A priori estimates for viscosity solutions of fully nonlinear parabolic equations

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Abstract

We study existence and regularity estimates of viscosity solutions of the fully nonlinear parabolic equation

$$F(D^2u) - \partial_t u = \gamma u^{\gamma-1}\Gamma(|\nabla u|)\chi_{\{u>0\}} \ \text{ in } Q_1 = B_1(0)\times(-1,0) \subset \mathbb{R}^{d+1}, \tag{1}$$

with initial and Dirichlet boundary conditions, under natural hypothesis on F and on the non-isotropic term Γ to be specified, where $1 < \gamma < 2$ and χ_A is the characteristic function of the set A.

The presence of the (non-isotropic) strong reaction term $\gamma u^{\gamma-1}\Gamma(|\nabla u|)\chi_{\{u>0\}}$ may forces the solution exhibits a *free boundary,* that is, $\partial\{u>0\}\cap Q_1 \neq \emptyset$. We are interested in analytic and geometric properties of $\partial\{u>0\}$. However, when we consider free boundary problems, optimal regularity results and sharp non-degeneracy are crucial for further analysis of the set $\partial\{u>0\}$.

Equation (1) is the fully nonlinear version of the parabolic problem

$$\Delta u - \partial_t u = \gamma u^{\gamma-1}\chi_{\{u>0\}} \ \text{ in } \ Q_1, \tag{2}$$

studied by Choe & Weiss. They proved optimal regularity and non-degeneracy estimates for the solution. As a consequence, they were able to obtain finite speed of propagation of the set $\{u>0\}$ and also Hausdorff measure estimates of the free boundary $\partial\{u>0\}$ with respect to the parabolic metric.

Besides the existence and regularity results, we are interested in sharp estimates close to the free boundary that allows us to obtain fine analytical and geometric properties of this set. Notice that, in the limit case $\gamma = 1$, we have a sort of parabolic obstacle problem with nonlinear dependence on the gradient. It is also interesting to observe that the model solution become degenerate when $\gamma \to 2^-$.

Let us now describe in more details our results and hypothesis.

We will be considering uniformly parabolic equations. Thus, we assume that the operator $F\colon \mathcal{S}(d) \to \mathbb{R}$ is (λ,Λ)-elliptic, *i.e.,* there are two constants $\Lambda \geq \lambda > 0$ such that, for any $M \in \mathcal{S}(d)$,

$$\lambda\|N\| \leq F(M+N) - F(M) \leq \Lambda\|N\| \tag{3}$$

holds for an arbitrary positive definite matrix N, where $\|N\|$ is, for instance, $\max|n_{ij}|$, $N = (n_{ij})_{i,j}$.

Also, it is assumed that

$$F(0) = 0, \quad F \in C^{1,\alpha}(\mathcal{S}(d)), \tag{4}$$

for some $\alpha > 0$.

The function F will always be assumed to be convex, this means that there is a supporting hyperplane (from below) to the graph of F at $0 \in \mathcal{S}$.

For the non-isotropic term $\Gamma : [0,\infty) \to \mathbb{R}$ we are going to assume that

$$\Gamma \in C^1((0,\infty)); \tag{5}$$

$$\Gamma'(s) \geq 0; \tag{6}$$

$$0 < \Gamma(0) \leq \Gamma(s) \leq \Gamma_0(1+s^m), \quad 0 \leq m < 2-\gamma, \tag{7}$$

for some constant $\Gamma_0 > 0$. For instance, we could use $\Gamma(s) = (1+s^m)$ and $\Gamma(s) = (1+s^2)^{m/2}$, for m in the above range.

Our existence theorem relies on a singularly perturbed analysis. To this end, we shall define the perturbed term

$$f_\varepsilon(u) = \begin{cases} \dfrac{\gamma u}{(u+\varepsilon)^{2-\gamma}} & \text{for } u \geq 0, \\ 0 & \text{for } u < 0, \end{cases} \tag{8}$$

and the corresponding approximating problem

$$\begin{cases} F(D^2u) - \partial_t u = f_\varepsilon(u)\Gamma(|\nabla u|) & \text{in } Q_1, \\ u = g & \text{on } \partial_p Q_1 = (B_1 \times \{-1\}) \cup (\partial B_1 \times [-1,0)), \end{cases} \tag{9}$$

where $g \in C^{2,\alpha}(\overline{Q}_1),\ g \geq 0$, with

$$1 + \sup_{Q_1} g + \sup_{B_1 \times \{0\}} |g^{1-\gamma} F(D^2 g)| + \sup_{(-1,0)\times \partial B_1} |g^{1-\gamma}\partial_t g| \leq G_0 < \infty. \tag{10}$$

For each $\varepsilon > 0$ there exists a viscosity solution u_ε of (9). In order to pass to the limit as $\varepsilon \to 0^+$ to obtain a solution of (1) we need sharp estimates uniform in $\varepsilon \in (0,1]$.

Theorem 0.1. *Let $0 < \varepsilon$ be fixed. Suppose that F is (λ,Λ)-elliptic and convex satisfying (4), Γ satisfies (5)–(7) and that $g \in C^{2,\alpha}(\overline{Q}_1),\ g \geq 0$, satisfying (10). Let u_ε be a viscosity solution of (9). Then, for any $Q \subset\subset Q_1$, there exists a constant $C > 0$, $C = C(d,\gamma,G_0,\lambda,\Lambda,Q)$ (in particular, not depending on ε), such that*

$$\left\| \nabla \left(u_\varepsilon^{(2-\gamma)/2} \right) \right\|_{L^\infty(Q)} + \left\| \partial_t \left(u_\varepsilon^{(2-\gamma)} \right) \right\|_{L^\infty(Q)} \leq C. \tag{11}$$

Theorem 0.1 gives us the necessary compactness to pass to the limit as $\varepsilon \to 0^+$ obtaining a viscosity solution of (1). As a by-product, we also have sharp regularity.

Theorem 0.2. *Suppose that F is (λ,Λ)-elliptic and convex satisfying (4), Γ satisfies (5)–(7) and that $g \in C^{2,\alpha}(\overline{Q}_1),\ g \geq 0$, satisfying (10). Then for a sequence $\varepsilon_j \to 0^+$, the associated sequence of solutions $u_j = u_{\varepsilon_j}$ converges to a viscosity solution of (1). Furthermore, the following estimate holds for any $Q \subset\subset Q_1$:*

$$\left\| u^{(2-\gamma)/2} \right\|_{C^{1,1/2}(Q)} \leq C, \tag{12}$$

for a constant $C > 0$, $C = C(d,\gamma,G_0,\lambda,\Lambda,Q)$.

The sharp regularity obtained in Theorem 0.2 gives us a control from above for the growth of the solution. We now start to present the first results in the study of the free boundary.

Proposition 0.3. *Suppose that F is (λ,Λ)-elliptic and convex satisfying (4), Γ satisfies (5)–(7) and that $g \in C^{2,\alpha}(\overline{Q}_1),\ g \geq 0$, satisfying (10). Let u be the viscosity solution of (1) given by Theorem 0.2 and $(x_0,t_0) \in \overline{\{u>0\}}$. Then there exists a constant $C = C(n,\gamma,\Lambda,\Gamma) > 0$ such that*

$$\sup_{Q_r(x_0,t_0)} u \geq C r^{2/(2-\gamma)} \tag{13}$$

for any $Q_r(x_0,t_0) \subset Q_1$, where $Q_r(x_0,t_0) = B_r(x_0) \times (t_0 - r^2, t_0) \subset \mathbb{R}^{d+1}$.

Two interesting consequences of Proposition 0.3 are the positive Lebesgue density of the set $\{u > 0\}$ and the finite speed of propagation of the free boundary $\partial\{u > 0\}$.

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