

# Absolute continuity and band gaps of the spectrum of the Dirichlet Laplacian in periodic waveguides

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June 12, 2017

## Abstract

Consider the Dirichlet Laplacian operator  $-\Delta^D$  in a periodic waveguide  $\Omega$ . Under the condition that  $\Omega$  is sufficiently thin, we show that its spectrum  $\sigma(-\Delta^D)$  is absolutely continuous (in each finite region).

**MSC (2010):** primary 35J10. Secondary: 47F05, 81V99.

## 1 Introduction

During the last years the Dirichlet Laplacian operator  $-\Delta^D$  restricted to strips (in  $\mathbb{R}^2$ ) or tubes (in  $\mathbb{R}^3$ ) has been studied under various aspects. We highlight the particular case where the geometry of these regions are periodic [1, 2, 3, 4, 5, 7]. In this situation, an interesting point is to know under what conditions the spectrum  $\sigma(-\Delta^D)$  is purely absolutely continuous. On the other hand, since  $\sigma(-\Delta^D)$  is an union of bands, another question is about the existence of gaps in its structure.

In the case of planar periodically curved strips, the absolute continuity was proved by Sobolev [5] and the existence and location of band gaps was studied by Yoshitomi [7]. Our goal is to prove similar results to those in the three dimensional case. In the following paragraphs, we explain the details.

Let  $r : \mathbb{R} \rightarrow \mathbb{R}^3$  be a simple  $C^3$  curve in  $\mathbb{R}^3$  parametrized by its arc-length parameter  $s$ . Suppose that  $r$  is periodic, i.e., there exists  $L > 0$  and a nonzero vector  $u \in \mathbb{R}^3$  so that  $r(s + L) = u + r(s)$ ,  $\forall s \in \mathbb{R}$ . Denote by  $k(s)$  and  $\tau(s)$  the curvature and torsion of  $r$  at the position  $s$ , respectively. Pick  $S \neq \emptyset$ ; an open, bounded, smooth and connected subset of  $\mathbb{R}^2$ . Build a tube (waveguide) in  $\mathbb{R}^3$  by properly moving the region  $S$  along  $r(s)$ ; at each point  $r(s)$  the cross-section region  $S$  may present a (continuously differentiable) rotation angle  $\alpha(s)$ . Suppose that  $\alpha(s)$  is  $L$ -periodic. For  $\varepsilon > 0$  small enough, one can realize this same construction with the region  $\varepsilon S$  and so obtaining a thin waveguide which is denoted by  $\Omega_\varepsilon$ .

Let  $-\Delta_{\Omega_\varepsilon}^D$  be the Dirichlet Laplacian on  $\Omega_\varepsilon$ . Conventionally,  $-\Delta_{\Omega_\varepsilon}^D$  is the Friedrichs extension of the Laplacian operator  $-\Delta$  in  $L^2(\Omega_\varepsilon)$  with domain  $C_0^\infty(\Omega_\varepsilon)$ . Denote by  $\lambda_0 > 0$  the first eigenvalue of the Dirichlet Laplacian  $-\Delta_S^D$  in  $S$ . Our result states that

**Theorem 1.** *For each  $E > 0$ , there exists  $\varepsilon_E > 0$  so that the spectrum of  $-\Delta_{\Omega_\varepsilon}^D$  is absolutely continuous in the interval  $[0, \lambda_0/\varepsilon^2 + E]$ , for all  $\varepsilon \in (0, \varepsilon_E)$ .*

In [1], the authors proved this result considering the particular case where the cross section of  $\Omega_\varepsilon$  is a ball  $\mathcal{B}_\varepsilon = \{y \in \mathbb{R}^2 : |y| < \varepsilon\}$  (this fact eliminates the twist effect). Covering the case where  $\Omega_\varepsilon$  can be simultaneously curved and twisted is our main contribution on the theme.

Ahead, we summarize the main steps to prove Theorem 1. In particular, we call attention to Corollary 1, which are our main tools to generalize the result of [1].

Fix a number  $c > \|k^2/4\|_\infty$ . Denote by  $\mathbf{1}$  the identity operator. For technical reasons, we start to study the operator  $-\Delta_{\Omega_\varepsilon}^D + c\mathbf{1}$ .

A change of coordinates shows that  $-\Delta_{\Omega_\varepsilon}^D + c\mathbf{1}$  is unitarily equivalent to the operator

$$T_\varepsilon \psi := -\frac{1}{\beta_\varepsilon} (\partial_{sy}^R \beta_\varepsilon^{-1} \partial_{sy}^R) \psi - \frac{1}{\varepsilon^2 \beta_\varepsilon} \operatorname{div}(\beta_\varepsilon \nabla_y \psi) + c \psi, \quad (1)$$

where

$$\begin{aligned} \beta_\varepsilon(s, y) &:= 1 - \varepsilon k(s)(y_1 \cos \alpha(s) - y_2 \sin \alpha(s)) \\ \partial_{sy}^R \psi &:= \psi' + \langle \nabla_y \psi, Ry \rangle (\tau + \alpha')(s), \end{aligned} \quad (2)$$

$\psi' := \partial\psi/\partial s$ ,  $\nabla_y \psi := (\partial\psi/\partial y_1, \partial\psi/\partial y_2)$  and  $R$  is the rotation matrix  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . The domain  $\operatorname{dom} T_\varepsilon$  is a subspace of the Hilbert space  $L^2(\mathbb{R} \times S, \beta_\varepsilon \operatorname{d}sd y)$  where the measure  $\beta_\varepsilon \operatorname{d}sd y$  comes from the Riemannian metric.

Since the coefficients of  $T_\varepsilon$  are periodic with respect to  $s$ , we utilize the Floquet-Bloch reduction under the Brillouin zone  $\mathcal{C} := [-\pi/L, \pi/L]$ . More precisely, we show that  $T_\varepsilon$  is unitarily equivalent to the operator  $\int_{\mathcal{C}}^\oplus T_\varepsilon^\theta \operatorname{d}\theta$ , where

$$T_\varepsilon^\theta \psi := \frac{1}{\beta_\varepsilon} (-i\partial_{sy}^R + \theta) \beta_\varepsilon^{-1} (-i\partial_{sy}^R + \theta) \psi - \frac{1}{\varepsilon^2 \beta_\varepsilon} \operatorname{div}(\beta_\varepsilon \nabla_y \psi) + c \psi. \quad (3)$$

Now, the domain of  $T_\varepsilon^\theta$  is a subspace of  $L^2((0, L) \times S, \beta_\varepsilon \operatorname{d}sd y)$  and the functions in  $\operatorname{dom} T_\varepsilon^\theta$  satisfy the boundary conditions  $\psi(0, y) = \psi(L, y)$  and  $\psi'(0, y) = \psi'(L, y)$  in  $L^2(S)$ . Furthermore, each  $T_\varepsilon^\theta$  is self-adjoint.

Each  $T_\varepsilon^\theta$  has compact resolvent and is bounded from below. Thus,  $\sigma(T_\varepsilon^\theta)$  is discrete. Denote by  $E_n(\varepsilon, \theta)$  the  $n$ th eigenvalue of  $T_\varepsilon^\theta$  counted with multiplicity and  $\psi_n(\varepsilon, \theta)$  the corresponding normalized eigenfunction, i.e.,

$$T_\varepsilon^\theta \psi_n(\varepsilon, \theta) = E_n(\varepsilon, \theta) \psi_n(\varepsilon, \theta), \quad n = 1, 2, 3, \dots, \quad \theta \in \mathcal{C}.$$

We have

$$\begin{aligned} E_1(\varepsilon, \theta) &\leq E_2(\varepsilon, \theta) \leq \dots \leq E_n(\varepsilon, \theta) \leq \dots, \quad \theta \in \mathcal{C}, \\ \sigma(-\Delta_{\Omega_\varepsilon}^D) &= \cup_{n=1}^\infty \{E_n(\varepsilon, \mathcal{C})\}, \quad \text{where } E_n(\varepsilon, \mathcal{C}) := \cup_{\theta \in \mathcal{C}} \{E_n(\varepsilon, \theta)\}; \end{aligned} \quad (4)$$

each  $E_n(\varepsilon, \mathcal{C})$  is called  $n$ th band of  $\sigma(-\Delta_{\Omega_\varepsilon}^D)$ .

We begin with the following result.

**Lemma 1.**  $\{T_\varepsilon^\theta : \theta \in \mathcal{C}\}$  is a type A analytic family.

This lemma ensures that the functions  $E_n(\varepsilon, \theta)$  are real analytic; consequently, each  $E_n(\varepsilon, \mathcal{C})$  is either a closed interval or an one point set.

Another important point to prove Theorem 1 is to know an asymptotic behavior of the eigenvalues  $E_n(\varepsilon, \theta)$  as  $\varepsilon$  tends to 0. For this characterization, for each  $\theta \in \mathcal{C}$ , consider the one dimensional self-adjoint operator

$$T^\theta w := (-i\partial_s + \theta)^2 w + \left[ C(S)(\tau + \alpha')^2(s) + c - \frac{k^2(s)}{4} \right] w,$$

acting in  $L^2(0, L)$ , where the functions in  $\text{dom } T^\theta$  satisfy the conditions  $w(0) = w(L)$  and  $w'(0) = w'(L)$ . The constant  $C(S)$  depends on the cross section  $S$ .

The spectrum of  $T^\theta$  is purely discrete; denote by  $\kappa_n(\theta)$  its  $n$ th eigenvalue counted with multiplicity. Recall  $\lambda_0 > 0$  denotes the first eigenvalue of the Dirichlet Laplacian  $-\Delta_S^D$  in  $S$ . We have

**Corollary 1.** *For each  $n_0 \in \mathbb{N}$ , there exists  $\varepsilon_{n_0} > 0$  so that, for all  $\varepsilon \in (0, \varepsilon_{n_0})$ ,*

$$E_n(\varepsilon, \theta) = \frac{\lambda_0}{\varepsilon^2} + \kappa_n(\theta) + O(\varepsilon), \quad (5)$$

*holds for each  $n = 1, 2, \dots, n_0$ , uniformly in  $\mathcal{C}$ .*

With all these tools in hands, we have

**Proof of Theorem 1:** Let  $E > 0$ , without loss of generality, we can suppose that, for all  $\theta \in \mathcal{C}$ , the spectrum of  $T_\varepsilon^\theta$  below  $E$  consists of exactly  $n_0$  eigenvalues  $\{E_n(\varepsilon, \theta)\}_{n=1}^{n_0}$ . Lemma 1 ensures that  $E_n(\varepsilon, \theta)$  are real analytic functions. To conclude the theorem, it remains to show that each  $E_n(\varepsilon, \theta)$  is nonconstant.

Consider the functions  $\kappa_n(\theta)$ ,  $\theta \in \mathcal{C}$ . By Theorem XIII.89 in [6], they are nonconstant. By Corollary 1, there exists  $\varepsilon_E > 0$  so that (5) holds true for  $n = 1, 2, \dots, n_0$ , uniformly in  $\theta \in \mathcal{C}$ , for all  $\varepsilon \in (0, \varepsilon_E)$ . Note that  $\varepsilon_E > 0$  depends on  $n_0$ , i.e., the thickness of the tube depends on the length of the energies to be covered. By Section XIII.16 in [6], the conclusion follows.  $\square$

## References

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