Characterization of Curves that lie on a Geodesic Sphere in Hyperbolic and Spherical Riemannian Manifolds

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The geometry of spheres is certainly one of the most important topic of investigation in differential geometry, the search for necessary and/or sufficient conditions for a submanifold be a sphere being one of its major pursuit. A related and interesting problem then is: how can we characterize the curves $\alpha : I \to \mathbf{R}^{m+1}$ that belong to the surface of a sphere? After equipping a curve with its Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$, i.e., $m+1 = 3$, it is possible to prove that spherical curves are characterized by $\frac{\kappa}{\tau} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\dot{\kappa}}{\tau\kappa^2}\right) = 0$ [6, 7], where κ and τ are the curvature and torsion of a spatial curve, respectively (it is possible to write a similar relation in higher dimensions, but it is too cumbersome to be put here).

On the order hand, by equipping a curve with a *rotation minimizing* (RM) *frame*, one is able to characterize spherical curves by means of a simple and elegant algebraic equation involving the coefficients that dictate the frame motion: *a curve is spherical if and only if* $(\kappa_1(s), \ldots, \kappa_m(s))$ *lies on a line not passing through the origin* [1]. A RM frame $\{\mathbf{t}, \mathbf{n}_1, ..., \mathbf{n}_m\}$ along $\alpha : I \to \mathbf{R}^{m+1}$ is characterized by the equations $\mathbf{t}'(s) = \sum_{i=1}^{m} \kappa_i(s)\mathbf{n}_i(s)$ and $\mathbf{n}_i'(s) = -\kappa_i(s)\mathbf{t}(s)$, where s is an arc-length parameter. The basic idea of a RM frame is that $\mathbf{n}_i$ rotates only the necessary amount to remain normal to $\mathbf{t}$: in fact, $\mathbf{n}_i$ is parallel transported along α with respective to the normal connection [2]. Due to their minimal twist, RM frames are of importance in applications [3] and also in geometrical studies [5].

The goal of this work is to extend these investigations for curves on geodesic spheres in $\mathbf{S}^{m+1}(r)$ and $\mathbf{H}^{m+1}(r)$, the $(m+1)$-dimensional sphere and hyperbolic space of radius r, respectively. For spherical curves in $\mathbf{R}^{m+1}$, an important observation is that, up to a translation, the position vector lies on the normal plane to the curve:

$\langle \alpha - p, \alpha - p\rangle = R^2 \Leftrightarrow \langle \mathbf{t}, \alpha - p\rangle = 0$ (one calls α a *normal curve*). This makes sense due to the double nature of $\mathbf{R}^{m+1}$ as both a manifold and as a tangent space. In a Riemannian setting one should replace the line segment $\alpha(s) - p$ by a geodesic connecting p to a point $\alpha(s)$, as pointed out by Lucas and Ortega-Yagües [4] in the study of rectifying curves, i.e., $\alpha - p \in \mathrm{span}\{\mathbf{t}, \mathbf{b}\}$. Here we show that in $\mathbf{S}^{m+1}(r)$, or $\mathbf{H}^{m+1}(r)$, a regular curve is as a normal curve if and only if it lies on a geodesic sphere. Using this, we establish our main result: a regular curve α lies on a geodesic sphere in $\mathbf{S}^{m+1}(r)$ or $\mathbf{H}^{m+1}(r)$ if and only if it satisfies $\displaystyle\sum_{i=1}^{m} a_i\,\kappa_i + \frac{1}{r}\cot\left(\frac{z_0}{r}\right) = 0$ or $\displaystyle\sum_{i=1}^{m} a_i\,\kappa_i + \frac{1}{r}\coth\left(\frac{z_0}{r}\right) = 0$, respectively, where a_i and z_0 are constant.

References

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