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Variation-diminishing maximal operators and the argument of L-functions

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Variation-diminishing maximal operators and the argument of L-functions

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Abstract

This thesis is rooted on the first two reseach articles of the author, both in analysis, but with quite distinct themes. One is about estimates related to the argument of L-functions on the critical line; the other shows that, in some cases, the action of a maximal operator diminishes the norm of the derivative of the operand.

Contents

Chapter 1

Introduction

The reader is going to learn about the content of two following articles.

- *On the argument of L-functions* (with E. Carneiro), Bulletin of the Brazilian Mathematical Society 46, no. 4 (2015), 601 – 620.
- *On the variation of maximal operators of convolution type II* (with E. Carneiro and M. Sousa), preprint at arXiv:1512.02715.

In this introduction, we attempt to give a panoramic view of the mathematical landscape surrounding them and briefly present our contribution.

1.1 L-functions

Historically, the first example of an L-function is the Riemann zeta-function. Bernhard Riemann defined it for any complex number s such that $\operatorname{Re} s > 1$ by the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

With his restriction on s, this series is absolutely convergent. He then showed that ζ has a meromorphic extension to the whole complex plane and discovered a relation between $\zeta(s)$ and $\zeta(1-s)$, known as its *functional equation*. Using the Euler product formula

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1},$$

where p is a variable prime, and methods of complex analysis, he raised evidence that there is a connection between the zeros of ζ and the distribution of the prime numbers. In doing so, he assumed that all the zeros of ζ are on the line $\operatorname{Re} s = \frac{1}{2}$.

In fact, ζ and other meromorphic functions may be used to obtain number-theoretic information. Lejeune Dirichlet found a family of such functions – the series of the form

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where χ is a character (i.e. a periodic function from $\mathbb{Z}$ to $\mathbb{C}$ such that $\chi(mn) = \chi(m)\chi(n)$ for any $m, n \in \mathbb{Z}$ and $\chi(n) \neq 0$ if and only if n is coprime to its period). These functions can be used to obtain results about the distribution of prime numbers on arithmetic progressions. It was shown that each of them can be extended to the whole complex plane and, if χ is primitive[1], the corresponding function satisfies a functional equation.

Another interesting family is that of Dedekind zeta-functions. There is one of them for any field $K \subset \mathbb{C}$ that has finite degree over $\mathbb{Q}$ and it is built out of the ring of algebraic integers in K. These functions have Euler product formulas (because any nonzero ideal in the ring of integers of K is a product of prime ideals in a unique way, ignoring the order of the factors) and functional equations.

There are more families of meromorphic functions, related to different objects, that can be represented by a series and by an Euler product and satisfy a functional equation. These common properties were held together to define an L-function. There are some minor variations in different versions of the definition. We leave our version to Chapter 2. We denote the value of an L-function at s by $L(s,\pi)$, where π is thought of as an object that gives rises to the function, such as a Dirichlet character or a number field.

In our article listed above, we considered the function $S(t,\pi)$, which is the argument (i.e. the imaginary part of the logarithm) of $L(\frac{1}{2} + it, \pi)$ divided by the real number π, and its primitive $S_1(t,\pi) = \frac{1}{\pi}\int_{1/2}^{\infty} \log|L(\sigma + it, \pi)|\, \mathrm{d}\sigma$. When $L(\cdot,\pi) = \zeta$, we shall use the shorter notations $S(t)$ and $S_1(t)$. Carneiro, Chandee and Milinovich had obtained a bound for $S_1(t)$ in [6], assuming Riemann's hypothesis that the zeros of ζ are on a single line. We generalized their proof for any L-function, giving a bound whose main term is

$$\frac{(1+2\vartheta)^2\,\pi}{48} \frac{\log C(t,\pi)}{\left(\log\left(\frac{3}{m}\log C(t,\pi)\right)\right)^2},$$

[1] Let χ_1 and χ_2 be characters of periods q_1 and q_2. We say that χ_1 induces χ_2 if $q_1|q_2$ and $\chi_2(n) = \chi_1(n)$ for any n coprime to q_2. A character is said to be primitive if it is induced by no character other than itself.

where $C(t, \pi)$ is the *conductor* of $L(\cdot, \pi)$. All the notation is explained in Chapter 2. Then we used this to give an alternative proof of another theorem of Carneiro, Chandee and Milinovich, still depending on the Riemann hypothesis for $L(\cdot, \pi)$. This one, published in [7], is a bound for $S(t, \pi)$ whose leading term is

$$\frac{1+2\vartheta}{4}\frac{\log C(t,\pi)}{\log\left(\frac{3}{m}\log C(t,\pi)\right)}.$$

We also observed that the bound for $S(t, \pi)$ implies bounds for the multiplicity of a zero and for the distance between consecutive zeros of $L(\cdot, \pi)$. This is related to the fact that $S(t, \pi)$ changes by one unit whenever $\frac{1}{2} + it$ crosses a zero of $L(\cdot, \pi)$.

1.2 Sobolev spaces and maximal operators

We begin with the Hardy-Littlewood maximal operator M. Let u_0 be a locally integrable function on $\mathbb{R}^d$ and x be a point of $\mathbb{R}^d$. Then $Mu_0(x)$ is defined as the supremum of all the averages of $|u_0|$ over balls centered at x. This makes Mu_0 a function from $\mathbb{R}^d$ to $[0, \infty]$. It is well known that, if u_0 is in $L^p(\mathbb{R}^d)$ for some real number $p > 1$, then Mu_0 is also in $L^p(\mathbb{R}^d)$. This result has many applications, of which the most famous may be in a proof of Lebesgue's differentiation theorem. In a similar way, there are other operators defined as supremums of averages, known generically as *maximal operators*, that were employed in proofs of convergence results, such as Carleson's theorem on Fourier series and Birkhoff's theorem in ergodic theory.

More recently some questions concerning the derivative of maximal operators appeared. Here we consider derivatives in the weak sense: a function g is a derivative of f with respect to the kth variable if the equality

$$\int g\psi = -\int f\frac{\partial\psi}{\partial x_k}$$

holds for any smooth compactly supported ψ. Let $W^{1,p}(\mathbb{R}^d)$ be the space of functions in $L^p(\mathbb{R}^d)$ that have partial derivatives in $L^p(\mathbb{R}^d)$ for every k between 1 and d. Kinnunen's [25] initiated the investigation of maximal functions in $W^{1,p}(\mathbb{R}^d)$, showing that, if u_0 belongs to this space, so does Mu_0. Moreover, it is proved there that, if p is a real number greater than 1, then $\|\nabla(Mu_0)\|_p \leq C\|\nabla u_0\|_p$ for some constant C.

Soon after, maximal operators given by

$$u^*(x) = \sup_{t>0} \left(\varphi(\cdot, t) * |u_0|\right)(x),$$

were considered, with different choices of the function φ (from $\mathbb{R}^d \times (0,\infty)$ to $\mathbb{R}_+$). Note that this is the Hardy-Littlewood maximal operator if $\varphi(\cdot,t) = \frac{1}{m(B_t)}\chi_{B_t}$, where B_t is the ball of radius t centered at the origin and $m(B_t)$ is its Lebesgue measure. In Carneiro and Svaiter's article [14], they selected φ at different times as the heat kernel and as the Poisson kernel. Taking advantage of the fact that convolution with these kernels produces the solution of a partial differential equation, they proved that $\|\nabla u^*\|_p \leq \|\nabla u_0\|_p$ if either $d = 1$ or $p = 2$. In the case $d = p = 1$, they went even further, showing that the variation of u^* is less or equal to the variation of u_0 even if u_0 is not differentiable. This is equivalent to $\|(u^*)'\|_1 \leq \|u_0'\|_1$ if it *is* differentiable.

We continued the work [14] of Carneiro and Svaiter, changing somewhat the context, but aiming at the same conclusions. One of our results replaces φ with a function that satisfies the elliptic partial differential equation

$$a\frac{\partial^2 \varphi}{\partial t^2} - b\frac{\partial \varphi}{\partial t} + \Delta\varphi = 0,$$

where a and b are positive constants. This function is not an approximation of the identity in the most usual sense, but its integral is still 1 for each fixed t and its mass becomes concentrated at the origin as $t \to 0$. If $p = 2$ or $d = 1$ and $p > 1$, we proved that $\|\nabla u^*\|_p \leq \|\nabla u_0\|_p$. If $d = 1$, we proved that $V(u^*) \leq V(u_0)$ for any u_0 of bounded variation, where V is the variation. We also approached the analogous problem on the torus.

In another theorem, we treat the Poisson equation in the $(d+1)$-dimensional unit ball. If any integrable function $u_0 : \mathbb{S}^d \to \mathbb{C}$ is given as the initial condition, there exists a solution of the Poisson equation given by a kind of convolution with $|u_0|$. We define $u^*(x)$ as the supremum of the values of this solution over the radius connecting the origin to x, for any $x \in \mathbb{S}^d$. A second maximal operator acting on the sphere is obtained when we take the heat kernel instead of the Poisson kernel. Equipping the sphere with the usual metric, we have a solution of the heat equation on $\mathbb{S}^d \times (0,\infty)$ and we define u^* as before. In any case, if $p > 1$ and $u_0 \in W^{1,p}(\mathbb{S}^d)$, then $u^* \in W^{1,p}(\mathbb{S}^d)$ and we can show the same properties as in the previous theorem.

In the examples we have just considered, the maximal operators are defined as maxima over lines of the domain of a partial differential equation. In the last case we studied, however, the maximal operator is over a cone of pairs in the domain. If $u_0 \in L^1(\mathbb{R}^d)$, let u be the harmonic function on the $(d+1)$-dimensional upper half-space defined as a convolution of u_0 and the Poisson kernel. Let $u^*(x)$ be the supremum of $u(y,t)$ for all pairs (y,t) such that $|y-x| \leq \alpha t$ for some constant α. For this choice of u^*, we have only attained our goal if $d = 1$. In greater dimensions, the strategy fails.

For all of these theorems, as in those of Carneiro and Svaiter's [14], the main step of the proofs is to show that u^* is subharmonic in the set $\{x; u^*(x) > u_0(x)\}$.

Chapter 2

The argument of L-functions

2.1 Introduction

2.1.1 Background

Let $S(t) = \frac{1}{\pi}\arg\zeta(\frac{1}{2}+it)$, where the argument is obtained by continuous variation along the ray $\{s \in \mathbb{C};\ \operatorname{Re} s \geq \frac{1}{2} \text{ and } \operatorname{Im} s = t\}$, starting from 0 at infinity. Let $N(t)$ be the number of zeros of ζ whose imaginary part is between 0 and t. It is well known that, for $t \geq 1$,

$$N(t) = \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O\left(\frac{1}{t}\right), \tag{2.1.1}$$

provided that $S(t)$ and $N(t)$ are defined in a consistent way when t is the imaginary part of a zero of ζ.

In his article [29], Littlewood considered the function

$$S_1(t) = \frac{1}{\pi}\int_{1/2}^{\infty} \log|\zeta(\sigma+it)|\,\mathrm{d}\sigma$$

and observed that

$$\int_t^u S(v)\,\mathrm{d}v = S_1(u) - S_1(t).$$

The proof of [29, Theorem 9] shows how it is possible to use (2.1.1) to derive a bound for S from a bound for S_1. The idea is that, since $N(t)$ is nondecreasing, $S(t)$ does not decrease much faster than $-\frac{t}{2\pi}\log\frac{t}{2\pi}$, therefore a large value of $S(t)$ would cause a large variation of

S_1 near t. In the same article, he assumed the Riemann hypothesis to conclude that

$$S(t) = O\left(\frac{\log t}{\log\log t}\right)$$

and

$$S_1(t) = O\left(\frac{\log t}{(\log\log t)^2}\right)$$

for $t \geq 3$.

The works [17, 20, 34] went further by finding numerical bounds for

$$\limsup_{t\to\infty}\left|S(t)\left(\frac{\log t}{\log\log t}\right)^{-1}\right|. \tag{2.1.2}$$

The article [20] introduces the use of extremal functions of exponential type in this problem. Ramachandra and Sankaranarayanan, in [34], also remark that a bound of the same kind is true for Dirichlet L-functions, assuming the corresponding Riemann hypothesis. On the other hand, [18, 24] exhibit numerical bounds for

$$\limsup_{t\to\infty}\left|S_1(t)\left(\frac{\log t}{(\log\log t)^2}\right)^{-1}\right|. \tag{2.1.3}$$

Currently, the best conditional bounds for (2.1.2) and (2.1.3) are due to Carneiro, Chandee and Milinovich. We are going to explain their proof method (from [6]). Their first movement was to show that, if the Riemann hypothesis holds,

$$S_1(t) = \frac{1}{4\pi}\log t - \frac{1}{\pi}\sum_{\gamma} f_1(t-\gamma) + O(1),$$

where the sum is over all γ such that $\zeta\left(\frac{1}{2}+i\gamma\right) = 0$ and $f_1(x) = 1 - x\arctan(\frac{1}{x})$. There is a formula to calculate sums over zeros of ζ, called *Guinand-Weil explicit formula*, but it only applies to analytic functions. To overcome this, they looked for real entire minorants and majorants of exponential type for f_1, using the tools of [11], an article where Carneiro, Littmann and Vaaler studied the problem of finding majorants and minorants in a more abstract setting. Finally, they obtained [6, Theorem 1]

$$S_1(t) \leq \frac{\pi}{48}\frac{\log t}{(\log\log t)^2}\left(1 + O\left(\frac{\log\log\log t}{\log\log t}\right)\right) \tag{2.1.4}$$

and

$$S_1(t) \geq -\frac{\pi}{24}\frac{\log t}{(\log\log t)^2}\left(1 + O\left(\frac{\log\log\log t}{\log\log t}\right)\right). \tag{2.1.5}$$

These inequalities bound any difference $S_1(u) - S_1(t) = \int_t^u S(v)\,\mathrm{d}v$. Also, (2.1.1) may be used to compare this difference to $S(t)$, and choosing appropriate values of u yields the inequality [6, Theorem 2]

$$|S(t)| \leq \frac{1}{4}\frac{\log t}{\log\log t} + \left(1 + O\left(\frac{\log\log\log t}{\log\log t}\right)\right) \tag{2.1.6}$$

for sufficiently large t.

A shorter proof of (2.1.6) was recently obtained in [7, Theorem 1] using the classical Beurling-Selberg majorants and minorants of characteristic functions of intervals and exploiting the fact that ζ is self-dual (i.e. $\zeta(s) = \overline{\zeta(\overline{s})}$).

2.1.2 A class of L-functions

Let

$$\Gamma_{\mathbb{R}}(z) = \pi^{-z/2}\,\Gamma\left(\frac{z}{2}\right),$$

where Γ is the usual Gamma function, i.e. the meromorphic extension of $z \mapsto \int_0^\infty x^{z-1}e^{-x}\,\mathrm{d}x$. We work with a meromorphic function $L(\cdot,\pi)$ on $\mathbb{C}$ which meets the following requirements (for some positive integer m and some $\vartheta \in [0,1]$).

(i) There exists a sequence $\{\lambda_\pi(n)\}_{n\geq 1}$ of complex numbers ($\lambda_\pi(1) = 1$) such that the series

$$\sum_{n=1}^{\infty}\frac{\lambda_\pi(n)}{n^s}$$

converges absolutely to $L(s,\pi)$ on $\{s \in \mathbb{C};\, \operatorname{Re} s > 1\}$.

(ii) For each prime number p, there exist $\alpha_{1,\pi}(p), \alpha_{2,\pi}(p), \ldots, \alpha_{m,\pi}(p)$ in $\mathbb{C}$ such that $|\alpha_{j,\pi}(p)| \leq p^\vartheta$, where $0 \leq \vartheta \leq 1$ is independent of p, and

$$L(s,\pi) = \prod_p \prod_{j=1}^{m}\left(1 - \frac{\alpha_{j,\pi}(p)}{p^s}\right)^{-1},$$

with absolute convergence on the half plane $\operatorname{Re} s > 1$.

(iii) For some positive integer N and some complex numbers $\mu_1, \mu_2, \ldots, \mu_m$ whose real parts are greater than -1 and such that $\{\mu_1, \mu_2, \ldots, \mu_m\} = \{\overline{\mu_1}, \overline{\mu_2}, \ldots, \overline{\mu_m}\}$, the completed L-function

$$\Lambda(s,\pi) = N^{s/2} \prod_{j=1}^{m} \Gamma_{\mathbb{R}}(s+\mu_j) L(s,\pi)$$

is a meromorphic function of order 1 that has no poles other than 0 and 1. The points 0 and 1 are poles with the same order $r(\pi) \in \{0, 1, \ldots, m\}$. Furthermore, the function $\Lambda(s,\tilde{\pi}) := \overline{\Lambda(\overline{s},\pi)}$ satisfies the functional equation

$$\Lambda(s,\pi) = \kappa\, \Lambda(1-s,\tilde{\pi}) \tag{2.1.7}$$

for some unitary complex number κ.

Except for the assumption $r(\pi) \leq m$, we are in the same framework as [23, Chapter 5], where many examples may be found.

2.1.3 Main results

The theorems we prove are analogues of (2.1.4), (2.1.5) and (2.1.6) for L-functions. They are based on the *generalized Riemann hypothesis*, which asserts that $\Lambda(s,\pi) \neq 0$ if $\operatorname{Re} s \neq \frac{1}{2}$. The product

$$C(t,\pi) = N \prod_{j=1}^{m} (|it+\mu_j| + 3),$$

called the *analytic conductor* of $L(\cdot,\pi)$, is used in their statements. This function appears commonly in the theory of L-functions, as in [23, Proposition 5.7]. If the generalized Riemann hypothesis holds, it may be viewed as the density of zeros of $L(\cdot,\pi)$, thanks to Lemma 7.

Theorem 1. *Let $L(\cdot,\pi)$ satisfy the generalized Riemann hypothesis. Let*

$$S_1(t,\pi) = \frac{1}{\pi} \int_{1/2}^{\infty} \log |L(\sigma+it,\pi)| \, \mathrm{d}\sigma.$$

Then, for every real number t,

$$S_1(t,\pi) \leq \frac{(1+2\vartheta)^2\, \pi}{48} \frac{\log C(t,\pi)}{\left(\log\left(\frac{3}{m}\log C(t,\pi)\right)\right)^2} \left(1 + O\left(\frac{\log\log\left(\frac{3}{m}\log C(t,\pi)\right)}{\log\left(\frac{3}{m}\log C(t,\pi)\right)}\right)\right)$$

and

$$S_1(t,\pi) \geq -\frac{(1+2\vartheta)^2\,\pi}{24}\frac{\log C(t,\pi)}{\left(\log\left(\frac{3}{m}\log C(t,\pi)\right)\right)^2}\left(1+O\left(\frac{\log\log\left(\frac{3}{m}\log C(t,\pi)\right)}{\log\left(\frac{3}{m}\log C(t,\pi)\right)}\right)\right).$$

Here and henceforth, $O(E)$ refers to a quantity whose absolute value is bounded by a universal constant times E.

If t is not the imaginary part of a zero of $L(\cdot,\pi)$ and $t \neq 0$, the argument function is defined by

$$S(t,\pi) = -\frac{1}{\pi}\int_{1/2}^{\infty} \mathrm{Im}\frac{L'}{L}(\sigma+it,\pi)\,\mathrm{d}\sigma.$$

Otherwise, it is

$$S(t,\pi) = \lim_{\eta\to 0}\frac{S(t+\eta,\pi)+S(t-\eta,\pi)}{2}.$$

We note that $S_1(t,\pi)$ is a primitive for the function $S(t,\pi)$ (details in Section 2.3 below). An extension of (2.1.6) to L-functions, with a good leading constant, was obtained by Carneiro, Chandee and Milinovich in [7, Theorem 5], via a direct approach using extremal majorants and minorants of exponential type for the odd function $f(x) = \arctan\left(\frac{2}{x}\right) - \frac{2x}{4+x^2}$, available in the framework of [10]. Here we give an alternative proof of this result, deriving it from our Theorem 1.

Theorem 2. *Let $L(\cdot,\pi)$ satisfy the generalized Riemann hypothesis. Then, for every real number t,*

$$|S(t,\pi)| \leq \frac{1+2\vartheta}{4}\frac{\log C(t,\pi)}{\log\left(\frac{3}{m}\log C(t,\pi)\right)} + O\left(\frac{\log C(t,\pi)\log\log\left(\frac{3}{m}\log C(t,\pi)\right)}{(\log\left(\frac{3}{m}\log C(t,\pi)\right))^2}\right).$$

The previous result gives information about the distribution of the zeros of L-functions. An example is the following corollary, related to [20, Corollary 1] and [7, Theorem 7].

Corollary 3. *Let $L(\cdot,\pi)$ satisfy the generalized Riemann hypothesis.*

(i) Let $m(\gamma,\pi)$ denote the multiplicity of the zero $\frac{1}{2}+i\gamma$ of $\Lambda(\cdot,\pi)$. Then,

$$m(\gamma,\pi) \leq \frac{1+2\vartheta}{2}\frac{\log C(\gamma,\pi)}{\log\left(\frac{3}{m}\log C(\gamma,\pi)\right)}\left(1+O\left(\frac{\log\log\left(\frac{3}{m}\log C(\gamma,\pi)\right)}{\log\left(\frac{3}{m}\log C(\gamma,\pi)\right)}\right)\right).$$

(ii) Let $\frac{1}{2}+i\gamma$ and $\frac{1}{2}+i\gamma'$ be consecutive zeros of $\Lambda(\cdot,\pi)$. Then $\gamma'-\gamma$ is bounded by some universal constant and if $C(\gamma,\pi)^{3/m}$ is sufficiently large,

$$\gamma'-\gamma \leq \frac{(1+2\vartheta)\pi}{\log\left(\frac{3}{m}\log C(\gamma,\pi)\right)}\left(1+O\left(\frac{\log\log\left(\frac{3}{m}\log C(\gamma,\pi)\right)}{\log\left(\frac{3}{m}\log C(\gamma,\pi)\right)}\right)\right).$$

We make no attempt here to estimate the universal bound for the gap between consecutive zeros of our general class of L-functions. Such a gap has been estimated (for a slightly different class of L-functions) in [3, Theorem 2.1] (see also [33] for bounds on low-lying zeros of L-functions).

Carneiro and Chirre generalized Theorems 1 and 2 for a family of functions $S_n(\cdot,\pi)$ such that $S'_{n+1}(\cdot,\pi)=S_n(\cdot,\pi)$. They proved in [8] that

$$S_n(t,\pi)\leq C_n(1+2\vartheta)^{n+1}\frac{\log C(t,\pi)}{\left(\log\left(\frac{3}{m}\log C(t,\pi)\right)\right)^{n+1}}\left(1+D_n\frac{\log\log\left(\frac{3}{m}\log C(t,\pi)\right)}{\log\left(\frac{3}{m}\log C(t,\pi)\right)}\right),$$

where C_n decreases exponentially as $n\to\infty$, and that an analogous inequality holds in the reverse direction. In addition to these functions, the theory of extremal functions of exponential type can be used to provide upper bounds for the modulus of an L-function on the critical line. This has been carried out in the work of Chandee and Soundararajan [15]:

$$\log\left|L\left(\tfrac{1}{2}+it,\pi\right)\right|\leq\frac{(1+2\vartheta)\log 2}{2}\frac{\log C(t,\pi)}{\log\left(\frac{3}{m}\log C(t,\pi)\right)}\left(1+O\left(\frac{\log\log\left(\frac{3}{m}\log C(t,\pi)\right)}{\log\left(\frac{3}{m}\log C(t,\pi)\right)}\right)\right).$$

Although they considered explicitly only the case $t=0$, their reasoning is general. Other examples of the use of bandlimited majorants to the theory of the Riemann zeta-function include [4, 5, 19].

2.2 Proof of Theorem 1

In this section we prove Theorem 1. We adapt the strategy of [6] described in Subsection 2.1.1.

Lemma 4. *Let $L(\cdot,\pi)$ satisfy the generalized Riemann hypothesis. For any real t,*

$$S_1(t,\pi)=\frac{1}{\pi}\left(-\sum_{\gamma}F_1(t-\gamma)+\log C(t,\pi)\right)+O(m),$$

where the sum is over all values of γ *such that* $L\left(\frac{1}{2}+i\gamma,\pi\right)=0$, *counted with multiplicity, and*

$$F_1(x)=\frac{1}{2}\int_{1/2}^{5/2}\log\frac{4+x^2}{(\sigma-\frac{1}{2})^2+x^2}\,\mathrm{d}\sigma=2-x\arctan\left(\frac{2}{x}\right). \tag{2.2.1}$$

Proof. By the product expansion of $L(\cdot,\pi)$ and the inequality $|\alpha_{j,\pi}(p)|\leq p$,

$$|\log|L(s,\pi)||\leq m\log\zeta(\operatorname{Re}s-1)=O\left(\frac{m}{2^{\operatorname{Re}s}}\right)$$

for any s such that $\operatorname{Re}s\geq\frac{5}{2}$. Because of this and of the fact that $L(\cdot,\pi)$ is meromorphic, $S_1(\cdot,\pi)$ is well defined and

$$\begin{aligned}
\pi S_1(t,\pi)&=\int_{1/2}^{5/2}\log|L(\sigma+it,\pi)|\,\mathrm{d}\sigma+O(m)\\
&=\int_{1/2}^{5/2}\left\{\log|L(\sigma+it,\pi)|-\log\left|L\left(\tfrac{5}{2}+it,\pi\right)\right|\right\}\mathrm{d}\sigma+O(m)\\
&=\int_{1/2}^{5/2}\left\{\log|\Lambda(\sigma+it,\pi)|-\log\left|\Lambda\left(\tfrac{5}{2}+it,\pi\right)\right|\right\}\mathrm{d}\sigma \qquad (2.2.2)\\
&\quad+\int_{1/2}^{5/2}\left\{\log\left|N^{(5/2+it)/2}\right|-\log\left|N^{(\sigma+it)/2}\right|\right\}\mathrm{d}\sigma\\
&\quad+\sum_{j=1}^{m}\int_{1/2}^{5/2}\left\{\log\left|\Gamma_{\mathbb{R}}\left(\tfrac{5}{2}+it+\mu_j\right)\right|-\log|\Gamma_{\mathbb{R}}(\sigma+it+\mu_j)|\right\}\mathrm{d}\sigma+O(m).
\end{aligned}$$

We treat each integral separately. For the first one, we use Hadamard's factorization formula

$$\Lambda(s,\pi)=s^{-r(\pi)}(s-1)^{-r(\pi)}e^{A+Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho},$$

where A and B are constants and the product is over all zeros of $\Lambda(\cdot,\pi)$. From the functional equation $\Lambda(s,\pi)=\kappa\,\Lambda(1-s,\tilde{\pi})$, one deduces that $\operatorname{Re}B=-\sum_{\rho}\operatorname{Re}\left(\frac{1}{\rho}\right)$ (see [23, Proposition 5.7]). Hence, for $\frac{1}{2}\leq\sigma\leq\frac{5}{2}$,

$$\left|\frac{\Lambda(\sigma+it,\pi)}{\Lambda\left(\frac{5}{2}+it,\pi\right)}\right|=\left|\frac{\sigma+it}{\frac{5}{2}+it}\right|^{-r(\pi)}\left|\frac{\sigma-1+it}{\frac{3}{2}+it}\right|^{-r(\pi)}\prod_{\rho}\left|\frac{\sigma+it-\rho}{\frac{5}{2}+it-\rho}\right|,$$

which implies, via the substitution $\rho = \frac{1}{2} + i\gamma$, that

$$\log\left|\frac{\Lambda(\sigma+it,\pi)}{\Lambda\left(\frac{5}{2}+it,\pi\right)}\right| = O(m) + r(\pi)\log\left|\frac{\frac{3}{2}+it}{\sigma-1+it}\right| + \sum_{\gamma}\frac{1}{2}\log\frac{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}{4+(t-\gamma)^2}.$$

Since $1 \le \left|\frac{\frac{3}{2}+it}{\sigma-1+it}\right| \le \frac{\frac{3}{2}}{|\sigma-1|}$, integrating we get

$$\int_{1/2}^{5/2}\left\{\log|\Lambda(\sigma+it,\pi)| - \log\left|\Lambda\left(\tfrac{5}{2}+it,\pi\right)\right|\right\}\mathrm{d}\sigma = -\sum_{\gamma}F_1(t-\gamma) + O(m).$$

Our considerations on the last m integrals in (2.2.2) use Stirling's formula

$$\frac{\Gamma'}{\Gamma}(z) = \log(1+z) - \frac{1}{z} + O(1)$$

in the form

$$\frac{\Gamma_{\mathbb{R}}'}{\Gamma_{\mathbb{R}}}(z) = \frac{1}{2}\log(2+z) - \frac{1}{z} + O(1), \tag{2.2.3}$$

valid for $\operatorname{Re} z > -\frac{1}{2}$. For any μ such that $\operatorname{Re}\mu > 0$, integration by parts yields

$$\begin{aligned}
&\int_{1/2}^{5/2}\left\{\log\left|\Gamma_{\mathbb{R}}\left(\tfrac{5}{2}+\mu+it\right)\right| - \log|\Gamma_{\mathbb{R}}(\sigma+\mu+it)|\right\}\mathrm{d}\sigma \\
&= \int_{1/2}^{5/2}\left(\sigma-\tfrac{1}{2}\right)\operatorname{Re}\frac{\Gamma_{\mathbb{R}}'}{\Gamma_{\mathbb{R}}}(\sigma+\mu+it)\,\mathrm{d}\sigma \\
&= \int_{1/2}^{5/2}\left(\sigma-\tfrac{1}{2}\right)\left(\tfrac{1}{2}\log|2+\sigma+\mu+it| - \operatorname{Re}\frac{1}{\sigma+\mu+it}\right)\mathrm{d}\sigma + O(1) \\
&= \frac{1}{2}\int_{1/2}^{5/2}\left(\sigma-\tfrac{1}{2}\right)\log|2+\sigma+\mu+it|\,\mathrm{d}\sigma + O(1) \\
&= \frac{1}{2}\int_{1/2}^{5/2}\left(\sigma-\tfrac{1}{2}\right)\log(|\mu+it|+3)\,\mathrm{d}\sigma + O(1) \\
&= \log(|\mu+it|+3) + O(1).
\end{aligned}$$

If $-1 < \operatorname{Re}\mu \le 0$, the relation $\Gamma_{\mathbb{R}}(z+2) = \frac{z}{2\pi}\Gamma_{\mathbb{R}}(z)$ brings us back to the previous case. Indeed,

$$\int_{1/2}^{5/2}\log\left\{\left|\Gamma_{\mathbb{R}}\left(\tfrac{5}{2}+\mu+it\right)\right| - \log|\Gamma_{\mathbb{R}}(\sigma+\mu+it)|\right\}\mathrm{d}\sigma$$

$$
\begin{aligned}
&= \int_{1/2}^{5/2} \left\{ \log \left| \Gamma_{\mathbb{R}} \left(\tfrac{9}{2} + \mu + it \right) \right| - \log \left| \Gamma_{\mathbb{R}}(2 + \sigma + \mu + it) \right| - \log \left| \frac{\frac{5}{2} + \mu + it}{\sigma + \mu + it} \right| \right\} \mathrm{d}\sigma \\
&= \log(|2 + \mu + it| + 3) + O(1) + O\left(\int_{1/2}^{5/2} \log \left| \frac{\frac{5}{2} + \operatorname{Re} \mu}{\sigma + \operatorname{Re} \mu} \right| \, \mathrm{d}\sigma \right) \\
&= \log(|\mu + it| + 3) + O(1),
\end{aligned}
$$

as before.

Finally, $\int_{1/2}^{5/2} \log |N^{(5/2+it)/2}| - \log |N^{(\sigma+it)/2}| \, \mathrm{d}\sigma = \log N$. Combining our computations we get

$$
\begin{aligned}
\pi S_1(t, \pi) &= -\sum_{\gamma} F_1(t - \gamma) + \log N + \sum_{j=1}^{m} \log(|\mu_j + it| + 3) + O(m) \\
&= -\sum_{\gamma} F_1(t - \gamma) + \log C(t, \pi) + O(m).
\end{aligned}
$$

This completes the proof of the lemma. $\square$

To estimate the infinite sum that appears in the preceding lemma, we employ the Guinand-Weil explicit formula. Its statement depends on noting that, by the product expansion of $L(\cdot, \pi)$,

$$
\frac{L'}{L}(s, \pi) = -\sum_{p} \sum_{j=1}^{m} \frac{\alpha_{j,\pi}(p)}{p^s} \left(1 - \frac{\alpha_{j,\pi}(p)}{p^s} \right)^{-1} \log p \,,
$$

where the right-hand side converges absolutely if $\operatorname{Re} s > 1$. This shows that the logarithmic derivative of $L(\cdot, \pi)$ has a Dirichlet series:

$$
\frac{L'}{L}(s, \pi) = -\sum_{n=2}^{\infty} \frac{\Lambda_\pi(n)}{n^s}, \tag{2.2.4}
$$

where $\Lambda_\pi(n) = 0$ if n is not a power of prime and $\Lambda_\pi(p^k) = \sum_{j=1}^{m} \alpha_{j,\pi}(p)^k \log p$ if p is prime and k is a positive integer.

Lemma 5. *Let h be an analytic function defined on a strip $\{z \in \mathbb{C};\ -\frac{1}{2} - \varepsilon < \operatorname{Im} z < \frac{1}{2} + \varepsilon\}$*

such that $h(z)(1+|z|)^{1+\delta}$ is bounded for some positive δ. Then

$$\begin{aligned}\sum_{\rho} h\left(\frac{\rho-\frac{1}{2}}{i}\right) =& r(\pi)\left\{h\left(\frac{1}{2i}\right)+h\left(-\frac{1}{2i}\right)\right\}+\frac{\log N}{2\pi}\int_{-\infty}^{\infty} h(u)\,\mathrm{d}u \\ &+\frac{1}{\pi}\sum_{j=1}^{m}\int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\,\frac{\Gamma_{\mathbb{R}}'}{\Gamma_{\mathbb{R}}}\left(\tfrac{1}{2}+\mu_j+iu\right)\mathrm{d}u \\ &-\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\left\{\Lambda_\pi(n)\,\widehat{h}\left(\frac{\log n}{2\pi}\right)+\overline{\Lambda_\pi(n)}\,\widehat{h}\left(\frac{-\log n}{2\pi}\right)\right\} \\ &-\sum_{-1<\mathrm{Re}\,\mu_j<-\frac{1}{2}}\left\{h\left(\frac{-\mu_j-\frac{1}{2}}{i}\right)+h\left(\frac{\mu_j+\frac{1}{2}}{i}\right)\right\} \\ &-\frac{1}{2}\sum_{\mathrm{Re}\,\mu_j=-\frac{1}{2}}\left\{h\left(\frac{-\mu_j-\frac{1}{2}}{i}\right)+h\left(\frac{\mu_j+\frac{1}{2}}{i}\right)\right\},\end{aligned} \tag{2.2.5}$$

where the sum runs over all zeros ρ of $\Lambda(\cdot,\pi)$, the coefficients $\Lambda_\pi(n)$ are defined by (2.2.4) *and $\widehat{h}(\xi)=\int_{-\infty}^{\infty} h(x)e^{-2\pi i x\xi}\,\mathrm{d}x$.*

Proof. Let $T_1, T_2 > 0$. Let γ be the rectangular countour with vertices at the points $1+\eta+iT_1, -\eta+iT_1, -\eta-iT_2$ and $1+\eta-iT_2$, for $\eta=\varepsilon/2$. By the residue theorem we have

$$\sum_{\rho\in\gamma} h\left(\frac{\rho-\frac{1}{2}}{i}\right)-r(\pi)\left\{h\left(\frac{1}{2i}\right)+h\left(-\frac{1}{2i}\right)\right\}=\frac{1}{2\pi i}\int_{\gamma} h\left(\frac{s-\frac{1}{2}}{i}\right)\frac{\Lambda'(s,\pi)}{\Lambda(s,\pi)}\,\mathrm{d}s,$$

where the first sum is over the zeros of $\Lambda(\cdot,\pi)$ inside the countour and the second sum is given by the poles.

Step 1. Sending $T_1, T_2 \to \infty$

We choose T_1 and T_2 as "away" as possible from the zeros. At height T, we have $O(\log C(T,\pi))$ zeros (see [23, Proposition 5.7]), and hence we may choose T_1 so that the vertical spacing to any zero is virtually greater than $\frac{1}{\log C(T_1,\pi)}$ (and the same for T_2). From Hadamard's factorization of Λ'/Λ and basic estimates on certain sums over zeros, one can show that the contribution over the horizontal lines approaches zero as $T_1, T_2 \to \infty$.

Step 2. Applying the functional equation

We have then reduced matters to

$$\begin{aligned}&\sum_\rho h\Big(\frac{\rho-\frac12}{i}\Big)-r(\pi)\left\{h\left(\frac{1}{2i}\right)+h\left(-\frac{1}{2i}\right)\right\}\\ =&\frac{1}{2\pi i}\left[\int_{\gamma_1}h\Big(\frac{s-\frac12}{i}\Big)\frac{\Lambda'(s,\pi)}{\Lambda(s,\pi)}\,\mathrm{d}s+\int_{\gamma_2}h\Big(\frac{s-\frac12}{i}\Big)\frac{\Lambda'(s,\pi)}{\Lambda(s,\pi)}\,\mathrm{d}s\right],\end{aligned}$$

where $\gamma_1=\{s\in\mathbb{C};\operatorname{Re}s=-\eta\}$ and $\gamma_2=\{s\in\mathbb{C};\operatorname{Re}s=1+\eta\}$, with the appropriate orientation. From the functional equation we have

$$\frac{\Lambda'(s,\pi)}{\Lambda(s,\pi)}=-\frac{\Lambda'(1-s,\tilde\pi)}{\Lambda(1-s,\tilde\pi)}.$$

Hence

$$\begin{aligned}\int_{\gamma_1}h\Big(\frac{s-\frac12}{i}\Big)\frac{\Lambda'(s,\pi)}{\Lambda(s,\pi)}\,\mathrm{d}s&=-\int_{\gamma_1}h\Big(\frac{s-\frac12}{i}\Big)\frac{\Lambda'(1-s,\tilde\pi)}{\Lambda(1-s,\tilde\pi)}\,\mathrm{d}s\\&=\int_{\gamma_2}h\Big(\frac{\frac12-w}{i}\Big)\frac{\Lambda'(w,\tilde\pi)}{\Lambda(w,\tilde\pi)}\,\mathrm{d}w,\end{aligned}$$

and we arrive at

$$\begin{aligned}&\sum_\rho h\Big(\frac{\rho-\frac12}{i}\Big)-r(\pi)\left\{h\left(\frac{1}{2i}\right)+h\left(-\frac{1}{2i}\right)\right\}\\&=\frac{1}{2\pi i}\left(\int_{\gamma_2}h\Big(\frac{s-\frac12}{i}\Big)\frac{\Lambda'(s,\pi)}{\Lambda(s,\pi)}\,\mathrm{d}s+\int_{\gamma_2}h\Big(\frac{\frac12-s}{i}\Big)\frac{\Lambda'(s,\tilde\pi)}{\Lambda(s,\tilde\pi)}\,\mathrm{d}s\right)\\&=\frac{1}{2\pi i}\left(\int_{\gamma_2}h\Big(\frac{s-\frac12}{i}\Big)\frac{L'(s,\pi)}{L(s,\pi)}\,\mathrm{d}s+\int_{\gamma_2}h\Big(\frac{\frac12-s}{i}\Big)\frac{L'(s,\tilde\pi)}{L(s,\tilde\pi)}\,\mathrm{d}s\right)+\\&\qquad+\frac{1}{2\pi i}\left(\int_{\gamma_2}h\Big(\frac{s-\frac12}{i}\Big)\frac{L'(s,\pi_\infty)}{L(s,\pi_\infty)}\,\mathrm{d}s+\int_{\gamma_2}h\Big(\frac{\frac12-s}{i}\Big)\frac{L'(s,\tilde\pi_\infty)}{L(s,\tilde\pi_\infty)}\,\mathrm{d}s\right),\end{aligned}$$

where $L(s,\pi_\infty)=N^{s/2}\prod_{j=1}^m\Gamma_\mathbb{R}(s+\mu_j)$.

Step 3. Analyzing the first two integrals

Note that, if $\operatorname{Re} s = 1+\eta$,

$$\left|\frac{L'}{L}(s,\pi)\right| = \left|\sum_{n=1}^{\infty} \frac{\Lambda_\pi(n)}{n^s}\right| \leq \sum_{n=1}^{\infty} \frac{|\Lambda_\pi(n)|}{n^{1+\eta}} < \infty,$$

from the hypothesis that this Dirichlet series is absolutely convergent on $\operatorname{Re} s > 1$. Since h has good decay as $|s| \to \infty$, the first integral

$$I_1 = \frac{1}{2\pi i}\int_{\gamma_2} h\left(\frac{s-\frac{1}{2}}{i}\right)\frac{L'(s,\pi)}{L(s,\pi)}\,\mathrm{d}s = -\frac{1}{2\pi i}\int_{\gamma_2} h\left(\frac{s-\frac{1}{2}}{i}\right)\sum_{n=1}^{\infty}\frac{\Lambda_\pi(n)}{n^s}\,\mathrm{d}s$$

converges absolutely. Therefore, we may interchange summation and integral to arrive at

$$I_1 = -\frac{1}{2\pi i}\sum_{n=1}^{\infty}\Lambda_\pi(n)\int_{\gamma_2} h\left(\frac{s-\frac{1}{2}}{i}\right)\frac{1}{n^s}\,\mathrm{d}s.$$

Now it is clear that we can move each integration from the vertical line $\operatorname{Re} s = 1+\eta$ to the vertical line $\operatorname{Re} s = 1/2$. We then change the variables $s = \frac{1}{2}+it$ to get

$$\begin{aligned} I_1 &= -\frac{1}{2\pi}\sum_{n=1}^{\infty}\Lambda_\pi(n)\int_{-\infty}^{\infty} h(t)\frac{1}{n^{\frac{1}{2}+it}}\,\mathrm{d}t \\ &= -\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\Lambda_\pi(n)}{\sqrt{n}}\widehat{h}\left(\frac{\log n}{2\pi}\right). \end{aligned}$$

Analogously, we get

$$I_2 = \frac{1}{2\pi i}\int_{\gamma_2} h\left(\frac{\frac{1}{2}-s}{i}\right)\frac{L'(s,\tilde{\pi})}{L(s,\tilde{\pi})}\,\mathrm{d}s = -\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\overline{\Lambda_\pi(n)}}{\sqrt{n}}\widehat{h}\left(-\frac{\log n}{2\pi}\right)$$

This term appears in our explicit formula.

Step 4. Analyzing the remaining two integrals

We are left with the analysis of

$$I_3+I_4 = \frac{1}{2\pi i}\left(\int_{\gamma_2} h\left(\frac{s-\frac{1}{2}}{i}\right)\frac{L'(s,\pi_\infty)}{L(s,\pi_\infty)}\,\mathrm{d}s + \int_{\gamma_2} h\left(\frac{\frac{1}{2}-s}{i}\right)\frac{L'(s,\tilde{\pi}_\infty)}{L(s,\tilde{\pi}_\infty)}\,\mathrm{d}s\right).$$

We again aim to move the line of integration from γ_2 to the line $\{s \in \mathbb{C}; \operatorname{Re} s = 1/2\}$. In doing so, we may pick up residues coming from some poles of $L(s, \pi_\infty)$. These occur when we have a local parameter μ_j such that $-1 < \operatorname{Re} \mu_j \leq -1/2$ (note that when we have $\operatorname{Re} \mu_j = -1/2$ we shall pick half of the residue). Also, recall that $L(s, \pi_\infty) = L(s, \tilde{\pi}_\infty)$. Hence we have:

$$\begin{aligned} I_3 + I_4 = &\frac{1}{2\pi i} \left(\int_{\operatorname{Re} s = 1/2} h\left(\frac{s - \frac{1}{2}}{i}\right) \frac{L'(s, \pi_\infty)}{L(s, \pi_\infty)} \, \mathrm{d}s + \int_{\operatorname{Re} s = 1/2} h\left(\frac{\frac{1}{2} - s}{i}\right) \frac{L'(s, \tilde{\pi}_\infty)}{L(s, \tilde{\pi}_\infty)} \, \mathrm{d}s \right) \\ &- \sum_{-1 < \operatorname{Re} \mu_j < -\frac{1}{2}} \left[h\left(\frac{-\mu_j - \frac{1}{2}}{i}\right) + h\left(\frac{\mu_j + \frac{1}{2}}{i}\right) \right] \\ &- \frac{1}{2} \sum_{\operatorname{Re} \mu_j = -\frac{1}{2}} \left[h\left(\frac{-\mu_j - \frac{1}{2}}{i}\right) + h\left(\frac{\mu_j + \frac{1}{2}}{i}\right) \right] \end{aligned}$$

Observe that when we have some μ_j with $\operatorname{Re} \mu_j = -\frac{1}{2}$, then $L(s, \pi_\infty)$ has a pole at $s = -\mu_j = \frac{1}{2} + ib_j$. The integral

$$\frac{1}{2\pi i} \int_{\operatorname{Re} s = 1/2} h\left(\frac{s - \frac{1}{2}}{i}\right) \frac{L'(s, \pi_\infty)}{L(s, \pi_\infty)} \, \mathrm{d}s$$

should then be interpreted as the limit

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} h(t) \frac{L'(\frac{1}{2} + it, \pi_\infty)}{L(\frac{1}{2} + it, \pi_\infty)} \, \mathrm{d}t = \lim_{r \to 0^+} \frac{1}{2\pi} \int_{\mathbb{R} \setminus [b_j - r, b_j + r]} h(t) \frac{L'(\frac{1}{2} + it, \pi_\infty)}{L(\frac{1}{2} + it, \pi_\infty)} \, \mathrm{d}t$$

We can then relabel the variables in order to get

$$\begin{aligned} I_3 + I_4 = &\frac{1}{\pi} \int_{-\infty}^{\infty} h(t) \operatorname{Re} \frac{L'(\frac{1}{2} + it, \pi_\infty)}{L(\frac{1}{2} + it, \pi_\infty)} \, \mathrm{d}t \\ &- \sum_{-1 < \operatorname{Re} \mu_j < -\frac{1}{2}} \left[h\left(\frac{-\mu_j - \frac{1}{2}}{i}\right) + h\left(\frac{\mu_j + \frac{1}{2}}{i}\right) \right] \\ &- \frac{1}{2} \sum_{\operatorname{Re} \mu_j = -\frac{1}{2}} \left[h\left(\frac{-\mu_j - \frac{1}{2}}{i}\right) + h\left(\frac{\mu_j + \frac{1}{2}}{i}\right) \right]. \end{aligned}$$

□

We are going to take h equal to an analytic minorant or majorant of F_1. To have a good estimate of the right-hand side of (2.2.5), it is convenient to choose $\widehat{h}$ compactly supported and to minimize the L^1-norm of $h - F_1$. This problem was dealt with in [6]. The following lemma is a rescaling of the obtained conclusion.

Lemma 6. *For every $\Delta \geq 1$, there is a unique pair of real entire functions $G_\Delta^- : \mathbb{C} \to \mathbb{C}$ and $G_\Delta^+ : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:*

(i) For real x we have

$$\frac{-c}{1+x^2} \leq G_\Delta^-(x) \leq F_1(x) \leq G_\Delta^+(x) \leq \frac{c}{1+x^2},$$

for some positive constant c. Moreover, for any complex number z we have

$$|G_\Delta^\pm(z)| = O\left(\frac{\Delta^2}{1+\Delta|z|} e^{2\pi\Delta|\operatorname{Im} z|}\right).$$

(ii) The Fourier transforms of $G_\Delta^\pm$ are continuous functions supported on the interval $[-\Delta, \Delta]$ and satisfy

$$|\widehat{G}_\Delta^\pm(\xi)| = O(1)$$

for all $\xi \in [-\Delta, \Delta]$.

(iii) The L^1-distances of $G_\Delta^\pm$ to F_1 are given by

$$\int_{-\infty}^{\infty} \left\{F_1(x) - G_\Delta^-(x)\right\} \mathrm{d}x = \frac{2}{\Delta} \int_{1/2}^{3/2} \left\{\log\left(1+e^{-4\pi\Delta(\sigma-1/2)}\right) - \log\left(1+e^{-4\pi\Delta}\right)\right\} \mathrm{d}\sigma$$

and

$$\int_{-\infty}^{\infty} \left\{G_\Delta^+(x) - F_1(x)\right\} \mathrm{d}x = -\frac{2}{\Delta} \int_{1/2}^{3/2} \left\{\log\left(1-e^{-4\pi\Delta(\sigma-1/2)}\right) - \log\left(1-e^{-4\pi\Delta}\right)\right\} \mathrm{d}\sigma.$$

Proof. This is a slightly different version of [6, Lemma 4]. The definitions imply that $F_1(x) = 2f_1\left(\frac{x}{2}\right)$, so that one only needs to take $G_\Delta^\pm(z) = 2g_{2\Delta}^\pm\left(\frac{z}{2}\right)$ in the notation of [6, Lemma 4]. □

Observe that the L^1-distances given in Lemma 6 (iii) are of magnitude $1/\Delta^2$. Indeed,

$$\begin{aligned}
\int_{-\infty}^{\infty} \left\{F_1(x) - G_\Delta^-(x)\right\} \mathrm{d}x &= \frac{1}{2\pi\Delta^2} \int_0^{4\pi\Delta} \left\{\log\left(1+e^{-x}\right) - \log\left(1+e^{-4\pi\Delta}\right)\right\} \mathrm{d}x \\
&\leq \frac{1}{2\pi\Delta^2} \int_0^{\infty} \log\left(1+e^{-x}\right) \mathrm{d}x \\
&= \frac{1}{2\pi\Delta^2} \int_0^{1} \frac{\log(1+y)}{y} \mathrm{d}y \\
&= \frac{1}{2\pi\Delta^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} y^{n-1} \mathrm{d}y \\
&= \frac{\pi}{24\Delta^2},
\end{aligned} \tag{2.2.6}$$

because the Taylor series of $\frac{\log(1+y)}{y}$ converges uniformly in $[0,1]$. Similarly,

$$\begin{aligned}
\int_{-\infty}^{\infty} \left\{G_\Delta^+(x) - F_1(x)\right\} \mathrm{d}x &= \frac{1}{2\pi\Delta^2} \int_0^{4\pi\Delta} \left\{\log\left(1-e^{-x}\right)^{-1} - \log\left(1-e^{-4\pi\Delta}\right)^{-1}\right\} \mathrm{d}x \\
&\leq \frac{1}{2\pi\Delta^2} \int_0^{\infty} \log\left(1-e^{-x}\right)^{-1} \mathrm{d}x \\
&= \frac{1}{2\pi\Delta^2} \int_0^{1} \frac{\log(1-y)^{-1}}{y} \mathrm{d}y \\
&= \frac{1}{2\pi\Delta^2} \int_0^{1} \sum_{n=1}^{\infty} \frac{1}{n} y^{n-1} \mathrm{d}y \\
&= \frac{\pi}{12\Delta^2},
\end{aligned} \tag{2.2.7}$$

because all the terms of the sum are positive.

We are now ready to prove Theorem 1. The strategy is to apply Lemma 5 to the functions $G_\Delta^-(t-\cdot)$ and $G_\Delta^+(t-\cdot)$, to find bounds for $S_1(t,\pi)$ that depend on Δ and to optimize the choice of Δ.

Proof of Theorem 1. We first prove the upper bound. For each $\Delta \geq 1$, take G_Δ^- as in Lemma 6 and let $h(z) = G_\Delta^-(t-z)$. By Lemma 4,

$$S_1(t,\pi) \leq \frac{1}{\pi}\left(-\sum_{\gamma} h(\gamma) + \log C(t,\pi)\right) + O(m). \tag{2.2.8}$$

By Lemma 6 (i), the function $G_\Delta^-(z)(1+z^2)$ is bounded on the real line and $G_\Delta^-(z) =$

$O(\Delta^2 e^{2\pi\Delta|\mathrm{Im} z|})$. An application of the Phragmén-Lindelöf principle for the function $G_\Delta^-(z)(1+z^2)e^{2\pi\Delta iz}$ tells us that this function is bounded on the upper half plane. Hence $z \mapsto G_\Delta^-(z)(1+z^2)$ is bounded on the strip $0 \le \mathrm{Im}\, z \le \frac{1}{2} + \varepsilon$ (for any $\varepsilon > 0$), and since it is real entire, it is bounded on the strip $-\frac{1}{2} - \varepsilon \le \mathrm{Im}\, z \le \frac{1}{2} + \varepsilon$. Therefore, h satisfies the hypotheses of Lemma 5 and we obtain

$$\begin{aligned}\sum_\gamma h(\gamma) =& \frac{\log N}{2\pi} \int_{-\infty}^{\infty} h(u)\,\mathrm{d}u + \frac{1}{\pi} \sum_{j=1}^{m} \int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\,\frac{\Gamma'_\mathbb{R}}{\Gamma_\mathbb{R}}\left(\tfrac{1}{2} + \mu_j + iu\right)\mathrm{d}u \\ &- \frac{1}{2\pi}\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \left\{ \Lambda_\pi(n)\,\widehat{h}\left(\frac{\log n}{2\pi}\right) + \overline{\Lambda_\pi(n)}\,\widehat{h}\left(\frac{-\log n}{2\pi}\right)\right\} \\ &+ O(m\Delta^2 e^{\pi\Delta}).\end{aligned} \tag{2.2.9}$$

For each index $j = 1, 2, \dots, m$, Stirling's formula (2.2.3) yields

$$\begin{aligned}\int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\,\frac{\Gamma'_\mathbb{R}}{\Gamma_\mathbb{R}}\left(\tfrac{1}{2} + \mu_j + iu\right)\mathrm{d}u = \frac{1}{2}\int_{-\infty}^{\infty} & G_\Delta^-(t-u)\log\left|\tfrac{5}{2} + \mu_j + iu\right|\,\mathrm{d}u \\ - \int_{-\infty}^{\infty} & G_\Delta^-(t-u)\,\mathrm{Re}\left(\frac{1}{\mu_j + \frac{1}{2} + iu}\right)\mathrm{d}u + O(1).\end{aligned}$$

Combining this with the inequality

$$\begin{aligned}\left|\int_{-\infty}^{\infty} G_\Delta^-(t-u)\,\mathrm{Re}\left(\frac{1}{\mu_j + \frac{1}{2} + iu}\right)\mathrm{d}u\right| &\le c\int_{-\infty}^{\infty} \left|\mathrm{Re}\left(\frac{1}{\mu_j + \frac{1}{2} + iu}\right)\right|\,\mathrm{d}u \\ &= c\int_{-\infty}^{\infty} \frac{|\mathrm{Re}\,\mu_j + \frac{1}{2}|}{(\mathrm{Re}\,\mu_j + \frac{1}{2})^2 + u^2}\,\mathrm{d}u \\ &\le \pi c\end{aligned}$$

we find that

$$
\begin{aligned}
&\int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\,\frac{\Gamma_{\mathbb{R}}'}{\Gamma_{\mathbb{R}}}\left(\tfrac{1}{2}+\mu_j+iu\right)\mathrm{d}u\\
=&\frac{1}{2}\int_{-\infty}^{\infty} G_{\Delta}^{-}(t-u)\log\left|\tfrac{5}{2}+\mu_j+iu\right|\,\mathrm{d}u+O(1)\\
=&\frac{1}{2}\int_{-\infty}^{\infty} G_{\Delta}^{-}(u)\log\left|\tfrac{5}{2}+\mu_j+it-iu\right|\,\mathrm{d}u+O(1)\\
=&\frac{1}{2}\int_{-\infty}^{\infty} G_{\Delta}^{-}(u)\big\{\log(|\mu_j+it|+3)+O(\log(|u|+2))\big\}\mathrm{d}u+O(1)\\
=&\frac{1}{2}\log(|\mu_j+it|+3)\int_{-\infty}^{\infty} G_{\Delta}^{-}(u)\,\mathrm{d}u+O(1).
\end{aligned}
\tag{2.2.10}
$$

By Lemma 6 (ii), the Fourier transform $\widehat{h}(\xi)=e^{-2\pi it\xi}\,\widehat{G}_{\Delta}^{-}(-\xi)$ is supported on $[-\Delta,\Delta]$ and is uniformly bounded. Also, $|\Lambda_\pi(n)|\leq m\Lambda(n)n^{\vartheta}$, and therefore

$$
\begin{aligned}
\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\left\{\Lambda_\pi(n)\,\widehat{h}\left(\frac{\log n}{2\pi}\right)+\overline{\Lambda_\pi(n)}\,\widehat{h}\left(\frac{-\log n}{2\pi}\right)\right\}&=O\left(m\sum_{n\leq e^{2\pi\Delta}}\Lambda(n)n^{\vartheta-\frac{1}{2}}\right)\\
&=O\left(me^{(1+2\vartheta)\pi\Delta}\right),
\end{aligned}
\tag{2.2.11}
$$

where the last equality follows by the Prime Number Theorem and summation by parts.

In view of (2.2.10) and (2.2.11), equation (2.2.9) becomes

$$
\sum_{\gamma}h(\gamma)=\frac{\log C(t,\pi)}{2\pi}\int_{-\infty}^{\infty}G_{\Delta}^{-}(u)\,\mathrm{d}u+O\left(me^{(1+2\vartheta)\pi\Delta}\right)+O\left(m\Delta^2e^{\pi\Delta}\right).
$$

Inserting this in (2.2.8) and using (2.2.6) together with the fact that $\int_{-\infty}^{\infty}F_1(x)\,\mathrm{d}x=2\pi$, we obtain

$$
S_1(t,\pi)\leq\frac{\log C(t,\pi)}{48\pi\Delta^2}+O(m\Delta^2e^{(1+2\vartheta)\pi\Delta})
$$

for any t and any $\Delta\geq1$. Choosing

$$
\Delta=\max\left\{\frac{\log\left(\frac{3}{m}\log C(t,\pi)\right)-5\log\log\left(\frac{3}{m}\log C(t,\pi)\right)}{(1+2\vartheta)\pi},1\right\},
$$

we arrive at the desired conclusion.

The proof of the lower bound follows the same lines, using $h(z)=G_{\Delta}^{+}(t-z)$ and inequality (2.2.7). □

2.3 Theorem 1 implies Theorem 2

For real numbers $t < u$ let us denote by $N(t,u,\pi)$ the number of nontrivial zeros of $L(\cdot,\pi)$ with ordinates γ such that $t \le \gamma \le u$, counted with multiplicity (zeros with ordinates equal to the endpoints t or u are counted with half of their multiplicities). The following fact connects the variation of $S(\cdot,\pi)$ to the nontrivial zeros of $L(\cdot,\pi)$, like equation (2.1.1) in the case of ζ.

Lemma 7. *Let t and u be real numbers such that $t < u \le t+5$. Then*

$$N(t,u,\pi) = S(u,\pi) - S(t,\pi) + \frac{u-t}{2\pi}\log C(t,\pi) + O(m). \tag{2.3.1}$$

Proof. If $v \neq 0$ and v is not the ordinate of a zero of $L(\cdot,\pi)$, then $S'(v,\pi) = \frac{1}{\pi}\mathrm{Re}\frac{L'}{L}(\frac{1}{2}+iv,\pi)$. For each non-trivial zero $\rho = \frac{1}{2}+i\gamma$ of $L(\cdot,\pi)$ of multiplicity η, the function $S(\cdot,\pi)$ jumps by η at this ordinate; for each non-trivial zero $\rho = \sigma + i\gamma$ of $L(\cdot,\pi)$, with $\frac{1}{2} < \sigma \le 1$, of multiplicity η, the functional equation (2.1.7) implies that $\rho = (1-\sigma)+i\gamma$ is a non-trivial zero with the same multiplicity η, and the function $S(\cdot,\pi)$ jumps by 2η at this ordinate (this jump comes from the zero with $\frac{1}{2} < \sigma$ in the contour integration defining the argument function); at 0 it jumps by $-2r(\pi)$; for each j such that $\mathrm{Re}\,\mu_j = -\frac{1}{2}$, it jumps by 1 at $-\mathrm{Im}\,\mu_j$; and for each j such that $-1 < \mathrm{Re}\,\mu_j < -\frac{1}{2}$, it jumps by 2 at $-\mathrm{Im}\,\mu_j$. Therefore

$$N(t,u,\pi) = S(u,\pi) - S(t,\pi) - \frac{1}{\pi}\int_t^u \mathrm{Re}\,\frac{L'}{L}\left(\tfrac{1}{2}+iv,\pi\right)\mathrm{d}v + O(m).$$

By the definition of $\Lambda(\cdot,\pi)$,

$$\frac{\Lambda'}{\Lambda}(s,\pi) = \frac{\log N}{2} + \sum_{j=1}^{m}\frac{\Gamma_{\mathbb{R}}'}{\Gamma_{\mathbb{R}}}(s+\mu_j) + \frac{L'}{L}(s,\pi).$$

By the functional equation (2.1.7), the real part of $\frac{\Lambda'}{\Lambda}(\cdot,\pi)$ vanishes on the line $\frac{1}{2}+iv$. Therefore

$$\begin{aligned} -\int_t^u \mathrm{Re}\,\frac{L'}{L}\left(\tfrac{1}{2}+iv,\pi\right)\mathrm{d}v &= \int_t^u \left\{\frac{\log N}{2} + \sum_{j=1}^{m}\mathrm{Re}\,\frac{\Gamma_{\mathbb{R}}'}{\Gamma_{\mathbb{R}}}\left(\mu_j+\tfrac{1}{2}+iv\right)\right\}\mathrm{d}v \\ &= \frac{1}{2}\int_t^u\left\{\log N + \sum_{j=1}^{m}\log\left|\tfrac{5}{2}+\mu_j+iv\right|\right\}\mathrm{d}v + O(m) \\ &= \frac{u-t}{2}\log C(t,\pi) + O(m). \end{aligned}$$

$\square$

To derive Theorem 2 from Theorem 1, we recall the fact that $S_1(\cdot, \pi)$ is a primitive for $S(\cdot, \pi)$. Indeed, for almost every real v,

$$S(v, \pi) = -\frac{1}{\pi} \int_{1/2}^{\infty} \mathrm{Im} \frac{L'}{L} (\sigma + iv, \pi) \, \mathrm{d}\sigma.$$

The function $\frac{L'}{L}(\sigma + iv, \pi)$ is absolutely integrable in the region $\{s \in \mathbb{C}; \mathrm{Re}\, s \geq \frac{1}{2} \text{ and } t \leq \mathrm{Im}\, s \leq u\}$ since it has only simple poles and decays exponentially as $\sigma \to \infty$. So we can apply Fubini's theorem to get

$$\begin{aligned}
\int_t^u S(v, \pi) \, \mathrm{d}v &= -\frac{1}{\pi} \int_t^u \int_{1/2}^{\infty} \mathrm{Im} \frac{L'}{L} (\sigma + iv, \pi) \, \mathrm{d}\sigma \, \mathrm{d}v \\
&= -\frac{1}{\pi} \int_{1/2}^{\infty} \int_t^u \mathrm{Im} \frac{L'}{L} (\sigma + iv, \pi) \, \mathrm{d}v \, \mathrm{d}\sigma \\
&= \frac{1}{\pi} \int_{1/2}^{\infty} \Big\{ \log |L (\sigma + iu, \pi)| - \log |L (\sigma + it, \pi)| \Big\} \, \mathrm{d}\sigma \\
&= S_1(u, \pi) - S_1(t, \pi),
\end{aligned}$$

as claimed.

Proof of Theorem 2. Let ν be a real number such that $0 < \nu \leq 5$ to be chosen later. The inequality

$$\left| \frac{3}{m} \log C(t + \nu, \pi) - \frac{3}{m} \log C(t, \pi) \right| \leq 3 \log 3$$

implies that

$$\frac{\frac{3}{m} \log C(t + \nu, \pi)}{\left(\log \left(\frac{3}{m} \log C(t + \nu, \pi)\right)\right)^2} = \frac{\frac{3}{m} \log C(t, \pi)}{\left(\log \left(\frac{3}{m} \log C(t, \pi)\right)\right)^2} + O(1).$$

Therefore, by Theorem 1 at heights t and $t + \nu$,

$$\begin{aligned}
|S_1(t + \nu, \pi) - S_1(t, \pi)| \leq &\frac{(1 + 2\vartheta)^2 \pi}{16} \frac{\log C(t, \pi)}{\left(\log \left(\frac{3}{m} \log C(t, \pi)\right)\right)^2} \\
&+ O\left(\frac{\log C(t, \pi) \log \log \left(\frac{3}{m} \log C(t, \pi)\right)}{\left(\log \left(\frac{3}{m} \log C(t, \pi)\right)\right)^3} \right).
\end{aligned}$$

Applying Lemma 7 we see that

$$\begin{aligned} S_1(t+\nu,\pi) - S_1(t,\pi) &= \int_t^{t+\nu} S(u,\pi)\,\mathrm{d}u \\ &\geq \int_t^{t+\nu} \left\{ S(t,\pi) - \frac{u-t}{2\pi}\, \log C(t,\pi) + O(m) \right\} \mathrm{d}u \\ &= \nu S(t,\pi) - \frac{\nu^2}{4\pi}\, \log C(t,\pi) + O(m) \end{aligned}$$

and thus

$$\begin{aligned} S(t,\pi) \leq & \frac{(1+2\vartheta)^2\pi}{16\nu} \frac{\log C(t,\pi)}{\left(\log\left(\frac{3}{m}\log C(t,\pi)\right)\right)^2} \left(1 + O\left(\frac{\log\log\left(\frac{3}{m}\log C(t,\pi)\right)}{\log\left(\frac{3}{m}\log C(t,\pi)\right)}\right)\right) \\ & + \frac{\nu}{4\pi} \log C(t,\pi). \end{aligned}$$

The upper bound for $S(t,\pi)$ is obtained with the choice

$$\nu = \frac{(1+2\vartheta)\pi}{2\log\left(\frac{3}{m}\log C(t,\pi)\right)}$$

(note that $0 < \nu \leq 5$) and the lower bound can be established by the same method, considering $S_1(t,\pi) - S_1(t-\nu,\pi)$. □

Proof of Corollary 3. (i) If $\rho = \frac{1}{2} + i\gamma$ is a zero of $\Lambda(\cdot,\pi)$, part (i) follows directly from Theorem 2 and identity (2.3.1) with $t = \gamma^-$ and $u = \gamma^+$.

(ii) By Lemma 7, if t and u are real numbers such that $t < u \leq t+5$ and $\Lambda(\cdot,\pi)$ has no zeros between $\frac{1}{2} + it$ and $\frac{1}{2} + iu$,

$$\frac{u-t}{2\pi} \log C(t,\pi) = -S(u,\pi) + S(t,\pi) + O(m).$$

By Theorem 2,

$$u - t \leq \frac{(1+2\vartheta)\pi}{\log\left(\frac{3}{m}\log C(t,\pi)\right)} + a\,\frac{\log\log\left(\frac{3}{m}\log C(t,\pi)\right)}{\left(\log\left(\frac{3}{m}\log C(t,\pi)\right)\right)^2}$$

for some universal constant a. If $\gamma' - \gamma \leq 5$, it is enough to let $t \to \gamma$ and $u \to \gamma'$. Otherwise, we let $u = t + 5$ and $t \to \gamma$. The obtained inequality is possible only if $\frac{3}{m}\log C(\gamma,\pi) \leq$

$e^{\max\{a,3\}}$. Taking $\Lambda(\cdot, \tilde{\pi})$ in place of $\Lambda(\cdot, \pi)$, we get $\frac{3}{m} \log C(\gamma', \pi) \leq e^{\max\{a,3\}}$. Then

$$\frac{1}{m}(\log C(\gamma, \pi) + \log C(\gamma', \pi)) \leq \frac{2}{3} e^{\max\{a,3\}},$$

and for some index j we must have

$$\log(|i\gamma + \mu_j| + 3) + \log(|i\gamma' + \mu_j| + 3) \leq \frac{2}{3} e^{\max\{a,3\}}.$$

This implies that $i\gamma + \mu_j$ and $i\gamma' + \mu_j$ are bounded by some universal constant. $\square$

Chapter 3

Variation-diminishing maximal operators

3.1 Introduction

3.1.1 Background

Let $\varphi : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ be a nonnegative function such that

$$\int_{\mathbb{R}^d} \varphi(x,t)\,\mathrm{d}x = 1 \tag{3.1.1}$$

for each $t > 0$. Assume also that, when $t \to 0$, the family $\varphi(\cdot,t)$ is an approximation of the identity, in the sense that $\lim_{t\to 0} \varphi(\cdot,t) * f(x) = f(x)$ for a.e. $x \in \mathbb{R}^d$, if $f \in L^p(\mathbb{R}^d)$ for some $1 \le p \le \infty$. For an initial datum $u_0 : \mathbb{R}^d \to \mathbb{R}$ we consider the evolution $u : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ given by

$$u(x,t) = \varphi(\cdot,t) * |u_0|(x),$$

and the associated maximal function

$$u^*(x) = \sup_{t>0} u(x,t).$$

For a fixed time $t > 0$, due to (3.1.1), the convolution $\varphi(\cdot,t) * |u_0|$ is simply a weighted average of $|u_0|$, and hence it does not increase its variation (understood as the classical total variation or, more generally, as an L^p-norm of the gradient for some $1 \le p \le \infty$). One of the questions that interest us here is to know whether this smoothing behavior is preserved when we pass to the maximal function u^*. For instance, if $u_0 : \mathbb{R} \to \mathbb{R}$ is a function of bounded variation, do we have

$$V(u^*) \leq C\, V(u_0) \tag{3.1.2}$$

with $C = 1$?

The most natural example of an operator in this framework is the Hardy-Littlewood maximal operator, in which $\varphi(x,t) = \frac{1}{t^d m(B_1)}\chi_{B_1}(x/t)$, where B_1 is the unit ball centered at the origin and $m(B_1)$ is its d-dimensional Lebesgue measure. In this case, due to the work of Kurka [28], the one-dimensional estimate (3.1.2) is known to hold with constant $C = 240{,}004$, but the problem with $C = 1$ remains open. For the one-dimensional right Hardy-Littlewood maximal operator, i.e. when $\varphi(x,t) = \frac{1}{t}\chi_{[0,1]}(x/t)$, estimate (3.1.2) holds with $C = 1$ due to the work of Tanaka [40]. The sharp bound (3.1.2) with constant $C = 1$ also holds for the one-dimensional uncentered version of this operator, as proved by Aldaz and Pérez Lázaro [1]. Higher dimensional analogues of (3.1.2) for the Hardy- Littlewood maximal operator, centered or uncentered, are open problems (see, for instance, the work of Hajłasz and Onninen [22]). Other interesting works related to the regularity of the Hardy-Littlewood maximal operator and its variants, when applied to Sobolev and bounded variation functions, are [2, 9, 12, 13, 21, 25, 26, 27, 30, 31, 32, 39, 41].

In Theorems 1 and 2 of [14] (the precursor of our article), Carneiro and Svaiter proved the variation-diminishing property, i.e. inequality (3.1.2) with $C = 1$, for the maximal operators associated to the Poisson kernel

$$P(x,t) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}}\,\frac{t}{\left(|x|^2 + t^2\right)^{(d+1)/2}} \tag{3.1.3}$$

and the Gauss kernel

$$K(x,t) = \frac{1}{(4\pi t)^{d/2}}\, e^{-|x|^2/4t}. \tag{3.1.4}$$

Their proof is based on an interplay between the analysis of the maximal functions and the structure of the underlying partial differential equations (Laplace's equation and heat equation). The aforementioned examples are the only maximal operators of convolution type for which inequality (3.1.2) has been established (even allowing a constant $C > 1$).

3.1.2 Maximal operators associated to elliptic equations

A question that derives from [14] is whether the variation-diminishing property is a peculiarity of the smooth kernels (3.1.3) and (3.1.4) or if these can be seen as particular cases of a general family. One could, for example, look at the semigroup structure via the Fourier transforms [1] (in space) of these kernels:

$$\widehat{P}(\xi,t) = e^{-t(2\pi|\xi|)} \quad \text{and} \quad \widehat{K}(\xi,t) = e^{-t(2\pi|\xi|)^2}.$$

A reasonable way to connect these kernels would be to consider the one-parameter family

$$\widehat{\varphi}_\alpha(\xi,t) = e^{-t(2\pi|\xi|)^\alpha},$$

for $1 \leq \alpha \leq 2$. However, in this case, the function $u(x,t) = \varphi_\alpha(\cdot,t) * u_0(x)$ solves an evolution equation related to the fractional Laplacian

$$u_t + (-\Delta)^{\alpha/2}\, u = 0\,,$$

for which we do not have a local maximum principle, essential to run the argument of Carneiro and Svaiter in [14]. The problem of proving that the corresponding maximal operator is variation-diminishing seems more delicate and it is currently open.

A more suitable way to address this question is to consider the Gauss kernel as an appropriate limiting case. For $a > 0$ and $b \geq 0$ we define (motivated by the partial differential equation (3.1.9) below)

$$\widehat{\varphi}_{a,b}(\xi,t) := e^{-t\left(\frac{-b+\sqrt{b^2+16a\pi^2|\xi|^2}}{2a}\right)}. \tag{3.1.5}$$

Note that when $a = 1$ and $b = 0$ we have the Fourier transform of the Poisson kernel, and when $b = 1$ and $a \to 0^+$ the function (3.1.5) tends pointwise to the Fourier transform of the Gauss kernel by a Taylor expansion. For completeness, let us then define

$$\widehat{\varphi}_{0,b}(\xi,t) := e^{-\frac{t}{b}(2\pi|\xi|)^2}, \tag{3.1.6}$$

for $b > 0$. We will show that the inverse Fourier transform

$$\varphi_{a,b}(x,t) = \int_{\mathbb{R}^d} \widehat{\varphi}_{a,b}(\xi,t)\, e^{2\pi i x\cdot\xi}\, \mathrm{d}\xi \tag{3.1.7}$$

is a *nonnegative radial function* that has the desired properties of an approximation of the

[1] Our normalization of the Fourier transform is $\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi}\, f(x)\, \mathrm{d}x$.

identity. Let us consider the corresponding maximal operator

$$u^*(x) = \sup_{t>0} \varphi_{a,b}(\cdot, t) * |u_0|(x). \tag{3.1.8}$$

The fact that $u^*(x) \leq Mu_0(x)$ pointwise, where M denotes the Hardy-Littlewood maximal operator, follows as in [37, Chapter III, Theorem 2]. Hence, for $1 < p \leq \infty$, we have $\|u^*\|_{L^p(\mathbb{R}^d)} \leq C\,\|u_0\|_{L^p(\mathbb{R}^d)}$ for some $C > 1$. We also notice, from the work of Kinnunen [25, proof of Theorem 1.4], that the maximal operator of convolution type (3.1.8) is bounded on $W^{1,p}(\mathbb{R}^d)$ for $1 < p \leq \infty$, with $\|\nabla u^*\|_{L^p(\mathbb{R}^d)} \leq C\,\|\nabla u_0\|_{L^p(\mathbb{R}^d)}$ for some $C > 1$.

The first result of this chapter establishes that the corresponding maximal operator (3.1.8) is indeed *variation-diminishing* in multiple contexts. This extends [14, Theorems 1 and 2].

Theorem 8. *Let $a, b \geq 0$ with $(a,b) \neq (0,0)$, and let u^* be the maximal function defined in* (3.1.8)*. The following propositions hold.*

(i) Let $1 < p \leq \infty$ and $u_0 \in W^{1,p}(\mathbb{R})$. Then $u^* \in W^{1,p}(\mathbb{R})$ and

$$\|(u^*)'\|_{L^p(\mathbb{R})} \leq \|u_0'\|_{L^p(\mathbb{R})}.$$

(ii) Let $u_0 \in W^{1,1}(\mathbb{R})$. Then $u^* \in L^\infty(\mathbb{R})$ and has a weak derivative $(u^*)'$ that satisfies

$$\|(u^*)'\|_{L^1(\mathbb{R})} \leq \|u_0'\|_{L^1(\mathbb{R})}.$$

(iii) Let u_0 be of bounded variation on $\mathbb{R}$. Then u^* is of bounded variation on $\mathbb{R}$ and

$$V(u^*) \leq V(u_0).$$

(iv) Let $d > 1$ and $u_0 \in W^{1,p}(\mathbb{R}^d)$, for $p = 2$ or $p = \infty$. Then $u^* \in W^{1,p}(\mathbb{R}^d)$ and

$$\|\nabla u^*\|_{L^p(\mathbb{R}^d)} \leq \|\nabla u_0\|_{L^p(\mathbb{R}^d)}.$$

We shall see that the kernel (3.1.7) has an elliptic character (when $a > 0$) in the sense that $u(x,t) = \varphi_{a,b}(\cdot, t) * |u_0|(x)$ solves the equation

$$au_{tt} - bu_t + \Delta u = 0 \quad \text{in} \quad \mathbb{R}^d \times (0, \infty) \tag{3.1.9}$$

with

$$\lim_{t\to 0^+} u(x,t) = |u_0(x)| \quad \text{a.e. in } \mathbb{R}^d.$$

In particular, the corresponding maximum principle plays a relevant role in our analysis. By appropriate dilations in the space variable x and the time variable t, Theorem 8 essentially splits into three regimes: (i) the case $a = 1$ and $b = 0$ (which models all cases $a > 0$ and $b = 0$, corresponding to Laplace's equation) in which the level surfaces $|\xi| = \tau$ in (3.1.5) are *cones*; (ii) the case $a = 0$ and $b = 1$ (which models all cases $a = 0$ and $b > 0$, corresponding to the heat equation), in which the level surfaces $|\xi|^2 = \tau$ in (3.1.6) are *paraboloids*; (iii) the case $a = 1$ and $b = 1$ (which models all the remaining cases $a > 0$ and $b > 0$), in which the level surfaces $-1 + \sqrt{1 + 16\pi^2|\xi|^2} = \tau$ in (3.1.5) are *hyperboloids*. The first two cases were proved in [14, Theorems 1 and 2] (although here we provide a somewhat different and simpler proof than that of [14]) and the third regime is the novel contribution of this section.

3.1.3 Periodic analogues

We now address similar problems in the torus $\mathbb{T}^d \simeq \mathbb{R}^d/\mathbb{Z}^d$. For $a > 0$, $b \geq 0$, $t > 0$ and $n \in \mathbb{Z}^d$ let us now define

$$\widehat{\Psi}_{a,b}(n,t) := e^{-t\left(\frac{-b+\sqrt{b^2+16a\pi^2|n|^2}}{2a}\right)},$$

and when $a = 0$ and $b > 0$ we define

$$\widehat{\Psi}_{0,b}(n,t) := e^{-\frac{t}{b}(2\pi|n|)^2}.$$

We then consider the periodic kernel, for $x \in \mathbb{R}^d$,

$$\Psi_{a,b}(x,t) = \sum_{n\in\mathbb{Z}^d} \widehat{\Psi}_{a,b}(n,t)\, e^{2\pi i x\cdot n}.$$

It is clear that $\Psi_{a,b} \in C^\infty(\mathbb{R}^d \times (0,\infty))$. By Poisson summation formula, $\Psi_{a,b}$ is simply the periodization of $\varphi_{a,b}$ defined in (3.1.7), i.e.

$$\Psi_{a,b}(x,t) = \sum_{n\in\mathbb{Z}^d} \varphi_{a,b}(x+n,t).$$

Since $\varphi_{a,b}$ is nonnegative, and $\widehat{\Psi}_{a,b}(n,t)$ is also nonnegative, it follows that

$$0 \leq \Psi_{a,b}(x,t) \leq \Psi_{a,b}(0,t)$$

for all $x \in \mathbb{R}^d$ and $t > 0$. The approximate identity properties of the family $\varphi_{a,b}(\cdot, t)$, reviewed in Section 3.2.1, transfer to $\Psi_{a,b}(\cdot, t)$ in the periodic setting. For an initial datum $u_0 : \mathbb{T}^d \to \mathbb{R}$ (which we identify with its periodic extension to $\mathbb{R}^d$) we keep denoting the evolution $u(x,t) : \mathbb{T}^d \times (0, \infty) \to \mathbb{R}^+$ by

$$u(x,t) = \Psi_{a,b}(\cdot, t) * |u_0|(x) = \int_{\mathbb{T}^d} \Psi_{a,b}(x-y,t)\, |u_0(y)|\, \mathrm{d}y = \int_{\mathbb{R}^d} \varphi_{a,b}(x-y,t)\, |u_0(y)|\, \mathrm{d}y. \tag{3.1.10}$$

Also, we keep denoting the maximal function $u^* : \mathbb{T}^d \to \mathbb{R}^+$ by

$$u^*(x) = \sup_{t>0} u(x,t). \tag{3.1.11}$$

From (3.1.10) it follows that $u^*(x) \leq M u_0(x)$, where M denotes the Hardy-Littlewood maximal operator on $\mathbb{R}^d$, and hence the operator $u_0 \mapsto u^*$ is bounded on $L^p(\mathbb{T}^d)$ for $1 < p \leq \infty$ and maps $L^1(\mathbb{T}^d)$ into $L^1_{weak}(\mathbb{T}^d)$ (the case $p = \infty$ is trivial; the case $p = 1$ follows by the usual Vitali covering argument; the general case $1 < p < \infty$ follows by Marcinkiewicz interpolation). Then, it follows as in [25, proof of Theorem 1.4] that $u_0 \mapsto u^*$ is bounded on $W^{1,p}(\mathbb{T}^d)$ for $1 < p \leq \infty$, with $\|\nabla u^*\|_{L^p(\mathbb{T}^d)} \leq C\, \|\nabla u_0\|_{L^p(\mathbb{T}^d)}$ for some $C > 1$.

Our second result establishes the variation-diminishing property for the operator (3.1.11) in several cases.

Theorem 9. *Let $a, b \geq 0$ with $(a,b) \neq (0,0)$, and let u^* be the maximal function defined in* (3.1.11). *The following propositions hold.*

(i) Let $1 < p \leq \infty$ and $u_0 \in W^{1,p}(\mathbb{T})$. Then $u^* \in W^{1,p}(\mathbb{T})$ and

$$\|(u^*)'\|_{L^p(\mathbb{T})} \leq \|u_0'\|_{L^p(\mathbb{T})}.$$

(ii) Let $u_0 \in W^{1,1}(\mathbb{T})$. Then $u^* \in L^\infty(\mathbb{T})$ and has a weak derivative $(u^*)'$ that satisfies

$$\|(u^*)'\|_{L^1(\mathbb{T})} \leq \|u_0'\|_{L^1(\mathbb{T})}.$$

(iii) Let u_0 be of bounded variation on $\mathbb{T}$. Then u^* is of bounded variation on $\mathbb{T}$ and

$$V(u^*) \leq V(u_0).$$

(iv) Let $d > 1$ and $u_0 \in W^{1,p}(\mathbb{T}^d)$, for $p = 2$ or $p = \infty$. Then $u^* \in W^{1,p}(\mathbb{T}^d)$ and

$$\|\nabla u^*\|_{L^p(\mathbb{T}^d)} \leq \|\nabla u_0\|_{L^p(\mathbb{T}^d)}.$$

As in the case of $\mathbb{R}^d$, a relevant feature for proving Theorem 9 is the fact that $u(x,t) = \Psi_{a,b}(\cdot,t) * |u_0|(x)$ solves the partial differential equation

$$au_{tt} - bu_t + \Delta u = 0 \quad \text{in} \quad \mathbb{T}^d \times (0,\infty)$$

with

$$\lim_{t\to 0^+} u(x,t) = |u_0(x)| \quad \text{a.e. in } \mathbb{T}^d.$$

3.1.4 Maximal operators on the sphere

The set of techniques presented here allows us to address similar problems on other manifolds. We exemplify this by considering here the Poisson maximal operator and the heat flow maximal operator on the sphere $\mathbb{S}^d$.

Poisson maximal operator

Let $u_0 \in L^p(\mathbb{S}^d)$ with $1 \leq p \leq \infty$. For $\omega \in \mathbb{S}^d$ and $0 \leq \rho < 1$, let $u(\omega,\rho) = u(\rho\omega)$ be the function defined on the unit $(d+1)$-dimensional open ball $B_1 \subset \mathbb{R}^{d+1}$ as

$$u(\omega,\rho) = \int_{\mathbb{S}^d} \mathcal{P}(\omega,\eta,\rho)\,|u_0(\eta)|\,\mathrm{d}\sigma(\eta)\,, \tag{3.1.12}$$

where $\mathcal{P}(\omega,\eta,\rho)$ is the Poisson kernel defined for $\omega,\eta \in \mathbb{S}^d$ by

$$\mathcal{P}(\omega,\eta,\rho) = \frac{1-\rho^2}{\sigma_d\,|\rho\omega - \eta|^d} = \frac{1-\rho^2}{\sigma_d\,(\rho^2 - 2\rho\,\omega\cdot\eta + 1)^{d/2}}\,,$$

with σ_d being the surface area of $\mathbb{S}^d$. In this case, we know that $u \in C^\infty(B_1)$ and it solves the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } B_1\,; \\ \lim_{\rho\to 1} u(\omega,\rho) = |u_0(\omega)| & \text{for a.e. } \omega \in \mathbb{S}^d. \end{cases} \tag{3.1.13}$$

From [16, Chapter II, Theorem 2.3.6] we know that for each $0 \leq \rho < 1$ we have $u(\omega, \rho) \leq \mathcal{M}u_0(\omega)$, where $\mathcal{M}$ denotes de Hardy-Littlewood maximal operator on the sphere $\mathbb{S}^d$ (taken with respect to geodesic balls). Hence, we can define

$$u^*(\omega) = \sup_{0 \leq \rho < 1} u(\omega, \rho) \tag{3.1.14}$$

and we know that $u_0 \mapsto u^*$ is bounded on $L^p(\mathbb{S}^d)$ for $1 < p \leq \infty$ (see [16, Chapter II, Corollary 2.3.4]). Moreover, with an argument similar to [25, proof of Theorem 1.4], using (3.4.6) and (3.4.7) below to explore the convolution structure of the sphere at the gradient level, one can show that $u_0 \mapsto u^*$ is a bounded operator on $W^{1,p}(\mathbb{S}^d)$ for $1 < p \leq \infty$, with $\|\nabla u^*\|_{L^p(\mathbb{S}^d)} \leq C \, \|\nabla u_0\|_{L^p(\mathbb{S}^d)}$ for some $C > 1$.

Heat flow maximal operator

Let $u_0 \in L^p(\mathbb{S}^d)$ with $1 \leq p \leq \infty$. For $\omega \in \mathbb{S}^d$ and $t \in (0, \infty)$ let $u(\omega, t)$ be the function given by

$$u(\omega, t) = \int_{\mathbb{S}^d} \mathcal{K}(\omega, \eta, t) \, |u_0(\eta)| \, \mathrm{d}\sigma(\eta) \, , \tag{3.1.15}$$

where $\mathcal{K}(\omega, \eta, t)$ is the heat kernel on $\mathbb{S}^d$. As discussed in [35, Chapter III, Section 2], the kernel $\mathcal{K}$ verifies the following properties:

(P1) $\mathcal{K} : \mathbb{S}^d \times \mathbb{S}^d \times (0, \infty) \to \mathbb{R}$ is a nonnegative smooth function that verifies $\partial_t \mathcal{K} - \Delta_\omega \mathcal{K} = 0$, where Δ_ω denotes the Laplace-Beltrami operator with respect to the variable ω.

(P2) $\mathcal{K}(\omega, \eta, t) = \mathcal{K}(\nu, t)$, where $\nu = d(\omega, \eta) = \arccos(\eta \cdot \omega)$ is the geodesic distance between ω and η. Moreover, we also have $\frac{\partial \mathcal{K}}{\partial \nu} < 0$, which means that $\mathcal{K}$ is radially decreasing in the spherical sense.

(P3) (Approximate identity) For each $t > 0$ and $\omega \in \mathbb{S}^d$ we have

$$\int_{\mathbb{S}^d} \mathcal{K}(\omega, \eta, t) \, \mathrm{d}\sigma(\eta) = 1,$$

and the function $u(\omega, t)$ defined in (3.1.15) converges pointwise a.e. to $|u_0|$ as $t \to 0$ (if $u_0 \in C(\mathbb{S}^d)$ the convergence is uniform).

It then follows from (P1) and (P3) that $u(\omega, t)$ defined in (3.1.15) solves the heat equation

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathbb{S}^d \times (0,\infty)\,; \\ \lim\limits_{t\to 0^+} u(\omega,t) = |u_0(\omega)| & \text{for a.e. } \omega \in \mathbb{S}^d. \end{cases}$$

From (P2) and (P3) it follows from [16, Chapter II, Theorem 2.3.6] that $u(\omega, t) \leq \mathcal{M}u_0(\omega)$, for each $t > 0$. This allows us to define

$$u^*(\omega) = \sup_{t>0} u(\omega, t)\,, \tag{3.1.16}$$

and we see that $u_0 \mapsto u^*$ is bounded on $L^p(\mathbb{S}^d)$ for $1 < p \leq \infty$. As in the case of the Poisson maximal operator on $\mathbb{S}^d$ (or any maximal operator on the sphere associated to a smooth convolution kernel depending only on the inner product $\omega \cdot \eta$), using (3.4.6) below and [25, proof of Theorem 1.4], one can show that $u_0 \mapsto u^*$ is bounded on $W^{1,p}(\mathbb{S}^d)$ for $1 < p \leq \infty$, with $\|\nabla u^*\|_{L^p(\mathbb{S}^d)} \leq C\,\|\nabla u_0\|_{L^p(\mathbb{S}^d)}$ for some $C > 1$.

Variation-diminishing property

Our next result establishes the variation-diminishing property for these maximal operators on the sphere $\mathbb{S}^d$.

Theorem 10. *Let u^* be the maximal function defined in* (3.1.14) *or* (3.1.16)*. The following propositions hold.*

(i) Let $1 < p \leq \infty$ and $u_0 \in W^{1,p}(\mathbb{S}^1)$. Then $u^* \in W^{1,p}(\mathbb{S}^1)$ and

$$\|(u^*)'\|_{L^p(\mathbb{S}^1)} \leq \|u_0'\|_{L^p(\mathbb{S}^1)}.$$

(ii) Let $u_0 \in W^{1,1}(\mathbb{S}^1)$. Then $u^* \in L^\infty(\mathbb{S}^1)$ and has a weak derivative $(u^*)'$ that satisfies

$$\|(u^*)'\|_{L^1(\mathbb{S}^1)} \leq \|u_0'\|_{L^1(\mathbb{S}^1)}.$$

(iii) Let u_0 be of bounded variation on $\mathbb{S}^1$. Then u^* is of bounded variation on $\mathbb{S}^1$ and

$$V(u^*) \leq V(u_0).$$

(iv) Let $d > 1$ and $u_0 \in W^{1,p}(\mathbb{S}^d)$, for $p = 2$ or $p = \infty$. Then $u^* \in W^{1,p}(\mathbb{S}^d)$ and

$$\|\nabla u^*\|_{L^2(\mathbb{S}^d)} \leq \|\nabla u_0\|_{L^2(\mathbb{S}^d)}.$$

Remark: Since $\mathbb{S}^1 \sim \mathbb{T}$, in the case of the heat flow maximal operator, parts (i), (ii) and (iii) of Theorem 10 have already been considered in Theorem 9, and the novel part here is actually (iv).

3.1.5 Non-tangential maximal operators

The last operator considered here is the classical non-tangential maximal operator associated to the Poisson kernel (3.1.3). For $\alpha \geq 0$ we consider

$$u^*(x) = \sup_{\substack{t>0 \\ |y-x|\leq \alpha t}} P(\cdot, t) * |u_0|(y). \tag{3.1.17}$$

This operator is bounded on $L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$ (see [38, Chapter II, Equation (3.18)]). A modification of [25, proof of Theorem 1.4] (here one must discretize in time and in the set of possible directions) yields that this maximal operator is bounded on $W^{1,p}(\mathbb{R})$ for $1 < p \leq \infty$, with $\|\nabla u^*\|_{L^p(\mathbb{R}^d)} \leq C \|\nabla u_0\|_{L^p(\mathbb{R}^d)}$ for some $C > 1$. Here we establish the variation-diminishing property of this operator in dimension $d = 1$.

Theorem 11. *Let $\alpha \geq 0$ and let u^* be the maximal function defined in (3.1.17). The following propositions hold.*

(i) Let $1 < p \leq \infty$ and $u_0 \in W^{1,p}(\mathbb{R})$. Then $u^* \in W^{1,p}(\mathbb{R})$ and

$$\|(u^*)'\|_{L^p(\mathbb{R})} \leq \|u_0'\|_{L^p(\mathbb{R})}.$$

(ii) Let $u_0 \in W^{1,1}(\mathbb{R})$. Then $u^* \in L^\infty(\mathbb{R})$ and has a weak derivative $(u^*)'$ that satisfies

$$\|(u^*)'\|_{L^1(\mathbb{R})} \leq \|u_0'\|_{L^1(\mathbb{R})}.$$

(iii) Let u_0 be of bounded variation on $\mathbb{R}$. Then u^* is of bounded variation on $\mathbb{R}$ and

$$V(u^*) \leq V(u_0).$$

3.1.6 A brief strategy outline

The proofs of Theorems 8 - 11 follow the same broad outline, each with their own technicalities. One component of the proof is to establish that it is sufficient to consider a Lipschitz continuous initial datum u_0. The second and crucial component of the proof is to establish that, for a Lipschitz continuous initial datum u_0, the maximal function is *subharmonic in the detachment set*. The steps leading to these results are divided in several auxiliary lemmas in the proofs of each theorem.

We remark that the subharmonicity property for the non-tangential maximal function (3.1.17) in dimension $d > 1$ is not true. We present a counterexample after the proof of Theorem 11.

3.2 Proof of Theorem 8: Maximal operators and elliptic equations

3.2.1 Preliminaries on the kernel

Let $a > 0$ and $b > 0$. We first observe that the function $\widehat{\varphi}_{a,b}(\cdot, t) : \mathbb{R}^d \to \mathbb{R}$ defined in (3.1.5) belongs to the Schwartz class for each $t > 0$. Moreover, the function $g : [0, \infty) \to \mathbb{R}^+$ defined by

$$\widehat{\varphi}_{a,b}(\xi, t) =: g(|\xi|^2)$$

is *completely monotone*, in the sense that it verifies $(-1)^n g^{(n)}(s) \geq 0$ for $s > 0$ and $n = 0, 1, 2, \ldots$ and $g(0^+) = g(0)$. We may hence invoke a classical result of Schoenberg [36, Theorems 2 and 3] to conclude that there exists a finite nonnegative measure $\mu_{a,b,t}$ on $[0, \infty)$ such that

$$\widehat{\varphi}_{a,b}(\xi, t) = \int_0^\infty e^{-\pi\lambda|\xi|^2} \, \mathrm{d}\mu_{a,b,t}(\lambda). \tag{3.2.1}$$

It is convenient to record the explicit form of $\mu_{a,b,t}$. Starting from the identity [38, page 6], for $\beta > 0$,

$$e^{-\beta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \, e^{-\frac{\beta^2}{4u}} \, \mathrm{d}u = \frac{1}{2\pi} \int_0^\infty e^{-\pi\sigma\beta^2} \, e^{-\frac{1}{4\pi\sigma}} \, \sigma^{-\frac{3}{2}} \, \mathrm{d}\sigma,$$

we make $\beta = \frac{t}{2a} \left(b^2 + 16a\pi^2|\xi|^2\right)^{1/2}$ to obtain

$$\mathrm{d}\mu_{a,b,t}(\lambda) = \left(e^{\frac{tb}{2a}} \, \frac{t}{\sqrt{a}} \, e^{-\frac{\lambda b^2}{16\pi a}} \, e^{-\frac{\pi t^2}{a\lambda}} \, \lambda^{-\frac{3}{2}} \right) \mathrm{d}\lambda. \tag{3.2.2}$$

An application of Fubini's theorem gives us

$$\varphi_{a,b}(x,t) = \int_{\mathbb{R}^d} \widehat{\varphi}_{a,b}(\xi,t)\, e^{2\pi i x\cdot\xi}\, \mathrm{d}\xi = \int_0^\infty \lambda^{-\frac{d}{2}}\, e^{-\frac{\pi|x|^2}{\lambda}}\, \mathrm{d}\mu_{a,b,t}(\lambda). \tag{3.2.3}$$

In particular, (3.2.3) implies that $\varphi_{a,b}(\cdot,t) : \mathbb{R}^d \to \mathbb{R}$ is nonnegative and radial decreasing.

Taking $\xi = 0$ in (3.1.5) and (3.2.1), we get $\mu_{a,b,t}([0,\infty)) = 1$ for any t. Taking ξ as the first canonical basis vector (or any other fixed vector) and letting $t \to 0$, we can show that, for any positive Λ,

$$\lim_{t\to 0^+} \mu_{a,b,t}([0,\Lambda]) = 1.$$

From this fact and (3.2.3) we see that, for a fixed $\delta > 0$,

$$\lim_{t\to 0^+} \int_{|x|\geq\delta} \varphi_{a,b}(x,t)\, \mathrm{d}x = 0. \tag{3.2.4}$$

It follows that, for $f \in L^p(\mathbb{R}^d)$ with $1 \leq p < \infty$,

$$\lim_{t\to 0^+} \|\varphi_{a,b}(\cdot,t) * f - f\|_{L^p(\mathbb{R}^d)} = 0. \tag{3.2.5}$$

Moreover,

$$\varphi_{a,b}(\cdot,t) * f(x) = \int_0^\infty \left(\int_{\mathbb{R}^d} \lambda^{-\frac{d}{2}}\, e^{-\frac{\pi|y|^2}{\lambda}} f(x-y)\, \mathrm{d}y \right) \mathrm{d}\mu_{a,b,t}(\lambda).$$

Since the integrand is bounded by $Mf(x)$, the Dominated Convergence Theorem and the approximation of identity property for the Gaussian imply the pointwise convergence

$$\lim_{t\to 0^+} \varphi_{a,b}(\cdot,t) * f(x) = f(x) \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{3.2.6}$$

In (3.2.6) we may allow $f \in L^p(\mathbb{R}^d)$ with $1 \leq p \leq \infty$ and the convergence happens at every point in the Lebesgue set of f.

From (3.2.3) and (3.2.2) we see that $\varphi_{a,b} \in C^\infty(\mathbb{R}^d \times (0,\infty))$. Moreover its decay is strong enough to assure that, if the initial datum $u_0 \in L^p(\mathbb{R}^d)$ for some $1 \leq p \leq \infty$, then $u(x,t) = \varphi_{a,b}(\cdot,t) * u_0(x) \in C^\infty(\mathbb{R}^d \times (0,\infty))$, with $D^\alpha u(x,t) = (D^\alpha \varphi_{a,b}(\cdot,t)) * u_0(x)$ for any multi-index $\alpha \in (\mathbb{Z}^+)^{d+1}$. Finally, observe that $u(x,t)$ solves the partial differential equation

$$a u_{tt} - b u_t + \Delta u = 0 \quad \text{in} \quad \mathbb{R}^d \times (0,\infty). \tag{3.2.7}$$

This follows since the kernel $\varphi(x,t)$ solves the same equation, a fact that can be verified by differentiating under the integral sign the leftmost identity in (3.2.3). We also remark that if $u_0 \in C(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$, or if u_0 is bounded and Lipschitz continuous, then the function $u(x,t)$ is continuous up to the boundary $\mathbb{R}^d \times \{t = 0\}$ (this follows from (3.2.6) and (3.2.8) below).

3.2.2 Auxiliary lemmas

In order to prove Theorem 8, we may assume without loss of generality that $u_0 \geq 0$. In fact, if $u_0 \in W^{1,p}(\mathbb{R}^d)$ we have $|u_0| \in W^{1,p}(\mathbb{R}^d)$ and $|\nabla|u_0|| = |\nabla u_0|$ a.e. if u_0 is real-valued (in the general case of u_0 complex-valued we have $|\nabla|u_0|| \leq |\nabla u_0|$ a.e), and if u_0 is of bounded variation on $\mathbb{R}$ we have $V(|u_0|) \leq V(u_0)$. We adopt such assumption throughout the rest of this section.

The cases when $a = 0$ (heat kernel) or $b = 0$ (Poisson kernel) were already considered in [14, Theorems 1 and 2], so we focus in the remaining case $a > 0$, $b > 0$ [2]. We start with some auxiliary lemmas, following the strategy outlined in [14]. Throughout this section we write

$$\mathrm{Lip}(u) = \sup_{\substack{x,y \in \mathbb{R}^d \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|}$$

for the Lipschitz constant of a function $u : \mathbb{R}^d \to \mathbb{R}$. Let $B_r(x) \subset \mathbb{R}^d$ denote the open ball of radius r and center x, and let $\overline{B_r(x)}$ denote the corresponding closed ball. When $x = 0$ we shall simply write B_r.

Lemma 12 (Continuity). *Let $a, b > 0$ and u^* be the maximal function defined in* (3.1.8).

(i) If $u_0 \in C(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, for some $1 \leq p < \infty$, then $u^* \in C(\mathbb{R}^d)$.

(ii) If u_0 is bounded and Lipschitz continuous then u^* is bounded and Lipschitz continuous with $\mathrm{Lip}(u^*) \leq \mathrm{Lip}(u_0)$.

[2] By appropriate dilations in the space variable x and the time variable t, we could assume that $a = b = 1$. However, this reduction is mostly aesthetical and offers no major technical simplification.

Proof. Let us denote $\tau_h u_0 := u_0(x-h)$. Given $x \in \mathbb{R}^d$, we can choose $\delta > 0$ such that

$$
\begin{aligned}
&|\tau_h u_0 - u_0| * \varphi_{a,b}(\cdot,t)(x) \\
&= \int_{|y|<1} |\tau_h u_0 - u_0|(x-y)\, \varphi_{a,b}(y,t)\,\mathrm{d}y + \int_{|y|\geq 1} |\tau_h u_0 - u_0|(x-y)\, \varphi_{a,b}(y,t)\,\mathrm{d}y \\
&\leq \sup_{w\in B_1(x)} |\tau_h u_0 - u_0|(w) + \|\tau_h u_0 - u_0\|_p \, \|\chi_{\{|\cdot|\geq 1\}}\, \varphi_{a,b}(\cdot,t)\|_{p'} \\
&< \varepsilon
\end{aligned}
\tag{3.2.8}
$$

whenever $|h| < \delta$, for all $t > 0$. Above we have used the fact that $\|\chi_{\{|\cdot|\geq 1\}}\, \varphi_{a,b}(\cdot,t)\|_{p'}$ is uniformly bounded. Using the sublinearity, we then arrive at

$$\left|\tau_h u^*(x) - u^*(x)\right| \leq (\tau_h u_0 - u_0)^*(x) \leq \varepsilon$$

for $|h| < \delta$, which shows that u^* is continuous at the point x.

(ii) Observe that for each $t > 0$ the function $u(x,t) = \varphi_{a,b}(\cdot,t) * u_0(x)$ is bounded by $\|u_0\|_\infty$ and Lipschitz continuous with $\mathrm{Lip}(u(\cdot,t)) \leq \mathrm{Lip}(u_0)$. The result then follows since we are taking a pointwise supremum of uniformly bounded and Lipschitz functions. □

Lemma 13 (Behaviour at large times)**.** *Let $a, b > 0$ and $u(x,t) = \varphi_{a,b}(\cdot,t) * u_0(x)$.*

(i) If $u_0 \in L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$, then for a given $\varepsilon > 0$ there exists a time $t_\varepsilon < \infty$ such that $\|u(\cdot,t)\|_\infty < \varepsilon$ for all $t > t_\varepsilon$.

(ii) If u_0 is bounded and if $r > 0$ and $\varepsilon > 0$ are given, then there exists a time $t_{r,\varepsilon} < \infty$ such that $|u(x,t) - u(y,t)| < \varepsilon$ for all $x, y \in B_r$ and $t > t_{r,\varepsilon}$.

Proof. (i) From Hölder's inequality and the leftmost identity in (3.2.3),

$$
\begin{aligned}
\|u(\cdot,t)\|_\infty &\leq \|u_0\|_p \, \|\varphi_{a,b}(\cdot,t)\|_{p'} \\
&\leq \|u_0\|_{p'} \|\varphi_{a,b}(\cdot,t)\|_\infty^{\frac{1}{p}} \, \|\varphi_{a,b}(\cdot,t)\|_1^{\frac{1}{p'}} \\
&\leq \|u_0\|_{p'} \|\widehat{\varphi}_{a,b}(\cdot,t)\|_1^{\frac{1}{p}} .
\end{aligned}
$$

By monotone convergence, $\|\widehat{\varphi}_{a,b}(\cdot,t)\|_1 \to 0$ as $t \to \infty$. The statement follows.

(ii) Since $\varphi_{a,b}(\cdot,t)$ is in the Schwartz class, for every index $k \in \{1,\ldots,d\}$ we have

$$\frac{\partial u}{\partial x_k}(x,t) = \frac{\partial \varphi_{a,b}}{\partial x_k}(\cdot,t) * u_0(x).$$

This implies that $u(\cdot,t)$ is a Lipschitz function with constant bounded by $\|u_0\|_\infty \sum_{k=1}^d \left\|\frac{\partial \varphi_{a,b}}{\partial x_k}(\cdot,t)\right\|_1$. By (3.2.3), (3.2.2) and Fubini's theorem,

$$\begin{aligned}\left\|\frac{\partial \varphi_{a,b}}{\partial x_k}(\cdot,t)\right\|_1 &= \left(\int_{\mathbb{R}^d} 2\pi|x_k|e^{-\pi|x|^2}\mathrm{d}x\right)\left(\int_0^\infty \lambda^{-1/2}\,\mathrm{d}\mu_{a,b,t}(\lambda)\right)\\ &= \left(\int_{\mathbb{R}^d} 2\pi|x_k|e^{-\pi|x|^2}\mathrm{d}x\right)\left(\int_0^\infty \frac{t}{\sqrt{a}\lambda^2}e^{-\frac{\lambda}{16\pi a}\left(b-\frac{4\pi t}{\lambda}\right)^2}\mathrm{d}\lambda\right).\end{aligned}$$

Setting $\lambda = t\nu$ and applying dominated convergence, one concludes that the second factor converges to 0 as $t\to\infty$. The result plainly follows from this.

□

We now start to explore the qualitative properties of the underlying elliptic equation (3.2.7). We say that a continuous function f is *subharmonic* in an open set $A\subset\mathbb{R}^d$ if, for every $x\in A$, and every ball $\overline{B_r(x)}\subset A$ we have

$$f(x)\le \frac{1}{\sigma_{d-1}}\int_{\mathbb{S}^{d-1}} f(x+r\xi)\,\mathrm{d}\sigma(\xi),$$

where σ_{d-1} denotes the surface area of the unit sphere $\mathbb{S}^{d-1}$, and $\mathrm{d}\sigma$ denotes its surface measure.

Lemma 14 (Subharmonicity). *Let $a,b>0$ and u^* be the maximal function defined in* (3.1.8). *Let $u_0\in C(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$ for some $1\le p<\infty$ or u_0 be bounded and Lipschitz continuous. Then u^* is subharmonic in the open set $A=\{x\in\mathbb{R}^d;\, u^*(x)>u_0(x)\}$.*

Proof. From (3.2.6) we have $u^*(x)\ge u_0(x)$ for all $x\in\mathbb{R}^d$. From Lemma 12 we observe that u^* is a continuous function and hence the set A is indeed open. Let $x_0\in A$ and $\overline{B_r(x_0)}\subset A$. Let $h:\overline{B_r(x_0)}\to\mathbb{R}$ be the solution of the Dirichlet boundary value problem

$$\begin{cases}\Delta h = 0 & \text{in } B_r(x_0);\\ h = u^* & \text{in } \partial B_r(x_0).\end{cases}$$

Note that the auxiliary function $v(x,t)=u(x,t)-h(x)$ solves the equation

$$av_{tt}-bv_t+\Delta v = 0 \quad \text{in} \quad B_r(x_0)\times(0,\infty)$$

and it is continuous in $\overline{B_r(x_0)}\times[0,\infty)$, with $v(x,0)=u_0(x)-h(x)$. Let $y_0\in\overline{B_r(x_0)}$ be such that $M=\max_{x\in\overline{B_r(x_0)}} v(x,0)=v(y_0,0)$. We claim that $M\le 0$.

Assume that $M > 0$. Note that $v(x,t) \leq 0$ for every $x \in \partial B_r(x_0)$ and every $t > 0$. This implies that $y_0 \in B_r(x_0)$. By the maximum principle, observe that $h \geq 0$ in $\overline{B_r(x_0)}$ and let $x_1 \in \partial B_r(x_0)$ be such that $\min_{x \in \overline{B_r(x_0)}} h(x) = h(x_1)$. Given $\varepsilon > 0$, from Lemma 13 we may find a time t_0 such that $|u(x,t_1) - u(y,t_1)| \leq \varepsilon$ for all $x, y \in \overline{B_r(x_0)}$ and $t_1 > t_0$. In particular, for any $x \in \overline{B_r(x_0)}$, we have

$$v(x,t_1) \leq v(x,t_1) - v(x_1,t_1) = u(x,t_1) - u(x_1,t_1) - (h(x) - h(x_1)) \leq u(x,t_1) - u(x_1,t_1) \leq \varepsilon\,,$$

for $t_1 > t_0$. If we take $\varepsilon < M$, the maximum principle applied to the cylinder $\Gamma = \overline{B_r(x_0)} \times [0, t_1]$ with $t_1 > t_0$ gives us

$$v(y_0, t) \leq v(y_0, 0) = M$$

for all $0 \leq t \leq t_1$. This plainly implies that $u(y_0,t) \leq u_0(y_0)$ for all $0 \leq t \leq t_1$. Since t_1 is arbitrarily large, we obtain $u^*(y_0) = u_0(y_0)$, contradicting the fact that $y_0 \in A$. This proves our claim.

Once established that $M \leq 0$, given $\varepsilon > 0$ we apply again the maximum principle to the cylinder $\Gamma = \overline{B_r(x_0)} \times [0, t_1]$ with $t_1 > t_0$ as above to get $v(x_0,t) \leq \varepsilon$ for all $0 \leq t \leq t_1$. This implies that $u(x_0,t) \leq h(x_0) + \varepsilon$ for all $0 \leq t \leq t_1$, and since t_1 is arbitrarily large, we find that $u^*(x_0) \leq h(x_0) + \varepsilon$. Since $\varepsilon > 0$ is arbitrarily small, we conclude that

$$u^*(x_0) \leq h(x_0) = \frac{1}{\sigma_{d-1}} \int_{\mathbb{S}^{d-1}} h(x_0 + r\xi)\, \mathrm{d}\sigma(\xi) = \frac{1}{\sigma_{d-1}} \int_{\mathbb{S}^{d-1}} u^*(x_0 + r\xi)\, \mathrm{d}\sigma(\xi)\,,$$

by the mean value property of the harmonic function h. This concludes the proof. □

The next lemma is a general result of independent interest a bit more general than [14, Lemma 9]. We shall use it in the proof of Theorem 8 for the case $p = 2$.

Lemma 15. *Let $f, g \in C(\mathbb{R}^d) \cap W^{1,2}(\mathbb{R}^d)$ be real-valued functions. Suppose that $g \geq 0$ and that f is subharmonic in the open set $J = \{x \in \mathbb{R}^d;\ g(x) > 0\}$. Then*

$$\int_{\mathbb{R}^d} \nabla f(x) \,.\, \nabla g(x)\ \mathrm{d}x \leq 0.$$

Proof. At first, we assume that g is compactly supported and f is subharmonic in a neighborhood of $\bar{J}$. Let ψ_ε be a smooth nonnegative approximation of the identity. Since $f * \psi_\varepsilon$ is smooth,

$$\int_{\mathbb{R}^d} \nabla(f * \psi_\varepsilon) \,.\, \nabla g\ \mathrm{d}x = \int_{\mathbb{R}^d} (-\Delta(f * \psi_\varepsilon))\, g\, \mathrm{d}x.$$

Due to our assumption on f, if ε is sufficiently small, $f * \psi_\varepsilon$ is subharmonic in J. This implies that the right-hand side above is nonpositive. The claim then follows from the fact that $f * \psi_\varepsilon \to f$ in $W^{1,2}(\mathbb{R}^d)$.

Now we suppose that g is compactly supported, but we impose no restrictions on f. For any $\varepsilon > 0$, the support of the function $\max\{g - \varepsilon, 0\}$ is closed and contained in J. By the case we have already proved, the inner product of the gradients of f and $\max\{g - \varepsilon, 0\}$ is nonpositive. Now we let $\varepsilon \to 0$. By the explicit formula for the derivatives of $\max\{g - \varepsilon, 0\}$, this function goes to g in $W^{1,2}(\mathbb{R}^d)$.

For the general case, it is enough to approximate g by compactly supported functions in $W^{1,2}(\mathbb{R}^d)$ that vanish outside J. It is well known that $x \mapsto g(x)\eta(x/n)$ will do if $\eta(0) = 1$ and $n \to \infty$. □

Lemma 16 (Reduction to the Lipschitz case)**.** *In order to prove parts* (i)*,* (iii) *and* (iv) *of Theorem 8 it suffices to assume that the initial datum* $u_0 : \mathbb{R}^d \to \mathbb{R}^+$ *is bounded and Lipschitz.*

Proof. Parts (i) and (iv). For the case $p = \infty$, recall that any function $u_0 \in W^{1,\infty}(\mathbb{R}^d)$ can be modified in a set of measure zero to become bounded and Lipschitz continuous.

If $1 < p < \infty$, for $\varepsilon > 0$ we write $u_\varepsilon = \varphi_{a,b}(\cdot, \varepsilon) * u_0$. It is clear that u_ε is bounded, Lipschitz continuous and belongs to $W^{1,p}(\mathbb{R}^d)$. Assuming that the result holds for such u_ε, we would have $u_\varepsilon^* \in W^{1,p}(\mathbb{R}^d)$ with

$$\|\nabla u_\varepsilon^*\|_p \leq \|\nabla u_\varepsilon\|_p. \tag{3.2.9}$$

Note that

$$u_\varepsilon^*(x) = \sup_{t>0} \varphi_{a,b}(\cdot, t) * u_\varepsilon(x) = \sup_{t>\varepsilon} \varphi_{a,b}(\cdot, t) * u_0(x), \tag{3.2.10}$$

due to the semigroup property (3.1.5). Recall that there exists a universal $C > 1$ such that

$$\|u_\varepsilon^*\|_p \leq C\, \|u_\varepsilon\|_p. \tag{3.2.11}$$

From Young's inequality (and also Minkowski's inequality in the case of the gradients) we have

$$\|u_\varepsilon\|_p \leq \|u_0\|_p \quad \text{and} \quad \|\nabla u_\varepsilon\|_p \leq \|\nabla u_0\|_p. \tag{3.2.12}$$

From (3.2.9), (3.2.11) and (3.2.12) we see that u_ε^* is uniformly bounded in $W^{1,p}(\mathbb{R}^d)$. From (3.2.10) we have $u_\varepsilon^* \to u^*$ pointwise as $\varepsilon \to 0$. Hence, by the weak compactness of the space

$W^{1,p}(\mathbb{R}^d)$, we must have $u^* \in W^{1,p}(\mathbb{R}^d)$ and $u^*_\varepsilon \rightharpoonup u^*$ as $\varepsilon \to 0$. It then follows from the lower semicontinuity of the norm under weak limits, (3.2.9) and (3.2.12) that

$$\|\nabla u^*\|_p \leq \liminf_{\varepsilon\to 0} \|\nabla u^*_\varepsilon\|_p \leq \liminf_{\varepsilon\to 0} \|\nabla u_\varepsilon\|_p \leq \|\nabla u_0\|_p.$$

Part (iii). Let $u_0 : \mathbb{R} \to \mathbb{R}^+$ be of bounded variation. For $\varepsilon > 0$ write $u_\varepsilon = \varphi_{a,b}(\cdot, \varepsilon) * u_0$. Then $u_\varepsilon \in C^\infty(\mathbb{R})$ is bounded and Lipschitz continuous, and it is easy to see that $V(u_\varepsilon) \leq V(u_0)$. Assume that the result holds for such u_ε, i.e. that $V(u^*_\varepsilon) \leq V(u_\varepsilon)$. For any partition $\mathcal{P} = \{x_0 < x_1 < \ldots < x_N\}$ we then have

$$V_{\mathcal{P}}(u^*_\varepsilon) := \sum_{n=1}^{N} |u^*_\varepsilon(x_n) - u^*_\varepsilon(x_{n-1})| \leq V(u_\varepsilon) \leq V(u_0). \tag{3.2.13}$$

By (3.2.10), we recall that $u^*_\varepsilon \to u^*$ pointwise as $\varepsilon \to 0$. Passing this limit in (3.2.13) yields

$$V_{\mathcal{P}}(u^*) := \sum_{n=1}^{N} |u^*(x_n) - u^*(x_{n-1})| \leq V(u_0).$$

Since this holds for any partition $\mathcal{P}$, we conclude that $V(u^*) \leq V(u_0)$. This completes the proof. □

The next lemma will be used in the proof of part (i) of Theorem 8.

Lemma 17. *Let $[\alpha, \beta]$ be a compact interval. Let $f, g : [\alpha, \beta] \to \mathbb{R}$ be absolutely continuous functions with g convex. If $f(\alpha) = g(\alpha)$, $f(\beta) = g(\beta)$ and $f(x) < g(x)$ for all $x \in (\alpha, \beta)$, then*

$$\|g'\|_{L^p([\alpha,\beta])} \leq \|f'\|_{L^p([\alpha,\beta])} \tag{3.2.14}$$

for any $1 \leq p \leq \infty$.

Proof. Let us consider the case $1 \leq p < \infty$. The case $p = \infty$ follows by a passage to the limit in (3.2.14). Assume that the right-hand side of (3.2.14) is finite, otherwise there is nothing to prove. Let $X \subset (\alpha, \beta)$ be the set of points where g is differentiable and choose a sequence $\{x_n\}_{n=1}^\infty$ of elements of X that is dense in (α, β). For each x_n consider the affine function $L_n(x) := g(x_n) + g'(x_n)(x - x_n)$. Note that $L_n(x) \leq g(x)$ for all $x \in [\alpha, \beta]$. We set $f_0 = f$ and define inductively $f_{n+1} = \max\{f_n, L_{n+1}\}$. One can show that each f_n is absolutely continuous. Let $U_n = \{x \in (\alpha,\beta);\ L_{n+1}(x) > f_n(x)\}$. Then

$$\int_{[\alpha,\beta]} |f'_{n+1}(x)|^p \,\mathrm{d}x = \int_{[\alpha,\beta]\setminus U_n} |f'_n(x)|^p \,\mathrm{d}x + m(U_n)\,|g'(x_{n+1})|^p. \tag{3.2.15}$$

By Jensen's inequality, in each connected component $I = (r,s)$ of U_n we have

$$\begin{aligned}\int_I |f'_n(x)|^p \,\mathrm{d}x \geq& (s-r)\left(\frac{1}{s-r}\int_I |f'_n(x)|\,\mathrm{d}x\right)^p \\ &\geq (s-r)\left|\frac{f_n(s)-f_n(r)}{s-r}\right|^p \\ &= (s-r)\,|g'(x_{n+1})|^p.\end{aligned} \tag{3.2.16}$$

By (3.2.15) and (3.2.16) we conclude that

$$\|f'_{n+1}\|_{L^p([\alpha,\beta])} \leq \|f'_n\|_{L^p([\alpha,\beta])}. \tag{3.2.17}$$

Let $x \in X$. For sufficiently large N, there are indices $j,k \in \{1,2,\dots,N\}$ such that $x_j \leq x < x_k$. Take these indices such that x_j is as large as possible and x_k is as small as possible. Since $f(x) < g(x)$, for large values of N we have $f(x) < L_j(x)$ and $f(x) < L_k(x)$. Therefore $f_N(x) = \max\{f(x), L_1(x),\dots,L_N(x)\}$ is either equal to $L_j(x)$ or $L_k(x)$. In fact, the function f_N is differentiable in x with $f'_N(x) = g'(x_j)$ or $f'_N(x) = g'(x_k)$, except if $g'(x_j) \neq g'(x_k)$ and $L_j(x) = L_k(x)$, which only happens in a countable set Y. Assuming that $x \notin Y$ and that $g' : X \to \mathbb{R}$ is continuous at x (this is a set of full measure in (α,β)) we have $f'_N(x) \to g'(x)$ as $N \to \infty$. From (3.2.17) and Fatou's lemma we get

$$\|g'\|_{L^p([\alpha,\beta])} \leq \liminf_{N\to\infty} \|f'_N\|_{L^p([\alpha,\beta])} \leq \|f'\|_{L^p([\alpha,\beta])}.$$

□

Remark: If $f, g : [\alpha,\infty) \to \mathbb{R}$ are absolutely continuous functions with g convex, and $f(\alpha) = g(\alpha) \geq 0$, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ and $f(x) < g(x)$ for all $x \in (\alpha,\infty)$, the same proof of Lemma 17 gives

$$\|g'\|_{L^p([\alpha,\infty))} \leq \|f'\|_{L^p([\alpha,\infty))}$$

for any $1 \leq p < \infty$. Observe in (3.2.15) that either $g'(x_{n+1}) = 0$ or U_n is bounded. The same remark applies to the analogous situation on the interval $(-\infty,\beta]$.

3.2.3 Proof of Theorem 8

We are now in position to prove the main result of this section.

Proof of part (i)

We defer the case $p = \infty$ to part (iv). Let us consider here the case $1 < p < \infty$. From Lemma 16 we may assume that $u_0 \in L^p(\mathbb{R})$ is bounded and Lipschitz continuous. Then, from Lemma 12, we find that u^* is Lipschitz continuous and the detachment set $A = \{x \in \mathbb{R};\ u^*(x) > u_0(x)\}$ is open. Let us write A as a countable union of open intervals

$$A = \bigcup_j I_j = \bigcup_j (\alpha_j, \beta_j). \tag{3.2.18}$$

We allow the possibility of having $\alpha_j = -\infty$ or $\beta_j = \infty$, but note that, if $u_0 \not\equiv 0$, we must have $u^*(x_0) = u_0(x_0)$ at a global maximum x_0 of u_0, hence $A \neq (-\infty, \infty)$. From Lemma 14, u^* is subharmonic (hence convex) in each subinterval $I_j = (\alpha_j, \beta_j)$. Part (i) now follows from Lemma 17 (and the remark thereafter, since $u_0, u^* \in L^p(\mathbb{R})$).

Proof of part (ii)

Recall that a function $u_0 \in W^{1,1}(\mathbb{R})$ can be modified in a set of measure zero to become absolutely continuous. Then, from Lemma 12 we find that u^* is continuous and the detachment set $A = \{x \in \mathbb{R};\ u^*(x) > u_0(x)\}$ is open. Let us decompose A as in (3.2.18). From Lemma 14, u^* is subharmonic (hence convex) in each subinterval $I_j = (\alpha_j, \beta_j)$. Hence u^* is differentiable a.e. in A, with derivative denoted by v. It then follows from Lemma 17 (and the remark thereafter, since $u^* \in L^1_{weak}(\mathbb{R})$) that for each interval I_j we have

$$\int_{I_j} |v(x)|\, \mathrm{d}x \leq \int_{I_j} |u_0'(x)|\, \mathrm{d}x\,, \tag{3.2.19}$$

and since $u_0' \in L^1(\mathbb{R})$ we find that $v \in L^1(A)$.

We now claim that u^* is weakly differentiable with $(u^*)' = \chi_A.v + \chi_{A^c}.u_0'$. In fact, if $\psi \in C_c^\infty(\mathbb{R})$ we have

$$\int_{\mathbb{R}} u^*(x)\, \psi'(x)\, \mathrm{d}x = \int_{A^c} u_0(x)\, \psi'(x)\, \mathrm{d}x + \sum_j \int_{I_j} u^*(x)\, \psi'(x)\, \mathrm{d}x$$

$$= \int_{A^c} u_0(x)\,\psi'(x)\,\mathrm{d}x + \sum_j \left(u_0(\beta_j)\psi(\beta_j) - u_0(\alpha_j)\psi(\alpha_j) - \int_{I_j} v(x)\,\psi(x)\,\mathrm{d}x \right)$$

$$= \int_{A^c} u_0(x)\,\psi'(x)\,\mathrm{d}x$$

$$+ \sum_j \left(\int_{I_j} u_0(x)\,\psi'(x)\,\mathrm{d}x + \int_{I_j} u_0'(x)\,\psi(x)\,\mathrm{d}x - \int_{I_j} v(x)\,\psi(x)\,\mathrm{d}x \right)$$

$$= - \int_{A^c} u_0'(x)\,\psi(x)\,\mathrm{d}x - \int_{A} v(x)\,\psi(x)\,\mathrm{d}x,$$

as claimed. Finally, using (3.2.19) we arrive at

$$\int_{\mathbb{R}} |(u^*)'(x)|\,\mathrm{d}x = \int_A |v(x)|\,\mathrm{d}x + \int_{A^c} |u_0'(x)|\,\mathrm{d}x \leq \int_{\mathbb{R}} |u_0'(x)|\,\mathrm{d}x,$$

which concludes the proof of this part.

Proof of part (iii)

By Lemma 16 we may assume that $u_0 : \mathbb{R} \to \mathbb{R}^+$ of bounded variation is also Lipschitz continuous. By Lemma 14 the function u^* is subharmonic (hence convex) in the detachment set $A = \{x \in \mathbb{R};\ u^*(x) > u_0(x)\}$. This plainly leads to $V(u^*) \leq V(u_0)$, since the variation does not increase in each connected component of A.

Proof of part (iv)

We include here the case $d = 1$ as well. If $p = \infty$, a function $u_0 \in W^{1,\infty}(\mathbb{R}^d)$ can be modified on a set of measure zero to become Lipschitz continuous with $\mathrm{Lip}(u_0) \leq \|\nabla u_0\|_\infty$. From Lemma 12, the function u^* is also be bounded and Lipschitz continuous, with $\mathrm{Lip}(u^*) \leq \mathrm{Lip}(u_0)$, and the result follows, since in this case $u^* \in W^{1,\infty}(\mathbb{R}^d)$ with $\|\nabla u^*\|_\infty \leq \mathrm{Lip}(u^*)$.

If $p = 2$, from Lemma 16 it suffices to consider the case where $u_0 \in W^{1,2}(\mathbb{R}^d)$ is Lipschitz continuous. In this case, we have seen from the discussion in the introduction and from Lemma 12 that the maximal function $u^* \in W^{1,2}(\mathbb{R}^d)$ is also Lipschitz continuous. From Lemma 14, u^* is subharmonic in the detachment set $A = \{x \in \mathbb{R}^d;\, u^*(x) > u_0(x)\}$ and we may apply Lemma 15 with $f = u^*$ and $g = (u^* - u_0)$ to get

$$\|\nabla u_0\|_2^2 = \int_{\mathbb{R}^d} |\nabla u_0|^2\,\mathrm{d}x = \int_{\mathbb{R}^d} |\nabla u^* - \nabla(u^* - u_0)|^2\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} |\nabla(u^* - u_0)|^2 \,\mathrm{d}x - 2\int_{\mathbb{R}^d} \nabla u^* \,.\, \nabla(u^* - u_0)\,\mathrm{d}x + \int_{\mathbb{R}^d} |\nabla u^*|^2 \,\mathrm{d}x \\
&\geq \int_{\mathbb{R}^d} |\nabla u^*|^2 \,\mathrm{d}x = \|\nabla u^*\|_2^2.
\end{aligned}
$$

This concludes the proof.

3.3 Proof of Theorem 9: Periodic analogues

3.3.1 Auxiliary lemmas

We follow here the same strategy used in the proof of Theorem 8. We may assume in what follows that the initial datum u_0 is nonnegative. We now have to consider the whole range $a, b \geq 0$ with $(a, b) \neq (0, 0)$.

Lemma 18 (Continuity - periodic version)**.** *Let $a, b \geq 0$ with $(a, b) \neq (0, 0)$ and u^* be the maximal function defined in* (3.1.11)*.*

(i) If $u_0 \in C(\mathbb{T}^d)$ then $u^* \in C(\mathbb{T}^d)$.

(ii) If u_0 is Lipschitz continuous then u^* is Lipschitz continuous with $\mathrm{Lip}(u^*) \leq \mathrm{Lip}(u_0)$.

Proof. Part (i). If $u_0 \in C(\mathbb{T}^d)$ then u_0 is uniformly continuous in $\mathbb{T}^d$. Therefore, given $\varepsilon > 0$, there exists δ such that $|u_0(x-h) - u_0(x)| \leq \varepsilon$ whenever $|h| \leq \delta$. It follows that (recall that $\tau_h u_0 := u_0(x-h)$)

$$
|\tau_h u_0 - u_0| * \Psi_{a,b}(\cdot, t)(x) = \int_{\mathbb{T}^d} |\tau_h u_0 - u_0|(x-y)\, \Psi_{a,b}(y,t)\,\mathrm{d}y < \varepsilon
$$

if $|h| \leq \delta$, for every $t > 0$. Using the sublinearity, we then arrive at

$$
\big|\tau_h u^*(x) - u^*(x)\big| \leq (\tau_h u_0 - u_0)^*(x) \leq \varepsilon
$$

for $|h| < \delta$, which shows that u^* is continuous at the point x.

Part (ii). It follows since $\mathrm{Lip}(u(\cdot, t)) \leq \mathrm{Lip}(u_0)$ for each $t > 0$. □

Lemma 19 (Behaviour at large times - periodic version)**.** *Let $a, b \geq 0$ with $(a, b) \neq (0, 0)$ and $u(x,t) = \Psi_{a,b}(\cdot, t) * u_0(x)$. If $u_0 : \mathbb{T}^d \to \mathbb{R}^+$ is bounded and if $r > 0$ and $\varepsilon > 0$ are given, then there exists a time $t_{r,\varepsilon} < \infty$ such that $|u(x,t) - u(y,t)| < \varepsilon$ for all $x, y \in B_r$ and $t > t_{r,\varepsilon}$.*

Proof. It follows from (3.1.10) and Lemma 13 (ii). □

Lemma 20 (Subharmonicity)**.** *Let $a, b \geq 0$ with $(a, b) \neq (0, 0)$ and u^* be the maximal function defined in* (3.1.11)*. If $u_0 \in C(\mathbb{T}^d)$ then u^* is subharmonic in the open set $A = \{x \in \mathbb{T}^d;\ u^*(x) > u_0(x)\}$.*

Proof. Note initially that, by Lemma 18, the function u^* is continuous and the set $A \subset \mathbb{T}^d$ is indeed open. Moreover we have $A \neq \mathbb{T}^d$, since $u^*(x) = u_0(x)$ at a global maximum x of u_0. The rest of the proof is similar to the proof of Lemma 14, using the maximum principle for the heat equation in the case $a = 0$. □

Lemma 21. *Let $f, g \in C(\mathbb{T}^d) \cap W^{1,2}(\mathbb{T}^d)$. Suppose that $g \geq 0$ and that f is subharmonic in the open set $J = \{x \in \mathbb{T}^d;\ g(x) > 0\}$. Then*

$$\int_{\mathbb{T}^d} \nabla f(x) \,.\, \nabla g(x)\ \mathrm{d}x \leq 0.$$

Proof. This follows as Lemma 15. We omit the details. □

Lemma 22 (Reduction to the Lipschitz case - periodic version)**.** *In order to prove parts* (i)*,* (iii) *and* (iv) *of Theorem 9 it suffices to assume that the initial datum $u_0 : \mathbb{T}^d \to \mathbb{R}^+$ is Lipschitz.*

Proof. This follows as in the proof of Lemma 16. □

3.3.2 Proof of Theorem 9

Once we have established the lemmas of the previous subsection, together with Lemma 17, the proof of Theorem 9 follows essentially as in the proof of Theorem 8. We omit the details.

3.4 Proof of Theorem 10: Maximal operators on the sphere

3.4.1 Auxiliary lemmas

As before, we may assume that the initial datum u_0 is nonnegative.

In this section we denote by $B_r(\omega) \subset \mathbb{S}^d$ the geodesic ball of center ω and radius r, i.e.

$$B_r(\omega) = \{\eta \in \mathbb{S}^d;\ d(\eta, \omega) = \arccos(\eta \cdot \omega) < r\}.$$

We say that a continuous function $f : \mathbb{S}^d \to \mathbb{R}$ is *subharmonic* in a relatively open set $A \subset \mathbb{S}^d$ if, for every $\omega \in A$, and every geodesic ball $\overline{B_r(\omega)} \subset A$ we have

$$f(\omega) \leq \frac{1}{\sigma(\partial B_r(\omega))} \int_{\partial B_r(\omega)} f(\eta)\, \mathrm{d}\sigma(\eta),$$

where $\sigma(\partial B_r(\omega))$ denotes the surface area of $\partial B_r(\omega)$, and $\mathrm{d}\sigma$ denotes its surface measure. Throughout this section we write

$$\mathrm{Lip}(u) = \sup_{\substack{\omega,\eta \in \mathbb{S}^d \\ \omega \neq \eta}} \frac{|u(\omega) - u(\eta)|}{d(\omega, \eta)}$$

for the Lipschitz constant of a function $u : \mathbb{S}^d \to \mathbb{R}$.

Lemma 23 (Continuity - spherical version)**.** *Let u^* be the maximal function defined in* (3.1.14) *or* (3.1.16)*.*

(i) If $u_0 \in C(\mathbb{S}^d)$ then $u^* \in C(\mathbb{S}^d)$.

(ii) If u_0 is Lipschitz continuous then u^* is Lipschitz continuous with $\mathrm{Lip}(u^*) \leq \mathrm{Lip}(u_0)$*.*

Proof. (i) For the Poisson kernel this follows easily from the uniform continuity of u defined in (3.1.12) in the unit ball $\overline{B_1} \subset \mathbb{R}^{d+1}$. For the heat kernel we use the fact that the function $u(\omega, t)$ defined in (3.1.15) converges uniformly to the average value $M = \frac{1}{\sigma_d} \int_{\mathbb{S}^d} u_0(\eta)\, \mathrm{d}\sigma(\eta)$ as $t \to \infty$, which implies that u is uniformly continuous in $\mathbb{S}^d \times [0, \infty)$.

(ii) Let us consider the case of the Poisson kernel. The case of the heat kernel is analogous. Fix $0 < \rho < 1$ and consider two vectors ω_1 and ω_2 in $\mathbb{S}^d$. Let $E = \mathrm{span}\{\omega_1, \omega_2\}$ and F be the orthogonal complement of E in $\mathbb{R}^{d+1}$. Let R be an orthogonal transformation in $\mathbb{R}^{d+1}$ such that $R|_F = I$ and $R|_E$ is a rotation with $R\omega_1 = \omega_2$. It follows that for any $\eta \in \mathbb{S}^d$ we have $d(\eta, R\eta) \leq d(\omega_1, \omega_2)$. Using the fact that the Poisson kernel $\mathcal{P}(\omega, \eta, \rho)$ depends only on the inner product $\omega \cdot \eta$ (the same holds for the heat kernel) we have

$$\begin{aligned}
|u(\omega_1, \rho) - u(\omega_2, \rho)| &= \left| \int_{\mathbb{S}^d} \mathcal{P}(\omega_1, \eta, \rho)\, u_0(\eta)\, \mathrm{d}\sigma(\eta) - \int_{\mathbb{S}^d} \mathcal{P}(\omega_2, \eta, \rho)\, u_0(\eta)\, \mathrm{d}\sigma(\eta) \right| \\
&= \left| \int_{\mathbb{S}^d} \mathcal{P}(\omega_1, \eta, \rho)\, u_0(\eta)\, \mathrm{d}\sigma(\eta) - \int_{\mathbb{S}^d} \mathcal{P}(R^{-1}\omega_2, \eta, \rho)\, u_0(R\eta)\, \mathrm{d}\sigma(\eta) \right| \\
&\leq \int_{\mathbb{S}^d} \mathcal{P}(\omega_1, \eta, \rho) \left| u_0(\eta) - u_0(R\eta) \right| \mathrm{d}\sigma(\eta)
\end{aligned}$$

$$\leq \int_{\mathbb{S}^d} \mathcal{P}(\omega_1, \eta, \rho) \,\mathrm{Lip}(u_0)\, d(\eta, R\eta)\, \mathrm{d}\sigma(\eta)$$
$$\leq \mathrm{Lip}(u_0)\, d(\omega_1, \omega_2).$$

Hence $\mathrm{Lip}(u(\cdot,\rho)) \leq \mathrm{Lip}(u_0)$ and the pointwise supremum of Lipschitz functions with constants at most $\mathrm{Lip}(u_0)$ is also a Lipschitz function with constant at most $\mathrm{Lip}(u_0)$. □

Lemma 24 (Subharmonicity - spherical version)**.** *Let u^* be the maximal function defined in* (3.1.14) *or* (3.1.16)*. If $u_0 \in C(\mathbb{S}^d)$ then u^* is subharmonic in the open set $A = \{x \in \mathbb{S}^d;\, u^*(\omega) > u_0(\omega)\}$.*

Proof. First we deal with the maximal function associated to the Poisson kernel in (3.1.14). By Lemma 23 we know that u^* is continuous and the set A is indeed open. Take $\omega_0 \in A$ and consider a radius $r > 0$ such that the closed geodesic ball $\overline{B_r(\omega_0)}$ is contained in A. Let $h : \overline{B_r(\omega_0)} \to \mathbb{R}$ be the solution of the Dirichlet problem

$$\begin{cases} \Delta h = 0 & \text{in } B_r(\omega_0); \\ h = u^* & \text{in } \partial B_r(\omega_0), \end{cases}$$

where $\Delta = \Delta_{\mathbb{S}^d}$ is the Laplace-Beltrami operator with respect to the usual metric in $\mathbb{S}^d$. Since u^* is continuous, the unique solution h belongs to $C^2(B_r(\omega_0)) \cap C(\overline{B_r(\omega_0)})$. We now define the function

$$v(\omega, \rho) = u(\omega, \rho) - h(\omega),$$

which is harmonic (now with respect to the Euclidean Laplacian) in the open set $U = \{\rho\omega \in \mathbb{R}^d;\, \omega \in B_r(\omega_0),\ 0 < \rho < 1\}$. We claim that $v \leq 0$ in U. Assume that this is not the case and let

$$M = \sup_U v(\omega, \rho) > 0. \tag{3.4.1}$$

Let $\omega_1 \in \partial B_r(\omega_0)$ (by the maximum principle) be such that

$$\min_{\omega \in \overline{B_r(\omega_0)}} h(\omega) = h(\omega_1). \tag{3.4.2}$$

Since u is continuous in the unit Euclidean ball, let $\varepsilon > 0$ be such that (recall that we identify $u(\omega, \rho) = u(\rho\omega)$)

$$|u(\omega, \rho) - u(0)| \leq \frac{M}{2} \tag{3.4.3}$$

for $0 \le \rho \le \varepsilon$. Therefore, for $0 < \rho \le \varepsilon$, by (3.4.2) and (3.4.3) we have

$$v(\omega,\rho) = u(\omega,\rho) - h(\omega) \le \left(u(0) + \frac{M}{2}\right) - h(\omega_1) \le \left(u^*(\omega_1) + \frac{M}{2}\right) - h(\omega_1) = \frac{M}{2}. \quad (3.4.4)$$

Let $U_\varepsilon = \{\rho\omega \in \mathbb{R}^d;\, \omega \in B_r(\omega_0),\ \varepsilon < \rho < 1\}$. Note that v is continuous up to the boundary of U_ε and by (3.4.1) and (3.4.4) we have

$$M = \max_{\overline{U_\varepsilon}} v(\omega,\rho).$$

By the maximum principle, this maximum is attained at the boundary of U_ε. From (3.4.4) we may rule out the set where $\rho = \varepsilon$. Since $h = u^*$ in $\partial B_r(\omega_0)$, we have $v \le 0$ in the set $\{\rho\omega \in \mathbb{R}^d;\, \omega \in \partial B_r(\omega_0),\ \varepsilon \le \rho \le 1\}$. Hence the maximum M must be attained at a point $\eta \in B_r(\omega_0)$ (and $\rho = 1$). It follows that

$$u(\eta,\rho) - h(\eta) \le u_0(\eta) - h(\eta)$$

for every $0 < \rho < 1$, which implies that $u^*(\eta) = u_0(\eta)$, a contradiction. This establishes our claim.

It then follows that $u(\omega,\rho) \le h(\omega)$ for any $\omega \in B_r(\omega_0)$ and $0 < \rho < 1$, and this yields $u^* \le h$ in $B_r(\omega_0)$. Since this is true for any $\omega_0 \in A$ and any $r > 0$ such that $B_r(\omega_0) \subset A$, we conclude that u^* is subharmonic in A.

The proof for the maximal operator associated to the heat kernel (3.1.16) follows along the same lines (see the proof of [14, Lemma 8]), using the maximum principle for the heat equation. □

Lemma 25. *Let $f, g \in C(\mathbb{S}^d) \cap W^{1,2}(\mathbb{S}^d)$ be real-valued functions. Suppose that $g \ge 0$ and that f is subharmonic in the open set $J = \{\omega \in \mathbb{S}^d;\ g(\omega) > 0\}$. Then*

$$\int_{\mathbb{S}^d} \nabla f(\omega) \cdot \nabla g(\omega)\ \mathrm{d}\sigma(\omega) \le 0.$$

Proof. If both functions were smooth, by [16, Chapter I, Proposition 1.8.7], we would have

$$\int_{\mathbb{S}^d} \nabla f \cdot \nabla g\ \mathrm{d}\sigma(\omega) = \int_{\mathbb{S}^d} (-\Delta f)\ g\ \mathrm{d}\sigma(\omega) \le 0$$

and $-\Delta f \le 0$ in the set where $g > 0$. To prove the result, one can argue by approximaxion by smoother functions as in 15. We are only going to show the subtler part, which is to approximate f.

Let $O(d+1)$ be the group of rotations of $\mathbb{R}^{d+1}$ and let μ be its Haar probability measure. We consider a family ψ_ε of nonnegative C^∞-functions in $O(d+1)$ supported in an ε-neighborhood of the identity transformation with

$$\int_{O(d+1)} \psi_\varepsilon(R)\,\mathrm{d}\mu(R) = 1\,,$$

and we ask for each ε that $\psi_\varepsilon(S^t RS) = \psi_\varepsilon(R)$ for every $S \in O(d+1)$, i.e., that ψ_ε is invariant under conjugation. To construct such ψ_ε, it is enough to consider a smooth function of the trace in $O(d+1)$ which is concentrated in the set where the trace is in a neighborhood of $d+1$. We now define f_ε by

$$f_\varepsilon(\omega) = \int_{O(d+1)} f(R\omega)\,\psi_\varepsilon(R)\,\mathrm{d}\mu(R). \tag{3.4.5}$$

We now observe the following facts:

1. The function f_ε is in $C^\infty(\mathbb{S}^d)$. To see this we argue as follows. Let e_1 be the first canonical vector of $\mathbb{R}^{d+1}$ and define $F_\varepsilon : O(d+1) \to \mathbb{R}$ by

$$\begin{aligned} F_\varepsilon(S) &= \int_{O(d+1)} f(RSe_1)\,\psi_\varepsilon(R)\,\mathrm{d}\mu(R) \\ &= \int_{O(d+1)} f(Re_1)\,\psi_\varepsilon(RS^{-1})\,\mathrm{d}\mu(R). \end{aligned}$$

Since $\psi_\varepsilon(RS^{-1})$ is smooth as a function of R and S, the function F_ε is also smooth. Then the equality $F_\varepsilon(S) = f_\varepsilon(Se_1)$ and the fact that $S \mapsto Se_1$ is a smooth submersion from $O(d+1)$ to $\mathbb{S}^d$ imply that f_ε is also smooth.

2. The family f_ε approximates f in $W^{1,2}(\mathbb{S}^d)$ as $\varepsilon \to 0$. This can be verified directly from (3.4.5).

3. The function f_ε is subharmonic in the set $J_\varepsilon := \{\omega \in J;\, d(\omega, \partial J) > \varepsilon\}$. In fact, using the invariance of geodesic spheres under rotations and Fubini's theorem we find, for $\omega \in J_\varepsilon$,

$$\begin{aligned} f_\varepsilon(\omega) &= \int_{O(d+1)} f(R\omega)\,\psi_\varepsilon(R)\,\mathrm{d}\mu(R) \\ &\le \int_{O(d+1)} \left(\frac{1}{\sigma(\partial B_r(R\omega))} \int_{\partial B_r(R\omega)} f(\eta)\,\mathrm{d}\sigma(\eta) \right) \psi_\varepsilon(R)\,\mathrm{d}\mu(R) \end{aligned}$$

$$
\begin{aligned}
&= \int_{O(d+1)} \left(\frac{1}{\sigma(\partial B_r(\omega))} \int_{\partial B_r(\omega)} f(R\zeta)\, \mathrm{d}\sigma(\zeta) \right) \psi_\varepsilon(R)\, \mathrm{d}\mu(R) \\
&= \frac{1}{\sigma(\partial B_r(\omega))} \int_{\partial B_r(\omega)} f_\varepsilon(\zeta)\, \mathrm{d}\sigma(\zeta).
\end{aligned}
$$

Since f_ε is smooth, this implies that $(-\Delta f_\varepsilon) \leq 0$ in J_ε.

4. This is more a remark and is not strictly necessary for our proof. The function f_ε can be given as a convolution with a kernel that depends on the inner product of the entries. In fact, by the co-area formula one gets

$$
\begin{aligned}
f_\varepsilon(\omega) &= \int_{O(d+1)} f(R\omega)\, \psi_\varepsilon(R)\, \mathrm{d}\mu(R) \\
&= \int_{\mathbb{S}^d} f(\eta) \int_{\{R\omega=\eta\}} \psi_\varepsilon(R)\, [\![F_\omega(R)]\!]^{-1}\ \mathrm{d}\mathcal{H}^{d(d-1)/2}(R)\ \mathrm{d}\sigma(\eta) \\
&= \int_{\mathbb{S}^d} f(\eta)\, \Psi_\varepsilon(\omega, \eta)\, \mathrm{d}\sigma(\eta),
\end{aligned}
$$

where $[\![F_\omega(R)]\!]$ is the Jacobian of the submersion $F_\omega(R) = R\omega$ (this is just a constant) and $\mathcal{H}^{d(d-1)/2}$ is the $[d(d-1)/2]$-dimensional Hausdorff measure of $(O(d+1), \mathrm{d}\mu)$. From the invariance of ψ_ε by conjugation, it follows that $\Psi_\varepsilon(\omega, \eta)$ depends only on the inner product $\omega \cdot \eta$. The advantage of defining f_ε as in (3.4.5) is that we easily get the subharmonicity in $J_\varepsilon = \{\omega \in J;\, d(\omega, \partial J) > \varepsilon\}$ as shown in (3) above. In contrast to $\mathbb{R}^d$, there is no canonical way to move geodesic spheres that works in the same way as translation does in the Euclidean space, hence our choice to average over the whole group of rotations to arrive at this specific convolution kernel. □

The following lemma is a kind of integration by parts. We are going to need it to guarantee we can do a manipulation with a Sobolev function as if it were smooth.

Lemma 26. *Let $\omega, v \in \mathbb{S}^d$ be such that $\omega \cdot v = 0$ and let T be the linear transformation such that $T(\omega) = v$, $T(v) = -\omega$ and $T(\zeta) = 0$ whenever ζ is orthogonal to ω and v. If $f \in W^{1,1}(\mathbb{S}^d)$ and $g \in C^1(\mathbb{S}^d)$,*

$$
\int_{\mathbb{S}^d} f(\eta)\, (\nabla g(\eta) \cdot T(\eta))\, \mathrm{d}\sigma(\eta) = -\int_{\mathbb{S}^d} g(\eta)\, (\nabla f(\eta) \cdot T(\eta))\, \mathrm{d}\sigma(\eta).
$$

Proof. By a density argument, we may assume that $f \in C^1(\mathbb{S}^d)$. Now we observe that $e^{\lambda T}$

is a rotation on $\mathbb{R}^{d+1}$ for any $\lambda \in \mathbb{R}$. The equality we want to show is the same as

$$\lim_{\lambda \to 0} \int_{\mathbb{S}^d} f(\eta) \frac{g(e^{\lambda T}\eta) - g(\eta)}{\lambda} \mathrm{d}\sigma(\eta) = \lim_{\lambda \to 0} \int_{\mathbb{S}^d} g(\eta) \frac{f(\eta) - f(e^{\lambda T}\eta)}{\lambda} \mathrm{d}\sigma(\eta),$$

which holds because left-hand integral at λ equals the right-hand integral at $-\lambda$. □

Lemma 27 (Reduction to the continuous case - spherical version)**.** *In order to prove parts* (i)*,* (iii) *and* (iv) *of Theorem 10 it suffices to assume that the initial datum* $u_0 : \mathbb{S}^d \to \mathbb{R}^+$ *is continuous.*

Proof. We consider here the Poisson case and the heat flow case is analogous. For $0 < r < 1$ and $\omega \in \mathbb{S}^d$ let $u_r(\omega) = u(r\omega)$. It is clear that u_r is a continuous function (in fact it is smooth) and that the solution of the Dirichlet problem (3.1.13), with u_r replacing u_0 as the boundary condition, is a suitable dilation of u. Hence

$$u_r^*(\omega) = \sup_{0 \le \rho < r} u(\rho\omega),$$

which implies that $u_r^* \to u^*$ pointwise as $r \to 1$.

For any $\omega, v \in \mathbb{S}^d$ such that $\omega \cdot v = 0$, let T be as in Lemma 26. For any $\lambda \in \mathbb{R}$,

$$\begin{aligned} u_r(e^{\lambda T}\omega) &= \int_{\mathbb{S}^d} \mathcal{P}(e^{\lambda T}\omega, \eta, r)\, u_0(\eta)\, \mathrm{d}\sigma(\eta) \\ &= \int_{\mathbb{S}^d} \mathcal{P}(\omega, e^{-\lambda T}\eta, r)\, u_0(\eta)\, \mathrm{d}\sigma(\eta). \end{aligned}$$

Differentiating both sides with respect to λ, evaluating at $\lambda = 0$ and using Lemma 26 yields

$$\nabla u_r(\omega) \cdot v = \int_{\mathbb{S}^d} \mathcal{P}(\omega, \eta, r)\, (\nabla u_0(\eta) \cdot T(\eta))\, \mathrm{d}\sigma(\eta).$$

Since this holds for any v,

$$|\nabla u_r(\omega)| \le \int_{\mathbb{S}^d} \mathcal{P}(\omega, \eta, r)\, |\nabla u_0(\eta)|\, \mathrm{d}\sigma(\eta). \tag{3.4.6}$$

It follows that

$$|\nabla u_r(\omega)| \le |\nabla u_0|^*(\omega) \tag{3.4.7}$$

and, by (3.4.6) and Jensen's inequality, we obtain

$$\|\nabla u_r\|_{L^p(\mathbb{S}^d)} \leq \|\nabla u_0\|_{L^p(\mathbb{S}^d)}$$

for $1 \leq p \leq \infty$. The rest of the proof follows as in Lemma 16. □

3.4.2 Proof of Theorem 10

Combining the lemmas of the previous subsection with Lemma 17, the proof of Theorem 10 follows as in the proof of Theorem 8. We omit the details.

3.5 Proof of Theorem 11: Non-tangential maximal operators

3.5.1 Auxiliary lemmas

We keep the same strategy. The first step is still to note that the initial condition u_0 may be assumed to be nonnegative. In this section $u(x,t) = P(\cdot,t) * u_0(x)$ for $t > 0$ and $u(x,0) = u_0(x)$. The function u defined this way is harmonic in the open upper half-plane. We may restrict ourselves to the novel case $\alpha > 0$.

Lemma 28 (Continuity - non-tangential version)**.** *Let $\alpha > 0$ and u^* be the maximal function defined in* (3.1.17)*.*

(i) If $u_0 \in C(\mathbb{R}) \cap L^p(\mathbb{R})$, for some $1 \leq p < \infty$, then $u^* \in C(\mathbb{R})$.

(ii) If u_0 is bounded and Lipschitz continuous then u^* is bounded and Lipschitz continuous with $\mathrm{Lip}(u^*) \leq \mathrm{Lip}(u_0)$*.*

Proof. (i) From the hypothesis $u_0 \in C(\mathbb{R}) \cap L^p(\mathbb{R})$, we know that u is continuous up to the boundary. By Hölder's inequality, $|u(x,t)| \leq \|P(\cdot,t)\|_{p'}\|u_0\|_p$ and so $u(x,t)$ converges uniformly to zero as $t \to \infty$. These facts imply that $u^* \in C(\mathbb{R})$.

(ii) For any $t > 0$ and $y \in \mathbb{R}$, the function $x \mapsto u(x+y,t)$ is bounded by $\|u_0\|_\infty$ and is Lipschitz continuous with constant less than or equal to $\mathrm{Lip}(u_0)$. The claim follows since $u^*(x)$ is the supremum of these functions over all pairs (t,y) such that $|y| \leq \alpha t$. □

Lemma 29 (Subharmonicity - non-tangential version)**.** *Let* $\alpha > 0$ *and* u^* *be the maximal function defined in* (3.1.17)*. Let* $u_0 \in C(\mathbb{R}) \cap L^p(\mathbb{R})$ *for some* $1 \leq p < \infty$ *or* u_0 *be bounded and Lipschitz continuous. Then* u^* *is subharmonic in the open set* $A = \{x \in \mathbb{R};\, u^*(x) > u_0(x)\}$*.*

Proof. The set A is in fact open due to Lemma 28.

Step 1. We first prove the following claim: for any $x_0 \in A$ there exist arbitrarily small positive values of ε such that

$$u^*(x_0 + \varepsilon) + u^*(x_0 - \varepsilon) \geq 2u^*(x_0). \tag{3.5.1}$$

Case 1. Assume that u_0 is bounded and Lipschitz continuous and that

$$d = u^*(x_0) - \sup_{\substack{t>0 \\ |y-x_0|=\alpha t}} u(y,t) > 0. \tag{3.5.2}$$

Since $\mathrm{Lip}(u(\cdot,t)) \leq \mathrm{Lip}(u_0)$ for any positive t, we have

$$u^*(x_0) = \sup_{\substack{t>0 \\ |y-x_0| \leq \alpha t - \frac{d}{2\mathrm{Lip}(u_0)}}} u(y,t). \tag{3.5.3}$$

For $0 < \varepsilon < \frac{d}{2\mathrm{Lip}(u_0)}$ the region over which we take the supremum in (3.5.3) is contained in the region $|y - (x_0 + \varepsilon)| \leq \alpha t$ and so $u^*(x_0 + \varepsilon) \geq u^*(x_0)$. Similarly $u^*(x_0 - \varepsilon) \geq u^*(x_0)$, and this establishes (3.5.1).

Case 2. Let us define two operators: $u_R^*(x) = \sup_{t>0} u(x + \alpha t, t)$ and $u_L^*(x) = \sup_{t>0} u(x - \alpha t, t)$. If (3.5.2) does not happen then

$$u^*(x_0) = \max\{u_R^*(x_0), u_L^*(x_0)\}. \tag{3.5.4}$$

This is certainly the case when $u_0 \in C(\mathbb{R}) \cap L^p(\mathbb{R})$, since the function $u(x,t)$ converges to zero uniformly as $t \to \infty$ and (3.5.4) follows by the maximum principle. Let us assume without loss of generality that $u^*(x_0) = u_R^*(x_0)$.

Let $\theta = \arctan \alpha$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation of angle θ, given explicitly by $T(x,t) = (x\cos\theta - t\sin\theta, x\sin\theta + t\cos\theta)$. Letting $v = u \circ T^{-1}$, we get that v is continuous on $\{(x,t) \in \mathbb{R}^2;\, \alpha x \leq t\}$, $v(x\cos\theta, x\sin\theta) = u_0(x)$ and $u_R^*(x) = \sup_{t > x\sin\theta} v(x\cos\theta, t)$ for any $x \in \mathbb{R}$. Since rotations preserve harmonicity, if $t > x_0 \sin\theta$ and

$r < (t - x_0 \sin\theta)\cos\theta$ we have

$$v(x_0\cos\theta, t) = \frac{1}{\pi r^2}\int_{B_r(x_0\cos\theta,t)} v(y,s)\,\mathrm{d}y\,\mathrm{d}s \le \frac{1}{\pi r^2}\int_{-r}^{r} 2\sqrt{r^2-y^2}\; u_R^*\left(\frac{x_0\cos\theta + y}{\cos\theta}\right)\mathrm{d}y. \tag{3.5.5}$$

Since we are assuming that $x_0 \in A$ and $u^*(x_0) = u_R^*(x_0) > u_0(x_0)$, by the continuity of v there exists a $\delta = \delta(x_0)$ such that

$$v(x_0\cos\theta, t) < u^*(x_0) - \tfrac{1}{2}(u^*(x_0) - u_0(x_0))$$

for $x_0\sin\theta < t < x_0\sin\theta + \delta$. Hence the supremum in $u^*(x_0) = u_R^*(x_0) = \sup_{t > x_0\sin\theta} v(x_0\cos\theta, t)$ can be restricted to times $t \ge x_0\sin\theta + \delta$, and we can choose any $r < \delta\cos\theta$ in (3.5.5) to get

$$u^*(x_0) \le \frac{1}{\pi r^2}\int_{-r}^{r} 2\sqrt{r^2-y^2}\; u^*\left(x_0 + \frac{y}{\cos\theta}\right)\mathrm{d}y$$

and this implies the existence of $\varepsilon < \frac{r}{\cos\theta}$ verifying (3.5.1).

Step 2. If u^* were not subharmonic (i.e. convex in each connected component), we would be able to find an interval $[a,b] \subset A$ such that $u^*(a) + u^*(b) < 2u^*(\frac{a+b}{2})$. Let $h(x) = \frac{x-a}{b-a}u^*(b) + \frac{b-x}{b-a}u^*(a)$. Then $u^* - h$ vanishes at the endpoints a and b but is positive at their arithmetic mean. Choose $x_0 \in [a,b]$ as small as possible such that $(u^* - h)(x_0) = \sup_{x\in[a,b]}(u^* - h)(x)$. Then for all ε sufficiently small,

$$(u^* - h)(x_0 + \varepsilon) + (u^* - h)(x_0 - \varepsilon) < 2(u^* - h)(x_0),$$

which contradicts (3.5.1). This completes the proof. □

Lemma 30 (Reduction to the Lipschitz case - non-tangential version)**.** *In order to prove parts* (i) *and* (iii) *of Theorem 11 it suffices to assume that the initial datum $u_0 : \mathbb{R} \to \mathbb{R}^+$ is Lipschitz.*

Proof. It is the same as the proof of Lemma 16, replacing identity (3.2.10) with

$$u_\varepsilon^*(x) = \sup_{\substack{t>0 \\ |y-x|\le\alpha t}} P(\cdot,t) * u_\varepsilon(y) = \sup_{\substack{t>0 \\ |y-x|\le\alpha t}} u(y, t+\varepsilon).$$

Note that $u_\varepsilon^* \to u^*$ pointwise as $\varepsilon \to 0$. □

3.5.2 Proof of Theorem 11

Once we have established the lemmas of the previous subsection, together with Lemma 17, the proof of Theorem 11 follows essentially as in the proof of Theorem 8. We omit the details.

3.5.3 A counterexample in higher dimensions

If $\alpha > 0$ and $d > 1$, the non-tangential maximal function (3.1.17) in $\mathbb{R}^d$ is not necessarily subharmonic in the detachment set. We now present a counterexample.

Recall the explicit form of the Poisson kernel $P(x,t)$ as defined in (3.1.3). Let $u_0 : \mathbb{R}^d \to \mathbb{R}$ be given by

$$u_0(x) = (1+|x|^2)^{\frac{-d+1}{2}} = (d-1)\int_1^\infty \frac{s}{(s^2+|x|^2)^{\frac{d+1}{2}}}\,\mathrm{d}s.$$

Writing $C_d = \Gamma\left(\frac{d+1}{2}\right)\pi^{-(d+1)/2}$ we get

$$\begin{aligned}
u(x,t) &= \int_{\mathbb{R}^d} P(x-y,t)\,u_0(y)\,\mathrm{d}y \\
&= \frac{(d-1)}{C_d}\int_{\mathbb{R}^d}\int_1^\infty P(x-y,t)\,P(y,s)\,\mathrm{d}s\,\mathrm{d}y \\
&= \frac{(d-1)}{C_d}\int_1^\infty\int_{\mathbb{R}^d} P(x-y,t)\,P(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&= \frac{(d-1)}{C_d}\int_1^\infty P(x,t+s)\,\mathrm{d}s \\
&= \left((t+1)^2+|x|^2\right)^{\frac{-d+1}{2}}.
\end{aligned}$$

This is a translation of the fundamental solution of Laplace's equation on $\mathbb{R}^{d+1}$. A direct computation yields

$$u^*(x) = \begin{cases} u_0(x) & \text{if } |x| \le \frac{1}{\alpha}; \\ \left(\frac{(\alpha+|x|)^2}{\alpha^2+1}\right)^{\frac{-d+1}{2}} & \text{if } |x| > \frac{1}{\alpha}. \end{cases}$$

From this we obtain

$$-\Delta u^*(x) = (d-1)\,\frac{(\alpha^2+1)^{\frac{d-1}{2}}}{(\alpha+|x|)^{d+1}}\left(\frac{\alpha}{|x|}(d-1)-1\right)$$

for $|x| > \frac{1}{\alpha}$. This is strictly positive (hence u^* is superharmonic) for $\frac{1}{\alpha} < |x| < (d-1)\alpha$

(assuming that this interval is nonempty, i.e. that $(d-1)\alpha^2 > 1$).

Bibliography

[1] J. M. Aldaz and J. Pérez Lázaro, Functions of bounded variation, the derivative of the one dimensional maximal function, and applications to inequalities, Transactions of the American Mathematical Society 359, no. 5 (2007), 2443–2461.

[2] J. Bober, E. Carneiro, K. Hughes and L. B. Pierce, On a discrete version of Tanaka's theorem for maximal functions, Proceedings of the American Mathematical Society 140 (2012), 1669–1680.

[3] J. Bober, J. B. Conrey, D. Farmer, A. Fujii, S. Koutsolitas, S. Lemurell, M. Rubinstein and H. Yoshida, The highest lower zero of general L-functions, to appear in Journal of Number Theory, preprint at http://arxiv.org/abs/1211.5996.

[4] E. Carneiro and V. Chandee, Bounding $\zeta(s)$ in the critical strip, Journal of Number Theory 131 (2011), 363–384.

[5] E. Carneiro, V. Chandee, F. Littmann and M. B. Milinovich, Hilbert spaces and the pair correlation of zeros of the Riemann zeta-function, to appear in Journal für die reine und angewandte Mathematik, preprint at http://arxiv.org/abs/1406.5462.

[6] E. Carneiro, V. Chandee and M. B. Milinovich, Bounding $S(t)$ and $S_1(t)$ on the Riemann hypothesis, Mathematische Annalen 356, no. 3 (2013), 939–968.

[7] E. Carneiro, V. Chandee and M. B. Milinovich, A note on the zeros of zeta and L-functions, Mathematische Zeitschrift 281 (2015), 315–332.

[8] E. Carneiro and A. Chirre, Bounding $S_n(t)$ on the Riemann Hypothesis, preprint.

[9] E. Carneiro and K. Hughes, On the endpoint regularity of discrete maximal operators, Mathematical Research Letters 19, no. 6 (2012), 1245–1262.

[10] E. Carneiro and F. Littmann, Bandlimited approximation to the truncated Gaussian and applications, Constructive Approximation 38, no. 1 (2013), 19–57.

[11] E. Carneiro, F. Littmann and J. D. Vaaler, Gaussian subordination for the Beurling-Selberg extremal problem, Transactions of the American Mathematical Society 365, no. 7 (2013), 3493–3534.

[12] E. Carneiro and J. Madrid, Derivative bounds for fractional maximal functions, to appear in Transactions of the American Mathematical Society, preprint at http://arxiv.org/abs/1510.02965.

[13] E. Carneiro and D. Moreira, On the regularity of maximal operators, Proceedings of the American Mathematical Society 136, no. 12 (2008), 4395–4004.

[14] E. Carneiro and B. F. Svaiter, On the variation of maximal operators of convolution type, Journal of Functional Analysis 265 (2013), 837–865.

[15] V. Chandee and K. Soundararajan, Bounding $|\zeta(\frac{1}{2}+it)|$ on the Riemann Hypothesis, Bulletin of the London Mathematical Society 43 (2011), 243–250.

[16] F. Dai and Y. Xu, *Approximation theory and harmonic analysis on spheres and balls*, Springer (2013).

[17] A. Fujii, An explicit estimate in the theory of the distribution of the zeros of the Riemann zeta function, Commentarii Mathematici Universitatis Sancti Pauli 53 (2004), 85–114.

[18] A. Fujii, A note on the distribution of the argument of the Riemann zeta function, Commentarii Mathematici Universitatis Sancti Pauli 55, no. 2 (2006), 135–147.

[19] P. X. Gallagher, Pair correlation of zeros of the zeta function, Journal für die reine und angewandte Mathematik 362 (1985), 72–86.

[20] D. A. Goldston and S. M. Gonek, A note on $S(t)$ and the zeros of the Riemann zeta-function, Bulletin of the London Mathematical Society 39, no. 3 (2007), 482–486.

[21] P. Hajłasz and J. Maly, On approximate differentiability of the maximal function, Proceedings of the American Mathematical Society 138 (2010), 165–174.

[22] P. Hajłasz and J. Onninen, On boundedness of maximal functions in Sobolev spaces, Annales Academiae Scientiarum Fennicae Mathematica 29, no. 1 (2004), 167–176.

[23] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications, vol. 53 (2004).

[24] A. A. Karatsuba and M. A. Korolev, The argument of the Riemann zeta function, Russian Mathematical Surveys 60, no. 3 (2005), 433–488.

[25] J. Kinnunen, The Hardy-Littlewood maximal function of a Sobolev function, Israel Journal of Mathematics 100 (1997), 117–124.

[26] J. Kinnunen and P. Lindqvist, The derivative of the maximal function, Journal für die reine und angewandte Mathematik 503 (1998), 161–167.

[27] J. Kinnunen and E. Saksman, Regularity of the fractional maximal function, Bulletin of the London Mathematical Society 35, no. 4 (2003), 529–535.

[28] O. Kurka, On the variation of the Hardy-Littlewood maximal function, Annales Academiae Scientiarum Fennicae Mathematica 40 (2015), 109–133.

[29] J. E. Littlewood, On the zeros of the Riemann zeta-function, Proceedings of the Cambridge Philosophical Society 22 (1924), 295–318.

[30] H. Luiro, Continuity of the maximal operator in Sobolev spaces, Proceedings of the American Mathematical Society 135, no. 1 (2007), 243–251.

[31] H. Luiro, On the regularity of the Hardy-Littlewood maximal operator on subdomains of $\mathbb{R}^d$, Proceedings of the Edinburgh Mathematical Society 53, no. 1 (2010), 211–237.

[32] J. Madrid, Sharp inequalities for the variation of the discrete maximal function, preprint at http://arxiv.org/abs/1512.04112.

[33] S. D. Miller, The highest-lowest zero and other applications of positivity, Duke Mathematical Journal 112, no. 1 (2002), 83–116.

[34] K. Ramachandra and A. Sankaranarayanan, On some theorems of Littlewood and Selberg I, Journal of Number Theory 44 (1993), 281–291.

[35] R. Schoen and S-T. Yau, *Lectures in differential geometry*, International Press (2010).

[36] I. J. Schoenberg, Metric spaces and completely monotone functions, Annals of Mathematics (2) 39 (1938), 811–841.

[37] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press (1970).

[38] E. M. Stein and G. Weiss, *Fourier analysis on Euclidean spaces*, Princeton University Press (1971).

[39] S. Steinerberger, A rigidity phenomenon for the Hardy-Littlewood maximal function, to appear in Studia Mathematica, preprint at http://arxiv.org/abs/1410.0588.

[40] H. Tanaka, A remark on the derivative of the one-dimensional Hardy-Littlewood maximal function, Bulletin of the Australian Mathematical Society 65, no. 2 (2002), 253–258.

[41] F. Temur, On regularity of the discrete Hardy-Littlewood maximal function, preprint at http://arxiv.org/abs/1303.3993.