Graphs extensions and amenability

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Abstract

In this work, we characterise amenability of graphs through graph extensions of full Markov maps and Markov maps with embedded Gibbs-Markov structure. This research is inspired by Manuel's and Jaerisch's results for groups. The amenability criteria of Kesten (1959) and Day (1964) are classic results from probability theory. They relate amenability of a group with the behaviour of random walks on the group, that is to the exponential decay of return probabilities and the spectral radius of the associated Markov operator, respectively. Stadlbauer and Jaerisch generalised these results to group extensions of topological Markov chains.

We extend this approach to graph extensions of Markov maps with full branches. That is, we consider the time evolution of the second coordinate of

$$T : X \times \mathbf{V} \to X \times \mathbf{V}, (x, g) \mapsto (\theta(x), \kappa_x(g)),$$

where $\theta : X \to X$ is a Markov map with countably many full branches, $\kappa : X \times \mathbf{V} \to \mathbf{V}, (x, g) \mapsto \kappa_x(g)$ is a map such that κ_x is a bijection for all x and $\mathcal{G}$ is a graph with vertices $\mathbf{V}$ and edges $\mathbf{E}$.

The classical definition of amenable graph $\mathcal{G}$ is

$$\inf \left\{ \frac{|\partial K|}{|K|} : K \subset \mathbf{V}, |K| < \infty \right\} = 0$$

where $K \subset \mathbf{V}, |K| < \infty$, and the boundary of K is defined by $\partial K := \{v \in K : \exists e \in \mathbf{E} \text{ s.t. } s(e) = v, t(e) \notin K\}$.

Here, we would like to consider graphs who might contain vertices with infinitely many adjacent edges. Therefore, we introduce the notion of amenability for weighted graphs. We refer to a graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ as a weighted graph with weight $p : \mathbf{E} \to [0, 1]$ if for all $v \in \mathbf{V}$, we have $\sum_{e:s(e)=v} p(e) = 1$. This then gives rise to the definition of ϵ-boundary, this is, for $\epsilon > 0$ and $K \subset \mathbf{V}$,

$$\partial^\epsilon K := \{v \in K : \exists e \in \mathbf{E} \text{ s.t. } s(e) = v, t(e) \notin K, p(e) > \epsilon\}.$$

Definition 0.1. *The graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ is p-amenable if*

$$\lim_{\epsilon \to 0} \inf \left\{ \frac{|\partial^\epsilon K|}{|K|} : K \subset \mathbf{V}, |K| < \infty \right\} = 0.$$

We extend Day's result to this setting.

Theorem 1. *Suppose that $(X \times \mathbf{V}, T)$ is a topologically transitive extension with uniform loops of a Gibbs-Markov map (X, θ, μ) with full branches and invariant probability μ. Then the following assertions are equivalent.*

1. *The graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ is μ-amenable.*
2. $\rho(\widehat{T}) = 1$.
3. *For each $\epsilon > 0$, there exists a $A \subset \mathbf{V}$ finite such that $\int |\widehat{T}(\mathbf{1}_{X\times A}) - \mathbf{1}_{X\times A}| d\mu \leq \epsilon \cdot \#(A)$.*
4. *For each $\epsilon > 0$, there exists a $A \subset \mathbf{V}$ finite such that $[\![\widehat{T}(\mathbf{1}_{X\times A}) - \mathbf{1}_{X\times A}]\!]_1 \leq \epsilon [\![\mathbf{1}_{X\times A}]\!]_1$.*

In addition, we characterise the amenability of the graph through the extension by the graph of an embedded Gibbs-Markov structure.

Theorem 2. *Suppose that (Y, T, κ) is a graph extension of (X, θ, μ, α) with embedded topologically transitive Gibbs-Markov structure with uniform loops and that $\mu(X) < \infty$. Then the following holds.*

1. *If $R(T) = 1$, $\hat{\kappa}$ finitely covers κ and η is the first return time, then $\mathcal{G}$ is μ-amenable.*
2. *If $\mathcal{G}$ is μ-amenable, θ is ergodic and η is integrable, then $\rho(\widehat{T}) = 1$.*

In here, $R(T) := \limsup_{n\to\infty} \sqrt[n]{\mu(\{x \in X : \kappa_x^n(\mathbf{o}) = \mathbf{o}\})}$, with $o \in \mathbf{V}$. We say that $\hat{\kappa}$ finitely covers κ, if there exists a finite set $\mathcal{K} \subset \hat{\mathcal{W}}^\infty$ such that, for all $v \in \mathcal{W}^1$ and $g \in \mathbf{V}$, there exists $w \in \mathcal{K}$ such that $\kappa_v(g) = \hat{\kappa}_w(g)$. And we refer to $\rho(\widehat{T})$ as the spectral radius of operator transfer of the extension graphs.

This is joint work with Johannes Jaerisch and Manuel Stadlbauer.