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Singular Traveling Waves in Systems of Balance Laws

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A mis padres, Lucas y Zulma

Abstract

In this work we study traveling wave solutions to 2×2 systems of balance laws with an internal singularity. The internal singularity is characterized as an equilibrium of an implicit system of ordinary differential equations. This type of waves are found in models of enhanced oil recovery by air injection and in other problems of applied mathematics. We develop a classification of this type of waves in the 2×2 case based on the stability type of the equilibrium points on the profile. In this work we focus on two generic cases from the total of seven; the others are left for future study, needing extra conditions to be analyzed thoroughly (e.g., stability or a viscous regularity procedure). We provide a proof of the differentiability of the profile at the internal singularity. In contrast to other proofs found in the literature, our approach yields explicit formulas for the derivative. We develop a perturbation theory for the profile of singular traveling waves, and show that the two types of waves under study are structurally stable, given that a non-degeneracy condition is satisfied by the terms of the balance laws. We apply the results on differentiability to a model of enhanced oil recovery.

Contents

Chapter 1

Introduction

In this work we study a special type of traveling wave solution to 2×2 systems of balance laws: *singular traveling waves.* These waves have a profile with an internal singularity, which is found as a singularity of an associated vector field. Waves with an internal singular state, often called *resonant waves*, are important in models of enhanced oil recovery by air injection [8], and are also found in detonation problems [1,7]. Despite being important in applications, there are few works in the literature dealing with the theoretical aspects of these waves, see, e.g., [3, 4]. We consider waves that connect two different zeros of the source terms. Similar solutions to systems of balance laws are studied in [11], the difference being that in that paper the orbits considered follow a single zero of the source terms.

In the rest of this introductory chapter, we review some classical results on scalar conservation laws that provide a context for this work. First we consider the single conservation law without a source term, and find a discontinuous traveling wave solution. Then, we study a scalar conservation law with a source term, also called a balance law. We find a continuous traveling wave solution, in contrast to the discontinuous wave found for a conservation law. The generalization of this continuous traveling wave to systems of balance laws is the subject of this study. In the last section of this chapter we give an overview of the rest of the dissertation.

1.1 Scalar Conservation Laws

In this section we review some well known results on scalar conservation laws. First we study traveling wave solutions of Burgers equation. We obtain a discontinuous traveling wave solution of the equation without viscoscity. Our aim is to contrast this solution with the continuous traveling wave solutions of balance laws. We review the definition of weak solutions, necessary to make the results rigorous. The results of this section are classical and a general reference is [14].

1.1.1 Burgers Equation

Consider the equation

$$u_t + \left(\frac{u^2}{2}\right)_x = \nu u_{xx}, \quad x \in \mathbb{R}, t \geq 0, \tag{1.1}$$

where $\nu > 0$. It is called *Burgers equation*. It models wave phenomena with nonlinear propagation and diffusive effects [15]. We are interested in *traveling wave solutions* to this equation. A traveling wave solution $u(x,t)$ satisfies

$$u(x,t) = \tilde{u}(\xi), \quad \xi = x - st, \tag{1.2}$$

$$\lim_{\xi \to -\infty} \tilde{u}(\xi) = u_-, \quad \lim_{\xi \to +\infty} \tilde{u}(\xi) = u_+. \tag{1.3}$$

Here s denotes the *wave speed*, u_- is called the *left state* and u_+ is called the *right state*. By abuse of notation, we will drop tildes. Equation (1.1) and the boundary conditions (1.3) imply that u and s should satisfy the boundary value problem

$$-su' + \left(\frac{u^2}{2}\right)' = \nu u'', \tag{1.4}$$

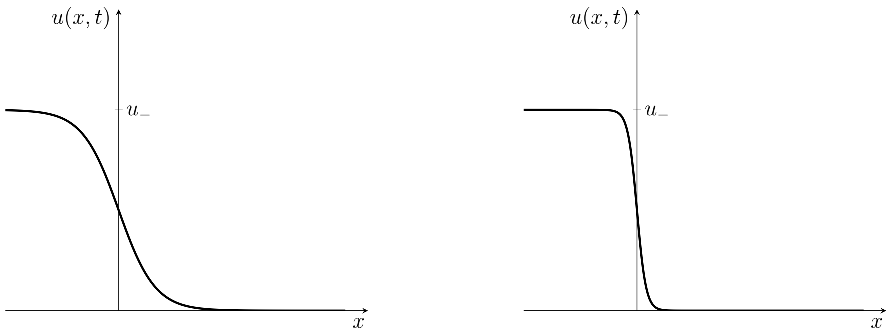
$$u(-\infty) = u_-, \quad u(+\infty) = u_+, \tag{1.5}$$

where $'$ denotes the derivative with respect to ξ. Equation (1.4) can be integrated to yield

$$-su + \frac{u^2}{2} = \nu u' + A, \tag{1.6}$$

where A is a constant of integration. The constants s and A are determined from the boundary conditions (1.5), together with the requirement that

$$\lim_{\xi \to \pm\infty} u'(\xi) = 0. \tag{1.7}$$



(a) (b)

Figure 1.1: Traveling wave profile for Burgers equation (1.1) with $u_+ = 0$. (a) $\nu = 0.4$, (b) $\nu = 0.1$.

These conditions imply

$$s = \frac{u_- + u_+}{2}, \tag{1.8}$$

$$A = -\frac{u_- u_+}{2}. \tag{1.9}$$

Using (1.8) and (1.9), (1.6) becomes

$$u' = \frac{1}{2\nu}(u - u_-)(u - u_+). \tag{1.10}$$

Equation (1.10) can be integrated to yield

$$u(\xi) = \frac{u_- - u_+ \exp(\frac{u_- - u_+}{2\nu}(\xi - \xi_0))}{1 - \exp(\frac{u_- - u_+}{2\nu}(\xi - \xi_0))}, \tag{1.11}$$

where $\xi_0 \in \mathbb{R}$ is an arbitrary constant responsible for a shift of the solution along the ξ axis. Expression (1.11) gives the profile of a traveling wave solution of (1.1) when $u_- > u_+$. This profile is shown in Figure 1.1, for different values of ν. A particular solution of (1.1) is this fixed profile moving to the right with speed s given by (1.8).

It can be seen from Figure 1.1 that the step from u_- to u_+ is steeper when ν is smaller. In fact, the profile given by (1.11) converges pointwise to a discontinuous traveling wave solution of the *inviscid* Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \tag{1.12}$$

when $\nu \to 0^+$. Let us show this. First, consider $\xi > 0$ in (1.11) and take $\nu \to 0^+$. L'Hôpital's rule yields that $u(\xi, \nu) \to u_+$. When $\xi < 0$ and $\nu \to 0^+$, the exponentials vanish and $u(\xi, \nu) \to u_-$. Therefore, when $\nu \to 0^+$, the profile given by (1.11) converges pointwise to the function

$$u(\xi) = \begin{cases} u_-, & \xi < \xi_0, \\ u_+, & \xi > \xi_0. \end{cases} \tag{1.13}$$

To show that (1.13) is indeed a solution of (1.12), we need to consider *weak solutions*, as the function is discontinuous and therefore can only satisfy the equation in a generalized sense.

1.1.2 Weak Solutions

Consider the initial value problem for a scalar conservation law [14]:

$$u_t + f(u)_x = 0, \quad u(x, 0) = u_0(x). \tag{1.14}$$

Suppose that u is a C^1 solution of (1.14). Let C_0^1 be the class of C^1 functions with compact support in $x \in \mathbb{R}$, $t \geq 0$. Let $\phi \in C_0^1$ and let D be a rectangle $0 \leq t \leq T$, $a \leq x \leq b$, such that $\phi = 0$ outside of D, and on the lines $t = T$, $x = a$ and $x = b$. We multiply (1.14) by ϕ and integrate over $t \geq 0$ to get

$$\iint_{t\geq 0} (u_t + f_x)\phi \, dx \, dt = \iint_{D} (u_t + f_x)\phi \, dx \, dt = \int_a^b \int_0^T (u_t + f_x)\phi \, dx \, dt = 0. \tag{1.15}$$

Integration by parts gives

$$\begin{aligned} \int_a^b \int_0^T u_t \phi &= \int_a^b u\phi \Big|_0^T dx - \int_a^b \int_0^T u\phi_t \, dx \, dt \\ &= \int_a^b -u_0(x)\phi(x, 0) \, dx - \int_a^b \int_0^T u\phi_t \, dx \, dt, \end{aligned}$$

and

$$\int_0^T \int_a^b f_x \phi \, dx \, dt = \int_0^T f\phi \Big|_a^b dt - \int_0^T \int_a^b f\phi_x \, dx \, dt.$$

Thus we finally obtain

$$\iint_{t\geq 0} (u\phi_t + f\phi_x) \, dx \, dt + \int_{t=0} u_0 \phi \, dx = 0. \tag{1.16}$$

We have shown that if u is a C^1 solution of (1.14), then (1.16) holds for all $\phi \in C_0^1$. But (1.16) makes sense if u and u_0 are merely bounded and measurable. We thus say that a bounded and measurable function $u(x,t)$ is a *weak solution* of the initial value problem (1.14) with bounded and measurable initial data u_0 if (1.16) holds for all $\phi \in C_0^1$.

Consider now the initial value problem

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \tag{1.17}$$

$$u(x,0) = u_0(x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases} \tag{1.18}$$

A straightforward computation shows that (1.13) with $\xi_0 = 0$ is a weak solution to (1.17), (1.18) (see [14]).

1.2 The Scalar Balance Law

In the previous section we found a discontinuous traveling wave solution to a scalar conservation law. In fact, any traveling wave solution to a scalar conservation law without diffusion must be constant or discontinuous [14]. In contrast with conservation laws, some balance laws do admit continuous traveling wave solutions. A balance law is an equation of the form

$$u_t + f(u)_x = g(u), \quad x \in \mathbb{R}, t \geq 0, f \in C^3, g \in C^2. \tag{1.19}$$

We follow [3] and assume the following about the nonlinear functions f and g:

(G) g possesses exactly three simple zeros $u_- < u_m < u_+$ with $g'(u_-) < 0$, $g'(u_m) > 0$ and $g'(u_+) < 0$.

(F) $f''(u_m) > 0$ and for all $u_l < u_m < u_r$, we have $f'(u_l) < f'(u_m) < f'(u_r)$.

As was done in the previous section, we consider traveling wave solutions $u(x,t)$ such that

$$u(x,t) = \tilde{u}(\xi), \quad \xi = x - st, \tag{1.20}$$

$$\lim_{\xi \to -\infty} \tilde{u}(\xi) = u_-, \quad \lim_{\xi \to +\infty} \tilde{u}(\xi) = u_+. \tag{1.21}$$

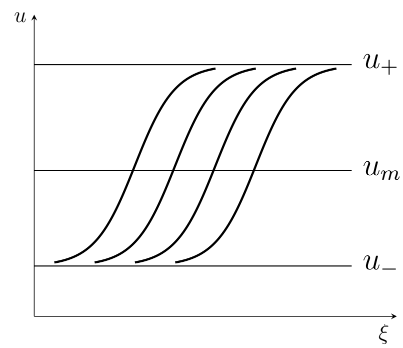


Figure 1.2: Solutions of equation (1.24). Orbits between the horizontal lines $u = u_-$ and $u = u_+$ reach u_- monotonically as $\xi \to -\infty$ and u_+ as $\xi \to +\infty$.

As usual, we drop tildes. Replacing in (1.19), we see that u and s should satisfy the boundary value problem

$$(f'(u) - s)\, u' = g(u), \tag{1.22}$$

$$u(-\infty) = u_-, \quad u(+\infty) = u_+, \tag{1.23}$$

where $'$ denotes the derivative with respect to ξ. For the wave speed $s_0 := f'(u_m)$ the traveling wave equation (1.22) can be put in the form

$$u' = \begin{cases} \frac{g(u)}{f'(u) - f'(u_m)}, & u \neq u_m, \\ \frac{g'(u_m)}{f''(u_m)}, & u = u_m. \end{cases} \tag{1.24}$$

Here we applied L'Hôpital's rule to remove the singularity for $u = u_m$. The wave speed s_0 is the only one for which the singularity of the function $g(u)(f'(u) - s)^{-1}$ can be removed. Using the assumptions (F) and (G) one can obtain a family of solutions of the ordinary differential equation (1.24), which are shown in Figure 1.2; these solutions differ only by a shift along the ξ axis. In particular, we conclude that there exists a monotone heteroclinic orbit from u_- to u_+ and crossing u_m precisely for the wave speed s_0.

The continuous traveling wave with the internal singularity u_m was studied in [3]. In that paper it is shown that this wave admits a viscous profile, and the influence of a viscous term on the wave speed s is analyzed.

1.3 Overview of the Following Chapters

In what follows, we will study traveling wave solutions with an internal singularity in systems of two balance laws. We call this type of waves *singular traveling waves*. It was shown in the previous section that a scalar balance law can have a traveling wave solution whose profile has three singularities: u_-, u_m and u_+. In the case of systems, traveling wave profiles with three singularities are also possible. We will denote the singularities as $\mathbf{w}_-$, $\mathbf{w}_m$ and $\mathbf{w}_+$. The extreme points $\mathbf{w}_-$ and $\mathbf{w}_+$ are equilibria of an associated vector field, but the internal singularity $\mathbf{w}_m$ is characterized as a pseudo-equilibrium of the same vector field. This point is also called a resonant point because the speed of the wave equals one of the characteristic speeds at that point, as will be shown in the next chapter. Orbits can reach the internal singular point $\mathbf{w}_m$ in a finite time because it is not a true equilibrium. A generalization of assumptions (F) and (G) to 2×2 systems yields a classification of this type of waves. In Chapter 3 we study the differentiability of the wave profile. In the case of systems, differentiability of the profile is not immediate as in the case of a scalar equation. In Chapter 4 we develop a perturbation theory for singular traveling wave profiles. This theory is based on Melnikov's method [2], which we generalize to the case of an orbit with an internal singularity. In Chapter 5 we present the model that served as initial motivation for this work. Although the system of balance laws in the model is not 2×2, after some manipulations the results of the previous chapters are applicable to it. In Chapter 6 we present the conclusions of this work.

Chapter 2

Singular Traveling Wave Solutions of Systems of Two Balance Laws

2.1 Systems of Two Balance Laws

We consider a general system of two balance laws of the form

$$\begin{aligned}\frac{\partial u}{\partial t} + \frac{\partial f_1(u,v)}{\partial x} &= g_1(u,v),\\ \frac{\partial v}{\partial t} + \frac{\partial f_2(u,v)}{\partial x} &= g_2(u,v),\end{aligned} \tag{2.1}$$

where u and v are real differentiable functions of a space coordinate $x \in \mathbb{R}$ and time $t \geq 0$. The *flux functions* f_1 and f_2 are of class C^k, and the *source functions* g_1 and g_2 are of class C^{k-1}, where $k \geq 2$. We are particularly interested in finding solutions in the form of traveling waves depending only on the traveling coordinate $\xi = x - st$. A solution $u(x,t), v(x,t)$ is a traveling wave if there exists a *speed* $s \in \mathbb{R}$, a *left state* $(u_-, v_-) \in \mathbb{R}^2$, a *right state* $(u_+, v_+) \in \mathbb{R}^2$ and functions $\tilde{u}$ and $\tilde{v}$ such that

$$u(x,t) = \tilde{u}(\xi), \quad v(x,t) = \tilde{v}(\xi), \quad \xi = x - st, \tag{2.2}$$

$$\lim_{\xi \to \pm\infty} \tilde{u}(\xi) = u_\pm, \quad \lim_{\xi \to \pm\infty} \tilde{v}(\xi) = v_\pm. \tag{2.3}$$

By abuse of notation, we drop tildes from now on. Substituting (2.2) into (2.1) and using $'$ to denote derivatives with respect to ξ, we get the following system of ordinary differential equations

$$\left(-s+\frac{\partial f_1}{\partial u}\right)u'+\frac{\partial f_1}{\partial v}v'=g_1, \tag{2.4}$$

$$\frac{\partial f_2}{\partial u}u'+\left(-s+\frac{\partial f_2}{\partial v}\right)v'=g_2. \tag{2.5}$$

From now on we will use bold letters to denote matrices and vectors. Introducing the column vectors $\mathbf{w}=(u,v)^T$, $\mathbf{g}(\mathbf{w})=(g_1,g_2)^T$, and the 2×2 matrix

$$\mathbf{A}(\mathbf{w},s)=\begin{pmatrix}-s+\dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v}\\ \dfrac{\partial f_2}{\partial u} & -s+\dfrac{\partial f_2}{\partial v}\end{pmatrix}, \tag{2.6}$$

we can express system (2.4), (2.5) as

$$\mathbf{A}(\mathbf{w},s)\mathbf{w}'=\mathbf{g}(\mathbf{w}). \tag{2.7}$$

The limiting state condition (2.3) can thus be written as

$$\lim_{\xi\to\pm\infty}\mathbf{w}(\xi)=\mathbf{w}_{\pm}. \tag{2.8}$$

Condition (2.8) implies that $\mathbf{g}(\mathbf{w}_{\pm})=0$. That is, the left state and the right state must be equilibrium points of (2.7). Therefore, the traveling wave profile is given by a heteroclinic orbit of system (2.7) that connects the equilibrium points $\mathbf{w}_-$ and $\mathbf{w}_+$. Notice that (2.7) is a system of implicit ordinary differential equations.

2.2 The Impasse Surface and Singular Traveling Waves

In the generic case, for fixed s, the plane $\mathbf{w}\in\mathbb{R}^2$ can be divided into a set of open connected regions with boundaries. In the interior of each region, the matrix $\mathbf{A}(\mathbf{w},s)$ is nonsingular, so that (2.7) yields an explicit differential equation for the wave profile. The regions are separated by the curves where $\mathbf{A}(\mathbf{w},s)$ is singular. When the limiting states $\mathbf{w}_{\pm}$ of the wave belong to different regions, the wave profile must contain an intermediate state with a singular matrix $\mathbf{A}(\mathbf{w}, s)$. In some applications, this internal state on the wave is called a *resonant* state, because the condition $\det \mathbf{A} = 0$ for the matrix (2.6) implies that the wave speed s is equal to one of the characteristic speeds, which are given by the eigenvalues of the jacobian $\partial(f, g)/\partial(u, v)$ [8].

The matrix $\mathbf{A}(\mathbf{w}, s)$ is singular on the set

$$\Sigma_s = \left\{\mathbf{w} \in \mathbb{R}^2 \,;\, \det \mathbf{A}(\mathbf{w}, s) = 0\right\}, \tag{2.9}$$

which represents a smooth curve on the plane $\mathbf{w} \in \mathbb{R}^2$, possibly with singularities. Rabier [12] studied solutions in a neighbourhood of Σ_s and showed that they typically reach Σ_s for finite (forward or backward) values of ξ and cannot be continued, because Σ_s attracts or repels the solution from both sides. For this reason Σ_s is often referred to as the impasse surface [4]. However, there may be parts of Σ_s where it is possible to cross from one side to the other.

To identify points on Σ_s where crossing is possible, we will introduce a change of variables that transforms the problem into an explicit differential equation. To this end, one uses the *adjugate matrix* $\operatorname{adj} \mathbf{A}(\mathbf{w}, s)$, also called the *classical adjoint*. In our case,

$$\operatorname{adj} \mathbf{A}(\mathbf{w}, s) = \begin{pmatrix} -s + \dfrac{\partial f_2}{\partial v} & -\dfrac{\partial f_1}{\partial v} \\ -\dfrac{\partial f_2}{\partial u} & -s + \dfrac{\partial f_1}{\partial u} \end{pmatrix}. \tag{2.10}$$

The 2×2 matrix $\operatorname{adj} \mathbf{A}(\mathbf{w}, s)$ depends on $\mathbf{w}$ and s as smoothly as $\mathbf{A}(\mathbf{w}, s)$ and satisfies the identity

$$\mathbf{A}(\mathbf{w}, s)\left(\operatorname{adj} \mathbf{A}(\mathbf{w}, s)\right) = \left(\operatorname{adj} \mathbf{A}(\mathbf{w}, s)\right)\mathbf{A}(\mathbf{w}, s) = \det \mathbf{A}(\mathbf{w}, s)\mathbf{I}. \tag{2.11}$$

Multiplying (2.7) by $\operatorname{adj} \mathbf{A}(\mathbf{w}, s)$ from the left, we get the equation

$$\det \mathbf{A}(\mathbf{w}, s)\mathbf{w}' = \operatorname{adj} \mathbf{A}(\mathbf{w}, s)\mathbf{g}(\mathbf{w}). \tag{2.12}$$

We introduce a new variable τ, which is related to ξ by the equation

$$\frac{d\xi}{d\tau} = \det \mathbf{A}(\mathbf{w}, s). \tag{2.13}$$

Using (2.13) in (2.12) we get the *desingularized system*

$$\dot{\mathbf{w}} = \operatorname{adj} \mathbf{A}(\mathbf{w}, s)\mathbf{g}(\mathbf{w}), \tag{2.14}$$

where the dot denotes the derivative with respect to τ. Outside of Σ_s, orbits of (2.7) are in one-to-one correspondence with those of (2.14). However, the direction of the orbits is reversed in the region of the plane $\mathbf{w} \in \mathbb{R}^2$ where $\det \mathbf{A}(\mathbf{w}, s) < 0$.

In addition to the equilibria of (2.7), given by $\mathbf{g}(\mathbf{w}) = 0$, equation (2.14) may possess additional fixed points. Since they are not equilibria of the original system, they are called *pseudo-equilibria*. By definition, if $\mathbf{w} \in \mathbb{R}^2$ is a pseudo-equilibrium, $\operatorname{adj} \mathbf{A}(\mathbf{w}, s)\mathbf{g}(\mathbf{w}) = 0$ and $\mathbf{g}(\mathbf{w}) \neq 0$. This implies that $\operatorname{adj} \mathbf{A}(\mathbf{w}, s)$ is singular, and by (2.11), so is $\mathbf{A}(\mathbf{w}, s)$. In other words, pseudo-equilibria belong to the impasse surface Σ_s.

In the case in which the orbit reaches the impasse surface at a point $\mathbf{w}_0$ that is not a pseudo-equilibrium, it can be proven that the orbit does not cross the impasse surface. In fact, in this case the point $\mathbf{w}_0$ is a *standard singular point* of equation (2.7), as defined in [12]. In that paper it is shown that these points are either *attracting* or *repelling*. Solutions through a standard singular point either *reach* or *go out of* Σ_s on both sides, but they do not cross the surface.

On the other hand, solutions that reach a pseudo-equilibrium $\mathbf{w}_m$ can cross Σ_s. The rescaling (2.13) is singular at the impasse surface Σ_s, so trajectories of (2.14) that reach $\mathbf{w}_m$ at an infinite value of τ, correspond to solutions of (2.7) that reach $\mathbf{w}_m$ for a finite value of ξ [4]. A solution of (2.7) may therefore consist of a concatenation of orbits of (2.14) on both sides, which allows crossing the impasse surface Σ_s through a pseudo-equilibrium. An orbit $\mathbf{w}(\xi)$ of (2.7) that crosses the impasse surface is represented schematically in Figure 2.1a. The orbits depicted in Figure 2.1b correspond to two solutions of the desingularized system (2.14), $\mathbf{w}_1(\tau)$ and $\mathbf{w}_2(\tau)$, obtained after the change of variables $\xi \mapsto \tau$. The orbit $\mathbf{w}_1(\tau)$ is on the region where $\det \mathbf{A} > 0$ and follows the same path as the corresponding part of $\mathbf{w}(\xi)$, while $\mathbf{w}_2(\tau)$ is on the region $\det \mathbf{A} < 0$, follows the same path as the correspondig part of $\mathbf{w}(\xi)$, but with the opposite direction, see (2.13).

2.3 Classification of Singular Traveling Waves

We limit our study to the case in which the singular traveling wave profile crosses the impasse surface once, but our ideas may be extended to the case of multiple intersections with Σ_s. The profile contains three singularities: the initial point $\mathbf{w}_-$, the final point $\mathbf{w}_+$, and the point on Σ_s. We denote the

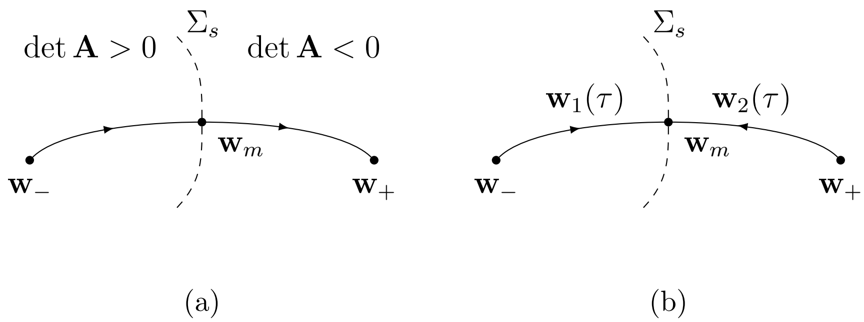


Figure 2.1: Solutions that pass through a pseudo-equilibrium. (a) Solution $\mathbf{w}(\xi)$ of system (2.7). (b) Solutions $\mathbf{w}_1(\tau)$ and $\mathbf{w}_2(\tau)$ of the desingularized system (2.14). The orbit direction is reversed when $\det \mathbf{A} < 0$, see (2.13).

point on the impasse surface by $\mathbf{w}_m$. Note that $\mathbf{w}_-$ and $\mathbf{w}_+$ are equilibria of (2.7), and $\mathbf{w}_m$ is a pseudo-equilibrium, as defined in the previous section.

We suppose, without loss of generality, that $\det \mathbf{A}(\mathbf{w}_-, s) > 0$ and that $\det \mathbf{A}(\mathbf{w}_+, s) < 0$, that is, the orbit starts in the region $\det \mathbf{A} > 0$ and ends in the region $\det \mathbf{A} < 0$. In fact, the other possible case, namely, $\det \mathbf{A}(\mathbf{w}_-, s) < 0$ and $\det \mathbf{A}(\mathbf{w}_+, s) > 0$, can be reduced to the first case by the change of coordinate $x \mapsto -x$ in the original equations (2.1). This change is equivalent to making $f_1 \mapsto -f_1$, $f_2 \mapsto -f_2$ in (2.1). Considering the corresponding change of speed $s \mapsto -s$, we obtain that the matrix in (2.6) changes as $\mathbf{A} \mapsto -\mathbf{A}$, so the vector field in (2.7) changes sign. As matrix $\mathbf{A}$ is 2×2, the determinant does not change, but the initial point now becomes $\mathbf{w}_+$, and the final point $\mathbf{w}_-$. Thus the orbit in the new system starts on the region $\det \mathbf{A} > 0$ and ends on $\det \mathbf{A} < 0$.

We consider the generic case in which the Jacobian matrix of the vector field defined by the right-hand side of (2.14) has eigenvalues with non-zero real part at the fixed points $\mathbf{w}_-$, $\mathbf{w}_m$ and $\mathbf{w}_+$. As the system is on the plane, each fixed point can be of one of the following types: source, sink or saddle [5]. Since $\mathbf{w}_-$ is the initial point of the profile, it cannot be a sink. Therefore, we have two possible types for $\mathbf{w}_-$, source or saddle. The orbit of the desingularized system has to reach $\mathbf{w}_+$ as $\tau \to -\infty$, as it is shown in Figure 2.1b. Thus, $\mathbf{w}_+$ can be either a saddle or a source too. The two orbits that reach $\mathbf{w}_m$ from both sides do so as $\tau \to +\infty$, as it is shown in Figure 2.1b. From this, we infer that $\mathbf{w}_m$ can be either a saddle or a sink, but not a source. Combining this, we obtain eight possible cases in total, which are

	$\mathbf{w}_-$	$\mathbf{w}_m$	$\mathbf{w}_+$	Observation
1	Source	Saddle	Source	More conditions needed
2	Source	Saddle	Saddle	
3	Source	Sink	Source	More conditions needed
4	Source	Sink	Saddle	More conditions needed
5	Saddle	Saddle	Source	
6	Saddle	Saddle	Saddle	Higher codimension
7	Saddle	Sink	Source	More conditions needed
8	Saddle	Sink	Saddle	More conditions needed

Table 2.1: Possible stability types of the singularities, assuming that $\mathbf{w}_-$ is inside the region where $\det \mathbf{A}(\mathbf{w}, s) > 0$ and $\mathbf{w}_+$ is inside the region where $\det \mathbf{A}(\mathbf{w}, s) < 0$.

listed in Table 2.1.

Consider the first case in the table, in which $\mathbf{w}_-$ is a source, $\mathbf{w}_m$ is a saddle and $\mathbf{w}_+$ is a source. The orbits of the desingularized system (2.14) are represented schematically in Figure 2.2. Recall that the system of differential equations (2.14) has the wave speed s as a parameter. The phase portrait shown in Figure 2.2 does not change qualitatively if s is perturbed slightly. In other words, a traveling wave exists for any speed s belonging to an interval. Thus, we have an infinite set of traveling wave solutions parametrized by the wave speed s. Extra conditions, which may be mathematical (e.g, stability) or physical (e.g., a viscous regularization procedure), are necessary in this case in order to select a single solution with a specific value of the speed s. The same observations are valid for cases 3, 4, 7 and 8 from Table 2.1. In all these cases, finding a traveling wave as an orbit does not provide a unique solution with the unique wave speed s, but instead yields an infinite set of possible wave profiles and speeds. For this reason, we do not consider these cases in this work, leaving them for a future study, and focus on the remaining cases 2, 5 and 6.

Consider now the case 6, in which all singularities are of saddle type. If we introduce a small perturbation in the system (e.g., a small change of the flux and source functions), both the connections from $\mathbf{w}_-$ to $\mathbf{w}_m$ and from $\mathbf{w}_+$ to $\mathbf{w}_m$ are destroyed in the generic case. One of these connections can be restored by adjusting the parameter s, but not the two simultaneously [2]. In other words, in the generic case this type of connection is not possible.

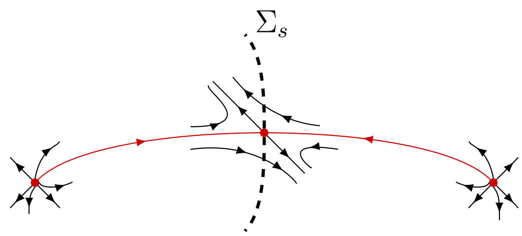


Figure 2.2: Profile of a singular traveling wave with fixed points of type 1, as listed in Table 2.1. $\mathbf{w}_-$ is a source, $\mathbf{w}_m$ a saddle and $\mathbf{w}_+$ a source.

The cases that remain are 2 and 5. In these cases, a small perturbation destroys either the connection from $\mathbf{w}_-$ to $\mathbf{w}_m$ or from $\mathbf{w}_+$ to $\mathbf{w}_m$, but it can be restored by adjusting the parameter s. This yields a structurally stable and unique (at least locally) traveling wave solution. We focus on these two cases from now on.

Chapter 3

Regularity of the Singular Traveling Wave Profile

In this chapter we prove some results about the regularity of the singular traveling wave profile. First, we prove the regularity of the profile as a geometric curve. We show that there is a smooth local parametrization in a neighborhood of the singularity. This will be a direct consequence of the Stable Manifold Theorem [2]. Then, we turn our attention to the differentiability of the curve with the original parametrization in ξ. We show that the profile $\mathbf{w}(\xi)$ is smooth given some non-degeneracy conditions. We also deduce explicit formulas for the derivative at the singular point. In this chapter we omit the parameter s in all expressions, as we consider only a fixed speed and a corresponding traveling wave.

3.1 Smooth Parametrization of the Profile

Suppose that a solution to the system (2.7) exists, connecting the initial point $\mathbf{w}_-$ to the final point $\mathbf{w}_+$, with internal singularity $\mathbf{w}_m$ on the impasse curve Σ. We suppose that $\mathbf{w}_m$ is a regular point of Σ, that is, that $(\det \mathbf{A})'(\mathbf{w}_m) \neq 0$. As a consequence, Σ is a regular curve near $\mathbf{w}_m$. We also suppose that $\det \mathbf{A}(\mathbf{w}_-) > 0$ and $\det \mathbf{A}(\mathbf{w}_+) < 0$, as justified in Section 2.3. The profile is represented in Figure 3.1a. After the change of variables $\xi \mapsto \tau$ given by (2.13), we obtain two solutions of the desingularized system (2.14), $\mathbf{w}_1(\tau)$ and $\mathbf{w}_2(\tau)$. The orbit in the region where $\det \mathbf{A} < 0$, $\mathbf{w}_2(\tau)$, reverses direction, because increasing ξ corresponds to decreasing τ, see (2.13). Otherwise,

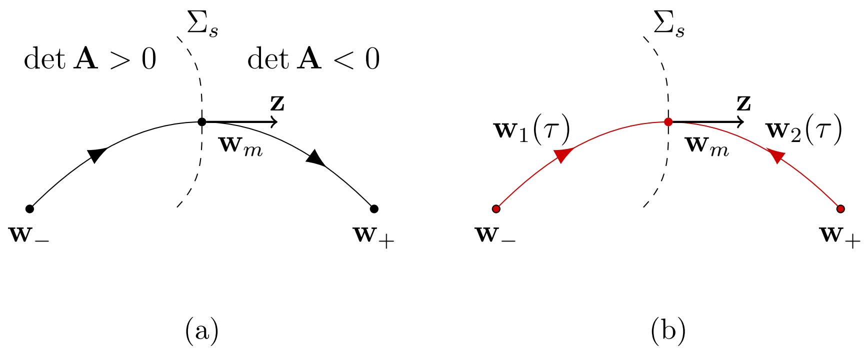


Figure 3.1: Profile of the singular traveling wave. (a) Solution $\mathbf{w}(\xi)$ of system (2.7). (b) Solutions $\mathbf{w}_1(\tau)$ and $\mathbf{w}_2(\tau)$ of the desingularized system (2.14). The orbit direction is reversed when $\det \mathbf{A} < 0$, see (2.13).

$\mathbf{w}_1(\tau)$ and $\mathbf{w}_2(\tau)$ follow the same path of the corresponding part of $\mathbf{w}(\xi)$. This is shown schematically in Figure 3.1b.

As $\mathbf{w}_1(\tau) \to \mathbf{w}_m$ and $\mathbf{w}_2(\tau) \to \mathbf{w}_m$ as $\tau \to +\infty$, the profile is contained in the stable manifold of $\mathbf{w}_m$, which we will denote as W. As $\mathbf{w}_m$ is a hyperbolic saddle (we consider cases 2 and 5 from the classification given in Section 2.3), the Stable Manifold Theorem implies that W is one-dimensional and smooth. We consider the generic case in which W is transversal to the impasse curve Σ. As the profile is contained in W, it also is smooth. To be more precise, there is a parametrization $\tilde{\mathbf{w}}(\eta)$ of the profile around the singularity $\mathbf{w}_m$. This parametrization is as smooth as the vector field defined by the right-hand side of (2.14). Supposing the flux functions f_1 and f_2 are of class C^k, and the source functions are of class C^{k-1}, with $k \geq 2$, the vector field is of class C^{k-1}, as it can be seen from (2.14). Hence, the profile is a curve of class C^k.

Note that the previous discussion does not consider the regularity of the profile with the original parametrization in ξ. This is the focus of the next section.

3.2 Parametrization in the Traveling Coordinate

Consider the profile of the traveling wave as a curve $\mathbf{w} = \mathbf{w}(\xi)$ parametrized by the traveling coordinate $\xi = x - st$. The parametrization $\mathbf{w}(\xi)$ is smooth for $\xi \neq 0$, away from the impasse surface, because $d\mathbf{w}/d\xi$ is given by (2.12) with a nonsingular matrix $\mathbf{A}(\mathbf{w})$. In this section we will show that this parametrization is smooth for all ξ.

In the previous section we showed that there is a smooth parametrization $\mathbf{w} = \tilde{\mathbf{w}}(\eta)$ of the profile. We can suppose that the parametrizations $\mathbf{w}$ and $\tilde{\mathbf{w}}$ are in one to one correspondence and that $\mathbf{w}(0) = \tilde{\mathbf{w}}(0) = \mathbf{w}_m$. For $\xi \neq 0$, we have that

$$\frac{d\mathbf{w}}{d\xi}(\xi) = \frac{d\tilde{\mathbf{w}}}{d\eta}(\eta)\frac{d\eta}{d\xi}(\xi). \tag{3.1}$$

Multiplying equation (3.1) by $\mathbf{A}(\mathbf{w}(\xi))$ and using equation (2.7), we obtain

$$\frac{d\eta}{d\xi}(\xi)\mathbf{A}(\mathbf{w}(\xi))\frac{d\tilde{\mathbf{w}}}{d\eta}(\eta) = \mathbf{g}(\mathbf{w}(\xi)). \tag{3.2}$$

Because the parametrizations are in one to one correspondence, we can replace $\mathbf{w}(\xi)$ by $\tilde{\mathbf{w}}(\eta)$ in (3.2) to obtain

$$\frac{d\eta}{d\xi}(\xi)\mathbf{A}(\tilde{\mathbf{w}}(\eta))\frac{d\tilde{\mathbf{w}}}{d\eta}(\eta) = \mathbf{g}(\tilde{\mathbf{w}}(\eta)). \tag{3.3}$$

Multiplying both sides of (3.3) by a non-zero row vector $\mathbf{h}^T$ we obtain

$$\frac{d\eta}{d\xi}(\xi)\mathbf{h}^T\mathbf{A}(\tilde{\mathbf{w}}(\eta))\frac{d\tilde{\mathbf{w}}}{d\eta}(\eta) = \mathbf{h}^T\mathbf{g}(\tilde{\mathbf{w}}(\eta)). \tag{3.4}$$

We can write the previous equation as

$$\frac{d\eta}{d\xi}(\xi) = \frac{\mathbf{h}^T\mathbf{g}(\tilde{\mathbf{w}}(\eta))}{\mathbf{h}^T\mathbf{A}(\tilde{\mathbf{w}}(\eta))\frac{d\tilde{\mathbf{w}}}{d\eta}(\eta)}, \tag{3.5}$$

provided that the denominator in the right-hand side does not vanish. We write

$$\frac{d\tilde{\mathbf{w}}}{d\eta}(0) = \mathbf{z}, \tag{3.6}$$

where $\mathbf{z}$ is a vector tangent to W, the stable manifold of $\mathbf{w}_m$. The vector $\mathbf{z}$ is represented schematically in Figure 3.1. We assume that the kernel of $\mathbf{A}(\mathbf{w}_m)$ is one-dimensional and transversal to W. This implies that

$$\mathbf{A}(\mathbf{w}_m)\mathbf{z} \neq 0. \tag{3.7}$$

We choose the vector

$$\mathbf{h} = \mathbf{A}(\mathbf{w}_m)\mathbf{z}. \tag{3.8}$$

For $\xi = 0$ and $\mathbf{h}$ given by (3.8), the denominator of the right-hand side of (3.5) becomes

$$(\mathbf{A}(\mathbf{w}_m)\mathbf{z})^T\mathbf{A}(\mathbf{w}_m)\mathbf{z} = \|\mathbf{A}(\mathbf{w}_m)\mathbf{z}\|^2 \neq 0. \tag{3.9}$$

As $\mathbf{A}(\mathbf{w}(\xi))\frac{d\tilde{\mathbf{w}}}{d\eta}(\eta)$ is a continuous function of ξ, the denominator of the right-hand side of (3.5) does not vanish for ξ in a neighborhood of zero. With $\mathbf{h}$ given by (3.8), equation (3.5) becomes

$$\frac{d\eta}{d\xi}(\xi) = \frac{(\mathbf{A}(\mathbf{w}_m)\mathbf{z})^T\mathbf{g}(\tilde{\mathbf{w}}(\eta))}{(\mathbf{A}(\tilde{\mathbf{w}}(\eta))\mathbf{z})^T\mathbf{A}(\tilde{\mathbf{w}}(\eta))\frac{d\tilde{\mathbf{w}}}{d\eta}(\eta)}. \tag{3.10}$$

Expression (3.10) is valid in a neighborhood of $\xi = 0$. As all functions on the righ-hand side are of at least class C^{k-1}, this shows that $\eta(\xi)$ is of class C^k. Replacing (3.10) in (3.1) we obtain the formula

$$\frac{d\mathbf{w}}{d\xi}(\xi) = \frac{(\mathbf{A}(\mathbf{w}_m)\mathbf{z})^T\mathbf{g}(\tilde{\mathbf{w}}(\eta))}{(\mathbf{A}(\mathbf{w}_m)\mathbf{z})^T\mathbf{A}(\tilde{\mathbf{w}}(\eta))\frac{d\tilde{\mathbf{w}}}{d\eta}(\eta)}\frac{d\tilde{\mathbf{w}}}{d\eta}(\eta). \tag{3.11}$$

The right-hand side of (3.11) is of class C^{k-1}, so this implies that $\mathbf{w}(\xi)$ is C^k. Replacing $\xi = 0$ in (3.11) we obtain the formula

$$\frac{d\mathbf{w}}{d\xi}(0) = \frac{(\mathbf{A}(\mathbf{w}_m)\mathbf{z})^T\mathbf{g}(\mathbf{w}_m)}{||\mathbf{A}(\mathbf{w}_m)\mathbf{z}||^2}\mathbf{z}. \tag{3.12}$$

Notice that $\mathbf{z}$ on the right-hand side of (3.12) can be replaced by any multiple of $\mathbf{z}$.

3.3 Tangent Vector to the Profile at the Singular Point

In this section we obtain an alternative formula for the derivative of the profile at the singular point, see (3.12). Droping the dependence on s, equation (2.7) can be written as

$$\mathbf{A}(\mathbf{w})\mathbf{w}' = \mathbf{g}(\mathbf{w}), \tag{3.13}$$

where $\mathbf{w} = (u, v)^T$ and $\mathbf{g} = (g_1, g_2)^T$. Outside of the singular point, we can invert matrix $\mathbf{A}(\mathbf{w})$, and if we consider each component of the vector equation, we obtain the following scalar equations

$$u'(\xi) = \frac{G_1(u(\xi), v(\xi))}{\det \mathbf{A}(u, v)}, \tag{3.14}$$

$$v'(\xi) = \frac{G_2(u(\xi), v(\xi))}{\det \mathbf{A}(u, v)}, \tag{3.15}$$

where

$$G_1(u, v) = \left(\frac{\partial f_2}{\partial v}(u, v) - s\right) g_1(u, v) - \frac{\partial f_1}{\partial v}(u, v) g_2(u, v), \tag{3.16}$$

$$G_2(u, v) = -\frac{\partial f_2}{\partial u}(u, v) g_1(u, v) + \left(\frac{\partial f_1}{\partial u}(u, v) - s\right) g_2(u, v). \tag{3.17}$$

The desingularized system (2.14) can be written as

$$\dot{u}(\tau) = G_1(u(\tau), v(\tau)), \tag{3.18}$$

$$\dot{v}(\tau) = G_2(u(\tau), v(\tau)). \tag{3.19}$$

The functions $G_1(u, v)$ and $G_2(u, v)$ vanish at the singular point $\mathbf{w}_m = (u_m, v_m)^T$ at $\xi = \xi_m$, because $\mathbf{w}_m$ is a pseudo-equilibrium, see Section 2.2. We want to find the derivative $\mathbf{w}'(\xi_m) = (u'(\xi_m), v'(\xi_m))^T$ of the profile at the singular point, which cannot be calculated with (3.14) and (3.15) because both the numerator and denominator vanish. However, $\mathbf{w}'(\xi_m)$ exists because of what has been discussed in the previous section. In fact, we have that $\mathbf{w}'(\xi_m) = \lim_{\xi \to \xi_m} \mathbf{w}'(\xi)$. To evaluate this limit, we apply L'Hôpital's rule to equations (3.14) and (3.15) to obtain that

$$u'(\xi_m) = \frac{\dfrac{\partial G_1}{\partial u} u'(\xi_m) + \dfrac{\partial G_1}{\partial v} v'(\xi_m)}{\dfrac{\partial \det \mathbf{A}}{\partial u} u'(\xi_m) + \dfrac{\partial \det \mathbf{A}}{\partial v} v'(\xi_m)}, \tag{3.20}$$

$$v'(\xi_m) = \frac{\dfrac{\partial G_2}{\partial u}u'(\xi_m) + \dfrac{\partial G_2}{\partial v}v'(\xi_m)}{\dfrac{\partial \det \mathbf{A}}{\partial u}u'(\xi_m) + \dfrac{\partial \det \mathbf{A}}{\partial v}v'(\xi_m)}, \tag{3.21}$$

where derivatives are evaluated at $\mathbf{w}_m$. We define the vector

$$\mathbf{n} = \begin{pmatrix} \frac{\partial \det \mathbf{A}}{\partial u} \\ \frac{\partial \det \mathbf{A}}{\partial v} \end{pmatrix}_{\mathbf{w}=\mathbf{w}_m}, \tag{3.22}$$

and introduce the following notation for the Jacobian matrix of the vector field defined by the desingularized system (3.18), (3.19),

$$\mathbf{J} = \begin{pmatrix} \frac{\partial G_1}{\partial u} & \frac{\partial G_1}{\partial v} \\ \frac{\partial G_2}{\partial u} & \frac{\partial G_2}{\partial v} \end{pmatrix}_{\mathbf{w}=\mathbf{w}_m}. \tag{3.23}$$

Equations (3.20) and (3.21) can be written as

$$\mathbf{w}'(\xi_m) = \frac{\mathbf{J}\mathbf{w}'(\xi_m)}{\mathbf{n}^T\mathbf{w}'(\xi_m)}. \tag{3.24}$$

Note that $\mathbf{n}$ is normal to the impasse surface at the singular point. Recall also that $\mathbf{J}$ has a negative eigenvalue and a positive eigenvalue, becuase $\mathbf{w}_m$ is a saddle point. As was shown in the previous section, the profile is contained in the stable manifold of $\mathbf{w}_m$, W. Therefore, the tangent to the profile at the singular point must have the direction of an eigenvector of $\mathbf{J}$ corresponding to its negative eigenvalue. Let $\mathbf{z}$ be any such eigenvector. By the assumption of transversality, we have that

$$\mathbf{n}^T\mathbf{z} \neq 0. \tag{3.25}$$

This implies $\mathbf{n}^T\mathbf{w}'(\xi_m) \neq 0$. To solve equation (3.24), take $\mathbf{w}'(\xi_m) = \alpha\mathbf{z}$ on the right-hand side to obtain

$$\mathbf{w}'(\xi_m) = \frac{\mathbf{J}(\alpha\mathbf{z})}{\mathbf{n}^T(\alpha\mathbf{z})} = \frac{\alpha\mathbf{J}\mathbf{z}}{\alpha\mathbf{n}^T\mathbf{z}} = \frac{\lambda\mathbf{z}}{\mathbf{n}^T\mathbf{z}}, \tag{3.26}$$

where $\lambda < 0$ is the negative eigenvalue of $\mathbf{J}$. This shows that the tangent $\mathbf{w}'(\xi_m)$ to the profile at the singular point can be computed given the normal $\mathbf{n}$, the negative eigenvalue of $\mathbf{J}$ and a corresponding eigenvector. Note that the arguments presented only relied on the fact that $\mathbf{w}_m$ is a saddle point, so the conclusions are valid for the two cases considered at the end of Section 2.3. Comparing (3.26) with (3.12) we see that we have two expressions for the tangent vector at the singularity:

$$\mathbf{w}'(\xi_m) = \frac{\lambda \mathbf{z}}{\mathbf{n}^T \mathbf{z}} = \frac{(\mathbf{A}(\mathbf{w}_m)\mathbf{z})^T \mathbf{g}(\mathbf{w}_m)}{||\mathbf{A}(\mathbf{w}_m)\mathbf{z}||^2}\mathbf{z}. \tag{3.27}$$

3.4 Regularity

In this section we collect all the results in the previous sections of this chapter. First we showed that the profile is a smooth curve applying the Stable Manifold Theorem. We showed that the parametrization of the curve with the traveling coordinate $\xi = x - st$ is smooth given some transversality conditions. We then deduced two expressions for the tangent to the profile curve at the singular point $\mathbf{w}_m$. These results are sumarized in the following theorem.

Theorem 1. *Suppose that the flux functions f_1, f_2 are of class C^k and the source functions g_1, g_2 of class C^{k-1}, with $k \geq 2$. Let $\mathbf{w}(\xi)$ be the profile of a singular traveling wave of type 2 or 5, as classified in Section 2.3, with internal point $\mathbf{w}_m \in \Sigma$. Suppose $\mathbf{w}_m$ is a saddle equilibrium of the desingularized vector field (3.18), (3.19), and let W be its stable manifold. Let $\mathbf{z}$ be an eigenvector of the matrix $\mathbf{J}$ defined by (3.23), corresponding to its negative eigenvalue $\lambda < 0$. If W is transversal both to the impasse curve Σ and to the kernel of $\mathbf{A}(\mathbf{w}_m)$, then the profile $\mathbf{w}(\xi)$ is of class C^k, and the derivative of the profile at the singularity is given by the formulas*

$$\mathbf{w}'(\xi_m) = \frac{(\mathbf{A}(\mathbf{w}_m)\mathbf{z})^T \mathbf{g}(\mathbf{w}_m)}{||\mathbf{A}(\mathbf{w}_m)\mathbf{z}||^2}\mathbf{z} = \frac{\lambda \mathbf{z}}{\mathbf{n}^T \mathbf{z}}. \tag{3.28}$$

A different proof of the regularity of a singular traveling wave profile is given in [10]. The proof given in that paper does not require W to be transversal to the kernel of $\mathbf{A}(\mathbf{w}_m)$. The regularity of the profile can also be obtained as a consequence of normal form theory. For example, a classification of local phase portraits for systems in the form of (2.7) is given in [16]. An advantage of our approach is that it yields explicit formulas for the derivative of the profile.

Chapter 4

Perturbation Theory for Singular Traveling Wave Profiles

In this chapter we develop a perturbation theory for singular traveling waves in 2×2 systems of inviscid balance laws. Our technique is based on Melnikov's method [2]. Similar techniques were considered in [6, 9] in the context of traveling waves in viscous conservation laws.

4.1 The Perturbed System

Consider a perturbed version of the system of balance laws (2.1)

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial \bar{f}_1(u,v,\varepsilon)}{\partial x} &= \bar{g}_1(u,v,\varepsilon), \\ \frac{\partial v}{\partial t} + \frac{\partial \bar{f}_2(u,v,\varepsilon)}{\partial x} &= \bar{g}_2(u,v,\varepsilon), \end{aligned} \tag{4.1}$$

where $\varepsilon \in \mathbb{R}$ is a perturbation parameter, the perturbed flux functions $\bar{f}_1$ and $\bar{f}_2$ are of class C^k with respect to (u,v) and C^1 with respect to ε, the perturbed source functions $\bar{g}_1$ and $\bar{g}_2$ are of class C^{k-1} with respecto to (u,v) and C^1 with respect to ε, with $k \geq 2$, and satisfy

$$\begin{aligned} \bar{f}_1(u,v,0) = f_1(u,v), \quad \bar{f}_2(u,v,0) = f_2(u,v), \\ \bar{g}_1(u,v,0) = g_1(u,v), \quad \bar{g}_2(u,v,0) = g_2(u,v). \end{aligned} \tag{4.2}$$

Let $\mathbf{w}(\xi)$ be a singular traveling wave solution to the unperturbed system (2.1), connecting $\mathbf{w}_-$ to $\mathbf{w}_+$, with internal singularity $\mathbf{w}_m \in \Sigma_s$, and with speed s_0. Suppose that $\mathbf{w}(\xi)$ is of either type 2 or 5, as classified in Section 2.3. We will show that this traveling wave solution persists as a solution $\mathbf{w}(\xi, \varepsilon)$ to the perturbed system (4.1) for small values of ε, supposing a non-degeneracy condition is met. We will also obtain first order perturbation formulas for the profile, the limiting states and the speed s of the perturbed wave.

The perturbed form of matrix (2.6) is

$$\bar{\mathbf{A}}(\mathbf{w}, s, \varepsilon) = \begin{pmatrix} -s + \dfrac{\partial \bar{f}_1}{\partial u}(u, v, \varepsilon) & \dfrac{\partial \bar{f}_1}{\partial v}(u, v, \varepsilon) \\ \dfrac{\partial \bar{f}_2}{\partial u}(u, v, \varepsilon) & -s + \dfrac{\partial \bar{f}_2}{\partial v}(u, v, \varepsilon) \end{pmatrix}, \tag{4.3}$$

and the corresponding traveling wave equation (2.7) becomes

$$\bar{\mathbf{A}}(\mathbf{w}, s, \varepsilon)\mathbf{w}' = \bar{\mathbf{g}}(\mathbf{w}, \varepsilon), \tag{4.4}$$

where $\bar{\mathbf{g}}(\mathbf{w}, \varepsilon) = (\bar{g}_1(\mathbf{w}, \varepsilon), \bar{g}_2(\mathbf{w}, \varepsilon))^T$. The corresponding desingularized system (2.14) becomes

$$\dot{\mathbf{w}} = \operatorname{adj} \bar{\mathbf{A}}(\mathbf{w}, s, \varepsilon)\bar{\mathbf{g}}(\mathbf{w}, \varepsilon). \tag{4.5}$$

We now prove results concerning existence and uniqueness of equilibria and pseudo-equilibria of the perturbed system.

Proposition 1. *Let $\mathbf{w}_\pm \in \mathbb{R}^2$ be a nondegenerate equilibrium (i.e., source, saddle or sink) of the unperturbed system (2.7) for $s = s_0$. Then, for $|\varepsilon|$ sufficiently small and s sufficiently close to s_0, there is a unique equilibrium $\bar{\mathbf{w}}_\pm(s, \varepsilon)$ of the perturbed system (4.4) such that $\bar{\mathbf{w}}_\pm(s_0, 0) = \mathbf{w}_\pm$. Furthermore, $\bar{\mathbf{w}}_\pm$ has the same stability type as $\mathbf{w}_\pm$.*

Proof. In a neighborhood of $\mathbf{w}_\pm$ we can write

$$\mathbf{w}' = \bar{\mathbf{A}}^{-1}(\mathbf{w}, s, \varepsilon)\bar{\mathbf{g}}(\mathbf{w}, \varepsilon). \tag{4.6}$$

The right side of equation (4.6) vanishes at $(\mathbf{w}, s, \varepsilon) = (\mathbf{w}_\pm, s_0, 0)$ and its Jacobian matrix with respect to the $\mathbf{w}$ variable is invertible. By the implicit function theorem, there is a unique equilibrium $\bar{\mathbf{w}}_\pm(s, \varepsilon)$ in a neighborhood of $(s_0, 0)$ with $\bar{\mathbf{w}}_\pm(s_0, 0) = \mathbf{w}_\pm$. By continuity of the derivative, this equilibrium has the same stability type as $\mathbf{w}_\pm$. □

Proposition 2. *Let $\mathbf{w}_m \in \mathbb{R}^2$ be a saddle pseudo-equilibrium of the system (4.4) for $s = s_0$. Then, for $|\varepsilon|$ sufficiently small, and for s sufficiently close to s_0, there is a unique pseudo-equilibrium $\bar{\mathbf{w}}_m(s, \varepsilon)$ of the perturbed desingularized system (4.4) such that $\bar{\mathbf{w}}_m(s_0, 0) = \mathbf{w}_m$. Furthermore, $\bar{\mathbf{w}}_m$ has the same stability type as $\mathbf{w}_m$.*

Proof. We have that $\operatorname{adj} \bar{\mathbf{A}}(\mathbf{w}_m, s_0, 0)\bar{g}(\mathbf{w}_m, 0) = 0$ and $\bar{\mathbf{g}}(\mathbf{w}_m, 0) \neq 0$, by definition of pseudo-equilibrium. The right side of (4.5) vanishes for $(\mathbf{w}, s, \varepsilon) = (\mathbf{w}_m, s_0, 0)$, and its derivative with respect to $\mathbf{w}$ is invertible because we assume that $\mathbf{w}_m$ is nondegenerate. The implicit function theorem guarantees the existence of a point $\bar{\mathbf{w}}_m(s, \varepsilon)$ on which the right side of (4.5) vanishes. By continuity, $\bar{\mathbf{g}}$ does not vanish at this point, so $\bar{\mathbf{w}}_m(s, \varepsilon)$ is a pseudo-equilibrium. The claim about the stability type follows from the continuity of the derivative. □

Figure 4.1 depicts the situation given by propositions 1 and 2, in the case $\mathbf{w}(\xi)$ is of type 2. We see that for the perturbed system (4.4) the singularities and their stability types are preserved, but the heteroclinic connection between them is not neccesarily preserved. In the case the wave is of type 5, the saddle-saddle connection is not guaranteed to be preserved between $\mathbf{w}_-$ and $\mathbf{w}_m$.

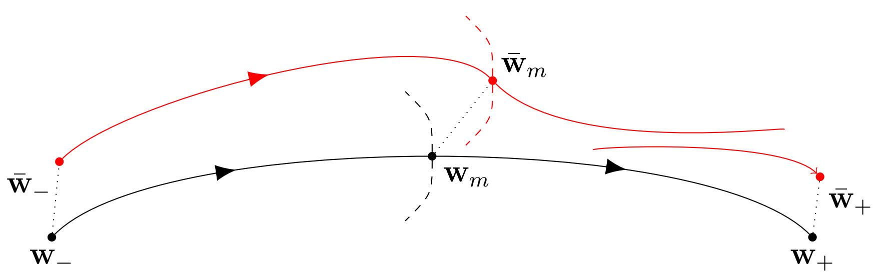


Figure 4.1: The traveling wave profile $\mathbf{w}(\xi)$ and its perturbation for ε small and s close to s_0. The fixed points and their stability types are preserved, but the heteroclinic connection bewteen them does not neccesarily persist.

We are going to show that, for each fixed small perturbation parameter ε, there exists a unique velocity $s(\varepsilon)$ close to s_0 such that the heteroclinic connection between the perturbed singularities exists. See Figure 4.2. This implies that a singular traveling wave solution of (2.1) persists for small perturbations of the flux functions and sources. We also calculate a first order perturbation formula for the speed of the wave $s(\varepsilon)$ and the perturbed profile. Our main tool for this task is Melnikov's method [2], which we need to generalize to the case of an orbit with an internal singularity.

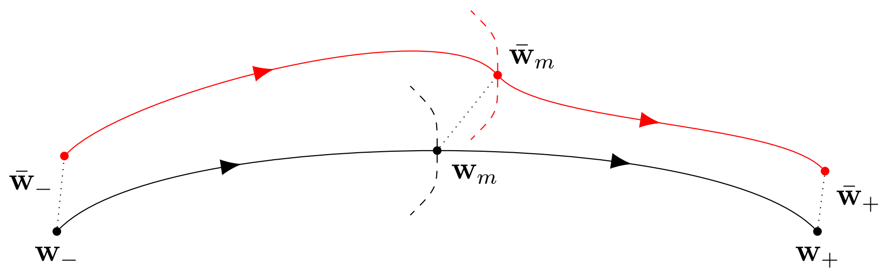


Figure 4.2: For each small perturbation ε, there is a unique velocity $s(\varepsilon)$ such that the heteroclinic connection between the three singularities persists.

4.2 The Separation Function and Melnikov's Method

We first study the effects of the perturbation on the heteroclinic connection between the two saddle equilibria on the wave. We will see that it is this connection that determines the speed of the perturbed wave. In the case the wave is of type 2, these are $\mathbf{w}_m$ and $\mathbf{w}_+$, in the case the wave is of type 5, $\mathbf{w}_-$ and $\mathbf{w}_m$. To simplify notation, in what follows we will work with type 2 waves, but all arguments are valid by changing the pair of saddle points $\mathbf{w}_m$ and $\mathbf{w}_+$ by the pair $\mathbf{w}_-$ and $\mathbf{w}_m$.

We suppose there is a singular heteroclinic connection between $\mathbf{w}_-$ and $\mathbf{w}_+$ for $s = s_0$, $\varepsilon = 0$, with singularity at $\mathbf{w}_m$. In particular, there is a connection from $\mathbf{w}_m$ to $\mathbf{w}_+$, that is, a curve $\mathbf{y}^0(\tau)$ with $\tau \in \mathbb{R}$ such that

$$\frac{d\mathbf{y}^0}{d\tau} = \mathbf{F}(\mathbf{y}^0, s_0, 0), \tag{4.7}$$

$$\lim_{\tau \to -\infty} \mathbf{y}^0(\tau) = \mathbf{w}_+, \quad \lim_{\tau \to +\infty} \mathbf{y}^0(\tau) = \mathbf{w}_m, \tag{4.8}$$

where

$$\mathbf{F}(\mathbf{w}, s, \varepsilon) = \operatorname{adj} \bar{\mathbf{A}}(\mathbf{w}, s, \varepsilon)\bar{\mathbf{g}}(\mathbf{w}, \varepsilon). \tag{4.9}$$

Note that defining $\xi \in \mathbb{R}$ by

$$\frac{d\xi}{d\tau}(\tau) = \det \bar{\mathbf{A}}(\mathbf{y}^0(\tau), s_0, 0), \tag{4.10}$$

and

$$\mathbf{w}^0(\xi) = \mathbf{y}^0(\tau(\xi)), \tag{4.11}$$

we get that $\mathbf{w}^0(\xi)$ is a connection from $\mathbf{w}_m$ to $\mathbf{w}_+$ in the original system (4.4). See Figure 4.3. Recall that the direction of orbits will be reversed when we go back to the variable ξ, because of (2.13).

Let's consider the point $\mathbf{x}_0 = \mathbf{y}^0(0)$. A normal vector to the orbit at that point is given by

$$\mathbf{u}_0 = \frac{1}{\|\mathbf{F}(\mathbf{x}_0, s_0, 0)\|^2} \begin{pmatrix} -F_2(\mathbf{x}_0, s_0, 0) \\ F_1(\mathbf{x}_0, s_0, 0) \end{pmatrix}, \tag{4.12}$$

where we use the notation $\mathbf{F} = (F_1, F_2)^T$. Consider a line segment Γ passing through $\mathbf{x}_0$ in the direction of $\mathbf{u}_0$. By continuity, for small ε and s close to s_0, the unstable manifold of $\bar{\mathbf{w}}_+(s, \varepsilon)$ and the stable manifold of $\bar{\mathbf{w}}_m(s, \varepsilon)$ intersect Γ. Let $\mathbf{y}_a$ be an orbit contained in the unstable manifold of $\bar{\mathbf{w}}_+$ such that $\mathbf{y}_a(0, s, \varepsilon) \in \Gamma$ and $\mathbf{y}_a(\tau, s_0, 0) = \mathbf{y}^0(\tau)$, and let $\mathbf{y}_b$ be an orbit contained in the stable manifold of $\bar{\mathbf{w}}_m$ such that $\mathbf{y}_b(0, s, \varepsilon) \in \Gamma$ and $\mathbf{y}_b(\tau, s_0, 0) = \mathbf{y}^0(\tau)$. See Figure 4.3.

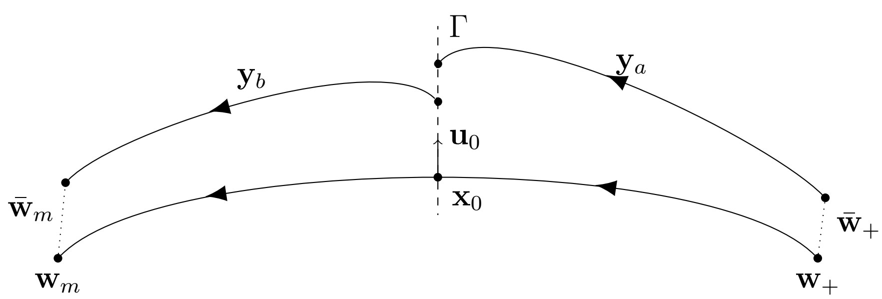


Figure 4.3: Orbits for the unperturbed and perturbed system.

We have that

$$\mathbf{y}_a(0, s, \varepsilon) = \mathbf{x}_0 + \eta_a(s, \varepsilon)\mathbf{u}_0, \quad \mathbf{y}_b(0, s, \varepsilon) = \mathbf{x}_0 + \eta_b(s, \varepsilon)\mathbf{u}_0, \tag{4.13}$$

where $\eta_a(s,\varepsilon)$ and $\eta_b(s,\varepsilon)$ are smooth functions satisfying

$$\eta_a(s_0,0)=\eta_b(s_0,0)=0. \tag{4.14}$$

We define the separation function as

$$S(s,\varepsilon)=\eta_a(s,\varepsilon)-\eta_b(s,\varepsilon). \tag{4.15}$$

If $S(s,\varepsilon)=0$, then there is an orbit that connects $\bar{\mathbf{w}}_m$ to $\bar{\mathbf{w}}_+$ for those particular values of ε and s. We have that $S(s_0,0)=0$. If $\frac{\partial S}{\partial s}(s_0,0)\neq 0$, the implicit function theorem will imply the existence a local function $s(\varepsilon)$ such that $S(s(\varepsilon),\varepsilon)=0$ and $s(0)=s_0$. This shows that for the parameter value $s(\varepsilon)$, there is a connection from $\bar{\mathbf{w}}_m$ to $\bar{\mathbf{w}}_+$.

In the next proposition we use Melnikov's method to calculate the partial derivatives of the separation function S. Let $D_{\mathbf{w}}\mathbf{F}$ be the Jacobian matrix of $\mathbf{F}$ with respect to the variables (u,v).

Proposition 3. *Let $S(s,\varepsilon)$ be defined by (4.15) on a small neighborhood of $(s_0,0)$. Let $\boldsymbol{\psi}(\tau)$ denote the solution of the linear differential equation*

$$\dot{\boldsymbol{\psi}}=-D_{\mathbf{w}}\mathbf{F}(\mathbf{y}^0(\tau),s_0,0)^T\boldsymbol{\psi}, \tag{4.16}$$

with initial condition $\boldsymbol{\psi}(0)=(-F_2(\mathbf{x}_0,s_0,0),F_1(\mathbf{x}_0,s_0,0))^T$. Then,

$$\frac{\partial S}{\partial s}(s_0,0)=\int_{-\infty}^{+\infty}\boldsymbol{\psi}(\alpha)^T\frac{\partial \mathbf{F}}{\partial s}(\mathbf{y}^0(\alpha),s_0,0)\,d\alpha, \tag{4.17}$$

$$\frac{\partial S}{\partial \varepsilon}(s_0,0)=\int_{-\infty}^{+\infty}\boldsymbol{\psi}(\alpha)^T\frac{\partial \mathbf{F}}{\partial \varepsilon}(\mathbf{y}^0(\alpha),s_0,0)\,d\alpha. \tag{4.18}$$

Proof. Notice that $\boldsymbol{\psi}(0)^T\mathbf{u}_0=1$ due to (4.12). Then, using (4.13) we get

$$\frac{\partial \eta_a}{\partial \varepsilon}(s_0,0)=\boldsymbol{\psi}(0)^T\frac{\partial \eta_a}{\partial \varepsilon}(s_0,0)\mathbf{u}_0=\boldsymbol{\psi}(0)^T\frac{\partial \mathbf{y}_a}{\partial \varepsilon}(0,s_0,0). \tag{4.19}$$

Now we exchange the order of partial derivatives to perform the following calculation

$$\begin{aligned}\frac{\partial^2\mathbf{y}_a}{\partial\tau\partial\varepsilon}(\tau,s_0,0)&=\frac{\partial}{\partial\varepsilon}\mathbf{F}\left(\mathbf{y}^0(\tau),s_0,0\right)=\\&=D_{\mathbf{w}}\mathbf{F}(\mathbf{y}^0(\tau),s_0,0)\frac{\partial \mathbf{y}^a}{\partial\varepsilon}(\tau,s_0,0)+\frac{\partial \mathbf{F}}{\partial\varepsilon}(\mathbf{y}^0(\tau),s_0,0).\end{aligned} \tag{4.20}$$

This shows that $\frac{\partial \mathbf{y}_a}{\partial \varepsilon}(\tau, s_0, 0)$ satisfies the inhomogeneous linear differential equation

$$\dot{\mathbf{v}} = D_{\mathbf{w}}\mathbf{F}(\mathbf{y}^0(\tau), s_0, 0)\mathbf{v} + \frac{\partial \mathbf{F}}{\partial \varepsilon}(\mathbf{y}^0(\tau), s_0, 0). \tag{4.21}$$

Given $\frac{\partial \mathbf{y}_a}{\partial \varepsilon}(-h, s_0, 0)$, the solution to (4.21) is given by the variation of parameters formula (see Appendix)

$$\frac{\partial \mathbf{y}_a}{\partial \varepsilon}(\tau, s_0, 0) = \mathbf{\Phi}(\tau, -h)\frac{\partial \mathbf{y}_a}{\partial \varepsilon}(-h, s_0, 0) + \int_{-h}^{\tau} \mathbf{\Phi}(\tau, \alpha)\frac{\partial \mathbf{F}}{\partial \varepsilon}(\mathbf{y}^0(\alpha), s_0, 0)\, d\alpha, \tag{4.22}$$

where $\mathbf{\Phi}(\tau, -h)$ is the state transition matrix of the corresponding homegeneous equation. For a definition and properties of the state transition matrix see the Appendix. Left-multiplying (4.22) by $\boldsymbol{\psi}(0)^T$ with $\tau = 0$, and using (4.19) and $\boldsymbol{\psi}(0)^T\mathbf{\Phi}(0, \alpha) = \boldsymbol{\psi}(\alpha)^T$, we get

$$\begin{aligned}\frac{\partial \eta_a}{\partial \varepsilon}(s_0, 0) &= \boldsymbol{\psi}(0)^T\frac{\partial \mathbf{y}_a}{\partial \varepsilon}(0, s_0, 0) = \\ &= \boldsymbol{\psi}(-h)^T\frac{\partial \mathbf{y}_a}{\partial \varepsilon}(-h, s_0, 0) + \int_{-h}^{0} \boldsymbol{\psi}(\alpha)^T\frac{\partial \mathbf{F}}{\partial \varepsilon}(\mathbf{y}^0(\alpha), s_0, 0)\, d\alpha.\end{aligned} \tag{4.23}$$

Note that $\boldsymbol{\psi}(\tau) \to 0$ as $\tau \to \pm\infty$, see, e.g. [13]. So, taking $h \to +\infty$, we get that

$$\frac{\partial \eta_a}{\partial \varepsilon}(s_0, 0) = \int_{-\infty}^{0} \boldsymbol{\psi}(\alpha)^T\frac{\partial \mathbf{F}}{\partial \varepsilon}(\mathbf{y}^0(\alpha), s_0, 0)\, d\alpha. \tag{4.24}$$

A similar calculation yields

$$\frac{\partial \eta_b}{\partial \varepsilon}(s_0, 0) = \int_{+\infty}^{0} \boldsymbol{\psi}(\alpha)^T\frac{\partial \mathbf{F}}{\partial \varepsilon}(\mathbf{y}^0(\alpha), s_0, 0)\, d\alpha. \tag{4.25}$$

So, using (4.15) we get that

$$\frac{\partial S}{\partial \varepsilon}(s_0, 0) = \int_{-\infty}^{+\infty} \boldsymbol{\psi}(\alpha)^T\frac{\partial \mathbf{F}}{\partial \varepsilon}(\mathbf{y}^0(\tau), s_0, 0)\, d\alpha. \tag{4.26}$$

A similar calculation yields formula (4.17). □

4.3 Perturbation of the Saddle-Saddle Connection

As was mentioned before, now we can prove the existence of the connection given that a non-degeneracy condition is met.

Proposition 4. *Let $\boldsymbol{\psi}(\tau)$ be defined as in Proposition 3. Suppose that*

$$\int_{-\infty}^{+\infty} \boldsymbol{\psi}(\alpha)^T \frac{\partial \mathbf{F}}{\partial s}(\mathbf{y}^0(\alpha), s_0, 0)\, d\alpha \neq 0. \tag{4.27}$$

Then, for each small ε, there is a unique wave speed $s(\varepsilon)$ and a solution $\mathbf{y}(\tau)$ of (4.4) with $s = s(\varepsilon)$ such that,

$$\lim_{\tau\to-\infty} \mathbf{y}(\tau) = \bar{\mathbf{w}}_+, \quad \lim_{\tau\to+\infty} \mathbf{y}(\tau) = \bar{\mathbf{w}}_m. \tag{4.28}$$

Furthermore, the speed $s(\varepsilon)$ satisfies the following perturbation formula

$$s(\varepsilon) = s_0 + s_1\varepsilon + O(\varepsilon^2), \tag{4.29}$$

where

$$s_1 = -\frac{\displaystyle\int_{-\infty}^{+\infty} \boldsymbol{\psi}(\alpha)^T \frac{\partial \mathbf{F}}{\partial \varepsilon}(\mathbf{y}^0(\alpha), s_0, 0)\, d\alpha}{\displaystyle\int_{-\infty}^{+\infty} \boldsymbol{\psi}(\alpha)^T \frac{\partial \mathbf{F}}{\partial s}(\mathbf{y}^0(\alpha), s_0, 0)\, d\alpha}. \tag{4.30}$$

Proof. Let $S(s, \varepsilon)$ be the separation function defined by (4.15). We have that $S(s_0, 0) = 0$ because we are supposing there is a connection for those parameters. Now, the left side of (4.27) equals the partial derivative $\frac{\partial S}{\partial s}(s_0, 0)$ by Proposition 3. Then, by the implicit function theorem, there exists a function $s(\varepsilon)$ defined for small $|\varepsilon|$, such that $S(s(\varepsilon), \varepsilon) = 0$. This function is unique. This implies the existence of an orbit $\mathbf{y}(\tau)$ of (4.4) with $s = s(\varepsilon)$ such that (4.28) holds. Also by the implicit function theorem, we have that

$$s_1 = \frac{ds}{d\varepsilon}(0) = -\frac{\frac{\partial S}{\partial \varepsilon}(s_0, 0)}{\frac{\partial S}{\partial s}(s_0, 0)}, \tag{4.31}$$

which by Proposition 3 results in (4.51). □

The next proposition gives the perturbation formula for the heteroclinic orbit.

Proposition 5. *Let $\mathbf{y}(\tau,\varepsilon)$ be the heteroclinic orbit connecting $\bar{\mathbf{w}}_+$ to $\bar{\mathbf{w}}_m$ guaranteed to exist by Proposition 4. Let $\mathbf{X}(\tau)$ be the fundamental matrix solution of the linear differential equation*

$$\dot{\boldsymbol{\phi}} = D_{\mathbf{w}}\mathbf{F}(\mathbf{y}^0(\tau), s_0, 0)\boldsymbol{\phi}, \tag{4.32}$$

with $\mathbf{X}(0) = \mathbf{I}$. Then,

$$\mathbf{y}(\tau,\varepsilon) = \mathbf{y}^0(\tau) + \mathbf{y}^1(\tau)\varepsilon + \mathbf{O}(\varepsilon^2), \tag{4.33}$$

where $\mathbf{y}^1(\tau)$ is given by

$$\begin{aligned}\mathbf{y}^1(\tau) &= \mathbf{X}(\tau)\mathbf{y}^1(0)+\\ &+ \mathbf{X}(\tau)\int_0^\tau \mathbf{X}^{-1}(\alpha)\left[\frac{\partial \mathbf{F}}{\partial s}(\mathbf{y}^0(\alpha), s_0, 0)s_1 + \frac{\partial \mathbf{F}}{\partial \varepsilon}(\mathbf{y}^0(u), s_0, 0)\right] d\alpha,\end{aligned} \tag{4.34}$$

$$\mathbf{y}^1(0) = \left[\int_{-\infty}^0 \boldsymbol{\psi}(\alpha)^T \left(\frac{\partial \mathbf{F}}{\partial \varepsilon}(\mathbf{y}^0(\alpha), s_0, 0) + \frac{\partial \mathbf{F}}{\partial s}(\mathbf{y}^0(\alpha), s_0, 0)s_1\right) d\alpha\right]\mathbf{u}_0. \tag{4.35}$$

Proof. Recall that $\mathbf{y}(\tau,\varepsilon)$ is a solution of (4.4), with $s = s(\varepsilon)$ given by Proposition 4. We thus have that

$$\frac{\partial \mathbf{y}(\tau,\varepsilon)}{\partial \tau} = \mathbf{F}(\mathbf{y}(\tau,\varepsilon), s(\varepsilon), \varepsilon). \tag{4.36}$$

Expanding both sides of equation (4.36) as Taylor series with respect to ε and collecting the terms of order ε, we get that

$$\begin{aligned}\frac{\partial}{\partial \tau}\frac{\partial \mathbf{y}}{\partial \varepsilon}(\tau, 0) &= D_{\mathbf{w}}\mathbf{F}(\mathbf{y}^0(\tau), s_0, 0)\frac{\partial \mathbf{y}}{\partial \varepsilon}(\tau, 0) + \frac{\partial \mathbf{F}}{\partial s}(\mathbf{y}^0(\tau), s_0, 0)s_1+\\ &+ \frac{\partial \mathbf{F}}{\partial \varepsilon}(\mathbf{y}^0(\tau), s_0, 0).\end{aligned} \tag{4.37}$$

We define

$$\mathbf{y}^1(\tau) = \frac{\partial \mathbf{y}}{\partial \varepsilon}(\tau, 0), \tag{4.38}$$

$$\tilde{\mathbf{f}}_s(\tau) = \frac{\partial \mathbf{F}}{\partial s}(\mathbf{y}^0(\tau), s_0, 0), \tag{4.39}$$

$$\tilde{\mathbf{f}}_\varepsilon(\tau) = \frac{\partial \mathbf{F}}{\partial \varepsilon}(\mathbf{y}^0(\tau), s_0, 0). \tag{4.40}$$

From (4.37) we conclude that $\mathbf{y}^1(\tau)$ is a solution to the inhomogeneous differential equation

$$\dot{\boldsymbol{\beta}} = D_{\mathbf{w}}\mathbf{F}(\mathbf{y}^0(\tau), s_0, 0)\boldsymbol{\beta} + \tilde{\mathbf{f}}_s(\tau)s_1 + \tilde{\mathbf{f}}_\varepsilon(\tau). \tag{4.41}$$

Given an initial condition, the variation of parameters formula (see Appendix) yields a solution to this equation in the form (4.34). We then need to find $\mathbf{y}^1(0)$. Expanding $\mathbf{y}(\tau, \varepsilon)$ in Taylor series we get equation (4.33). Recall equation (4.13), which can be rewritten as

$$\mathbf{y}(\tau, \varepsilon) = \mathbf{x}_0 + \eta(s(\varepsilon), \varepsilon)\mathbf{u}_0, \tag{4.42}$$

where $\eta = \eta_a = \eta_b$. Expanding both sides as Taylor series with respect to ε, and collecting the coefficients of order ε we get that

$$\mathbf{y}^1(0) = \left(\frac{\partial \eta}{\partial \varepsilon}(0) + \frac{\partial \eta}{\partial s}(0)s_1 \right) \mathbf{u}_0, \tag{4.43}$$

where the partial derivatives can be obtained with (4.24) and another similar equation with ε substituted by s. □

We also give a perturbation formula for the fixed points.

Proposition 6. *Let $\bar{\mathbf{w}}_+(\varepsilon)$, $\bar{\mathbf{w}}_m(\varepsilon)$ be the extreme points of the heteroclinic orbit given by Proposition 4. Then,*

$$\bar{\mathbf{w}}_+(\varepsilon) = \mathbf{w}_+ + \bar{\mathbf{w}}_+^1\varepsilon + \mathbf{O}(\varepsilon^2), \tag{4.44}$$

$$\bar{\mathbf{w}}_m(\varepsilon) = \mathbf{w}_m + \bar{\mathbf{w}}_m^1\varepsilon + \mathbf{O}(\varepsilon^2), \tag{4.45}$$

where

$$\bar{\mathbf{w}}_+^1 = -D_{\mathbf{w}}\mathbf{F}(\mathbf{w}_+, s_0, 0)^{-1} \left(\frac{\partial \mathbf{F}}{\partial s}(\mathbf{w}_+, s_0, 0)s_1 + \frac{\partial \mathbf{F}}{\partial \varepsilon}(\mathbf{w}_+, s_0, 0) \right), \tag{4.46}$$

$$\bar{\mathbf{w}}_m^1 = -D_{\mathbf{w}}\mathbf{F}(\mathbf{w}_m, s_0, 0)^{-1} \left(\frac{\partial \mathbf{F}}{\partial s}(\mathbf{w}_m, s_0, 0)s_1 + \frac{\partial \mathbf{F}}{\partial \varepsilon}(\mathbf{w}_m, s_0, 0) \right). \tag{4.47}$$

Proof. As $\bar{\mathbf{w}}_+(\varepsilon)$ is an equilibrium, we must have

$$\mathbf{F}(\bar{\mathbf{w}}_+(\varepsilon), s(\varepsilon), \varepsilon) = 0. \tag{4.48}$$

Expanding the left side in Taylor series with respect to ε and collecting the first order terms we get

$$D_{\mathbf{w}}\mathbf{F}(\mathbf{w}_+, s_0, 0)\frac{d\bar{\mathbf{w}}_+}{d\varepsilon}(0) + \frac{\partial \mathbf{F}}{\partial s}(\mathbf{w}_+, s_0, 0)s_1 + \frac{\partial \mathbf{F}}{\partial \varepsilon}(\mathbf{w}_+, s_0, 0) = 0. \tag{4.49}$$

Defining $\bar{\mathbf{w}}_+^1 = \frac{d\bar{\mathbf{w}}_+}{d\varepsilon}(0)$ and rearrenging the equation, we obtain formula (4.46). The Taylor expansion of $\bar{\mathbf{w}}_+$ yields formula (4.44). An analogous computation yields formulas (4.45) and (4.47). □

4.4 Structural Stability

We have proved that the saddle-saddle connection in singular traveling waves of type 2 and 5, as defined in Section 2.3, is structurally stable. The other connection of the profile is between a source point and a saddle point. It is known that this type of heteroclinic connection is structurally stable, see, e.g., [2]. This implies that the whole profile of the singular wave is structurally stable under small perturbations of the flux and source functions of the balance law. In other words, singular traveling waves do not dissapear by small perturbations in the equations. We summarize the main results in the next theorem.

Theorem 2. *Let $\mathbf{w}(\xi)$ be a singular traveling wave solution to system (2.1) with speed s_0, either of type 2 or 5, as defined in Section 2.3. Suppose that the non-degeneracy inequality (4.27) holds. Then, for small $\varepsilon \in \mathbb{R}$, there is a locally unique singular traveling wave solution $\mathbf{w}(\xi, \varepsilon)$ of the perturbed system (4.1) with speed smoothly dependent on ε and given by*

$$s(\varepsilon) = s_0 + s_1\varepsilon + O(\varepsilon^2), \tag{4.50}$$

where

$$s_1 = -\frac{\displaystyle\int_{-\infty}^{+\infty} \boldsymbol{\psi}(\alpha)^T \frac{\partial \mathbf{F}}{\partial \varepsilon}(\mathbf{y}^0(\alpha), s_0, 0)\, d\alpha}{\displaystyle\int_{-\infty}^{+\infty} \boldsymbol{\psi}(\alpha)^T \frac{\partial \mathbf{F}}{\partial s}(\mathbf{y}^0(\alpha), s_0, 0)\, d\alpha}. \tag{4.51}$$

This is a result of structural stability for singular traveling waves, which is of course a very desirable property. Furthermore, the theorem guarantees that the speed of the wave depends smoothly on the perturbation. Note that the proof is valid for any system of implicit differential equations with two parameters. Here, the parameters were called s and ε. The specific structure of the matrix $\bar{\mathbf{A}}$ in (4.4) was not used in the proofs. Therefore, the results are independent of the fact that the system (4.4) comes from a system of balance laws.

Chapter 5

Application To a Model of Oil Recovery by Air Injection

5.1 Oxidation Waves in Porous Media

We review the model used in [8] and present the essential mathematical features of the problem. The model is given by a system of multi-phase flow equations with additional terms that describe reaction and vaporization rates, and an energy balance equation. We consider flows possesing a combustion front when a gaseous oxidizer (air) is injected into a porous medium, a rock cylinder thermally insulated preventing lateral heat losses, filled with liquid fuel (gasoline or light oil). When oxygen reacts with hydrocarbons at low temperatures, a series of reactions occur that convert a part of the hydrobarbons into oxygenated hydrocarbons and gaseous products.

We consider motion in one space direction x. The fuel has saturation S. We follow the notations of [8], so no subscript is used for quantities related to the liquid fuel phase, and the subscript g for quantities in the gaseous phase. In the gaseous phase, we distinguish between the fraction of gaseous fuel X and of oxygen Y. Neglecting parabolic terms originating from gas mass diffusion and capillarity effects, the process is modeled by a system of balance laws for energy, fuel, total gas, gaseous fuel and oxygen. In dimensionless form the system is

$$\frac{\partial}{\partial t}(1+\alpha+\alpha_g S_g)\theta + \frac{\partial}{\partial x}(\alpha f + \alpha_g F_g)u\theta = \gamma w_r - \frac{\sigma\gamma w_v}{\varepsilon}, \tag{5.1}$$

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}uf = -\frac{\beta w_v}{\varepsilon}, \tag{5.2}$$

$$\frac{\partial S_g}{\partial t} + \frac{\partial}{\partial x}uF_g = (\nu_g - 1)w_r + \frac{w_v}{\varepsilon}, \tag{5.3}$$

$$\frac{\partial}{\partial t}XS_g + \frac{\partial}{\partial x}uXF_g = \frac{w_v}{\varepsilon}, \tag{5.4}$$

$$\frac{\partial}{\partial t}YS_g + \frac{\partial}{\partial x}uYF_g = -w_r. \tag{5.5}$$

The dependent variables in system (5.1)-(5.5) are the liquid fuel saturation S, the fraction of gaseous fuel X, the fraction of oxygen Y (all defined in the interval $[0,1]$), the temperature θ and the velocity u. The fuel and gas fractional flow functions are $f(S,\theta)$ and $f_g(S,\theta)$. $S_g(S,\theta)$ and $F_g(S,\theta)$ are temperature-corrected gas saturation and flow function. Function $w_r(S,Y,\theta)$ is the consumption rate of oxygen in the reaction, and $w_v(S,X,\theta)$ is the vaporization rate of liquid fuel. The other quantities are constant dimensionless parameters. The ratio between reaction rate and evaporation rate, ε, is assumed to be small.

Some of the equations can be rewritten as conservation laws. Using w_r and w_v from (5.4), (5.5), equations (5.1)-(5.3) are written as

$$\begin{aligned} &\frac{\partial}{\partial t}[(1+\alpha S+\alpha_g S_g)\theta + \gamma(Y+\sigma X)S_g] + \\ &+\frac{\partial}{\partial x}\left[(\alpha f+\alpha_g F_g)u\theta + \gamma(Y+\sigma X)uF_g\right] = 0, \end{aligned} \tag{5.6}$$

$$\frac{\partial}{\partial t}(S+\beta XS_g) + \frac{\partial}{\partial x}(uf+\beta XuF_g) = 0, \tag{5.7}$$

$$\frac{\partial}{\partial t}(1-X+(\nu_g-1)Y)S_g + \frac{\partial}{\partial x}(1-X+(\nu_g-1)Y)uF_g = 0. \tag{5.8}$$

5.2 Traveling Wave Solutions

We look for solutions in the form of waves traveling with constant speed $s > 0$. All the variables in these waves depend on a single traveling coordinate $\xi = x - st$. The equations for these waves are obtained by replacing $\partial/\partial x$ by

$d/d\xi$ and $\partial/\partial t$ by $-sd/d\xi$ in (5.6)-(5.8), (5.4), (5.5). This procedure yields

$$\frac{d}{d\xi}\left[(-s+\alpha\psi+\alpha_g\psi_g)\theta+\gamma\psi_Y+\sigma\gamma\psi_g X\right]=0, \tag{5.9}$$

$$\frac{d}{d\xi}(\psi+\beta\psi_g X)=0, \tag{5.10}$$

$$\frac{d}{d\xi}\left[(1-X)\psi_g+(\nu_g-1)\psi_Y\right]=0, \tag{5.11}$$

$$\frac{d}{d\xi}X\psi_g=\frac{w_v}{\varepsilon}, \tag{5.12}$$

$$\frac{d\psi_Y}{d\xi}=-w_r, \tag{5.13}$$

where we introduced notations for the fluxes of fuel, total gas and oxigen in the moving coordinate frame parametrized by ξ as

$$\psi=uf-sS,\quad \psi_g=uF_g-sS_g,\quad \psi_Y=Y(uF_g-sS_g). \tag{5.14}$$

These fluxes are functions of the dependent variables (S,X,Y,θ,u).

We look for a traveling wave solution, such that certain conditions for the dependent variables are satisfied at the limiting states ($\xi\to-\infty$ and $\xi\to+\infty$). For more details refer to [8]. Three algebraic equations for the wave profile can be found by integrating equations (5.9)-(5.11) from $-\infty$ to ξ. Using these equations and considering certain parameter values that simplify the model, the independent variables of the problem, (S,X,Y,θ,u), can be expressed in terms of only (ψ_Y,S). Therefore, equations (5.12), (5.13) determine a system of two ordinary differential equations on the (ψ_Y,S) plane that completelly determines the wave profile

$$\frac{d\psi_Y}{d\xi}=-w_r \tag{5.15}$$

$$\left(u\frac{\partial f}{\partial S}-s\right)\frac{ds}{d\xi}=\left(\frac{f/\theta_0}{1-X}+u\frac{\partial f}{\partial\theta}\right)\frac{\gamma w_r}{s}-\left(f+f\theta/\theta_0+\beta f_g\right)\frac{w_v}{\varepsilon}, \tag{5.16}$$

where X, u, θ can be expressed as functions of the two variables ψ_Y and S. Notice that this system can be expressed in the form of (2.7), that is

$$\mathbf{A}(\mathbf{w},s)\mathbf{w}'=\mathbf{g}(\mathbf{w}), \tag{5.17}$$

where

$$\mathbf{w} = (\psi_Y, S)^T, \tag{5.18}$$

$$\mathbf{A}(\mathbf{w}, s) = \begin{pmatrix} 1 & 0 \\ 0 & u\frac{\partial f}{\partial s} - s \end{pmatrix}, \tag{5.19}$$

and $\mathbf{g}(\mathbf{w})$ is given by the right sides of (5.15), (5.16). The impasse curve is given by the equation

$$\det \mathbf{A}(\mathbf{w}, s) = u\frac{\partial f}{\partial S} - s = 0. \tag{5.20}$$

Introducting the new variable τ defined by (2.13), we obtain the desingularized system in the form of (2.14),

$$\frac{d\psi_Y}{d\tau} = \left(u\frac{\partial f}{\partial s} - v \right) w_r, \tag{5.21}$$

$$\frac{ds}{d\tau} = -\left(\frac{f/\theta_0}{1-X} + u\frac{\partial f}{\partial \theta} \right) \frac{\gamma w_r}{v} + (f + f\theta/\theta_0 + \beta f_g)\, \frac{w_v}{\varepsilon}. \tag{5.22}$$

It is shown in [8] that, for a unique value of velocity s_r, an heteroclinic connection that crosses the impasse curve and that connects two limiting states exists. This solution is a singular (called resonant in [8]) traveling wave, with internal singularity of saddle type, as the wave we studied in the previous chapters. There are also non-resonant traveling wave solutions, but it is shown by numerical examples that this resonant traveling wave is the most interesting in applications.

5.3 Regularity of the Resonant Wave

We now prove the regularity of the resonant traveling wave solution. According to Theorem 1 from Chapter 3, the condition needed is the transversality of the stable manifold W to the impasse surface Σ and the kernel of $\mathbf{A}(\mathbf{w}_m)$. The direction of W is given by any eigenvector of matrix $\mathbf{J}$, defined by (3.23), corresponding to its negative eigenvalue. The Jacobian matrix $\mathbf{J}$ of the vector field (5.21), (5.22) was calculated in [8], and was shown to be of the form

$$\mathbf{J} = \begin{pmatrix} a & b \\ \frac{c}{\varepsilon} + \delta & d \end{pmatrix}, \tag{5.23}$$

where $b > 0$, $c > 0$ and $\varepsilon > 0$ can be assumed sufficiently small. The eigenvalues of this matrix are found to be

$$\frac{a+d}{2} \pm \sqrt{\left(\frac{a+d}{2}\right)^2 - ad + b\delta + \frac{bc}{\varepsilon}}. \tag{5.24}$$

Let λ be the negative eigenvalue. Making the parameter $\varepsilon > 0$ sufficiently small, we have that approximately

$$\lambda \approx -\sqrt{\frac{bc}{\varepsilon}}. \tag{5.25}$$

The corresponding eigenvector $\mathbf{z} = (\tilde{u}, \tilde{v})^T$ has the form

$$\tilde{v} \approx \lambda \tilde{u} \tag{5.26}$$

for $\varepsilon > 0$ small. As $|\lambda| \gg 1$, W approaches the vertical direction $(0,1)^T$ when $\varepsilon \to 0$, but has a non-zero horizontal component $1/\lambda$ for $\varepsilon > 0$. Using (5.19) and (5.20), we obtain that

$$\mathbf{A}(\mathbf{w}_m) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.27}$$

The kernel of $\mathbf{A}(\mathbf{w}_m)$ is parallel to the vertical axis, thus W is transversal to it. This proves the first transversality hypothesis required by Theorem 1. Now, we have that $\det \mathbf{A} = u\frac{\partial f}{\partial S} - s$ does not depend on the parameter ε. For any $\varepsilon > 0$, its gradient with respect to $\mathbf{w}$, which equals vector $\mathbf{n}$ defined by (3.22), will have a fixed direction. The direction of W approaches the vertical axis when $\varepsilon \to 0$, therefore we just need to make $\varepsilon > 0$ small enough to ensure W is transversal to the fixed vector $\mathbf{n}$. Thus, the conditions of Theorem 1 are met. This proves the differentiability of the resonant traveling wave at the fold point.

Chapter 6

Conclusions

We studied traveling wave solutions to 2×2 systems of balance laws with an internal singularity. The internal singularity was characterized as a pseudo-equilibrium of an associated vector field. It was shown that orbits can reach the internal singular point in a finite time because it is not a true equilibrium. We called this type of solutions *singular traveling waves*, although we proved that the profile is differentiable at the internal singularity. A classification of singular waves was developed, according to the stability types of the equilibrium points on the profile of the wave. We proved the C^k regularity of the profile at the singular point, and we deduced explicit formulas for the derivative. The effect on the profile when the source and flux functions of the balance laws are perturbed was studied. We showed that a singular traveling wave persists under small perturbations, and deduced formulas for the perturbed speed and profile. Our results are applicable to the model of enhanced oil recovery by air injection presented in [8]. As an example, we proved the regularity of the resonant traveling wave found on that paper. There are a few directions for future study of singular traveling waves:

- Study of the remaining cases from our classification in Section 2.3, including the necessary stability conditions.

- Generalization of the results of this work to $n \times n$ systems of balance laws.

- Study the effect of a viscous term on waves with an internal singularity. This was done in [3] for a single equation; the problem of finding viscous profiles of singular traveling wave solutions to systems of balance laws remains unsolved, see [4].

Appendix A

The Adjoint Equation, State Transition Matrix, and Variation of Parameters

A.1 The Adjoint Equation

Consider the linear differential equation $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$, where $\mathbf{x}$ is a column n-vector and $\mathbf{A}(t)$ is a $n \times n$ matrix. The *adjoint equation* is defined as $\mathbf{y}' = -\mathbf{A}(t)^T\mathbf{y}$, where $\mathbf{y}$ is a column n-vector. An equivalent equation is $\mathbf{w}' = -\mathbf{w}\mathbf{A}(t)$, where $\mathbf{w}$ is a row n-vector.

Proposition 7. *If $\mathbf{\Phi}(t)$ is a fundamental matrix solution of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$, i.e., a solution to the equation $\mathbf{X}' = \mathbf{A}(t)\mathbf{X}$ with $\mathbf{\Phi}(0) = \mathbf{I}$, then $\mathbf{\Phi}^{-1}(t)$ is a fundamental matrix solution of the adjoint equation $\mathbf{w}' = -\mathbf{w}\mathbf{A}(t)$.*

Proof.

$$
\begin{aligned}
0 = \frac{d}{dt}\mathbf{I} = \frac{d}{dt}\left(\mathbf{\Phi}^{-1}(t)\mathbf{\Phi}(t)\right) &= \left(\frac{d}{dt}\mathbf{\Phi}^{-1}(t)\right)\mathbf{\Phi}(t) + \mathbf{\Phi}^{-1}(t)\left(\frac{d}{dt}\mathbf{\Phi}(t)\right) \\
&= \left(\frac{d}{dt}\mathbf{\Phi}^{-1}(t)\right)\mathbf{\Phi}(t) + \mathbf{\Phi}^{-1}(t)\mathbf{A}(t)\mathbf{\Phi}(t) \Rightarrow \frac{d}{dt}\mathbf{\Phi}^{-1}(t) = -\mathbf{\Phi}^{-1}(t)\mathbf{A}(t).
\end{aligned}
$$

□

Corollary 1. *In the case $n = 2$, let*

$$\mathbf{\Phi}(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$$

be a fundamental matrix solution of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$*. Then, a fundamental matrix solution of* $\mathbf{w}' = -\mathbf{w}\mathbf{A}(t)$ *is*

$$\mathbf{\Phi}^{-1}(t) = \frac{1}{a(t)d(t) - b(t)c(t)} \begin{pmatrix} d(t) & -b(t) \\ -c(t) & a(t) \end{pmatrix}$$

The point of the previous corollary is that for $n = 2$, if we know just one solution of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$, and use Liouville's formula to calculate $\det \mathbf{\Phi}(t)$, then we know one row of $\mathbf{\Phi}^{-1}(t)$, that is, one solution of $\mathbf{w}' = -\mathbf{w}\mathbf{A}(t)$.

A.2 The State Transition Matrix

Let $\mathbf{\Phi}(t)$ be a fundamental matrix solution of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$. We define the *state transition matrix* $\mathbf{\Phi}(t, \tau)$ as

$$\mathbf{\Phi}(t, \tau) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(\tau). \tag{A.1}$$

The following proposition gives an important property of the state transition matrix.

Proposition 8. *For any solution* $\mathbf{x}(t)$ *of* $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$*, we have that*

$$\mathbf{x}(t) = \mathbf{\Phi}(t, \tau)\mathbf{x}(\tau) \tag{A.2}$$

for all t *and* τ*.*

Proof. Let $\mathbf{y}(t) = \mathbf{\Phi}(t, \tau)\mathbf{x}(\tau)$. We have that

$$\frac{d\mathbf{y}}{dt} = \frac{d}{dt}\left(\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(\tau)\mathbf{x}(\tau)\right) = \mathbf{A}(t)\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(\tau)\mathbf{x}(\tau) = \mathbf{A}(t)\mathbf{y}(t),$$

and $\mathbf{y}(\tau) = \mathbf{x}(\tau)$ because of (A.1). By the uniqueness of solutions of linear ordinary differential equations, we have that $\mathbf{x}(t) = \mathbf{y}(t)$ for all t. □

A.3 The Variations of Parameters Formula

Consider the linear inhomogeneous equation

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t). \tag{A.3}$$

Let $\mathbf{\Phi}(t)$ be a fundamental matrix solution of the corresponding homogeneous equation $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ with $\mathbf{\Phi}(t_0) = \mathbf{I}$. The *variations of parameters formula* expresses a solution of (A.3) in the form (see, e.g., [5])

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(\tau)\mathbf{b}(\tau)\, d\tau, \tag{A.4}$$

where $(t_0, \mathbf{x}(t_0))$ is the initial condition. Using (A.1), equation (A.4) becomes

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{b}(\tau)\, d\tau. \tag{A.5}$$

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