Abstract

In this work we study constraint qualification (CQ) and optimality conditions for the generalized Nash equilibrium problems (GNEP) analogous relations as the case of optimization are discussed. GNEPs are a generalization of the classic Nash equilibrium problem (NEP) where each player’s strategy sets depend on the choices of other players. Using these conditions of qualification and optimality we define and prove global convergence of an augmented Lagrangian-type algorithm that calculates a Karush-Kuhn-Tucker (KKT) point of a GNEP.

Key words: Augmented Lagrangian methods, Constraint qualifications, generalized Nash equilibrium problem, KKT conditions, optimality conditions.

1 Introduction

We consider the generalized Nash equilibrium problem (GNEP) where, given $N$ players $v = 1, \ldots, N$, every player aims at minimizing

$$P^v(x^{-v}) : \min_{x^v} f^v(x^v, x^{-v}) \quad s.t. \quad g^v(x^v, x^{-v}) \leq 0 \quad (1.0.1)$$

by controlling his own variables $x^v \in \mathbb{R}^{n_v}$. Here, $f^v : \mathbb{R}^n \to \mathbb{R}$ denotes the objective or utility function of player $v$, $g^v : \mathbb{R}^n \to \mathbb{R}^{m_v}$ defines the constraints. As usual in the context of GNEPs, we often write $(x^v, x^{-v})$ instead of $x \in \mathbb{R}^n$, where the vector $x^{-v}$ is defined by $x^{-v} = (x^u)_{u=1, u \neq v}^N$. Note that we have $n = n_1 + \cdots + n_N$; furthermore, we set $m := m_1 + \cdots + m_N$ for the total number of constraints.

The GNEP is called \textit{player convex} if all functions $f^v$ and $g^v$ are continuous and, as a function of $x^v$ alone, convex. whereas the GNEP is called \textit{jointly convex} if $g^1 = \cdots = g^N = g$ is convex as a function of the entire vector $x$. Note that the GNEP reduces to the standard Nash equilibrium problem (NEP) in the special case where $g^v$ depends on the subvector $x^v$ only, i.e;

$$\min_{x^v} f^v(x^v, x^{-v}) \quad s.t. \quad g^v(x^v) \leq 0 \quad (1.0.2)$$
Using this notation, we recall that \( x^* = (x^*_1, \ldots, x^*_N) \) is a generalized Nash equilibrium (GNE) or simply a solution of the GNEP if \( x^* \) satisfies all the constraints and, in addition, for each player \( v = 1, \ldots, N \), it holds that
\[
  f^v(x^*) \leq f^v(x^v, x^{*-v}) \quad \forall x^v : g^v(x^v, x^{*-v}) \leq 0
\]
i.e., \( x^* \) is a solution if and only if no player \( v \) can improve his situation by unilaterally changing his strategy.

Note that we do not include equality constraints in our GNEP simply for the sake of notational convenience; our subsequent approach can easily be extended to equality and inequality constraints. Apart from this, the above setting is very general since, so far, we do not assume any convexity assumptions on the mappings \( f^v \) and \( g^v \) as is done in many other GNEP papers where only the player convex or jointly convex case is considered, cf. [6,13] for more details. It follows that our framework can, in principle, be applied to very general classes of GNEPs.

For problems of nonlinear programming conditions of optimality are conditions that are satisfied in an global solution of the problem, in general are of the form

\[
\text{global minimizer } \Rightarrow \text{ stationary point }
\]

One of the main subjects in the theory of nonlinear optimization is the characterization of optimality, which is often achieved through conditions that use the derivatives of the constraints at a prospective optimum. Among such conditions, arguably the most important is the Karush-Kuhn-Tucker (KKT) condition, which is extensively used in the development of algorithms to solve optimization problems. In order to ensure that the KKT conditions are necessary for optimality a constraint qualification (CQ) is needed. In optimization a constraint qualification (CQ) for the problem are hypotheses made about the functions that define the problem that when satisfied by a local minimizer make it to be stationary. Among the most common qualification conditions in literature are linear independence (LICQ), Mangasarian-Fromovitz (MFCQ) and constant positive linear dependence (CPLD) and other weak qualification conditions discovered in recent years as the recent cone-continuity property (CCP). For a detailed study on qualification conditions and the relationships between these a good reference is [12].

In this work we study optimality conditions and qualification constraints for GNEPs. It turns out, however, that some results are different from those that are known for standard optimization problems.

In the meantime, there exist a variety of methods for the solution of GNEPs, though most of them are designed for player or jointly convex GNEPs and therefore do not cover the GNEP in its full generality. We refer the interested reader once again to the two survey papers [7, 10] and the references therein for a quite complete overview of the existing approaches.

Penalty-type schemes belong to this class of methods. The first penalty method for GNEPs that we are aware of is due to Fukushima [11]. A related penalty algorithm was also proposed in [8], and a modification of this algorithm is described in [9] where only some of the constraints are penalized. While all these approaches prove exactness results under suitable assumptions, they suffer from the drawback that the resulting penalized subproblems are nonsmooth Nash equilibrium problems and therefore generally difficult to be solved numerically. Taking this into account, it is natural to apply an augmented Lagrangian-type approach in order to solve GNEPs because the resulting subproblems then have a higher degree of smoothness and should therefore be easier to solve. This idea is not completely new since Pang
Fukushima [17] applied this idea to quasi-variational inequalities (QVIs). An improved version of that method can be found in [14], also for QVIs. Since the GNEP is a special instance of a QVI, these two papers also discuss briefly the GNEP within their general QVI-framework.

Here, we developed an augmented Lagrangian method for GNEPs that was almost at the same time developed in [15] with similar results almost all of them based on the book [4]. Recall that the augmented Lagrangian (or multiplier-penalty) method is one of the traditional methods for the solution of constrained optimization problems [18] which have also been the subject of some recent research with several improved convergence results, see, e.g., [1] and references therein. We therefore try to adapt these recent improvements to GNEPs in order to get a better understanding of the augmented Lagrangian approach applied to GNEPs especially in dealing with subproblems and the analysis of convergence. This paper is organized as follows. In Section 2, we deal with GNEP-tailored optimality conditions and constraint qualifications (CQs), prove some basic results many of them similar to the case of optimization.

Section 3 we extend the concept of Approximate-KKT (AKKT) for NEP and GNEP by showing that this property is uniquely satisfied for the first class of problems. Section 4 gives a precise statement of the Augmented Lagrangian method for GNEPs using the results of section 3 for the analysis of the subproblems together with a refined convergence analysis that generalizes the corresponding result from [15].

References


