asymptotic configurations appears in ([7], [5]) and the mean curvature direction configurations for surfaces in $I\!R^4$ described by LF. Mello [9].

The equation in $\mathbb{R}^5$ around a semiumbilic point. Let M be a surface in $\mathbb{R}^5$. Given a orthonormal reference (w_1, w_2) in a neighborhood U of a point $m \in M$, let us denote by H, B, C the sections of the normal bundle of m over U given by $H = \frac{1}{2}(\alpha(w_1, w_1) + \alpha(w_2, w_2)), B = \frac{1}{2}(\alpha(w_1, w_1) - \alpha(w_2, w_2))$ and $C = \alpha(w_1, w_2)$, where α denotes the second fundamental form of M.

Let us denote by $bb = B \cdot B,\ bc = B \cdot C$, etc. Then, the binary equation of the asymptotic lines on M can be written in an isothermal chart as

$$c_1 + c_2 \operatorname{sen}(2w) + c_3 \cos(2w) = 0, \tag{1}$$

where $c_1 = bb\,cc - bc^2,\ c_2 = bb\,hc - bc\,hb,\ c_3 = cc\,hb - bc\,hc$ and w is the angle of the line with the first coordinate vector field of that chart.

Assume now that M is generic in the sense of the paper by GonÃ§alves, MartÃnez Alfaro, Montesinos-Amilibia and Romero Fuster, and that m is a generic semiumbilic ([1]). Then, by means of Mathematica, it can be shown that c_1 and c_3 are infinitesimals of second order around m, and that c_2 is an infinitesimal of first order at m. Thus, we are interested in the configuration of the discriminant of the binary equation in a small neighborhood of m. We will use polar coordinates, so that we assume that the coordinates of m are $(0, 0)$ and that we put $u = r \cos t,\ v = r \sin t$. Then, if we truncate the Taylor expansion of the coefficients c_i at the second order, we may write $c_1 = r^2 P(t),\ c_2 = r(a \cos t + b \operatorname{sen} t + rQ(t)),\ c_1 = r^2 R(t)$, where $P, Q, R : \mathbb{R} \to \mathbb{R}$ are periodic functions with period π. Thus, the equation of the asymptotic lines around m may be written in a second order approximation as

$$rP(t) + (a \cos t + b \operatorname{sen} t + rQ(t)) \operatorname{sen}(2w) + rR(t) \cos(2w) = 0, \tag{2}$$

where we have dropped a common factor r. The appearance of that factor means obviously that all tangent directions at m are asymptotic, because at a semiumbilic point that equation has a singularity.

The discriminant. We are interested in the discriminant of equation 2, that separates the region where there are asymptotic lines from the region where they are not. Now, if we have an equation in w as 1, then there is some value β such that $c_2 = \sqrt{c_2^2+c_3^2}\cos\beta,\ c_3 = -\sqrt{c_2^2+c_3^2}\sin\beta$, so that the equation reads

$$\frac{c_1}{\sqrt{c_2^2+c_3^2}} + \operatorname{sen}(2w-\beta) = 0.$$

It is clear then that it has a solution at all iff $c_2^2+c_3^2-c_1^2 \geq 0$. Therefore, we can write equation 2 as

$$rP(t) + (\operatorname{sen}(t-\theta) + rQ(t))\operatorname{sen}(2w) + rR(t)\cos(2w) = 0, \tag{3}$$

for some value $\alpha \in \mathbb{R}$, and it has a solution at (r,t) iff

$$\Delta(t,r) = (\operatorname{sen}(t-\theta) + rQ(t))^2 + r^2(R(t)^2 - P(t)^2) \tag{4}$$

is non-negative. Let $D = \Delta^{-1}(0)$. This set is the *discriminant* of the binary equation 1. A point in U will be called *hyperbolic, parabolic, elliptic* according to the value of Δ being positive, zero or negative. It is obvious that, being the functions P, Q, R bounded, if m is an accumulation point of D, the limit of the direction t of points in D when r goes to zero is θ. Hence, any curve in D which pass by m is tangent at m to the direction θ.

If $R(\theta)^2 > P(\theta)^2$, then, there is some $\epsilon > 0$ such that $R(\theta+\delta)^2 - P(\theta+\delta)^2 > 0$ for any $\delta \in \mathbb{R}$ such that $|\delta| < \epsilon$. Then $\Delta(\theta+\delta, r) > 0$, $\forall r > 0$. Therefore, there is a finite sector around the direction given by θ consisting of elliptic points. Therefore m is not a point of accumulation of D. Thus, there is a neighborhood of m consisting of elliptic points (m excluded, of course).

By the genericity of m we do not consider the case of $R(\theta)^2 = P(\theta)^2$. Assume thus that $R(\theta)^2 < P(\theta)^2$. Then, for the direction t near enough to the direction θ, equation 4 may be written as

$$\big(\operatorname{sen}(t-\theta) + rQ(t) + rS(t)\big)\big(\operatorname{sen}(t-\theta) + rQ(t) - rS(t)\big) = 0, \tag{5}$$

where $S(t) = \sqrt{P(t)^2 - R(t)^2}$. Its solutions for r are

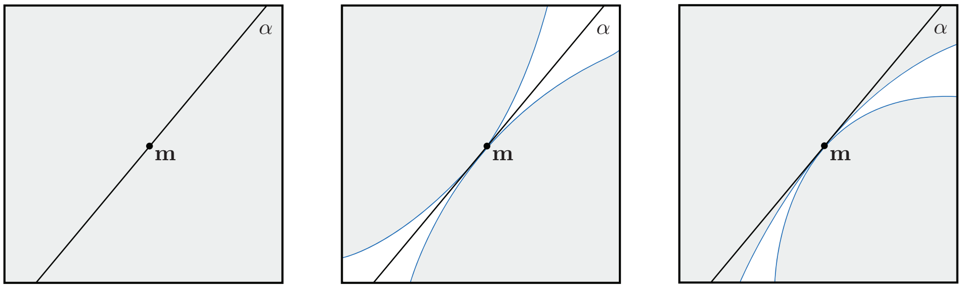
$$r = -\frac{\operatorname{sen}(t-\theta)}{Q(t) \pm S(t)}.$$

The right hand side may be equivalently written as

$$\frac{\operatorname{sen}(t+\pi-\theta)}{Q(t+\pi) \pm S(t+\pi)}$$

by the periodicity of Q and S. Note that the direction t is the same as the direction $t+\pi$. Therefore, for each choice of the sign in the expression $\pm S(t)$, there is a choice of "sense" in the direction given by t such that the solution r be positive as it should. Hence, m is an accumulation point of D. Since D is approximately a quartic algebraic curve, we have in this case that in a neighborhood of m that curve consists of two arcs that meet at m with tangents in the direction θ.

Now, $\Delta(\theta, r) = r^2(Q(\theta)^2 + R(\theta)^2 - P(\theta)^2)$. Thus, if $Q(\theta)^2 + R(\theta)^2 - P(\theta)^2 > 0$ (resp. < 0) there is a curve by m tangent to the direction θ at m whose points are hyperbolic (resp. elliptic) in a neighborhood of m. This completes our study.



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