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Regularity of Fractional Maximal Functions and Extremal Functions of Exponential Type

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Regularity of Fractional Maximal Functions and Extremal Functions of Exponential Type

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To my family

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Abstract

In this Ph.D. thesis we discuss three different problems in analysis: (a) regularity properties of fractional maximal operators acting on BV–functions, both in the continuous and in the discrete settings; (b) sharp inequalities for the variation of the discrete maximal function; (c) extremal approximations of exponential type for the Gaussian function and for a class of radial functions in the Euclidean space, with applications to analytic number theory.

Contents

Chapter 1

Introduction

This Ph.D. thesis is composed of three chapters that describe the advances obtained in the following research projects:

[A] E. Carneiro and J. Madrid, Derivative Bounds for Fractional Maximal Functions, to appear in Transactions of the American Mathematical Society (2016), 30 pp.

[B] J. Madrid, Sharp Inequalities for the Variation of the Discrete Maximal Function, preprint submitted, 12 pp.

[C] F. Gonçalves, M. Kelly and J. Madrid, One-sided Band-limited Approximations of Some Radial Functions, Bulletin of the Brazilian Mathematical Society 46, No. 4 (2015), 563–599.

The first two of them are related to the regularity theory of maximal operators. The third one deals with the construction of certain extremal functions of exponential type.

Chapter 2 discusses new bounds for the derivative of the fractional maximal function, both in the continuous and in the discrete settings. This was a joint work with my advisor Emanuel Carneiro (IMPA). The behavior of the maximal operator with respect to weak derivatives was first studied by J. Kinnunen [45] in 1997, when he proved that the classical Hardy-Littlewood maximal operator M is bounded on the Sobolev space $W^{1,p}(\mathbb{R}^d)$, for $p > 1$. In [47] Kinnunen and Saksman studied the regularity properties of the fractional maximal operator M_β. One of the results they proved is that $M_\beta : W^{1,p}(\mathbb{R}^d) \to W^{1,q}(\mathbb{R}^d)$ is bounded if $1 < p < \infty$, $0 < \beta < d/p$ and $q = dp/(d - \beta p)$, extending Kinunnen's original result [45] for the case $\beta = 0$. Due to the lack of reflexivity of L^1, results for $p = 1$ are subtler and progress on this direction has been restricted to dimension $d = 1$. In [74] Tanaka showed that the operator $f \to \nabla \widetilde{M} f$ from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$ is bounded, where $\widetilde{M}$ denotes the uncentered maximal operator. This result was later refined by Aldaz and Pérez Lázaro [2]. The analogous result in the discrete setting was obtained by Bober, Carneiro, Hughes and Pierce in [8]. In Chapter 2 we extend these results to the fractional context. The strategy is to study first the discrete setting. The heart of the proof in this case is to establish a local control for the q-variation of the discrete fractional uncentered maximal operator. In the continuous setting, we reduce the problem to consider Lipschitz functions, we adapt some lemmas from the discrete setting and then use a classical theorem of F. Riesz to conclude. In the discrete setting we also obtain an interesting family of inequalities that approximate the conjectured result, improving the main results of Carneiro and Hughes in [16]. These results are collected in the paper [A].

Chapter 3 summarizes new optimal results for certain inequalities related to the variation of discrete maximal operators. In [8] it was proved that, in the one-dimensional discrete setting, the variation of the centered maximal function Mf is less than or equal to $2(1 + 146/315)$ times the ℓ^1-norm of the function f. The authors of [8] conjectured that the optimal constant for this inequality would be 2. In this chapter we give an affirmative answer to this conjecture and I extend this optimal result to higher dimensions in two distinct ways, finding the optimal constants of the main results presented in [16] for some special discrete maximal operators. This takes into consideration the geometry of the associated sets. The results of this chapter are collected in the paper [B].

Chapter 4 describes a project done in collaboration with Felipe Gonçalves (IMPA) and Michael Kelly (University of Michigan) on extremal entire approximations of prescribed exponential type for real-valued functions. The most well-known example of this theory is the problem of approximating $f(x) = \mathrm{sgn}(x)$ that was considered by A. Beurling and A. Selberg in the 1930's and 1970's. Most of the work on these sorts of problems has been focused on solving Selberg's problem, but with $\mathrm{sgn}(x)$ replaced by a different single variable function, such as $\log|x|$, $|x|^\beta$, $\beta > -1$ and $e^{-\lambda|x|}$, providing applications to analytic number theory and equidistribution theory. In [20], Carneiro, Littmann and Vaaler solved the extremal problem for the Gaussian $f(x) = e^{-\pi\lambda x^2}$, $\lambda > 0$. In Chapter 4 we extend this construction to higher dimensions, for the Gaussian function and a class of radial functions. The majorants that we construct are shown to be extremal and our minorants are shown to be asymptotically extremal in an appropriate sense. We also obtain periodic analogues of the main results and applications to Hilbert-type inequalities. These results are collected in the paper [C].

Chapter 2

Regularity of fractional maximal functions

2.1 Preliminaries

2.1.1 History of the problem

Let M denote the centered Hardy-Littlewood maximal operator on $\mathbb{R}^d$, i.e. for $f \in L^1_{loc}(\mathbb{R}^d)$,

$$Mf(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)| \, \mathrm{d}y \,, \tag{2.1.1}$$

where $B_r(x)$ is the ball centered at x with radius r and $m(B_r(x))$ is its d-dimensional Lebesgue measure. One of the cornerstones of harmonic analysis is the celebrated theorem of Hardy-Littlewood-Wiener that asserts that $M : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is bounded for $1 < p \leq \infty$. For $p = 1$ we have $M : L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$ bounded. The process of averaging a function is, in essence, a variation-diminishing process (e.g. for a fixed radius r in (2.1.1)), so it is natural to wonder if such behavior is preserved when taking a pointwise supremum over averages instead. This leads one to consider maximal operators acting on functions of bounded variation and Sobolev functions. The first result in this direction is due to Kinunnen [45], who showed that $M : W^{1,p}(\mathbb{R}^d) \to W^{1,p}(\mathbb{R}^d)$ is bounded for $1 < p \leq \infty$, elegantly combining basic tools of functional analysis. This paradigm has been extended to multilinear, local and fractional contexts in [22, 46, 47]. Due to the lack of reflexivity of L^1, results for $p = 1$ are subtler. Examples of such results were motivated by the following question posed in [38]:

Question A. (Hajłasz and Onninen [38]) Is the operator $f \mapsto |\nabla Mf|$ bounded from

$W^{1,1}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$?

A standard dilation argument reveals the true nature of this question: whether the variation of the maximal function is controlled by the variation of the original function, i.e. if we have

$$\|\nabla Mf\|_{L^1(\mathbb{R}^d)} \leq C\|\nabla f\|_{L^1(\mathbb{R}^d)}.$$

Progress on this problem has been restricted to dimension $d = 1$. For the uncentered maximal operator (defined similarly as in (2.1.1), with the supremum taken over all balls containing the point x in its closure), which we denote here by $\widetilde{M}$, Tanaka [74] showed that if $f \in W^{1,1}(\mathbb{R})$ then $\widetilde{M}f$ is weakly differentiable and

$$\big\|\big(\widetilde{M}f\big)'\big\|_{L^1(\mathbb{R})} \leq 2\,\|f'\|_{L^1(\mathbb{R})}. \tag{2.1.2}$$

This result was later refined by Aldaz and Pérez Lázaro [2], who showed that if f is of bounded variation then $\widetilde{M}f$ is in fact absolutely continuous and

$$\mathrm{Var}\,\big(\widetilde{M}f\big) \leq \mathrm{Var}\,(f), \tag{2.1.3}$$

where $\mathrm{Var}\,(f)$ denotes the total variation of f. Observe that inequality (2.1.3) is sharp. More recently, in the remarkable paper [49], Kurka considered the centered maximal operator in dimension $d = 1$ and proved that

$$\mathrm{Var}\,(Mf) \leq 240{,}004\,\mathrm{Var}\,(f). \tag{2.1.4}$$

It is also shown in [49] that if $f \in W^{1,1}(\mathbb{R})$ then Mf is weakly differentiable and (2.1.2) also holds with constant $C = 240{,}004$. It is currently unknown if one can bring down the value of such constant to $C = 1$ in the centered case. The status of Question A in the general case $d > 1$ is widely open, even in establishing the weak differentiability of Mf or $\widetilde{M}f$ (related issues were considered by Hajłasz and Maly in [37]). In [26], Carneiro and Svaiter considered maximal operators of convolution type associated to smooth kernels (namely, the Gauss kernel and the Poisson kernel), and obtained the inequalities (2.1.2) and (2.1.3) with the sharp constant $C = 1$ by exploring the connections with the underlying partial differential equations. Other interesting works related to this theory are [1, 56, 57, 73].

For $0 \leq \beta < d$, we define the centered fractional maximal operator as

$$M_\beta f(x) = \sup_{r>0} \frac{1}{m(B_r(x))^{1-\frac{\beta}{d}}} \int_{B_r(x)} |f(y)|\,\mathrm{d}y.$$

When $\beta = 0$ we plainly recover (2.1.1). Such fractional maximal operators have applications in potential theory and partial differential equations. By comparison with an appropriate Riesz potential, one can show that if $1 < p < \infty$, $0 < \beta < d/p$ and $q = dp/(d - \beta p)$, then $M_\beta : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is bounded. When $p = 1$ we have again a weak-type bound (for details, see [70, Chapter V, Theorem 1]). In [47], Kinnunen and Saksman studied the regularity properties of such fractional maximal operators. One of the results they proved [47, Theorem 2.1] is that $M_\beta : W^{1,p}(\mathbb{R}^d) \to W^{1,q}(\mathbb{R}^d)$ is bounded for p, q, β, d as described above, extending Kinunnen's original result [45] for the case $\beta = 0$. It is then natural to consider the extension of Question A to the fractional case at the endpoint $p = 1$:

Question B. Let $0 \leq \beta < d$ and $q = d/(d-\beta)$. Is the operator $f \mapsto |\nabla M_\beta f|$ bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$?

In the case $1 \leq \beta < d$, Question B admits a positive answer, which follows from the main result of Kinnunen and Saksman in their aforementioned work [47]. In fact, [47, Theorem 3.1] states the following regularizing effect: if $f \in L^r(\mathbb{R}^d)$ with $1 < r < d$ and $1 \leq \beta < d/r$, then $M_\beta f$ is weakly differentiable and

$$|\nabla M_\beta f(x)| \leq C\, M_{\beta-1} f(x) \tag{2.1.5}$$

holds for almost all $x \in \mathbb{R}^d$, where $C = C(d, \beta)$ is a universal constant. In our case, given $1 \leq \beta < d$ and $f \in W^{1,1}(\mathbb{R}^d)$, by the Sobolev embedding we have $f \in L^{p^*}(\mathbb{R}^d)$, where $p^* = d/(d-1)$, and hence $f \in L^r(\mathbb{R}^d)$ for any $1 \leq r \leq p^*$. We may choose r with $1 < r < d$ such that $1 \leq \beta < d/r$ and hence (2.1.5) holds. Then

$$\|\nabla M_\beta f\|_{L^q(\mathbb{R}^d)} \leq C\, \|M_{\beta-1} f\|_{L^q(\mathbb{R}^d)} \leq C'\, \|f\|_{L^{p^*}(\mathbb{R}^d)} \leq C'' \|\nabla f\|_{L^1(\mathbb{R}^d)}.$$

Question B (in the case $0 \leq \beta < 1$) is the main motivation for this work. The presence of the fractional part introduces additional difficulties as we shall see in the course of the chapter (e.g. one does not necessarily have $M_\beta(f)(x) \geq |f(x)|$ a.e.). Here we give a positive answer to Question B in dimension $d = 1$ for the uncentered fractional maximal operator (which we denote by $\widetilde{M}_\beta$), both in the continuous and discrete settings. For general $d \geq 1$, in the discrete setting, we also obtain an interesting family of inequalities that approximate the conjectured bounds, for both the centered and uncentered versions (in a more general framework, where the balls are replaced by dilations of a given convex set). We now briefly state these results.

2.1.2 Main results

Continuous setting

In dimension $d = 1$, for $0 \leq \beta < 1$, the uncentered fractional maximal operator is

$$\widetilde{M}_\beta f(x) = \sup_{\substack{r,s \geq 0 \\ r+s>0}} \frac{1}{(r+s)^{1-\beta}} \int_{x-r}^{x+s} |f(y)| \, \mathrm{d}y. \tag{2.1.6}$$

For a function $f : \mathbb{R} \to \mathbb{R}$ and $1 \leq q < \infty$, motivated by the Riemann sums of a Riemann integrable function, we define its q-variation as

$$\mathrm{Var}_q(f) := \sup_{\mathcal{P}} \left(\sum_{n=1}^{N-1} \frac{|f(x_{n+1}) - f(x_n)|^q}{|x_{n+1} - x_n|^{q-1}} \right)^{1/q}, \tag{2.1.7}$$

where the supremum is taken over all finite partitions $\mathcal{P} = \{x_1 < x_2 < \ldots < x_N\}$. This is also known as Riesz q-variation of f, (see for instance [5]). Our first result is the extension of (2.1.2) and (2.1.3) for the uncentered fractional maximal operator.

Theorem 1. *Let $0 \leq \beta < 1$ and $q = 1/(1-\beta)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation such that $\widetilde{M}_\beta f \not\equiv \infty$. Then $\widetilde{M}_\beta f$ absolutely continuous and its derivative satisfies*

$$\left\| \big(\widetilde{M}_\beta f\big)' \right\|_{L^q(\mathbb{R})} = \mathrm{Var}_q\big(\widetilde{M}_\beta f\big) \leq 8^{1/q} \,\mathrm{Var}\,(f). \tag{2.1.8}$$

The constant $C = 8^{1/q}$ appearing in (2.1.8) is likely not sharp. The problem of finding the sharp constant in this inequality is certainly an interesting one. Another inviting possibility is the investigation of the validity of Theorem 1 for the centered fractional maximal function, which, if confirmed, would be an extension of Kurka's work [49]. It is worth mentioning that our strategy to approach the fractional case is very different from that of Tanaka [74] and Aldaz and Pérez Lázaro [2] for the case $\beta = 0$. In those papers the essential idea is to prove that the maximal function does not have any local maxima in the set where it disconnects from the original function. In the fractional case $\beta > 0$, the mere notion of the disconnecting set is ill-posed, since one does not necessarily have $M_\beta(f)(x) \geq |f(x)|$ a.e. anymore.

Discrete setting

We shall denote a vector $\vec{n} \in \mathbb{Z}^d$ by $\vec{n} = (n_1, n_2, \ldots, n_d)$. For a function $f : \mathbb{Z}^d \to \mathbb{R}$ (or, in general, for a vector-valued function $f : \mathbb{Z}^d \to \mathbb{R}^m$) we define its ℓ^p-norm as usual:

$$\|f\|_{\ell^p(\mathbb{Z}^d)} = \left(\sum_{\vec{n} \in \mathbb{Z}^d} |f(\vec{n})|^p \right)^{1/p}, \tag{2.1.9}$$

if $1 \leq p < \infty$, and

$$\|f\|_{\ell^\infty(\mathbb{Z}^d)} = \sup_{\vec{n} \in \mathbb{Z}^d} |f(\vec{n})|.$$

The gradient ∇f of a discrete function $f : \mathbb{Z}^d \to \mathbb{R}$ is the vector

$$\nabla f(\vec{n}) = \left(\frac{\partial f}{\partial x_1}(\vec{n}), \frac{\partial f}{\partial x_2}(\vec{n}), \ldots, \frac{\partial f}{\partial x_d}(\vec{n}) \right),$$

where

$$\frac{\partial f}{\partial x_i}(\vec{n}) = f(\vec{n} + \vec{e}_i) - f(\vec{n}),$$

and $\vec{e}_i = (0, 0, \ldots, 1, \ldots, 0)$ is the canonical $i-$th base vector. For $f : \mathbb{Z} \to \mathbb{R}$ and $1 \leq q < \infty$, the discrete analogue of (2.1.7) is the q-variation defined by

$$\mathrm{Var}_q(f) = \left(\sum_{n=-\infty}^{\infty} |f(n+1) - f(n)|^q \right)^{1/q} = \|f'\|_{\ell^q(\mathbb{Z})}.$$

For $0 \leq \beta < 1$ and $f : \mathbb{Z} \to \mathbb{R}$, we define the one-dimensional discrete uncentered fractional maximal operator by

$$\widetilde{M}_\beta f(n) = \sup_{r,s \geq 0} \frac{1}{(r+s+1)^{1-\beta}} \sum_{k=-r}^{s} |f(n+k)|. \tag{2.1.10}$$

Our next result is the discrete analogue of Theorem 1.

Theorem 2. *Let $0 \leq \beta < 1$ and $q = 1/(1-\beta)$. Let $f : \mathbb{Z} \to \mathbb{R}$ be a function of bounded variation such that $\widetilde{M}_\beta f \not\equiv \infty$. Then*

$$\big\|\big(\widetilde{M}_\beta f\big)'\big\|_{\ell^q(\mathbb{Z})} \leq 4^{1/q}\, \|f'\|_{\ell^1(\mathbb{Z})}.$$

In the case of the discrete uncentered Hardy-Littlewood maximal function ($\beta = 0$ in the setting above), Theorem 2 was proved by Bober, Carneiro, Hughes and Pierce in [8], with the sharp constant $C = 1$. The analogue of Kurka's inequality (2.1.4) for the discrete centered Hardy-Littlewood maximal function was established by Temur in [75] (with constant $C = 294,912,004$). As in the continuous case, the investigation of the validity of Theorem 2 for the discrete centered operator is also a very interesting problem.

Operators associated to convex sets

We now report progress related to Question B in the multidimensional discrete setting. We do this for a more general family of fractional maximal operators defined as follows. Let $\Omega \subset \mathbb{R}^d$ be a bounded open convex set with Lipschitz boundary. Let us assume that $\vec{0} \in \text{int}(\Omega)$ and that $\pm\vec{e}_i \in \overline{\Omega}$ for $1 \leq i \leq d$ (by renormalizing Ω if necessary)[1]. For $r > 0$ we write

$$\overline{\Omega}_r(\vec{x}_0) = \big\{\vec{x} \in \mathbb{R}^d;\, r^{-1}(\vec{x} - \vec{x}_0) \in \overline{\Omega}\big\},$$

and for $r = 0$ we consider

$$\overline{\Omega}_0(\vec{x}_0) = \{\vec{x}_0\}.$$

Whenever $\vec{x}_0 = \vec{0}$ we shall write $\overline{\Omega}_r = \overline{\Omega}_r\big(\vec{0}\big)$ for simplicity. This object plays the role of the "ball of center $\vec{x}_0$ and radius r" in our maximal operators below. For instance, to work with regular ℓ^p-balls, one should consider $\Omega = \{\vec{x} \in \mathbb{R}^d; \|\vec{x}\|_p < 1\}$, where $\|\vec{x}\|_p = (|x_1|^p + |x_2|^p + \ldots + |x_d|^p)^{\frac{1}{p}}$ for $\vec{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$.

Given $0 \leq \beta < d$ and $f : \mathbb{Z}^d \to \mathbb{R}$, we denote by $M_{\Omega,\beta}$ the discrete centered fractional maximal operator [2] associated to Ω, i.e.

$$M_{\Omega,\beta} f(\vec{n}) = \sup_{r \geq 0} \frac{1}{N(r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_r} |f(\vec{n} + \vec{m})|, \tag{2.1.11}$$

[1] This renormalization is merely aesthetic. For general Ω, one has to dilate the constants appearing in our results accordingly.

[2] We remark that the results in this section remain valid if we choose $N(r)^{-1}\, r^\beta$ instead of $N(r)^{-1+\frac{\beta}{d}}$ in the definition of our discrete fractional maximal functions.

and we denote by $\widetilde{M}_{\Omega,\beta}$ its uncentered version

$$\widetilde{M}_{\Omega,\beta}f(\vec{n}) = \sup_{\overline{\Omega}_r(\vec{x}_0)\ni\vec{n}} \frac{1}{N(\vec{x}_0,r)^{1-\frac{\beta}{d}}} \sum_{\vec{m}\in\overline{\Omega}_r(\vec{x}_0)} |f(\vec{m})|, \tag{2.1.12}$$

where $N(\vec{x},r)$ is the number of the lattice points in the set $\overline{\Omega}_r(\vec{x})$ (and $N(r) := N(\vec{0},r)$). It should be understood throughout the rest of the chapter that we always consider Ω-balls with at least one lattice point.

These convex Ω–balls have roughly the same behavior as the regular Euclidean balls from the geometric and arithmetic points of view. For instance, we have the following asymptotics [50, Chapter VI §2, Theorem 2], for the number of lattice points

$$N(\vec{x},r) = C_\Omega\, r^d + O\big(r^{d-1}\big) \tag{2.1.13}$$

as $r \to \infty$, where $C_\Omega = m(\Omega)$ is the d–dimensional volume of Ω, and the constant implicit in the big O notation depends only on the dimension d and on the set Ω (e.g. if Ω is the ℓ^∞-ball we have the exact expression $N(r) = (2\lfloor r\rfloor + 1)^d$). From (2.1.13) we can find a constant c_1 depending only on the dimension d and on the set Ω such that

$$N(\vec{x},r) \le C_\Omega(r+c_1)^d \tag{2.1.14}$$

and

$$N(\vec{x},r) \ge \max\{C_\Omega(\max\{r-c_1,0\})^d, 1\} =: C_\Omega(r-c_1)_+^d. \tag{2.1.15}$$

We define $c_2 > c_1$ as the constant such that

$$C_\Omega(c_2-c_1)^d = 1. \tag{2.1.16}$$

Since Ω is bounded, there exists $\lambda > 0$ (depending only on Ω) such that $\overline{\Omega} \subset \overline{B}_\lambda = \overline{B}_\lambda(\vec{0})$ (note that $\lambda \ge 1$, since we assume $\pm\vec{e}_i \in \overline{\Omega}$ for $1 \le i \le d$). This means that if $\vec{p} \in \overline{\Omega}_r(\vec{x}_0)$ then

$$|\vec{p} - \vec{x}_0| \le \lambda r. \tag{2.1.17}$$

If $0 < \beta < d$ and $f : \mathbb{Z}^d \to \mathbb{R}$, we consider the discrete fractional integral operator

$$I_\beta f(\vec{n}) = \sum_{\vec{m}\in\mathbb{Z}^d\setminus\{\vec{0}\}} \frac{f(\vec{n}-\vec{m})}{|\vec{m}|^{d-\beta}}.$$

As consequence of (2.1.15) we have that

$$\begin{aligned} M_{\Omega,\beta} f(\vec{n}) &\leq \sum_{\vec{m}\in\mathbb{Z}^d} \frac{1}{N\left(\frac{|\vec{m}|}{\lambda}\right)^{1-\frac{\beta}{d}}} f(\vec{n}-\vec{m}) \\ &\leq C\, I_\beta |f|(\vec{n}) + |f(\vec{n})| \end{aligned}$$

for all $\vec{n} \in \mathbb{Z}^d$, where $C = C(d, \Omega, \beta)$. It is known that if $1 < p < \infty$, $0 < \beta < d/p$ and $q = dp/(d-\beta p)$, then $I_\beta : \ell^p(\mathbb{Z}^d) \to \ell^q(\mathbb{Z}^d)$ is bounded (see, for instance, Pierce's thesis [63, Proposition 2.4]). Observe also that, if $p \leq q$, then

$$\|f\|_{\ell^q(\mathbb{Z}^d)} \leq \|f\|_{\ell^p(\mathbb{Z}^d)}. \tag{2.1.18}$$

This plainly implies that $M_{\Omega,\beta} : \ell^p(\mathbb{Z}^d) \to \ell^q(\mathbb{Z}^d)$ is bounded for p, q, β, d as above. One can also verify the pointwise inequality

$$\left| \frac{\partial}{\partial x_i} (M_{\Omega,\beta} f)(\vec{n}) \right| \leq M_{\Omega,\beta} \left(\frac{\partial f}{\partial x_i} \right)(\vec{n})$$

for all $\vec{n} \in \mathbb{Z}^d$. Hence, for $f \in \ell^p(\mathbb{Z}^d)$, we have [3]

$$\|\nabla M_{\Omega,\beta} f\|_{\ell^q(\mathbb{Z}^d)} \leq C\, \|M_{\Omega,\beta} |\nabla f|\|_{\ell^q(\mathbb{Z}^d)} \leq C'\, \|\nabla f\|_{\ell^p(\mathbb{Z}^d)}. \tag{2.1.19}$$

Moreover, if $f, g \in \ell^p(\mathbb{Z}^d)$, we have (recall that derivation is a bounded operator in $\ell^q(\mathbb{Z}^d)$, by the triangle inequality)

$$\|\nabla M_{\Omega,\beta} f - \nabla M_{\Omega,\beta} g\|_{\ell^q(\mathbb{Z}^d)} \leq C\, \|M_{\Omega,\beta} f - M_{\Omega,\beta} g\|_{\ell^q(\mathbb{Z}^d)} \leq C\, \|M_{\Omega,\beta} |f-g|\|_{\ell^q(\mathbb{Z}^d)} \leq C'\, \|f - g\|_{\ell^p(\mathbb{Z}^d)},$$

and we see that the operator $f \mapsto \nabla M_{\Omega,\beta} f$ is continuous from $\ell^p(\mathbb{Z}^d)$ to $\ell^q(\mathbb{Z}^d)$. Similar remarks apply to the uncentered version $\widetilde{M}_{\Omega,\beta}$. In relation to (2.1.19), the conjectured bound suggested by Question B in the discrete endpoint case $p = 1$ is the following:

Question C. Let $0 \leq \beta < d$ and $q = d/(d-\beta)$. For a discrete function $f : \mathbb{Z}^d \to \mathbb{R}$ do we have $\|\nabla M_{\Omega,\beta} f\|_{\ell^q(\mathbb{Z}^d)} \leq C(d, \Omega, \beta)\, \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}$?

Our next result is related to this question. In the case $0 \leq \beta < 1$ we present a family of estimates that approximate the conjectured bounds, whereas in the case $1 \leq \beta < d$ we give a positive answer (under the assumption that $f \in \ell^1(\mathbb{Z}^d)$) by adapting the methods of

[3] Throughout this chapter our constants may vary from line to line.

Kinnunen and Saksman [47] to the discrete setting. We complement these results with their corresponding continuity statements.

Theorem 3. .

(i) Let $0 \leq \beta < d$ and $0 \leq \alpha \leq 1$. Let $q \geq 1$ be such that

$$q > \frac{d}{d-\beta+\alpha}. \tag{2.1.20}$$

Then there exists a constant $C = C(d, \Omega, \alpha, \beta, q) > 0$ such that

$$\|\nabla M_{\Omega,\beta} f\|_{\ell^q(\mathbb{Z}^d)} \leq C \, \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{1-\alpha} \, \|f\|_{\ell^1(\mathbb{Z}^d)}^{\alpha} \quad \forall f \in \ell^1(\mathbb{Z}^d). \tag{2.1.21}$$

Moreover, the operator $f \mapsto \nabla M_{\Omega,\beta} f$ is continuous from $\ell^1(\mathbb{Z}^d)$ to $\ell^q(\mathbb{Z}^d)$.

(ii) Let $1 \leq \beta < d$ and $0 \leq \alpha < 1$. Let

$$q = \frac{d}{d-\beta+\alpha}. \tag{2.1.22}$$

Then there exists a constant $C = C(d, \Omega, \alpha, \beta) > 0$ such that

$$\|\nabla M_{\Omega,\beta} f\|_{\ell^q(\mathbb{Z}^d)} \leq C \, \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{1-\alpha} \, \|f\|_{\ell^1(\mathbb{Z}^d)}^{\alpha} \quad \forall f \in \ell^1(\mathbb{Z}^d). \tag{2.1.23}$$

Moreover, the operator $f \mapsto \nabla M_{\Omega,\beta} f$ is continuous from $\ell^1(\mathbb{Z}^d)$ to $\ell^q(\mathbb{Z}^d)$.

The same results hold for the discrete uncentered fractional maximal operator $\widetilde{M}_{\Omega,\beta}$.

Theorem 3 extends the result of Carneiro and Hughes in [16, Theorem 1], which corresponds to the case $\beta = 0$, $\alpha = 1$ and $q = 1$. Our approach here is different and simpler than that of [16]. The boundedness part of Theorem 3 (i) for the classical discrete Hardy-Littlewood maximal operator ($\beta = 0$) and $q = 1$ was first established by Carneiro and Rogers (unpublished manuscript). It is important to observe that inequality (2.1.21) (and its analogue for the uncentered case) can only hold if

$$q \geq \frac{d}{d-\beta+\alpha}. \tag{2.1.24}$$

This is due, essentially, to a dilation argument. To see this, let us consider, for instance, the uncentered case where $\Omega = (-1,1)^d$ is the unit open cube. Let $k \in \mathbb{N}$ and consider the cube $Q_k = [-k,k]^d$ and its characteristic function $f_k := \chi_{Q_k}$. One has $\|f_k\|_{\ell^1(\mathbb{Z}^d)} \sim_d k^d$, $\|\nabla f_k\|_{\ell^1(\mathbb{Z}^d)} \sim_d k^{d-1}$ and $\|\nabla \widetilde{M}_{\Omega,\beta} f_k\|_{\ell^q(\mathbb{Z}^d)} \gg_{\Omega,\beta,d} k^{\frac{d}{q}-1+\beta}$. One can see this last estimate by considering the region $H = \{\vec{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d;\ n_1 \geq 4dk\,;\ |n_i| \leq k, \text{ for } i = 2, 3, \ldots, d\}$ and showing that the maximal function at $\vec{n} \in H$ is realized by the cube of side $n_1 + k$ that contains the cube Q_k. Then we sum $|\widetilde{M}_{\Omega,\beta} f_k(\vec{n}+\vec{e}_1) - \widetilde{M}_{\Omega,\beta} f_k(\vec{n})|^q$ from $n_1 = 4dk$ to ∞, and then sum these contributions over the $\sim k^{d-1}$ possibilities for $(n_2, \ldots, n_d)$. Letting $k \to \infty$ we obtain the necessary condition (2.1.24).

We may collect the cases left open in Theorem 3 in our final question:

Question D. Does the inequality (2.1.21) (and its analogue for the uncentered case) hold for all $\alpha \leq \beta$ and $q = d/(d - \beta + \alpha)$?

We now proceed to the proofs of these results. In doing so, we opt to consider the discrete cases first, since they describe the essence of the main ideas with a little less technicalities than the continuous cases. In Section 2.2 we prove the boundedness part of Theorem 3. In Section 2.3 we prove the continuity part of Theorem 3, a nontrivial statement that does not follow directly from the boundedness, as the maximal operators are no longer sublinear at the derivative level. In Section 2.4 we prove Theorem 2 and, finally, in Section 2.5 we adapt some of our ideas used in the discrete setting to the continuous setting, and conclude by proving Theorem 1.

2.2 Proof of Theorem 3 - Boundedness

Throughout this section we work with the discrete maximal operators (2.1.11) and (2.1.12) associated to a convex set Ω as described in §2.1.2, and we remove the subscript Ω in some passages for simplicity. Given such a convex set Ω, let us fix the constants $C_\Omega = m(\Omega)$, c_1, c_2 and λ as defined in (2.1.13) - (2.1.17). In proving the boundedness statements of Theorem 3 we may assume, without loss of generality, that f is nonnegative since $\big|\nabla |f|(\vec{n})\big| \leq \big|\nabla f(\vec{n})\big|$ and $M_\beta |f| = M_\beta f$ (resp. $\widetilde{M}_\beta |f| = \widetilde{M}_\beta f$).

2.2.1 Centered case - part (i)

To prove (2.1.21) it is sufficient to show that

$$\left\| \frac{\partial M_\beta f}{\partial x_i} \right\|_{\ell^q(\mathbb{Z}^d)}^q \leq C\, \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{(1-\alpha)q}\, \|f\|_{\ell^1(\mathbb{Z}^d)}^{\alpha q} \tag{2.2.1}$$

for each $i = 1, 2, \ldots, d$. We will work with $i = d$ and the other cases are analogous. Since $f \in \ell^1(\mathbb{Z}^d)$, for each $\vec{n} \in \mathbb{Z}^d$ there exists $r \geq 0$ such that

$$M_\beta f(\vec{n}) = A_r f(\vec{n}) := \frac{1}{N(r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \Omega_r} f(\vec{n} + \vec{m}). \tag{2.2.2}$$

Let $\ell(\vec{n})$ be the minimum $\ell \in \mathbb{Z}^+$ such that there exists $r \geq 0$ which satisfies (2.2.2) and $\lceil r \rceil = \ell(\vec{n})$. Also, let $r(\vec{n}) \geq 0$ be such that $r(\vec{n})$ satisfies (2.2.2) and $\lceil r(\vec{n}) \rceil = \ell(\vec{n})$.

For all $k \in \mathbb{Z}^+$ we define the set X_k^+ by

$$X_k^+ := \{\vec{n} \in \mathbb{Z}^d : M_\beta f(\vec{n} + \vec{e}_d) \geq M_\beta f(\vec{n}) \text{ and } \ell(\vec{n} + \vec{e}_d) = k\} \tag{2.2.3}$$

and the set X_k^- to be

$$X_k^- := \{\vec{n} \in \mathbb{Z}^d : M_\beta f(\vec{n} + \vec{e}_d) < M_\beta f(\vec{n}) \text{ and } \ell(\vec{n}) = k\}. \tag{2.2.4}$$

Hence, we may write

$$\sum_{\vec{n} \in \mathbb{Z}^d} \left| \frac{\partial}{\partial x_d} M_\beta f(\vec{n}) \right|^q = \sum_{k \geq 0} \sum_{\vec{n} \in X_k^+} \big(M_\beta f(\vec{n} + \vec{e}_d) - M_\beta f(\vec{n})\big)^q + \sum_{k \geq 0} \sum_{\vec{n} \in X_k^-} \big(M_\beta f(\vec{n}) - M_\beta f(\vec{n} + \vec{e}_d)\big)^q.$$

We will prove that

$$\sum_{k \geq 0} \sum_{\vec{n} \in X_k^-} \big(M_\beta f(\vec{n}) - M_\beta f(\vec{n} + \vec{e}_d)\big)^q \leq C \, \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{q(1-\alpha)} \, \|f\|_{\ell^1(\mathbb{Z}^d)}^{q\alpha}, \tag{2.2.5}$$

and, analogously, we will have

$$\sum_{k \geq 0} \sum_{\vec{n} \in X_k^+} \big(M_\beta f(\vec{n} + \vec{e}_d) - M_\beta f(\vec{n})\big)^q \leq C \, \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{q(1-\alpha)} \, \|f\|_{\ell^1(\mathbb{Z}^d)}^{q\alpha}. \tag{2.2.6}$$

Inequalities (2.2.5) and (2.2.6) then imply (2.2.1) for $i = d$ [4]. To show (2.2.5), first we note

[4] Recall that our constants may vary from line to line.

that for $\vec{n} \in X_k^-$ and $r = r(\vec{n})$ we have

$$\begin{aligned} M_\beta f(\vec{n}) - M_\beta f(\vec{n}+\vec{e}_d) &\le A_r f(\vec{n}) - A_r f(\vec{n}+\vec{e}_d) \\ &= \frac{1}{N(r)^{1-\frac{\beta}{d}}} \sum_{\vec{m}\in\overline{\Omega}_r} f(\vec{n}+\vec{m}) - \frac{1}{N(r)^{1-\frac{\beta}{d}}} \sum_{\vec{m}\in\overline{\Omega}_r} f(\vec{n}+\vec{e}_d+\vec{m}) \\ &\le \frac{1}{N^+(k-1)^{1-\frac{\beta}{d}}} \sum_{\vec{m}\in\overline{\Omega}_k} \left| f(\vec{n}+\vec{m}) - f(\vec{n}+\vec{e}_d+\vec{m}) \right|, \end{aligned} \tag{2.2.7}$$

where $N^+(\vec{x},\tau) := \max\{N(\vec{x},\tau),1\}$, and $N^+(\tau) = N^+(\vec{0},\tau)$ [5]. On the other hand,

$$\begin{aligned} M_\beta f(\vec{n}+\vec{e}_d) &\ge \frac{1}{N(k+1)^{1-\frac{\beta}{d}}} \sum_{\vec{m}\in\overline{\Omega}_{k+1}} f(\vec{n}+\vec{e}_d+\vec{m}) \\ &\ge \frac{N(k+1)^{\frac{\beta}{d}}}{N(k+1)} \sum_{\vec{m}\in\overline{\Omega}_k} f(\vec{n}+\vec{m}) \\ &\ge \frac{N(r)}{N(k+1)} M_\beta f(\vec{n}). \end{aligned} \tag{2.2.8}$$

In the second inequality above we have used the convexity of Ω and the fact that $-\vec{e}_d \in \overline{\Omega}$ to conclude that $\overline{\Omega}_k(\vec{n}) \subset \overline{\Omega}_{k+1}(\vec{n}+\vec{e}_d)$. Using (2.2.8) we obtain

$$\begin{aligned} M_\beta f(\vec{n}) - M_\beta f(\vec{n}+\vec{e}_d) &\le \left(1 - \frac{N(r)}{N(k+1)}\right) M_\beta f(\vec{n}) \\ &\le \frac{N(k+1) - N^+(k-1)}{N(k+1)} \frac{1}{N^+(k-1)^{1-\frac{\beta}{d}}} \sum_{\vec{m}\in\overline{\Omega}_k} f(\vec{n}+\vec{m}). \end{aligned} \tag{2.2.9}$$

Putting together the estimates (2.2.7) and (2.2.9), we see that $\big(M_\beta f(\vec{n}) - M_\beta f(\vec{n}+\vec{e}_d)\big)^q$ is

[5] Here we formally include the possibility of having $\tau < 0$, with the understanding that, in this case, $N^+(\vec{x},\tau) = 1$.

bounded above by the product [6]

$$\left(\frac{1}{N^+(k-1)^{1-\frac{\beta}{d}}}\sum_{\vec m\in\overline{\Omega}_k}\left|f(\vec n+\vec m)-f(\vec n+\vec e_d+\vec m)\right|\right)^{q(1-\alpha)} \times\left(\frac{N(k+1)-N^+(k-1)}{N(k+1)}\frac{1}{N^+(k-1)^{1-\frac{\beta}{d}}}\sum_{\vec m\in\overline{\Omega}_k}f(\vec n+\vec m)\right)^{q\alpha}. \tag{2.2.10}$$

By (2.1.14) and (2.1.15), there is a positive constant C such that

$$\frac{N(\vec x,k+1)-N^+(\vec x,k-1)}{N(\vec x,k+1)}\le\frac{C}{N^+(\vec x,k-1)^{\frac{1}{d}}}\quad\forall k\in\mathbb{Z}^+\,;\,\forall\vec x\in\mathbb{R}^d. \tag{2.2.11}$$

Let

$$\left(1-\frac{\beta}{d}\right)q+\frac{\alpha q}{d}=1+\gamma. \tag{2.2.12}$$

From (2.1.20) we have $\gamma>0$. Then it follows from (2.2.10) and (2.2.11) that $\big(M_\beta f(\vec n)-M_\beta f(\vec n+\vec e_d)\big)^q$ is bounded above by the product

$$\left(\frac{1}{N^+(k-1)^{1+\gamma}}\left(\sum_{\vec m\in\overline{\Omega}_k}\left|f(\vec n+\vec m)-f(\vec n+\vec e_d+\vec m)\right|\right)^q\right)^{(1-\alpha)} \times\left(\frac{C}{N^+(k-1)^{1+\gamma}}\left(\sum_{\vec m\in\overline{\Omega}_k}f(\vec n+\vec m)\right)^q\right)^{\alpha}.$$

Thus, by Hölder's inequality with exponents $p=\frac{1}{1-\alpha}$ and $p'=\frac{1}{\alpha}$, we see that the left-hand side of (2.2.5) is bounded by

$$\left[\sum_{k\ge0}\sum_{\vec n\in X_k^-}\frac{1}{N^+(k-1)^{1+\gamma}}\left(\sum_{\vec m\in\overline{\Omega}_k}\left|f(\vec n+\vec m)-f(\vec n+\vec e_d+\vec m)\right|\right)^q\right]^{1-\alpha} \times\left[\sum_{k\ge0}\sum_{\vec n\in X_k^-}\frac{C}{N^+(k-1)^{1+\gamma}}\left(\sum_{\vec m\in\overline{\Omega}_k}f(\vec n+\vec m)\right)^q\right]^{\alpha}.$$

[6] If $\alpha=0$ or 1, it is understood that we only have one term in this product. The modifications for the rest of the proof are standard.

Since $q \geq 1$, this last product is bounded by

$$\|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)(1-\alpha)} \left[\sum_{k\geq 0} \sum_{\vec{n}\in X_k^-} \frac{1}{N^+(k-1)^{1+\gamma}} \left(\sum_{\vec{m}\in\overline{\Omega}_k} \left| f(\vec{n}+\vec{m}) - f(\vec{n}+\vec{e}_d+\vec{m}) \right| \right) \right]^{1-\alpha}$$
$$\times \|f\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)\alpha} \left[\sum_{k\geq 0} \sum_{\vec{n}\in X_k^-} \frac{C}{N^+(k-1)^{1+\gamma}} \left(\sum_{\vec{m}\in\overline{\Omega}_k} f(\vec{n}+\vec{m}) \right) \right]^{\alpha}.$$

By Fubini's theorem and (2.1.17), this is bounded by

$$\|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)(1-\alpha)} \left[\sum_{\vec{m}\in\mathbb{Z}^d} \sum_{k\geq \frac{|\vec{m}|}{\lambda}} \frac{1}{N^+(k-1)^{1+\gamma}} \sum_{\vec{n}\in X_k^-} \left| f(\vec{n}+\vec{m}) - f(\vec{n}+\vec{e}_d+\vec{m}) \right| \right]^{1-\alpha}$$
$$\times \|f\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)\alpha} \left[\sum_{\vec{m}\in\mathbb{Z}^d} \sum_{k\geq \frac{|\vec{m}|}{\lambda}} \frac{C}{N^+(k-1)^{1+\gamma}} \sum_{\vec{n}\in X_k^-} f(\vec{n}+\vec{m}) \right]^{\alpha},$$

which turns out to be bounded by

$$\|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)(1-\alpha)} \left[\sum_{\vec{m}\in\mathbb{Z}^d} \frac{1}{N^+\left(\frac{|\vec{m}|}{\lambda}-1\right)^{1+\gamma}} \sum_{k\geq \frac{|\vec{m}|}{\lambda}} \sum_{\vec{n}\in X_k^-} \left| f(\vec{n}+\vec{m}) - f(\vec{n}+\vec{e}_d+\vec{m}) \right| \right]^{1-\alpha}$$
$$\times \|f\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)\alpha} \left[\sum_{\vec{m}\in\mathbb{Z}^d} \frac{C}{N^+\left(\frac{|\vec{m}|}{\lambda}-1\right)^{1+\gamma}} \sum_{k\geq \frac{|\vec{m}|}{\lambda}} \sum_{\vec{n}\in X_k^-} f(\vec{n}+\vec{m}) \right]^{\alpha}. \tag{2.2.13}$$

Since the sets X_k^- are pairwise disjoint, this is bounded above by

$$\|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{q(1-\alpha)} \left[\sum_{\vec{m}\in\mathbb{Z}^d} \frac{1}{N^+\left(\frac{|\vec{m}|}{\lambda}-1\right)^{1+\gamma}} \right]^{1-\alpha} \times \quad \|f\|_{\ell^1(\mathbb{Z}^d)}^{q\alpha} \left[\sum_{\vec{m}\in\mathbb{Z}^d} \frac{C}{N^+\left(\frac{|\vec{m}|}{\lambda}-1\right)^{1+\gamma}} \right]^{\alpha}. \tag{2.2.14}$$

By (2.1.15) we know that

$$N^+\left(\frac{|\vec{m}|}{\lambda}-1\right) \geq \max\left\{ C_\Omega \left(\max\left\{ \frac{|\vec{m}|}{\lambda} - 1 - c_1, 0 \right\} \right)^d, 1 \right\} \quad \forall\, \vec{m}\in\mathbb{Z}^d.$$

This implies that both sums in brackets in (2.2.14) are finite and we arrive at the desired estimate (2.2.5). Using (2.1.14) we see that

$$\sum_{\vec{m}\in\mathbb{Z}^d} \frac{1}{N^+\left(\frac{|\vec{m}|}{\lambda}-1\right)^{1+\gamma}} \to \infty \quad \text{as} \quad \gamma \to 0^+\,,$$

which implies that our constant C in (2.1.21) blows up when we approach the case of equality in (2.1.24).

2.2.2 Uncentered case - part (i)

Here it suffices to show that

$$\left\|\frac{\partial \widetilde{M}_\beta f}{\partial x_i}\right\|^q_{\ell^q(\mathbb{Z}^d)} \leq C\, \|\nabla f\|^{(1-\alpha)q}_{\ell^1(\mathbb{Z}^d)}\, \|f\|^{\alpha q}_{\ell^1(\mathbb{Z}^d)} \tag{2.2.15}$$

for each $i = 1, 2, \ldots, d$. We work again with $i = d$ and the other cases are analogous. Since $f \in \ell^1(\mathbb{Z}^d)$, for each $\vec{n} \in \mathbb{Z}^d$ we can take a point $\vec{x} = \vec{x}(\vec{n}) \in \mathbb{R}^d$ and a radius $r = r(\vec{n})$ such that $\vec{n} \in \overline{\Omega}_r(\vec{x})$ and the fractional average over the set $\overline{\Omega}_r(\vec{x})$ realizes the supremum in the maximal function, i.e.

$$\widetilde{M}_\beta f(\vec{n}) = A_{(\vec{x},r)} f(\vec{n}) := \frac{1}{N(\vec{x},r)^{1-\frac{\beta}{d}}} \sum_{\vec{m}\in\overline{\Omega}_r(\vec{x})} f(\vec{m}). \tag{2.2.16}$$

Let $\tilde{\ell}(\vec{n})$ be the minimum $\tilde{\ell} \in \mathbb{Z}^+$ such that there is a pair $(\vec{x}, r)$ that verifies the equality (2.2.16) with $\lceil r \rceil = \tilde{\ell}(\vec{n})$. Let $r(\vec{n})$ be such a radius and $\vec{x}(\vec{n})$ be such a center.

For $k \in \mathbb{Z}^+$ we now define the set $\widetilde{X}_k^+$ by

$$\widetilde{X}_k^+ := \left\{\vec{n} \in \mathbb{Z}^d : \widetilde{M}_\beta f(\vec{n}+\vec{e}_d) \geq \widetilde{M}_\beta f(\vec{n}) \text{ and } \tilde{\ell}(\vec{n}+\vec{e}_d) = k\right\}$$

and the set $\widetilde{X}_k^-$ by

$$\widetilde{X}_k^- := \left\{\vec{n} \in \mathbb{Z}^d : \widetilde{M}_\beta f(\vec{n}+\vec{e}_d) < \widetilde{M}_\beta f(\vec{n}) \text{ and } \tilde{\ell}(\vec{n}) = k\right\}.$$

Hence,

$$\sum_{\vec{n}\in\mathbb{Z}^d} \left|\frac{\partial}{\partial x_d}\widetilde{M}_\beta f(\vec{n})\right|^q = \sum_{k\geq 0}\sum_{\vec{n}\in X_k^+} \left(\widetilde{M}_\beta f(\vec{n}+\vec{e}_d) - \widetilde{M}_\beta f(\vec{n})\right)^q + \sum_{k\geq 0}\sum_{\vec{n}\in X_k^-} \left(\widetilde{M}_\beta f(\vec{n}) - \widetilde{M}_\beta f(\vec{n}+\vec{e}_d)\right)^q.$$

We will prove that

$$\sum_{k\geq 0}\sum_{\vec{n}\in X_k^-}\left(\widetilde{M}_\beta f(\vec{n})-\widetilde{M}_\beta f(\vec{n}+\vec{e}_d)\right)^q\leq C\,\|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{q(1-\alpha)}\,\|f\|_{\ell^1(\mathbb{Z}^d)}^{q\alpha},\tag{2.2.17}$$

and, analogously, we will have

$$\sum_{k\geq 0}\sum_{\vec{n}\in X_k^+}\left(\widetilde{M}_\beta f(\vec{n}+\vec{e}_d)-\widetilde{M}_\beta f(\vec{n})\right)^q\leq C\,\|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{q(1-\alpha)}\,\|f\|_{\ell^1(\mathbb{Z}^d)}^{q\alpha}.\tag{2.2.18}$$

As a consequence of (2.2.17) and (2.2.18) we will obtain (2.2.15) for $i=d$, as desired.

For $\vec{n}\in X_k^-$ let $r=r(\vec{n})$ and $\vec{x}=\vec{x}(\vec{n})$. Then

$$\begin{aligned}\widetilde{M}_\beta f(\vec{n})-\widetilde{M}_\beta f(\vec{n}+\vec{e}_d)&\leq A_{(\vec{x},r)}f(\vec{n})-A_{(\vec{x}+\vec{e}_d,r)}f(\vec{n}+\vec{e}_d)\\&=\frac{1}{N(\vec{x},r)^{1-\frac{\beta}{d}}}\sum_{\vec{m}\in\Omega_r(\vec{x})}f(\vec{m})-\frac{1}{N(\vec{x}+\vec{e}_d,r)^{1-\frac{\beta}{d}}}\sum_{\vec{m}\in\Omega_r(\vec{x}+\vec{e}_d)}f(\vec{m})\\&\leq\frac{1}{N(\vec{x},r)^{1-\frac{\beta}{d}}}\sum_{\vec{m}\in\Omega_r(\vec{x})}\big|f(\vec{m})-f(\vec{m}+\vec{e}_d)\big|\\&\leq\frac{1}{N^+(\vec{x},k-1)^{1-\frac{\beta}{d}}}\sum_{\vec{m}\in\Omega_k(\vec{x})}\big|f(\vec{m})-f(\vec{m}+\vec{e}_d)\big|.\end{aligned}$$

On the other hand,

$$\begin{aligned}\widetilde{M}_\beta f(\vec{n}+\vec{e}_d)&\geq\frac{1}{N(\vec{x}+\vec{e}_d,k+1)^{1-\frac{\beta}{d}}}\sum_{\vec{m}\in\Omega_{k+1}(\vec{x}+\vec{e}_d)}f(\vec{m})\\&\geq\frac{N(\vec{x},k+1)^{\frac{\beta}{d}}}{N(\vec{x},k+1)}\sum_{\vec{m}\in\Omega_k(\vec{x})}f(\vec{m})\\&\geq\frac{N(\vec{x},r)}{N(\vec{x},k+1)}\,\widetilde{M}_\beta f(\vec{n}).\end{aligned}$$

In the second inequality above we have used the convexity of Ω and the fact that $-\vec{e}_d\in\overline{\Omega}$ to conclude that $\overline{\Omega}_k(\vec{x})\subset\overline{\Omega}_{k+1}(\vec{x}+\vec{e}_d)$. Therefore,

$$\begin{aligned}\widetilde{M}_\beta f(\vec{n})-\widetilde{M}_\beta f(\vec{n}+\vec{e}_d)&\leq\left(1-\frac{N(\vec{x},r)}{N(\vec{x},k+1)}\right)\widetilde{M}_\beta f(\vec{n})\\&\leq\frac{N(\vec{x},k+1)-N^+(\vec{x},k-1)}{N(\vec{x},k+1)}\frac{1}{N^+(\vec{x},k-1)^{1-\frac{\beta}{d}}}\sum_{\vec{m}\in\overline{\Omega}_k(\vec{x})}f(\vec{m}).\end{aligned}$$

With these two inequalities, we can see that $\big(\widetilde{M}_\beta f(\vec{n}) - \widetilde{M}_\beta f(\vec{n}+\vec{e}_d)\big)^q$ is bounded above by the product

$$\left(\frac{1}{N^+(\vec{x},k-1)^{1-\frac{\beta}{d}}} \sum_{\vec{m}\in\Omega_k(\vec{x})} \big|f(\vec{m}) - f(\vec{m}+\vec{e}_d)\big|\right)^{q(1-\alpha)} \times \left(\frac{N(\vec{x},k+1) - N^+(\vec{x},k-1)}{N(\vec{x},k+1)} \frac{1}{N^+(\vec{x},k-1)^{1-\frac{\beta}{d}}} \sum_{\vec{m}\in\overline{\Omega}_k(\vec{x})} f(\vec{m})\right)^{q\alpha}. \tag{2.2.19}$$

Let $\gamma > 0$ be defined as in (2.2.12). Using (2.2.11), we conclude that (2.2.19) is bounded above by the product

$$\left(\frac{1}{N^+(\vec{x},k-1)^{1+\gamma}} \left(\sum_{\vec{m}\in\Omega_k(\vec{x})} \big|f(\vec{m}) - f(\vec{m}+\vec{e}_d)\big|\right)^q\right)^{1-\alpha} \times \left(\frac{C}{N^+(\vec{x},k-1)^{1+\gamma}} \left(\sum_{\vec{m}\in\overline{\Omega}_k(\vec{x})} f(\vec{m})\right)^q\right)^{\alpha}.$$

For each $r \in \mathbb{R}$, let $\widetilde{N}^+(r) := \min\big\{N^+(\vec{x},r);\, \vec{x}\in\mathbb{R}^d\big\}$. Thus, by Hölder's inequality with exponents $p = \frac{1}{1-\alpha}$ and $p' = \frac{1}{\alpha}$ we see that the left-hand side of (2.2.17) is bounded by

$$\left[\sum_{k\geq 0}\sum_{\vec{n}\in X_k^-} \frac{1}{\widetilde{N}^+(k-1)^{1+\gamma}} \left(\sum_{\vec{m}\in\Omega_k(\vec{x}(\vec{n}))} \big|f(\vec{m}) - f(\vec{m}+\vec{e}_d)\big|\right)^q\right]^{1-\alpha} \times \left[\sum_{k\geq 0}\sum_{\vec{n}\in X_k^-} \frac{C}{\widetilde{N}^+(k-1)^{1+\gamma}} \left(\sum_{\vec{m}\in\overline{\Omega}_k(\vec{x}(\vec{n}))} f(\vec{m})\right)^q\right]^{\alpha}.$$

In turn, this is bounded by

$$\left[\sum_{k\geq 0}\sum_{\vec{n}\in X_k^-} \frac{1}{\widetilde{N}^+(k-1)^{1+\gamma}} \left(\sum_{\vec{m}\in\overline{B}_{2k\lambda}(\vec{n})} \big|f(\vec{m}) - f(\vec{m}+\vec{e}_d)\big|\right)^q\right]^{1-\alpha} \times \left[\sum_{k\geq 0}\sum_{\vec{n}\in X_k^-} \frac{C}{\widetilde{N}^+(k-1)^{1+\gamma}} \left(\sum_{\vec{m}\in\overline{B}_{2k\lambda}(\vec{n})} f(\vec{m})\right)^q\right]^{\alpha},$$

since $\overline{\Omega} \subset \overline{B}_\lambda$ and thus $\overline{\Omega}_k(\vec{x}(\vec{n})) \subset \overline{B}_{k\lambda}(\vec{x}(\vec{n})) \subset \overline{B}_{2k\lambda}(\vec{n})$. The remaining steps of the proof are analogous to the centered case in §2.2.1.

2.2.3 Centered case - part (ii)

In the case $1 \leq \beta < d$, the operator M_β has a certain regularizing effect. This was observed in [47, Theorem 3.1] in the continuous setting. In what follows we adapt their argument to the discrete setting. Let $\vec{n} \in \mathbb{Z}^d$ and assume that $M_\beta f(\vec{n}) \geq M_\beta f(\vec{n} + \vec{e}_d)$. Since we are assuming that $f \in \ell^1(\mathbb{Z}^d)$, there exists $r \geq 0$ such that

$$M_\beta f(\vec{n}) = A_r f(\vec{n}) = \frac{1}{N(r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \Omega_r} f(\vec{n} + \vec{m}).$$

Proceeding as in (2.2.8) we find that

$$\begin{aligned} M_\beta f(\vec{n} + \vec{e}_d) &\geq \frac{1}{N(r+1)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_{r+1}} f(\vec{n} + \vec{e}_d + \vec{m}) \\ &\geq \frac{1}{N(r+1)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_r} f(\vec{n} + \vec{m}). \end{aligned}$$

Hence

$$M_\beta f(\vec{n}) - M_\beta f(\vec{n} + \vec{e}_d) \leq \left(\frac{1}{N(r)^{1-\frac{\beta}{d}}} - \frac{1}{N(r+1)^{1-\frac{\beta}{d}}} \right) \sum_{\vec{m} \in \overline{\Omega}_r} f(\vec{n} + \vec{m}). \tag{2.2.20}$$

We claim that

$$\left(\frac{1}{N(r)^{1-\frac{\beta}{d}}} - \frac{1}{N(r+1)^{1-\frac{\beta}{d}}} \right) \leq C \, \frac{1}{N(r)^{1-\frac{\beta-1}{d}}} \, . \tag{2.2.21}$$

This is certainly true for small r, whereas for large r we may take c_1 in (2.1.14) and (2.1.15) to be strictly smaller than $\frac{1}{2}$ and use the mean value theorem (after clearing the denominators). From (2.2.20) and (2.2.21) we find that

$$M_\beta f(\vec{n}) - M_\beta f(\vec{n} + \vec{e}_d) \leq C \, M_{\beta-1} f(\vec{n}).$$

If $M_\beta f(\vec{n}) < M_\beta f(\vec{n} + \vec{e}_d)$ an analogous argument leads to

$$M_\beta f(\vec{n} + \vec{e}_d) - M_\beta f(\vec{n}) \leq C \, M_{\beta-1} f(\vec{n} + \vec{e}_d).$$

Moreover, we may replace $\vec{e}_d$ by any other basis vector $\vec{e}_j$. This leads to the following pointwise bound

$$\left|\nabla M_\beta f(\vec{n})\right| \leq C \left\{ M_{\beta-1} f(\vec{n}) + \sum_{j=1}^{d} M_{\beta-1} f(\vec{n}+\vec{e}_j) \right\}. \tag{2.2.22}$$

Finally, letting $r = d/(d+\alpha-1)$ and $p^* = d/(d-1)$, we use the boundedness of $M_{\beta-1}$, interpolation and the Sobolev embedding (for which the proof in the discrete setting is analogous to the proof in the continuous setting as in [70, Chapter V, §2.5]) to obtain

$$\begin{aligned}\|\nabla M_\beta f\|_{\ell^q(\mathbb{Z}^d)} &\leq C\|M_{\beta-1} f\|_{\ell^q(\mathbb{Z}^d)} \leq C'\|f\|_{\ell^r(\mathbb{Z}^d)} \\ &\leq C'\|f\|^{1-\alpha}_{\ell^{p^*}(\mathbb{Z}^d)}\|f\|^{\alpha}_{\ell^1(\mathbb{Z}^d)} \leq C''\|\nabla f\|^{1-\alpha}_{\ell^1(\mathbb{Z}^d)}\|f\|^{\alpha}_{\ell^1(\mathbb{Z}^d)}.\end{aligned}$$

This concludes the proof of this part.

2.2.4 Uncentered case - part (ii)

The proof in this case is analogous to §2.2.3, establishing the pointwise bound (2.2.22) for the uncentered operator $\widetilde{M}_\beta$. We omit the details.

2.3 Proof of Theorem 3 - Continuity

In this section we keep working with the discrete maximal operators $M_\beta = M_{\Omega,\beta}$ (resp. $\widetilde{M}_\beta = \widetilde{M}_{\Omega,\beta}$). Given the convex set Ω as described in §2.1.2, we fix the constants $C_\Omega = m(\Omega)$, c_1, c_2 and λ as defined in (2.1.13) - (2.1.17).

2.3.1 Centered case - part (i)

We want to show that if $f_j \to f$ in $\ell^1(\mathbb{Z}^d)$ then $\nabla M_\beta f_j \to \nabla M_\beta f$ in $\ell^q(\mathbb{Z}^d)$. From the fact that $||f_j| - |f|| \leq |f_j - f|$, we may assume without loss of generality that $f_j \geq 0$ for all j, and that $f \geq 0$. It suffices show that

$$\left\| \frac{\partial}{\partial x_i} M_\beta f_j - \frac{\partial}{\partial x_i} M_\beta f \right\|_{\ell^q(\mathbb{Z}^d)} \to 0 \tag{2.3.1}$$

as $j \to \infty$, for each $i = 1, 2, \ldots, d$. As before, we will prove it for $i = d$ and the other cases are analogous.

Convergence of radii

Given a function $g \in \ell^1(\mathbb{Z}^d)$ and a point $\vec{n} \in \mathbb{Z}^d$, we first study the set of radii that realize the supremum in the fractional maximal function at the point $\vec{n}$. Let us define

$$\mathcal{R}g(\vec{n}) = \left\{ r \in [0,\infty);\ M_\beta g(\vec{n}) = A_r|g|(\vec{n}) = \frac{1}{N(r)^{1-\frac{\beta}{d}}} \sum_{\vec{m}\in\overline{\Omega}_r} |g(\vec{n}+\vec{m})| \right\}.$$

We prove the following auxiliary lemma, which can be seen as a discrete fractional analogue of a lemma of Luiro [56, Lemma 2.2].

Lemma 4. *Let $f_j \to f$ in $\ell^1(\mathbb{Z}^d)$ and let $R > 0$. There exists j_0 such that for $j \geq j_0$ we have $\mathcal{R}f_j(\vec{n}) \subset \mathcal{R}f(\vec{n})$ for each $\vec{n} \in \overline{B}_R$.*

Proof. Fix $\vec{n} \in \overline{B}_R$ and consider the map $r \mapsto A_r f(\vec{n})$. Since $f \in \ell^1(\mathbb{Z}^d)$, there is only a finite number of values in the image set $\{A_r f(\vec{n}); r \geq 0\}$ such that $A_r f(\vec{n}) \geq \frac{1}{2} M_\beta f(\vec{n})$. Therefore, there exists $\varepsilon(\vec{n}) > 0$ such that, if $A_r f(\vec{n}) > M_\beta f(\vec{n}) - \varepsilon(\vec{n})$, then $A_r f(\vec{n}) = M_\beta f(\vec{n})$ and $r \in \mathcal{R}f(\vec{n})$. Let

$$\varepsilon = \tfrac{1}{3} \min\left\{\varepsilon(\vec{n});\ \vec{n} \in \overline{B}_R\right\}.$$

From (2.1.18) we see that $\|f_j - f\|_{\ell^1(\mathbb{Z}^d)} \to 0$ implies that $\|f_j - f\|_{\ell^a(\mathbb{Z}^d)} \to 0$ for each $a \geq 1$. In particular, there exists j_0 such that $\|f_j - f\|_{\ell^{\frac{d}{\beta}}(\mathbb{Z}^d)} < \varepsilon$ for all $j \geq j_0$ (if $\beta = 0$ then $d/\beta = \infty$). By Hölder's inequality,

$$\begin{aligned}
\left|A_r f_j(\vec{n}) - A_r f(\vec{n})\right| &\leq \frac{1}{N(r)^{1-\frac{\beta}{d}}} \sum_{\vec{m}\in\overline{\Omega}_r} \left|f_j(\vec{n}+\vec{m}) - f(\vec{n}+\vec{m})\right| \\
&\leq \frac{1}{N(r)^{1-\frac{\beta}{d}}} \left(\sum_{\vec{m}\in\overline{\Omega}_r} \left|f_j(\vec{n}+\vec{m}) - f(\vec{n}+\vec{m})\right|^{\frac{d}{\beta}} \right)^{\frac{\beta}{d}} N(r)^{1-\frac{\beta}{d}} \qquad (2.3.2)\\
&\leq \|f_j - f\|_{\ell^{\frac{d}{\beta}}(\mathbb{Z}^d)} \\
&\leq \varepsilon
\end{aligned}$$

for all $r \geq 0$ and $j \geq j_0$. Hence, for any $\vec{n} \in \overline{B}_R$ and $j \geq j_0$, if we take $s \in \mathcal{R}f(\vec{n})$ we get

$$M_\beta f(\vec{n}) = A_s f(\vec{n}) = A_s f_j(\vec{n}) + A_s(f - f_j)(\vec{n}) \leq M_\beta f_j(\vec{n}) + \varepsilon. \qquad (2.3.3)$$

For any $\vec{n} \in \overline{B}_R$, $j \geq j_0$ and $r_j \in \mathcal{R}f_j(\vec{n})$, we use (2.3.2) and (2.3.3) to obtain

$$A_{r_j} f(\vec{n}) = A_{r_j}(f - f_j)(\vec{n}) + A_{r_j} f_j(\vec{n}) \geq M_\beta f(\vec{n}) - 2\varepsilon.$$

From the definition of ε we must have $r_j \in \mathcal{R}f(\vec{n})$, which completes the proof of the lemma.

□

Brezis-Lieb reduction

For a fixed $\vec{n} \in \mathbb{Z}^d$, given $\varepsilon > 0$, we use Lemma 4 to find j_0 such that, if $j \geq j_0$, we have $\mathcal{R}f_j(\vec{n}) \subset \mathcal{R}f(\vec{n})$ and $\|f_j - f\|_{\ell^{\frac{d}{\beta}}(\mathbb{Z}^d)} < \varepsilon$. Taking $r_j \in \mathcal{R}f_j(\vec{n})$ and using (2.3.2) we obtain

$$|M_\beta f_j(\vec{n}) - M_\beta f(\vec{n})| = |A_{r_j} f_j(\vec{n}) - A_{r_j} f(\vec{n})| \leq \varepsilon$$

for $j \geq j_0$, and therefore $M_\beta f_j(\vec{n}) \to M_\beta f(\vec{n})$ as $j \to \infty$. Hence it follows that

$$\frac{\partial}{\partial x_d} M_\beta f_j(\vec{n}) \to \frac{\partial}{\partial x_d} M_\beta f(\vec{n}) \tag{2.3.4}$$

as $j \to \infty$. From the classical Brezis-Lieb lemma [9], in order to prove (2.3.1) it suffices to prove the convergence of the norms, i.e.

$$\lim_{j\to\infty} \left\| \frac{\partial}{\partial x_d} M_\beta f_j \right\|_{\ell^q(\mathbb{Z}^d)} = \left\| \frac{\partial}{\partial x_d} M_\beta f \right\|_{\ell^q(\mathbb{Z}^d)}.$$

From Fatou's lemma and the pointwise convergence in (2.3.4) we have

$$\left\| \frac{\partial}{\partial x_d} M_\beta f \right\|_{\ell^q(\mathbb{Z}^d)} \leq \liminf_{j\to\infty} \left\| \frac{\partial}{\partial x_d} M_\beta f_j \right\|_{\ell^q(\mathbb{Z}^d)}.$$

We prove the opposite inequality in the next subsection.

Upper bound

Let $\varepsilon_0 > 0$ be given. Our goal now is to find j_0 such that, for $j \geq j_0$,

$$\left\| \frac{\partial}{\partial x_d} M_\beta f_j \right\|_{\ell^q(\mathbb{Z}^d)} \leq \left\| \frac{\partial}{\partial x_d} M_\beta f \right\|_{\ell^q(\mathbb{Z}^d)} + \varepsilon_0. \tag{2.3.5}$$

Let R be a sufficiently large radius (to be properly chosen later) and let $Q_{2R} = \{\vec{x} \in \mathbb{R}^d : \|\vec{x}\|_\infty \leq 2R\}$ be the cube of side $4R$. We write

$$\begin{aligned}\left\|\frac{\partial}{\partial x_d} M_\beta f_j\right\|^q_{\ell^q(\mathbb{Z}^d)} &= \sum_{\|\vec{n}\|_\infty \leq 2R} \left|\frac{\partial}{\partial x_d} M_\beta f_j(\vec{n})\right|^q + \sum_{\|\vec{n}\|_\infty > 2R} \left|\frac{\partial}{\partial x_d} M_\beta f_j(\vec{n})\right|^q \\ &=: S_1 + S_2.\end{aligned} \tag{2.3.6}$$

We bound S_1 and S_2 separately.

Let $\varepsilon_1 > 0$ (to be properly chosen later). By Lemma 4, there exists j_1 such that, if $j \geq j_1$, we have $\mathcal{R}f_j(\vec{n}) \subset \mathcal{R}f(\vec{n})$ for each $\vec{n}$ with $\|\vec{n}\|_\infty \leq 2R + 1$ and

$$\|f_j - f\|_{\ell^{\frac{d}{\beta}}(\mathbb{Z}^d)} \leq \varepsilon_1.$$

By (2.3.2) we obtain that

$$\left|\frac{\partial}{\partial x_d} M_\beta f_j(\vec{n}) - \frac{\partial}{\partial x_d} M_\beta f(\vec{n})\right| \leq 2\varepsilon_1,$$

for any $\vec{n} \in Q_{2R}$. Hence, by the triangle inequality,

$$S_1 \leq \left(\left\|\frac{\partial}{\partial x_d} M_\beta f\right\|_{\ell^q(Q_{2R})} + 2\varepsilon_1 (4R+1)^{\frac{d}{q}}\right)^q. \tag{2.3.7}$$

In order to bound S_2, we recall the definition of the sets X_k^+ and X_k^- in (2.2.3) and (2.2.4). We then write

$$\begin{aligned}S_2 &= \sum_{k \geq 0} \sum_{\substack{\|\vec{n}\|_\infty > 2R \\ \vec{n} \in X_k^+}} \big(M_\beta f_j(\vec{n} + \vec{e}_d) - M_\beta f_j(\vec{n})\big)^q + \sum_{k \geq 0} \sum_{\substack{\|\vec{n}\|_\infty > 2R \\ \vec{n} \in X_k^-}} \big(M_\beta f_j(\vec{n}) - M_\beta f_j(\vec{n} + \vec{e}_d)\big)^q \\ &=: S_2^+ + S_2^-.\end{aligned}$$

From the calculations leading to (2.2.13) we have

$$S_2^- \;\le\; \|\nabla f_j\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)(1-\alpha)} \left[\sum_{\vec{m}\in\mathbb{Z}^d} \frac{1}{N^+\big(\frac{|\vec{m}|}{\lambda}-1\big)^{1+\gamma}} \sum_{k\ge\frac{|\vec{m}|}{\lambda}} \sum_{\substack{\|\vec{n}\|_\infty>2R\\ \vec{n}\in X_k^-}} \big|f_j(\vec{n}+\vec{m})-f_j(\vec{n}+\vec{e}_d+\vec{m})\big| \right]^{1-\alpha}$$
$$\times \|f_j\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)\alpha} \left[\sum_{\vec{m}\in\mathbb{Z}^d} \frac{C}{N^+\big(\frac{|\vec{m}|}{\lambda}-1\big)^{1+\gamma}} \sum_{k\ge\frac{|\vec{m}|}{\lambda}} \sum_{\substack{\|\vec{n}\|_\infty>2R\\ \vec{n}\in X_k^-}} f_j(\vec{n}+\vec{m}) \right]^{\alpha}.$$

We have already noted after (2.2.14) that the sum $\sum_{\vec{m}\in\mathbb{Z}^d} \frac{1}{N^+\big(\frac{|\vec{m}|}{\lambda}-1\big)^{1+\gamma}}$ is finite. We now consider two cases: (i) when $\|\vec{m}\|_\infty < R$, which implies that $\vec{n}+\vec{m}\in Q_R^c$; (ii) $\|\vec{m}\|_\infty \ge R$. We define the function

$$h(R) = \sum_{\|\vec{m}\|_\infty\ge R} \frac{1}{N^+\big(\frac{|\vec{m}|}{\lambda}-1\big)^{1+\gamma}}\,,$$

and obtain the following upper bound (recall that the sets X_k^- are pairwise disjoint)

$$\begin{aligned} S_2^- \;\le\; & \|\nabla f_j\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)(1-\alpha)} \big(C\,\|\nabla f_j\|_{\ell^1(Q_R^c)} + h(R)\|\nabla f_j\|_{\ell^1(\mathbb{Z}^d)}\big)^{(1-\alpha)} \\ & \times \|f_j\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)(\alpha)} \big(C\,\|f_j\|_{\ell^1(Q_R^c)} + h(R)\|f_j\|_{\ell^1(\mathbb{Z}^d)}\big)^{\alpha}. \end{aligned} \tag{2.3.8}$$

The same bound as in (2.3.8) holds for S_2^+. Putting together (2.3.6), (2.3.7) and (2.3.8) (this last one duplicated, for S_2^- and for S_2^+) we obtain

$$\begin{aligned} \left\|\frac{\partial}{\partial x_d} M_\beta f_j\right\|_{\ell^q(\mathbb{Z}^d)}^q \le & \left(\left\|\frac{\partial}{\partial x_d} M_\beta f\right\|_{\ell^q(Q_{2R})} + 2\varepsilon_1(4R+1)^{\frac{d}{q}}\right)^q \\ & + 2\Big\{\|\nabla f_j\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)(1-\alpha)} \big(C\,\|\nabla f_j\|_{\ell^1(Q_R^c)} + h(R)\|\nabla f_j\|_{\ell^1(\mathbb{Z}^d)}\big)^{(1-\alpha)} \\ & \times \|f_j\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)\alpha} \big(C\,\|f_j\|_{\ell^1(Q_R^c)} + h(R)\|f_j\|_{\ell^1(\mathbb{Z}^d)}\big)^{\alpha}\Big\}. \end{aligned}$$

The crucial point in our argument is the fact that $h(R)\to 0$ as $R\to\infty$. It is now clear that we can choose R sufficiently large to make the second term above very small (for j large), and then we choose ε_1 sufficiently small to arrive at (2.3.5). This completes the proof in the centered case.

2.3.2 Uncentered case - part (i)

Minor modifications are needed in comparison to the previous argument. For a function $g \in \ell^1(\mathbb{Z}^d)$ and a point $\vec{n} \in \mathbb{Z}^d$, we now define

$$\widetilde{\mathcal{R}}g(\vec{n}) = \left\{ (\vec{x}, r) \in \mathbb{R}^d \times \mathbb{R}^+ ; \ \widetilde{M_\beta}\, g(\vec{n}) = A_{(\vec{x},r)}|g|(\vec{n}) = \frac{1}{N(\vec{x}, r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_r(\vec{x})} |g(\vec{m})| \right\}.$$

The following lemma can be proved with the same ideas used in the proof of Lemma 4.

Lemma 5. *Let $f_j \to f$ in $\ell^1(\mathbb{Z}^d)$ and let $R > 0$. There exists j_0 such that for $j \geq j_0$ we have $\widetilde{\mathcal{R}}f_j(\vec{n}) \subset \widetilde{\mathcal{R}}f(\vec{n})$ for each $\vec{n} \in \overline{B}_R$.*

Once we have adjusted this auxiliary lemma, the remaining steps of the proof are analogous to the centered case.

2.3.3 Centered case - part (ii)

The proof follows along similar lines to §2.3.1 and we only present the minor modifications needed. Observe that Lemma 4 continues to hold [7] and we may arrive at (2.3.6) in the same way. We bound S_1 just as we did in (2.3.7). To bound S_2 we need a different argument, since γ would be zero in this case. In fact, we use the pointwise estimate (2.2.22) to get directly

$$S_2 = \sum_{\|\vec{n}\|_\infty > 2R} \left| \frac{\partial}{\partial x_d} M_\beta f_j(\vec{n}) \right|^q \leq C \sum_{\|\vec{n}\|_\infty \geq 2R} \left| M_{\beta-1} f_j(\vec{n}) \right|^q. \tag{2.3.9}$$

Since $f_j \to f$ in $\ell^1(\mathbb{Z}^d)$ we have $f_j \to f$ in $\ell^r(\mathbb{Z}^d)$ for $r = d/(d+\alpha-1)$. Since $M_{\beta-1} : \ell^r(\mathbb{Z}^d) \to \ell^q(\mathbb{Z}^d)$ is bounded and continuous we have $M_{\beta-1} f_j \to M_{\beta-1} f$ in $\ell^q(\mathbb{Z}^d)$. We can then choose R sufficiently large to make the right-hand side of (2.3.9) very small (for j large). The rest of the proof is analogous to §2.3.1.

2.3.4 Uncentered case - part (ii)

The proof is essentially analogous to §2.3.1, §2.3.2 and §2.3.3. One just has to use the analogue of (2.2.22) to the uncentered operator to bound S_2 as in (2.3.9). We omit the details.

[7] In fact, note that all we need to establish Lemma 4 is that $f \in \ell^s(\mathbb{Z}^d)$ and $f_j \to f$ in $\ell^s(\mathbb{Z}^d)$ for some $1 \leq s < d/\beta$.

2.4 Proof of Theorem 2

In this section we work with the one-dimensional discrete uncentered maximal operator defined in (2.1.10). To prove Theorem 2 we may assume without loss of generality that $f \geq 0$. For $n \in \mathbb{Z}$ and $r, s \in \mathbb{Z}^+ \times \mathbb{Z}^+$ we define the fractional average

$$A_{r,s}f(n) = \frac{1}{(r+s+1)^{1-\beta}} \sum_{k=-r}^{s} f(n+k).$$

We start with the following preliminary lemma.

Lemma 6. *Let $f : \mathbb{Z} \to \mathbb{R}^+$ be a bounded function such that $\widetilde{M}_\beta f \not\equiv \infty$.*

(i) We have $\widetilde{M}_\beta f(n) < \infty$ for all $n \in \mathbb{Z}$.

(ii) If $\widetilde{M}_\beta f(n)$ is not attained by any average $A_{r,s}f(n)$ with $r, s \in \mathbb{Z}^+ \times \mathbb{Z}^+$, then

$$\widetilde{M}_\beta f(m) \geq \widetilde{M}_\beta f(n) \tag{2.4.1}$$

for all $m \in \mathbb{Z}$.

Proof. (i) If there is $n \in \mathbb{Z}$ such that $\widetilde{M}_\beta f(n) = \infty$, there exists a sequence $\{r_j, s_j\}$ in $\mathbb{Z}^+ \times \mathbb{Z}^+$, with $r_j + s_j \to \infty$ such that $A_{r_j,s_j}f(n) \to \infty$ as $j \to \infty$. For each $m \in \mathbb{Z}$, letting $C = \|f\|_{\ell^\infty(\mathbb{Z})}$ we have

$$A_{r_j,s_j}f(m) \geq A_{r_j,s_j}f(n) - \frac{2C|m-n|}{(r_j+s_j+1)^{1-\beta}}\,, \tag{2.4.2}$$

which implies that $\widetilde{M}_\beta f(m) = \infty$, a contradiction.

(ii) If $\widetilde{M}_\beta f(n)$ is not attained, there exists a sequence $\{r_j, s_j\}$ in $\mathbb{Z}^+ \times \mathbb{Z}^+$, with $r_j + s_j \to \infty$ such that $A_{r_j,s_j}f(n) \to \widetilde{M}_\beta(n)$ as $j \to \infty$. The inequality (2.4.1) plainly follows from (2.4.2). □

The next lemma is the heart of the proof. It bounds the q-variation of $\widetilde{M}_\beta f$ in a monotone interval by the variation of f in a comparable interval.

Lemma 7. *Let $f : \mathbb{Z} \to \mathbb{R}^+$ be a function of bounded variation such that $\widetilde{M}_\beta f$ is non-constant (in particular, $\widetilde{M}_\beta f \not\equiv \infty$).*

(i) Let $a < b$ be integers such that $\widetilde{M}_\beta f$ is non-increasing in $[a,b]$, with $\widetilde{M}_\beta f(a) > \widetilde{M}_\beta f(a+1)$. Let r be the smallest nonnegative integer such that $\widetilde{M}_\beta f(a) = A_{r,0}f(a)$. Then we have

$$\sum_{n=a}^{b-1} \left|\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)\right|^q \leq 2\,\|f'\|_{\ell^1(\mathbb{Z})}^{q-1} \sum_{n=a-r}^{b-1} |f(n) - f(n+1)|. \tag{2.4.3}$$

(ii) Let $a < b$ be integers such that $\widetilde{M}_\beta f$ is non-decreasing in $[a,b]$, with $\widetilde{M}_\beta f(b-1) < \widetilde{M}_\beta f(b)$. Let s be the smallest nonnegative integer such that $\widetilde{M}_\beta f(b) = A_{0,s}f(b)$. Then we have

$$\sum_{n=a}^{b-1} \left|\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)\right|^q \leq 2\,\|f'\|_{\ell^1(\mathbb{Z})}^{q-1} \sum_{n=a}^{b+s-1} |f(n) - f(n+1)|. \tag{2.4.4}$$

Proof. We prove (i) and (ii) is analogous. Observe first that the existence of such minimal r is guaranteed by Lemma 6. Then note that

$$\begin{aligned}\left|\widetilde{M}_\beta f(a) - \widetilde{M}_\beta f(b)\right| \leq \left|A_{r,0}f(a) - A_{r,0}f(b)\right| &= \left|\sum_{k=0}^{b-a-1} A_{r,0}f(a+k) - A_{r,0}f(a+k+1)\right| \\ &\leq \frac{1}{(r+1)^{1-\beta}} \sum_{n=a-r}^{b-1} (r+1)|f(n) - f(n+1)| \\ &= (r+1)^{\beta} \sum_{n=a-r}^{b-1} |f(n) - f(n+1)|.\end{aligned} \tag{2.4.5}$$

Let m be the smallest integer in $[a, b-1]$ such that

$$\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1) = \max\left\{\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1);\, n \in [a, b-1]\right\} > 0\,,$$

and let $t = \min\{t \in \mathbb{Z}^+;\, \widetilde{M}_\beta f(m) = A_{t,0}f(m)\}$. The existence of such t is guaranteed by Lemma 6.

Step 1

Let us first consider the situation when $t \geq r$. In this case, using (2.4.5) we obtain

$$\begin{aligned}
\sum_{n=a}^{b-1} \big|\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)\big|^q &\leq \big|\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)\big|^{q-1} \sum_{n=a}^{b-1} \big|\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)\big| \\
&= \big|\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)\big|^{q-1} \big|\widetilde{M}_\beta f(a) - \widetilde{M}_\beta f(b)\big| \\
&\leq \big|\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)\big|^{q-1} \\
&\qquad \times (r+1)^\beta \sum_{n=a-r}^{b-1} |f(n) - f(n+1)| \\
&= \Big|(r+1)^{1-\beta}\big(\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)\big)\Big|^{q-1} \\
&\qquad \times \sum_{n=a-r}^{b-1} |f(n) - f(n+1)| \\
&\leq \Big|(r+1)^{1-\beta}\big(A_{t,0} f(m) - A_{t,0} f(m+1)\big)\Big|^{q-1} \\
&\qquad \times \sum_{n=a-r}^{b-1} |f(n) - f(n+1)| \\
&\leq \left|\frac{(r+1)^{1-\beta}}{(t+1)^{1-\beta}}\big(f(m-t) - f(m+1)\big)\right|^{q-1} \\
&\qquad \times \sum_{n=a-r}^{b-1} |f(n) - f(n+1)| \\
&\leq \|f'\|_{\ell^1(\mathbb{Z})}^{q-1} \sum_{n=a-r}^{b-1} |f(n) - f(n+1)|.
\end{aligned} \tag{2.4.6}$$

This establishes (2.4.3) in this case.

Step 2

Now assume that $t < r$ and $m - t \leq a$. Note that in this case we have $a - r < m - t$. We may proceed as in (2.4.5) and (2.4.6) to obtain

$$\begin{aligned}
&\sum_{n=a}^{b-1} \left|\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)\right|^q \\
&\leq (t+1)\left|\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)\right|^q + \left|\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)\right|^{q-1} \sum_{n=m}^{b-1} \left|\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)\right| \\
&\leq |f(m-t) - f(m+1)|^q + \left|\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)\right|^{q-1} \left|\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(b)\right| \\
&\leq |f(m-t) - f(m+1)|^q + \left|\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)\right|^{q-1} (t+1)^\beta \sum_{n=m-t}^{b-1} |f(n) - f(n+1)| \\
&\leq |f(m-t) - f(m+1)|^{q-1} \left(|f(m-t) - f(m+1)| + \sum_{n=m-t}^{b-1} |f(n) - f(n+1)| \right) \\
&\leq 2\|f'\|_{\ell^1(\mathbb{Z})}^{q-1} \sum_{n=a-r}^{b-1} |f(n) - f(n+1)|,
\end{aligned} \tag{2.4.7}$$

and we have again established (2.4.3).

Step 3

Finally, we consider the case $t < r$ and $a < m - t$. Reasoning as in (2.4.7) we obtain

$$\begin{aligned}
&\sum_{n=m-t}^{b-1} \left|\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)\right|^q \\
&\leq (t+1)\left|\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)\right|^q \\
&\quad + \left|\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)\right|^{q-1} \sum_{n=m}^{b-1} \left|\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)\right| \\
&\leq 2\|f'\|_{\ell^1(\mathbb{Z})}^{q-1} \sum_{n=m-t}^{b-1} |f(n) - f(n+1)|.
\end{aligned} \tag{2.4.8}$$

We then proceed inductively. Let $(m_1, t_1) = (m, t)$. Having defined $(m_1, t_1), (m_2, t_2), \ldots, (m_{l-1}, t_{l-1})$, if $t_{l-1} < r$ and $a < m_{l-1} - t_{l-1}$ we define m_l as the smallest integer in the interval $[a, m_{l-1} - t_{l-1} - 1]$ such that

$$\widetilde{M}_\beta f(m_l) - \widetilde{M}_\beta f(m_l+1) = \max\left\{\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1);\, n \in [a, m_{l-1} - t_{l-1} - 1]\right\} > 0\,,$$

and let $t_l = \min\{t \in \mathbb{Z}^+;\, \widetilde{M}_\beta f(m_l) = A_{t,0} f(m_l)\}$. If $t_l < r$ and $a < m_l - t_l$ we reboot Step 3 to obtain

$$\sum_{n=m_l-t_l}^{m_{l-1}-t_{l-1}-1} \left|\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)\right|^q \leq 2\,\|f'\|_{\ell^1(\mathbb{Z})}^{q-1} \sum_{n=m_l-t_l}^{m_{l-1}-t_{l-1}-1} |f(n) - f(n+1)|. \tag{2.4.9}$$

This process must terminate, i.e. there exists a smallest N such that either (i) $t_N \geq r$ or (ii) $t_N < r$ and $m_N - t_N \leq a$. In the first case we use Step 1 to bound the q-variation of $\widetilde{M}_\beta f$ on the interval $[a, m_{N-1} - t_{N-1}]$ and in the second case we use Step 2 to bound the q-variation of $\widetilde{M}_\beta f$ on the interval $[a, m_{N-1} - t_{N-1}]$. We then sum with all the previous inequalities (2.4.8) and (2.4.9) of the inductive process to arrive at the desired result. □

We now introduce the local maxima and minima of a discrete function $g : \mathbb{Z} \to \mathbb{R}$.[8] We say that an interval $[n, m]$ is a *string of local maxima* of g if

$$g(n-1) < g(n) = \ldots = g(m) > g(m+1).$$

If $n = -\infty$ or $m = \infty$ (but not both simultaneously) we modify the definition accordingly, eliminating one of the inequalities. The rightmost point m of such a string is a *right local maximum* of g, while the leftmost point n is a *left local maximum* of g. We define *string of local minima*, *right local minimum* and *left local minimum* analogously.

Proof of Theorem 2. Assume that $\widetilde{M}_\beta f$ is not constant (in case $\widetilde{M}_\beta f$ is constant the result is obviously true). Let $\{[a_j^-, a_j^+]\}_{j\in\mathbb{Z}}$ and $\{[b_j^-, b_j^+]\}_{j\in\mathbb{Z}}$ be the ordered strings of local maxima and local minima of $\widetilde{M}_\beta f$ (we allow the possibilities of a_j^- or $b_j^- = -\infty$ and a_j^+ or $b_j^+ = \infty$), i.e.

$$\ldots < a_{-1}^- \leq a_{-1}^+ < b_{-1}^- \leq b_{-1}^+ < a_0^- \leq a_0^+ < b_0^- \leq b_0^+ < a_1^- \leq a_1^+ < b_1^- \leq b_1^+ < \ldots \tag{2.4.10}$$

This sequence may terminate in one or both sides (in principle, we are not even ruling out the possibility of this sequence being empty, i.e. of $\widetilde{M}_\beta f$ being monotone), and we adjust the notation accordingly.

[8] The local extrema are defined slightly differently in [8, 16], but used with the meaning stated here.

Let a_j^+ be one of the right local maxima and r_j^+ be the smallest integer radius such that $\widetilde{M}_\beta f(a_j^+) = A_{r_j^+,0} f(a_j^+)$. The crucial observation is that $a_j^- \leq a_j^+ - r_j^+$ and Lemma 7 yields

$$\sum_j \sum_{n=a_j^+}^{b_j^- -1} \left|\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)\right|^q \leq 2 \, \|f'\|_{\ell^1(\mathbb{Z})}^{q-1} \sum_j \sum_{n=a_j^-}^{b_j^- -1} |f(n) - f(n+1)|.$$

Analogously,

$$\sum_j \sum_{n=b_j^+}^{a_{j+1}^- -1} \left|\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)\right|^q \leq 2 \, \|f'\|_{\ell^1(\mathbb{Z})}^{q-1} \sum_j \sum_{n=b_j^+}^{a_{j+1}^+ -1} |f(n) - f(n+1)|.$$

Adding up these inequalities we obtain the desired result (note the potential overlap in each $[a_j^-, a_j^+]$)

$$\sum_{n=-\infty}^{\infty} \left|\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)\right|^q \leq 4 \, \|f'\|_{\ell^1(\mathbb{Z})}^{q}.$$

Note that, in the exceptional cases when the sequence (3.4.1) terminates to one or both sides, or even when the sequence (3.4.1) is empty (if $\widetilde{M}_\beta f$ is monotone), the q-variation of $\widetilde{M}_\beta f$ on the intervals of length infinity where it is monotone can be bounded directly using Lemma 7. □

2.5 Proof of Theorem 1

We now move the discussion to the continuous setting. In this case, the one-dimensional uncentered fractional maximal operator $\widetilde{M}_\beta$ is defined as in (2.1.6). The ideas presented in the previous section (discrete setting) also play a relevant role here, while new technical details arise. When proving Theorem 1 we may assume without loss of generality that $f \geq 0$ since $\mathrm{Var}\,(f) \leq \mathrm{Var}\,(|f|)$. The case $\beta = 0$ of Theorem 1 was proved by Aldaz and Pérez Lázaro [2] (with the sharp constant $C = 1$), so throughout this section we restrict ourselves to the case $0 < \beta < 1$. For $r, s \geq 0$ we keep denoting the fractional averages by [9]

$$A_{r,s} f(x) = \frac{1}{(r+s)^{1-\beta}} \int_{x-r}^{x+s} f(y)\,\mathrm{d}y.$$

[9] We define $A_{0,0}(f)(x) = \limsup_{r,s\to 0^+} \frac{1}{(r+s)^{1-\beta}} \int_{x-r}^{x+s} f(y)\,\mathrm{d}y$. If f is locally bounded and $\beta > 0$ we have $A_{0,0}(f)(x) = 0$.

We start with the following preliminary lemma.

Lemma 8. *Let* $0 < \beta < 1$. *Let* $f : \mathbb{R} \to \mathbb{R}^+$ *be a bounded function such that* $\widetilde{M}_\beta f \not\equiv \infty$.

(i) We have $\widetilde{M}_\beta f(x) < \infty$ *for all* $x \in \mathbb{R}$.

(ii) If $\widetilde{M}_\beta f(x)$ *is not attained by any average* $A_{r,s}f(x)$ *with* $r, s \geq 0$, *then*

$$\widetilde{M}_\beta f(y) \geq \widetilde{M}_\beta f(x) \tag{2.5.1}$$

for all $y \in \mathbb{R}$.

Proof. (i) If $\widetilde{M}_\beta f(x) = \infty$ for some $x \in \mathbb{R}$, since f is bounded there exists a sequence $(r_j, s_j) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $(r_j + s_j) \to \infty$ such that $A_{r_j,s_j}f(x) \to \infty$ as $j \to \infty$. Letting $C = \|f\|_{L^\infty(\mathbb{R})}$, for any $y \in \mathbb{R}$ we have

$$A_{r_j,s_j}f(y) \geq A_{r_j,s_j}f(x) - \frac{2C|x-y|}{(r_j+s_j)^{1-\beta}}, \tag{2.5.2}$$

which implies that $\widetilde{M}_\beta f(y) = \infty$, a contradiction.
(ii) If $\widetilde{M}_\beta f(x)$ is not attained, there exists a sequence $(r_j, s_j) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $A_{r_j,s_j}f(x) \to \widetilde{M}_\beta f(x)$ as $j \to \infty$. If $(r_j + s_j)$ is bounded, passing to a subsequence if necessary, we can find $(r, s) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $r_j \to r$ and $s_j \to s$, which implies that $A_{r,s}f(x) = \widetilde{M}_\beta f(x)$, a contradiction. Hence we have $(r_j + s_j) \to \infty$ and inequality (2.5.1) plainly follows from (2.5.2). □

Our next proposition establishes the result for Lipschitz functions.

Proposition 9. *Let* $0 < \beta < 1$. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a Lipschitz function such that* $\widetilde{M}_\beta f \not\equiv \infty$. *Then*

$$\mathrm{Var}_q\big(\widetilde{M}_\beta f\big) \leq 8^{1/q}\,\mathrm{Var}\,(f). \tag{2.5.3}$$

The following proposition is a classical theorem of F. Riesz [62, Chapter IX §4, Theorem 7]. This will be useful in the proof of Theorem 1.

Proposition 10. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a function and* $1 < q < \infty$. *Then* $\mathrm{Var}_q(g) < \infty$ *if and only if* g *is an absolutely continuous function and its derivative* g' *belongs to* $L^q(\mathbb{R})$. *Moreover, in this case, we have that*

$$\|g'\|_{L^q(\mathbb{R})} = \mathrm{Var}_q(g). \tag{2.5.4}$$

We postpone the proof of these results until §2.5.2. For now, let us assume the validity of Propositions 9 and 10 and conclude the proof of Theorem 1.

2.5.1 Proof of Theorem 1

We start by showing the validity of (2.5.3) for all functions $f : \mathbb{R} \to \mathbb{R}^+$ of bounded variation such that $\widetilde{M}_\beta f \not\equiv \infty$.

Let $\varphi \in C_c^\infty(\mathbb{R})$ be a nonnegative smooth function with support in $[-1,1]$ and $\|\varphi\|_{L^1(\mathbb{R})} = 1$. For $\varepsilon > 0$ we define $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})$ and for a function $f : \mathbb{R} \to \mathbb{R}^+$ of bounded variation we write $f_\varepsilon = f * \varphi_\varepsilon$. Note that f_ε is Lipschitz continuous and $\mathrm{Var}\,(f_\varepsilon) \le \mathrm{Var}\,(f)$ for all $\varepsilon > 0$.

Fix a partition $\mathcal{P} = \{x_1 < x_2 < \ldots < x_N\}$. By Proposition 9 we have

$$\left(\sum_{n=1}^{N-1} \frac{\left|\widetilde{M}_\beta f_\varepsilon(x_{n+1}) - \widetilde{M}_\beta f_\varepsilon(x_n)\right|^q}{|x_{n+1} - x_n|^{q-1}}\right)^{1/q} \le 8^{1/q}\,\mathrm{Var}\,(f_\varepsilon) \le 8^{1/q}\,\mathrm{Var}\,(f) \tag{2.5.5}$$

for all $\varepsilon > 0$. We claim that

$$\lim_{\varepsilon\to 0} \widetilde{M}_\beta f_\varepsilon(x) = \widetilde{M}_\beta f(x) \tag{2.5.6}$$

for all $x \in \mathbb{R}$. It then follows from (2.5.5) and (2.5.6) that

$$\left(\sum_{n=1}^{N-1} \frac{\left|\widetilde{M}_\beta f(x_{n+1}) - \widetilde{M}_\beta f(x_n)\right|^q}{|x_{n+1} - x_n|^{q-1}}\right)^{1/q} \le 8^{1/q}\,\mathrm{Var}\,(f).$$

Since the original partition $\mathcal{P}$ was arbitrary, we arrive at (2.5.3).

We now prove (2.5.6). Recall that $\lim_{\varepsilon\to 0} f_\varepsilon(x) = f(x)$ at every point of continuity of f (in particular, almost everywhere). By Fatou's lemma, for any nontrivial interval $[x-s, x+r]$ containing x we have

$$A_{r,s}f(x) \le \liminf_{\varepsilon\to 0} A_{r,s}f_\varepsilon(x) \le \liminf_{\varepsilon\to 0} \widetilde{M}_\beta f_\varepsilon(x)\,,$$

from which we conclude that

$$\widetilde{M}_\beta f(x) \le \liminf_{\varepsilon\to 0} \widetilde{M}_\beta f_\varepsilon(x). \tag{2.5.7}$$

We prove the opposite inequality by contradiction. Given $x \in \mathbb{R}$, assume that for some $\eta > 0$

we have

$$\limsup_{\varepsilon\to 0} \widetilde{M}_\beta f_\varepsilon(x) > (1+2\eta)\,\widetilde{M}_\beta f(x) \tag{2.5.8}$$

(in particular, $f \not\equiv 0$ and $\widetilde{M}_\beta f(x) > 0$). Then, for a certain sequence of $\varepsilon \to 0$, there exist $y = y_\varepsilon$ and $r = r_\varepsilon > 0$ such that $x \in [y-r, y+r]$ and

$$\frac{1}{(2r)^{1-\beta}} \int_{y-r}^{y+r} f_\varepsilon(t)\,\mathrm{d}t > (1+\eta)\,\widetilde{M}_\beta f(x). \tag{2.5.9}$$

Since $\|f_\varepsilon\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^\infty(\mathbb{R})}$ and we are assuming that $0 < \beta < 1$, we note that we must have $r_\varepsilon > c > 0$ for this sequence of $\varepsilon \to 0$. On the other hand, observe that

$$\begin{aligned}
\frac{1}{(2r)^{1-\beta}} \int_{y-r}^{y+r} f_\varepsilon(t)\,\mathrm{d}t &= \frac{1}{(2r)^{1-\beta}} \int_{y-r}^{y+r} \int_{-\varepsilon}^{\varepsilon} f(t-u)\,\varphi_\varepsilon(u)\,\mathrm{d}u\,\mathrm{d}t \\
&= \frac{1}{(2r)^{1-\beta}} \int_{-\varepsilon}^{\varepsilon} \left(\int_{y-r}^{y+r} f(t-u)\,\mathrm{d}t \right) \varphi_\varepsilon(u)\,\mathrm{d}u \\
&\leq \frac{1}{(2r)^{1-\beta}} \int_{-\varepsilon}^{\varepsilon} \left(\int_{y-r-\varepsilon}^{y+r+\varepsilon} f(t)\,\mathrm{d}t \right) \varphi_\varepsilon(u)\,\mathrm{d}u \\
&\leq \frac{(2r+2\varepsilon)^{1-\beta}}{(2r)^{1-\beta}} \int_{-\varepsilon}^{\varepsilon} \widetilde{M}_\beta f(x)\,\varphi_\varepsilon(u)\,\mathrm{d}u \\
&\leq \frac{(2r+2\varepsilon)^{1-\beta}}{(2r)^{1-\beta}} \widetilde{M}_\beta f(x).
\end{aligned} \tag{2.5.10}$$

From (2.5.9) and (2.5.10) we conclude that

$$\frac{(2r+2\varepsilon)^{1-\beta}}{(2r)^{1-\beta}} > 1+\eta.$$

If we restrict ourselves to the range $\varepsilon \leq 1$, the inequality above implies that $r \leq N$ for some large $N = N(\beta,\eta)$ and then $|y| \leq |x| + N$. Therefore, there exists a subsequence $\{y_{\varepsilon_k}, r_{\varepsilon_k}\} \subset \{y_\varepsilon, r_\varepsilon\}$ and a pair (y_0, r_0) with $r_0 > 0$ such that $y_{\varepsilon_k} \to y_0$ and $r_{\varepsilon_k} \to r_0$ as $\varepsilon_k \to 0$. Then $x \in [y_0 - r_0, y_0 + r_0]$ and

$$\widetilde{M}_\beta f(x) \geq \frac{1}{(2r_0)^{1-\beta}} \int_{y_0-r_0}^{y_0+r_0} f(t)\,\mathrm{d}t = \lim_{\varepsilon_k\to 0} \frac{1}{(2r_{\varepsilon_k})^{1-\beta}} \int_{y_{\varepsilon_k}-r_{\varepsilon_k}}^{y_{\varepsilon_k}+r_{\varepsilon_k}} f_\varepsilon(t)\,\mathrm{d}t \geq (1+\eta)\,\widetilde{M}_\beta f(x),$$

which is a contradiction. Hence (2.5.8) cannot occur and we must have

$$\limsup_{\varepsilon\to 0} \widetilde{M}_\beta f_\varepsilon(x) \leq \widetilde{M}_\beta f(x),$$

which, together with (2.5.7), establishes the pointwise convergence (2.5.6) and concludes the proof of the extension of (2.5.3) for all functions $f:\mathbb{R}\to\mathbb{R}^+$ of bounded variation such that $\widetilde{M}_\beta f \not\equiv \infty$.

Using Proposition 10 we conclude that given a function $f:\mathbb{R}\to\mathbb{R}^+$ of bounded variation such that $\widetilde{M}_\beta f \not\equiv \infty$, we have that $\widetilde{M}_\beta f$ is absolutely continuous, its derivative belongs to $L^q(\mathbb{R})$ and $\big\|\big(\widetilde{M}_\beta f\big)'\big\|_{L^q(\mathbb{R})} = \operatorname{Var}_q\big(\widetilde{M}_\beta f\big)$. Combining these observations we obtain the desired result.

2.5.2 Proof of Propositions 9 and 10

We start by proving Proposition 9. Recall that we are assuming that $f\geq 0$. If $\operatorname{Var}(f)=\infty$, we understand that inequality (2.5.3) is true, so let us also assume that f is of bounded variation. The next lemma is the continuous analogue of Lemma 7 and is the core of this proof.

Lemma 11. *Let $0<\beta<1$. Let $f:\mathbb{R}\to\mathbb{R}^+$ be a Lipschitz function of bounded variation such that $\widetilde{M}_\beta f$ is non-constant (in particular, $\widetilde{M}_\beta f \not\equiv \infty$).*

(i) Let $x_1 < x_2 < \ldots < x_N$ be a sequence of real numbers such that

$$\widetilde{M}_\beta f(x_1) > \widetilde{M}_\beta f(x_2) \geq \ldots \geq \widetilde{M}_\beta f(x_{N-1}) \geq \widetilde{M}_\beta f(x_N).$$

Let $r,s\geq 0$ be such that $\widetilde{M}_\beta f(x_1) = A_{r,s}f(x_1)$, with $r+s$ minimal (and then with r minimal, if necessary). Then

$$\sum_{n=1}^{N-1} \frac{\big|\widetilde{M}_\beta f(x_{n+1}) - \widetilde{M}_\beta f(x_n)\big|^q}{|x_{n+1}-x_n|^{q-1}} \leq 4\,\|f'\|_{L^1(\mathbb{R})}^{q-1} \int_{x_1-r}^{x_N} |f'(x)|\,\mathrm{d}x. \tag{2.5.11}$$

(ii) Let $x_1 < x_2 < \ldots < x_N$ be a sequence of real numbers such that

$$\widetilde{M}_\beta f(x_1) \leq \widetilde{M}_\beta f(x_2) \leq \ldots \leq \widetilde{M}_\beta f(x_{N-1}) < \widetilde{M}_\beta f(x_N).$$

Let $r, s \geq 0$ be such that $\widetilde{M}_\beta f(x_N) = A_{r,s}f(x_N)$, with $r+s$ minimal (and then with s minimal, if necessary). Then

$$\sum_{n=1}^{N-1} \frac{\left|\widetilde{M}_\beta f(x_{n+1}) - \widetilde{M}_\beta f(x_n)\right|^q}{|x_{n+1}-x_n|^{q-1}} \leq 4\, \|f'\|_{L^1(\mathbb{R})}^{q-1} \int_{x_1}^{x_N+s} |f'(x)|\,\mathrm{d}x. \tag{2.5.12}$$

Proof. We prove (i) and (ii) is analogous. Note that the existence of such a pair (r, s) is guaranteed by Lemma 8 and the continuity properties of the averages. For such pair (r, s), since $\widetilde{M}_\beta f(x_1) > \widetilde{M}_\beta f(x_N)$, we must have $s < x_N - x_1$. Let $a \in [x_1 - r, x_1 + s]$ be a point such that

$$\frac{1}{(r+s)} \int_{x_1-r}^{x_1+s} f(x)\,\mathrm{d}x = f(a),$$

and let $b \in [x_N - r - s, x_N] \subset [x_1 - r, x_N]$ be a point such that

$$\frac{1}{r+s} \int_{x_N-r-s}^{x_N} f(x)\,\mathrm{d}x = f(b).$$

We then obtain

$$\begin{aligned} \left|\widetilde{M}_\beta f(x_1) - \widetilde{M}_\beta f(x_N)\right| &\leq \left|A_{r,s}f(x_1) - A_{r+s,0}f(x_N)\right| \\ &= (r+s)^\beta\, |f(a) - f(b)| \\ &\leq (r+s)^\beta \int_{x_1-r}^{x_N} |f'(x)|\,\mathrm{d}x. \end{aligned} \tag{2.5.13}$$

Now let m be the smallest integer with $1 \leq m \leq N-1$ such that

$$\frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{|x_m - x_{m+1}|} = \max\left\{\frac{\widetilde{M}_\beta f(x_n) - \widetilde{M}_\beta f(x_{n+1})}{|x_n - x_{n+1}|};\ 1 \leq n \leq N-1\right\} > 0.$$

Let $t, u \geq 0$ be such that $\widetilde{M}_\beta f(x_m) = A_{t,u}f(x_m)$, with $t+u$ minimal (and then with t minimal, if necessary). The existence of such a pair (t, u) is guaranteed by Lemma 8 and we have $0 \leq u < x_{m+1} - x_m$.

Step 1

Let us first consider the case when $t+u \geq r+s$. Using (2.5.13) we have

$$\begin{aligned}
&\sum_{n=1}^{N-1} \frac{\big|\widetilde{M}_\beta f(x_n) - \widetilde{M}_\beta f(x_{n+1})\big|^q}{|x_n - x_{n+1}|^{q-1}} \\
&\quad \leq \left|\frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}}\right|^{q-1} \sum_{n=1}^{N-1} \big|\widetilde{M}_\beta f(x_n) - \widetilde{M}_\beta f(x_{n+1})\big| \\
&\quad = \left|\frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}}\right|^{q-1} \big|\widetilde{M}_\beta f(x_1) - \widetilde{M}_\beta f(x_N)\big| \\
&\quad \leq \left|\frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}}\right|^{q-1} (r+s)^\beta \int_{x_1-r}^{x_N} |f'(x)|\,\mathrm{d}x \\
&\quad = \left|(r+s)^{1-\beta}\,\frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}}\right|^{q-1} \int_{x_1-r}^{x_N} |f'(x)|\,\mathrm{d}x \\
&\quad \leq \left|\left(\frac{r+s}{t+u}\right)^{1-\beta} \int_{x_m-t}^{x_m+u} \frac{f(y) - f(y+x_{m+1}-x_m-u)}{x_{m+1}-x_m}\,\mathrm{d}y\right|^{q-1} \int_{x_1-r}^{x_N} |f'(x)|\,\mathrm{d}x \\
&\quad \leq \left|\left(\frac{r+s}{t+u}\right)^{1-\beta} \frac{1}{x_{m+1}-x_m} \int_{x_m-t}^{x_m+u} \int_0^{x_{m+1}-x_m-u} |f'(y+z)|\,\mathrm{d}z\,\mathrm{d}y\right|^{q-1} \int_{x_1-r}^{x_N} |f'(x)|\,\mathrm{d}x \\
&\quad \leq \left|\left(\frac{r+s}{t+u}\right)^{1-\beta} \frac{1}{x_{m+1}-x_m} \int_0^{x_{m+1}-x_m-u} \int_{x_m-t}^{x_m+u} |f'(y+z)|\,\mathrm{d}y\,\mathrm{d}z\right|^{q-1} \int_{x_1-r}^{x_N} |f'(x)|\,\mathrm{d}x \\
&\quad \leq \|f'\|_{L^1(\mathbb{R})}^{q-1} \int_{x_1-r}^{x_N} |f'(x)|\,\mathrm{d}x\,,
\end{aligned}$$

which establishes (2.5.11) in this case.

Step 2

We now consider the case when $t+u < r+s$ and $x_m - t \leq x_1$. In this case note that $x_1 - r \leq x_m - t$ (which is clear if $m=1$, and if $m>1$ we note that $s < x_m - x_1$). Reasoning as before we have

$$\sum_{n=1}^{N-1} \frac{\big|\widetilde{M}_\beta f(x_n) - \widetilde{M}_\beta f(x_{n+1})\big|^q}{|x_n - x_{n+1}|^{q-1}}$$

$$
\begin{aligned}
&\leq \left|\frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}}\right|^q \sum_{n=1}^{m-1} |x_n - x_{n+1}| \\
&\qquad + \left|\frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}}\right|^{q-1} \sum_{n=m}^{N-1} \left|\widetilde{M}_\beta f(x_n) - \widetilde{M}_\beta f(x_{n+1})\right| \\
&\leq \left|\frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}}\right|^q |x_1 - x_m| \\
&\qquad + \left|\frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}}\right|^{q-1} \left|\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_N)\right| \\
&\leq t \left|\frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}}\right|^q \\
&\qquad + \left|\frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}}\right|^{q-1} (t+u)^\beta \int_{x_m - t}^{x_N} |f'(x)| \, \mathrm{d}x. \\
&\leq \left| \left(\frac{t}{t+u}\right)^{1-\beta} \int_{x_m - t}^{x_m + u} \frac{f(y) - f(y + x_{m+1} - x_m - u)}{x_{m+1} - x_m} \, \mathrm{d}y \right|^q \\
&\qquad + \left| \int_{x_m - t}^{x_m + u} \frac{f(y) - f(y + x_{m+1} - x_m - u)}{x_{m+1} - x_m} \, \mathrm{d}y \right|^{q-1} \int_{x_m - t}^{x_N} |f'(x)| \, \mathrm{d}x \\
&\leq \left| \frac{1}{x_{m+1} - x_m} \int_{x_m - t}^{x_m + u} \int_0^{x_{m+1} - x_m - u} |f'(y+z)| \, \mathrm{d}z \, \mathrm{d}y \right|^q \\
&\qquad + \left| \frac{1}{x_{m+1} - x_m} \int_{x_m - t}^{x_m + u} \int_0^{x_{m+1} - x_m - u} |f'(y+z)| \, \mathrm{d}z \, \mathrm{d}y \right|^{q-1} \int_{x_m - t}^{x_N} |f'(x)| \, \mathrm{d}x \\
&= \left| \frac{1}{x_{m+1} - x_m} \int_0^{x_{m+1} - x_m - u} \int_{x_m - t}^{x_m + u} |f'(y+z)| \, \mathrm{d}z \, \mathrm{d}y \right|^q \\
&\qquad + \left| \frac{1}{x_{m+1} - x_m} \int_0^{x_{m+1} - x_m - u} \int_{x_m - t}^{x_m + u} |f'(y+z)| \, \mathrm{d}z \, \mathrm{d}y \right|^{q-1} \int_{x_m - t}^{x_N} |f'(x)| \, \mathrm{d}x \\
&\leq \|f'\|_{L^1(\mathbb{R})}^{q-1} \left\{ \frac{1}{x_{m+1} - x_m} \int_0^{x_{m+1} - x_m - u} \int_{x_m - t}^{x_N} |f'(x)| \, \mathrm{d}x \, \mathrm{d}y + \int_{x_m - t}^{x_N} |f'(x)| \, \mathrm{d}x \right\} \\
&\leq 2 \, \|f'\|_{L^1(\mathbb{R})}^{q-1} \int_{x_m - t}^{x_N} |f'(x)| \, \mathrm{d}x \\
&\leq 2 \, \|f'\|_{L^1(\mathbb{R})}^{q-1} \int_{x_1 - r}^{x_N} |f'(x)| \, \mathrm{d}x,
\end{aligned}
$$

which establishes (2.5.11).

Step 3

Finally, we consider the case when $t + u < r + s$ and $x_1 < x_m - t$. We let k be the unique integer such that $x_{k-1} < x_m - t \leq x_k$. Arguing as in Step 2 we get

$$\sum_{n=k}^{N-1} \frac{\left|\widetilde{M}_\beta f(x_n) - \widetilde{M}_\beta f(x_{n+1})\right|^q}{|x_n - x_{n+1}|^{q-1}} \leq 2 \, \|f'\|_{L^1(\mathbb{R})}^{q-1} \int_{x_{k-1}}^{x_N} |f'(x)| \, \mathrm{d}x. \tag{2.5.14}$$

We then proceed inductively. Let $(m_1, t_1, u_1, k_1) = (m, t, u, k)$. Having defined (m_1, t_1, u_1, k_1), (m_2, t_2, u_2, k_2), $\ldots$, $(m_{l-1}, t_{l-1}, u_{l-1}, k_{l-1})$, if $t_{l-1} + u_{l-1} < r + s$ and $x_1 < x_{m_{l-1}} - t_{l-1}$ then k_{l-1} is the unique integer such that $x_{k_{l-1}-1} < x_{m_{l-1}} - t_{l-1} \leq x_{k_{l-1}}$. We then let m_l be the smallest integer with $1 \leq m_l \leq k_{l-1} - 1$ such that

$$\frac{\widetilde{M}_\beta f(x_{m_l}) - \widetilde{M}_\beta f(x_{m_l+1})}{|x_{m_l} - x_{m_l+1}|} = \max \left\{ \frac{\widetilde{M}_\beta f(x_n) - \widetilde{M}_\beta f(x_{n+1})}{|x_n - x_{n+1}|}; \, 1 \leq n \leq k_{l-1} - 1 \right\} > 0.$$

Let $t_l, u_l \geq 0$ be such that $\widetilde{M}_\beta f(x_{m_l}) = A_{t_l,u_l} f(x_{m_l})$, with $t_l + u_l$ minimal (and then with t_l minimal, if necessary). The existence of such a pair (t_l, u_l) is guaranteed by Lemma 8. If $t_l + u_l < r + s$ and $x_1 < x_{m_l} - t_l$ we let k_l be the unique integer such that $x_{k_l-1} < x_{m_l} - t_l \leq x_{k_l}$. We then reboot Step 3 to obtain

$$\sum_{n=k_l}^{k_{l-1}-1} \frac{\left|\widetilde{M}_\beta f(x_n) - \widetilde{M}_\beta f(x_{n+1})\right|^q}{|x_n - x_{n+1}|^{q-1}} \leq 2 \, \|f'\|_{L^1(\mathbb{R})}^{q-1} \int_{x_{k_l-1}}^{x_{k_{l-1}}} |f'(x)| \, \mathrm{d}x. \tag{2.5.15}$$

This process must terminate, i.e. there is a smallest integer L such that either: (i) $t_L + u_L \geq r + s$ or (ii) $t_L + u_L < r + s$ and $x_{m_L} - t_L \leq x_1$. In the first case (resp. second case) we use Step 1 (resp. Step 2) to bound the q-variation of $\widetilde{M}_\beta f$ (over the partition $\{x_n\}$) from x_1 to $x_{k_{L-1}}$. We then sum all the previous inequalities (2.5.14) and (2.5.15) to arrive at the desired conclusion (note that the sum of the integrals on right-hand sides of (2.5.14) and (2.5.15) has a two-fold overlap over each interval $[x_{k_l-1}, x_{k_l}]$). □

Proof of Proposition 9. Fix a partition $\mathcal{P} = \{x_1 < x_2 < \ldots < x_N\}$. For a generic function $g : \mathbb{R} \to \mathbb{R}$, we say that an interval $[x_n, x_m]$ is a *string of local maxima* of g (relative to the partition $\mathcal{P}$) if

$$g(x_{n-1}) < g(x_n) = \ldots = g(x_m) > g(x_{m+1}), \tag{2.5.16}$$

provided $n \neq 1$ and $m \neq N$. In case $n = 1$ (resp. $m = N$) we disregard the leftmost (resp.

rightmost) inequality in (2.5.16). When $m \neq N$, the rightmost point x_m of such a string is a *right local maximum* of g, and when $n \neq 1$ the leftmost point x_n is a *left local maximum* of g. We define *string of local minima*, *right local minimum* and *left local minimum* (relative to the partition $\mathcal{P}$) analogously.

Assume that $\widetilde{M}_\beta f$ is not constant (in case $\widetilde{M}_\beta f$ is constant the result is obviously true). The strategy is the same as in the discrete case, to bound the q-variation of $\widetilde{M}_\beta f$ between a right local maximum and the next left local minimum. The q-variation of $\widetilde{M}_\beta f$ between a right local minimum and the next left local maximum is treated analogously.

Let x_m be a right local maximum and x_l be the next local minimum of $\widetilde{M}_\beta f$ (i.e. with the smallest $l > m$). Let $r, s \geq 0$ be such that $\widetilde{M}_\beta f(x_m) = A_{r,s} f(x_m)$, with $r + s$ minimal (and then with r minimal, if necessary). By Lemma 11 we have

$$\sum_{n=m}^{l-1} \frac{\left|\widetilde{M}_\beta f(x_{n+1}) - \widetilde{M}_\beta f(x_n)\right|^q}{|x_{n+1} - x_n|^{q-1}} \leq 4 \, \|f'\|_{L^1(\mathbb{R})}^{q-1} \int_{x_m - r}^{x_l} |f'(x)| \, \mathrm{d}x. \tag{2.5.17}$$

In case there exists a right local minimum $x_{l'}$ before x_m (which is the case if x_m is not the first right local maximum), note that we have $x_{l'} < x_m - r$. We may therefore sum the inequalities (2.5.17) over all pairs of consecutive local extrema (the non-increasing and non-decreasing pieces). Noting the additional two-fold overlap on the right-hand side we arrive at

$$\sum_{n=1}^{N-1} \frac{\left|\widetilde{M}_\beta f(x_{n+1}) - \widetilde{M}_\beta f(x_n)\right|^q}{|x_{n+1} - x_n|^{q-1}} \leq 8 \, \|f'\|_{L^1(\mathbb{R})}^{q}.$$

Since the right-hand side is now independent of the partition $\mathcal{P}$, we may take the supremum over all such partitions to obtain the desired result. □

Proof of Proposition 10. We start by assuming that g is absolutely continuous and its derivative g' belongs to $L^q(\mathbb{R})$. Consider a finite interval $[a, b] \subset \mathbb{R}$. Given any partition $\mathcal{P} = \{a = x_1 < x_2 < \ldots < x_N = b\}$, by Jensen's inequality we have

$$\begin{aligned} \sum_{n=1}^{N-1} \frac{|g(x_{n+1}) - g(x_n)|^q}{|x_{n+1} - x_n|^{q-1}} &\leq \sum_{n=1}^{N-1} \frac{1}{|x_{n+1} - x_n|^{q-1}} \left(\int_{x_n}^{x_{n+1}} |g'(x)| \, \mathrm{d}x \right)^q \\ &\leq \sum_{n=1}^{N-1} \int_{x_n}^{x_{n+1}} |g'(x)|^q \, \mathrm{d}x = \int_a^b |g'(x)|^q \, \mathrm{d}x, \end{aligned}$$

and therefore

$$\mathrm{Var}_q(g)_{[a,b]} \leq \|g'\|_{L^q([a,b])}.$$

Here the notation $\mathrm{Var}_q(g)_{[a,b]}$ refers to the q-variation of f over the interval $[a,b]$, in which we consider only partitions $\mathcal{P} = \{a = x_1 < x_2 < \ldots < x_N = b\}$ in (2.1.7). Taking the supremum over $[a,b]$ on both sides we obtain that

$$\mathrm{Var}_q(g) \leq \|g'\|_{L^q(\mathbb{R})} < \infty\,. \tag{2.5.18}$$

On the other hand, we assume that $\mathrm{Var}_q(g) < \infty$. For an arbitrary finite system of pairwise disjoint open intervals (x_k, y_k) $(k = 1, 2, \ldots, N)$ contained in $[a,b]$, by Hölder's inequality we have that

$$\begin{aligned}
\sum_{k=1}^{N} |g(x_k) - g(y_k)| &= \sum_{n=1}^{N} \frac{|g(x_k) - g(y_k)|}{|x_k - y_k|^{\frac{q-1}{q}}} |x_k - y_k|^{\frac{q-1}{q}} \\
&\leq \left(\sum_{n=1}^{N} \frac{|g(x_k) - g(y_k)|^q}{|x_k - y_k|^{q-1}} \right)^{\frac{1}{q}} \left(\sum_{n=1}^{N} |x_k - y_k| \right)^{\frac{q-1}{q}} \\
&\leq \mathrm{Var}_q(g) \left(\sum_{n=1}^{N} |x_k - y_k| \right)^{\frac{q-1}{q}}.
\end{aligned}$$

This implies that g is absolutely continuous. It remains to show that its derivative g' belongs to $L^q(\mathbb{R})$. For each $N \geq 2$ let

$$x_k = a + \frac{(k-1)(b-a)}{(N-1)} \quad ; \quad 1 \leq k \leq N.$$

Define $g_N : [a,b] \to \mathbb{R}$ by fixing

$$g_N(x_k) = g(x_k) \quad ; \quad 1 \leq k \leq N,$$

and making g_N linear in between these nodes, i.e.

$$g_N((1-t)\, x_k + t\, x_{k+1}) = (1-t)\, g_N(x_k) + t\, g_N(x_{k+1})$$

for each $t \in (0,1)$ and $1 \leq k \leq N-1$. Then

$$\left(\int_a^b |g_N'(x)|^q \,\mathrm{d}x\right)^{1/q} = \left(\sum_{n=1}^{N-1} \frac{|g_N(x_{n+1}) - g_N(x_n)|^q}{|x_{n+1} - x_n|^{q-1}}\right)^{1/q} \leq \mathrm{Var}_q(g)_{[a,b]}. \tag{2.5.19}$$

We claim that

$$g_n'(x) \to g'(x) \tag{2.5.20}$$

almost everywhere. In fact if x is not a point of subdivision and if $g'(x)$ exists and is finite, then x lies in some open interval (x_{k_N}, x_{k_N+1}) of length $\frac{b-a}{N}$ for each natural number N. Since $x_{k_N+1} - x_{k_N} = \frac{b-a}{N} \to 0$ as $N \to \infty$, it follows that each of the expressions

$$\frac{g(x_{k_N+1}) - g(x)}{x_{k_N+1} - x}, \quad \frac{g(x) - g(x_{k_N})}{x - x_{k_N}} \tag{2.5.21}$$

converges to $g'(x)$ as $N \to \infty$. However $g_N'(x) = \frac{g(x_{k_N+1}) - g(x_{k_N})}{x_{k_N+1} - x_{k_N}}$ lies between the two numbers in (2.5.21), this implies the validity of (2.5.20). Then using Fatou's lemma and (2.5.19) we obtain that

$$\|g'\|_{L^q([a,b])} \leq \liminf_{N\to\infty} \|g_N'\|_{L^q([a,b])} \leq \mathrm{Var}_q(g)_{[a,b]}.$$

Taking the supremum over $[a,b]$ on both sides and combining this inequality with (2.5.18) we conclude the proof of the proposition. □

Chapter 3

Sharp inequalities for the variation of the discrete maximal function

3.1 Preliminaries

In this chapter we continue to study regularity properties of maximal operators, but now we focus on finding optimal estimates for inequalities involving discrete maximal operators.

Given $f : \mathbb{Z}^d \to \mathbb{R}$, we define its total variation $\operatorname{Var} f$ by

$$\operatorname{Var} f = \sum_{i=1}^{d} \sum_{\vec{n} \in \mathbb{Z}^d} \big| f(\vec{n} + \vec{e_i}) - f(\vec{n}) \big|,$$

where $\vec{e_i} = (0, 0, \dots, 1, \dots, 0)$ is the canonical $i-$th base vector. Also, we say that a function $f : \mathbb{Z}^d \to \mathbb{R}$ is a *delta function* if there exist $\vec{p} \in \mathbb{Z}^d$ and $k \in \mathbb{R}$, such that

$$f(\vec{p}) = k \quad \text{and} \quad f(\vec{n}) = 0 \;\; \forall\, \vec{n} \in \mathbb{Z}^d \setminus \{\vec{p}\}.$$

3.1.1 A sharp inequality in dimension one

For $f : \mathbb{Z} \to \mathbb{R}$ we define its centered Hardy-Littlewood maximal function $Mf : \mathbb{Z} \to \mathbb{R}^+$ as

$$Mf(n) = \sup_{r \in \mathbb{Z}^+} \frac{1}{(2r+1)} \sum_{k=-r}^{r} |f(n+k)|,$$

while the uncentered maximal function $\widetilde{M}f : \mathbb{Z} \to \mathbb{R}^+$ is given by

$$\widetilde{M}f(n) = \sup_{r,s\in\mathbb{Z}^+} \frac{1}{(r+s+1)} \sum_{k=-r}^{s} |f(n+k)|.$$

In [8], Bober, Carneiro, Hughes and Pierce proved the following inequalities

$$\mathrm{Var}\, \widetilde{M}f \leq \mathrm{Var}\, f \leq 2\|f\|_{\ell^1(\mathbb{Z})} \tag{3.1.1}$$

and

$$\mathrm{Var}\, Mf \leq \left(2 + \frac{146}{315}\right) \|f\|_{\ell^1(\mathbb{Z})}. \tag{3.1.2}$$

The leftmost inequality in (3.1.1) is the discrete analogue of (2.1.3). The rightmost inequality in (3.1.1) is simply the triangle inequality. Both inequalities in (3.1.1) are in fact sharp (e.g. equality is attained if f is a delta function). On the other hand, inequality (3.1.2) is not optimal, and it was asked in [8] whether the sharp constant for (3.1.2) is in fact $C = 2$. Our first result answers this question affirmatively, also characterizing the extremal functions.

Theorem 12. *Let $f : \mathbb{Z} \to \mathbb{R}$ be a function in $\ell^1(\mathbb{Z})$. Then*

$$\mathrm{Var}\, Mf \leq 2\, \|f\|_{\ell^1(\mathbb{Z})}, \tag{3.1.3}$$

and the constant $C = 2$ is the best possible. Moreover, the equality is attained if and only if f is a delta function.

Remark: In [75], Temur proved the analogue of (2.1.4) in the discrete setting, i.e.

$$\mathrm{Var}\, Mf \leq C\ \mathrm{Var}\, f \tag{3.1.4}$$

with constant $C = (72000)2^{12} + 4$. This inequality is qualitatively stronger that (3.1.3) (in fact, $\mathrm{Var}\, f$ should be seen as the natural object to be on the right-hand side), but it does not imply (3.1.3). By triangle inequality, inequality (3.1.3) suggests that it may be possible to prove (3.1.4) with constant $C = 1$, but this is currently an open problem.

3.1.2 Sharp inequalities in higher dimensions

We now aim to extend Theorem 12 to higher dimensions. In order to do so, we first recall the notion of maximal operators associated to regular convex sets as considered in [16].

Let $\Omega \subset \mathbb{R}^d$ be a bounded open convex set with Lipschitz boundary, such that $\vec{0} \in \text{int}(\Omega)$ and that $\pm\vec{e}_i \in \overline{\Omega}$ for $1 \leq i \leq d$. Given $f : \mathbb{Z}^d \to \mathbb{R}$, in the Subsection 2.1.2 we defined $M_{\Omega,\beta}$ to be the discrete centered fractional maximal operator associated to Ω. Taking $\beta = 0$ we obtain the discrete centered maximal operator associated to Ω

$$M_\Omega f(\vec{n}) = \sup_{r \geq 0} A_r f(\vec{n}) = \sup_{r \geq 0} \frac{1}{N(r)} \sum_{\vec{m} \in \overline{\Omega}_r} |f(\vec{n} + \vec{m})|. \tag{3.1.5}$$

Here $A_r f(\vec{n})$ is given by taking $\beta = 0$ on the right hand side of (2.2.2). Analogously, we have its uncentered version

$$\widetilde{M}_\Omega f(\vec{n}) = \sup_{\overline{\Omega}_r(\vec{x}_0) \ni \vec{n}} A_r f(\vec{x}_0) = \sup_{\overline{\Omega}_r(\vec{x}_0) \ni \vec{n}} \frac{1}{N(\vec{x}_0, r)} \sum_{\vec{m} \in \overline{\Omega}_r(\vec{x}_0)} |f(\vec{m})|, \tag{3.1.6}$$

where $N(\vec{x}, r)$ is the number of the lattice points in the set $\overline{\Omega}_r(\vec{x})$ (and $N(r) := N(\vec{0}, r)$).

For all $1 \leq p \leq \infty$ we denote by $\Omega_{\ell^p} = \{\vec{x} \in \mathbb{R}^d; \|\vec{x}\|_p < 1\}$, where $\|\vec{x}\|_p = (|x_1|^p + |x_2|^p + \ldots + |x_d|^p)^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|\vec{x}\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_d|\}$ for all $\vec{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$. Given $1 \leq p < \infty$ and $f \in \ell^1_{loc}(\mathbb{Z}^d)$, we denote by M_p the discrete centered maximal operator associated to Ω_p, i.e

$$M_p f(\vec{n}) = M_{\Omega_p} f(\vec{n}),$$

and for $p = \infty$, we denote

$$M f(\vec{n}) = M_{\Omega_\infty} f(\vec{n}).$$

Analogously, we denote by $\widetilde{M}_p f$ and $\widetilde{M} f$ the uncentered versions of the discrete maximal operators associated to Ω_p, for $1 \leq p \leq \infty$. Note that in dimension $d = 1$ we have $M_p = M$ and $\widetilde{M}_p = \widetilde{M}$ for all $1 \leq p \leq \infty$.

In [16], Carneiro and Hughes showed that, for any regular set Ω as above and $f : \mathbb{Z}^d \to \mathbb{R}$, there exist constants $C(\Omega, d)$ and $\widetilde{C}(\Omega, d)$ such that

$$\text{Var}\, M_\Omega f \leq C(\Omega, d) \|f\|_{\ell^1(\mathbb{Z}^d)} \tag{3.1.7}$$

and

$$\text{Var}\, \widetilde{M}_\Omega f \leq \widetilde{C}(\Omega, d) \|f\|_{\ell^1(\mathbb{Z}^d)}. \tag{3.1.8}$$

Inequalities (3.1.7) and (3.1.8) were extended to a fractional setting in [21, Theorem 3]. Here we extend Theorem 12 to higher dimensions in two distinct ways. We find the sharp form of (3.1.7), when $d \geq 1$ and $\Omega = \Omega_{\ell^1}$ (i.e. rombus), and the sharp form of (3.1.8), when $d \geq 1$

and $\Omega = \Omega_{\ell^\infty}$ (i.e. regular cubes). As we shall see below, we use different techniques in the proofs of these two extensions, taking into consideration the geometry of the chosen sets Ω.

For $d \geq 1$ and $k \geq 0$ we denote $N_{1,d}(k) = \left|\overline{(\Omega_1)_k}\right| = \left|\{\vec{x} \in \mathbb{Z}^d; \|\vec{x}\|_1 \leq k\}\right|$. Here is our next result.

Theorem 13. *Let $d \geq 2$ and $f : \mathbb{Z}^d \to \mathbb{R}$ be a function in $\ell^1(\mathbb{Z}^d)$. Then*

$$\operatorname{Var} M_1 f \leq 2d\left(1 + \sum_{k\geq 1} \frac{(N_{1,d-1}(k) - N_{1,d-1}(k-1))}{N_{1,d}(k)}\right)\|f\|_{\ell^1(\mathbb{Z}^d)} =: C(d)\|f\|_{\ell^1(\mathbb{Z}^d)}, \quad (3.1.9)$$

and this constant $C(d)$ is the best possible. Moreover, the equality is attained if and only if f is a delta function.

Remark: Note that $C(d) < \infty$, because there exists a constant C such that

$$N_{1,d}(k) = Ck^d + O(k^{d-1}),$$

where $C = m(\Omega_{\ell^1})$ (see [50, Chapter VI §2, Theorem 2]). Then, for sufficiently large k we have

$$\frac{N_{1,d-1}(k) - N_{1,d-1}(k-1)}{N_{1,d}(k)} \sim \frac{1}{k^2}.$$

In particular, for $d = 2$ we obtain

$$C(2) = 4 + 8\sum_{k\geq 1} \frac{1}{k^2 + (k+1)^2}.$$

Our proof of Theorem 13 is the natural extension of the proof of Theorem 12 but we decided to present Theorem 12 separately since it contains the essential idea with less technical details. The next result is the sharp version of (3.1.8) for the discrete uncentered maximal operator with respect to cubes (i.e. ℓ^∞-balls). This proof follows a different strategy from Theorems 12 and 13.

Theorem 14. *Let $d \geq 1$ and $f : \mathbb{Z}^d \to \mathbb{R}$ be a function in $\ell^1(\mathbb{Z}^d)$. Then*

$$\begin{aligned}\operatorname{Var} \widetilde{M} f \quad &\leq 2d\left(1 + \sum_{k\geq 1} \frac{1}{k}\left(\left(\frac{2}{k+1} + \frac{2k-1}{k}\right)^{d-1} - \left(\frac{2k-1}{k}\right)^{d-1}\right)\right)\|f\|_{\ell^1(\mathbb{Z}^d)} \\ &=: \widetilde{C}(d)\|f\|_{\ell^1(\mathbb{Z}^d)}, \qquad (3.1.10)\end{aligned}$$

and the constant $\widetilde{C}(d)$ is the best possible. Moreover, the equality is attained if and only if f is a delta function.

Remark: In particular $\widetilde{C}(1) = 2$ (and we recover (3.1.1)) and $\widetilde{C}(2) = 12$.

Remark: Using the idea of the proof of Theorem 13 we obtain an explicit bound as in Theorem 14 for the centered version of the discrete maximal operator with respect to cubes (i.e. ℓ^∞-balls). However, the corresponding inequality is not necessarily sharp in this case. This is presented in the last section of the chapter.

For the proofs of these three theorems we may assume throughout the rest of the chapter, without loss of generality, that $f \geq 0$.

3.2 Proof of Theorem 12

Since $f \in \ell^1(\mathbb{Z})$, we have that for all $n \in \mathbb{Z}$ there exists $r_n \in \mathbb{Z}$ such that $Mf(n) = A_{r_n} f(n)$. We define

$$X^- = \{n \in \mathbb{Z}; Mf(n) \geq Mf(n+1)\} \quad \text{and} \quad X^+ = \{n \in \mathbb{Z}; Mf(n+1) > Mf(n)\}.$$

Then we have

$$\begin{aligned} \operatorname{Var} Mf &= \sum_{n\in\mathbb{Z}} |Mf(n) - Mf(n+1)| \\ &= \sum_{n\in X^-} Mf(n) - Mf(n+1) + \sum_{n\in X^+} Mf(n+1) - Mf(n) \\ &\leq \sum_{n\in X^-} A_{r_n} f(n) - A_{r_n+1} f(n+1) \\ &\quad + \sum_{n\in X^+} A_{r_{n+1}} f(n+1) - A_{r_{n+1}+1} f(n). \end{aligned} \tag{3.2.1}$$

Given $p \in \mathbb{Z}$ fixed, we want to evaluate the maximal contribution of $f(p)$ to the right-hand side of (3.2.1).

Case 1: If $n \in X^-$ and $n \geq p$. In this situation we have that the contribution of $f(p)$ to $A_{r_n} f(n) - A_{r_n+1} f(n+1)$ is 0 (if $p < n - r_n$) or $\frac{1}{2r_n+1} - \frac{1}{2r_n+3}$ (if $n - r_n \leq p$). In the second case we have

$$\frac{1}{2r_n+1} - \frac{1}{2r_n+3} = \frac{2}{(2r_n+1)(2r_n+3)}$$

$$
\begin{aligned}
&\leq \frac{2}{(2(n-p)+1)(2(n-p)+3)}\\
= &= \frac{1}{2(n-p)+1} - \frac{1}{2(n-p)+3}.
\end{aligned}
$$

The equality is attained if and only if $r_n = n - p$.

Case 2: If $n \in X^+$ and $n \geq p$. Now we have that the contribution of $f(p)$ to $A_{r_{n+1}} f(n+1) - A_{r_{n+1}+1} f(n)$ is non-positive (if $p < n + 1 - r_{n+1}$) or $\frac{1}{2r_{n+1}+1} - \frac{1}{2r_{n+1}+3}$ (if $n + 1 - r_{n+1} \leq p$). In the second case we have

$$
\begin{aligned}
\frac{1}{2r_{n+1}+1} - \frac{1}{2r_{n+1}+3} &= \frac{2}{(2r_{n+1}+1)(2r_{n+1}+3)}\\
&\leq \frac{2}{(2(n+1-p)+1)(2(n+1-p)+3)}\\
&= \frac{1}{2(n+1-p)+1} - \frac{1}{2(n+1-p)+3}\\
&< \frac{1}{2(n-p)+1} - \frac{1}{2(n-p)+3}.
\end{aligned}
$$

Case 3: If $n \in X^-$ and $n < p$. In this situation we have that the contribution of $f(p)$ to $A_{r_n} f(n) - A_{r_n+1} f(n+1)$ is non-positive (if $p > n + r_n$) or $\frac{1}{2r_n+1} - \frac{1}{2r_n+3}$ (if $n + r_n \geq p$). In the second case we have

$$
\begin{aligned}
\frac{1}{2r_n+1} - \frac{1}{2r_n+3} &= \frac{2}{(2r_n+1)(2r_n+3)}\\
&\leq \frac{2}{(2(p-n)+1)(2(p-n)+3)}\\
&= \frac{1}{2(p-n)+1} - \frac{1}{2(p-n)+3}\\
&< \frac{1}{2(p-n-1)+1} - \frac{1}{2(p-n-1)+3}.
\end{aligned}
$$

Case 4: If $n \in X^+$ and $n < p$. Now we have that the contribution of $f(p)$ to $A_{r_{n+1}} f(n+1) - A_{r_{n+1}+1} f(n)$ is either 0 (if $p > n + 1 + r_{n+1}$) or $\frac{1}{2r_{n+1}+1} - \frac{1}{2r_{n+1}+3}$ (if $n + 1 + r_{n+1} \geq p$). In the second case we have

$$
\begin{aligned}
\frac{1}{2r_{n+1}+1} - \frac{1}{2r_{n+1}+3} &= \frac{2}{(2r_{n+1}+1)(2r_{n+1}+3)}\\
&\leq \frac{2}{(2(p-n-1)+1)(2(p-n-1)+3)}
\end{aligned}
$$

$$= \frac{1}{2(p-n-1)+1} - \frac{1}{2(p-n-1)+3}.$$

The equality is achieved if and only if $r_{n+1} = p - n - 1$.

Conclusion: Therefore the contribution of $f(p)$ to the right-hand side of (3.2.1) is bounded by

$$\sum_{n \geq p} \frac{1}{2(n-p)+1} - \frac{1}{2(n-p)+3} + \sum_{n<p} \frac{1}{2(p-n-1)+1} - \frac{1}{2(p-n-1)+3} = 2.$$

As p is an arbitrary point in $\mathbb{Z}$, this establishes (3.1.3). If f is a delta function we can easily see that

$$\mathrm{Var}\, Mf = 2\|f\|_{\ell^1(\mathbb{Z})}.$$

On the other hand, given a function $f : \mathbb{Z} \to \mathbb{R}$ such that $\mathrm{Var}\, Mf = 2\|f\|_{\ell^1(\mathbb{Z})}$ and $f \geq 0$, let us define $P = \{t \in \mathbb{Z}; f(t) \neq 0\}$. Then

$$\mathrm{Var}\, Mf = 2 \sum_{t \in P} f(t),$$

and, given $t_1 \in P$, the contribution of $f(t_1)$ to (3.2.1) is 2. Therefore, by the previous analysis we note that for all $n \geq t_1$ we must have that $n \in X^-$ and $r_n = n - t_1$. If we take $t_2 \in P$ the same should happen, which implies that $t_1 = t_2$ and therefore $P = \{t_1\}$. This proves that f is a delta function and the proof is concluded.

3.3 Proof of Theorem 13

3.3.1 Preliminaries

Since $f \in \ell^1(\mathbb{Z}^d)$, we have that for each $\vec{n} \in \mathbb{Z}^d$ there exists $r_{\vec{n}} \in \mathbb{Z}$ such that $M_1 f(\vec{n}) = A_{r_{\vec{n}}} f(\vec{n})$. For all $\vec{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$ we define

$$|\vec{m}|_1 = \sum_{i=1}^{d} |m_i|,$$

and for $1 \leq j \leq d$, we define

$$I_j = \{l \subset \mathbb{Z}^d;\ l \text{ is a line parallel to the vector } \vec{e}_j\}$$

and as in the subsection 2.2.1, we define

$$X_j^- = \{\vec{n} \in \mathbb{Z}^d; M_1 f(\vec{n}) \geq M_1 f(\vec{n}+\vec{e}_j)\} \ \text{ and } \ X_j^+ = \{\vec{n} \in \mathbb{Z}^d; M_1 f(\vec{n}+\vec{e}_j) > M_1 f(\vec{n})\}.$$

We then have

$$\begin{aligned}
\operatorname{Var} M_1 f &= \sum_{\vec{n}\in\mathbb{Z}^d}\sum_{j=1}^{d} |M_1 f(\vec{n}) - M_1 f(\vec{n}+\vec{e}_j)| \\
&= \sum_{j=1}^{d}\sum_{l\in I_j}\sum_{\vec{n}\in l\cap X_j^-} M_1 f(\vec{n}) - M_1 f(\vec{n}+\vec{e}_j) + \sum_{j=1}^{d}\sum_{l\in I_j}\sum_{\vec{n}\in l\cap X_j^+} M_1 f(\vec{n}+\vec{e}_j) - M_1 f(\vec{n}) \\
&\leq \sum_{j=1}^{d}\sum_{l\in I_j}\sum_{\vec{n}\in l\cap X_j^-} A_{r_{\vec{n}}} f(\vec{n}) - A_{r_{\vec{n}}+1} f(\vec{n}+\vec{e}_j) \qquad (3.3.1) \\
&\qquad + \sum_{j=1}^{d}\sum_{l\in I_j}\sum_{\vec{n}\in l\cap X_j^+} A_{r_{\vec{n}+\vec{e}_j}} f(\vec{n}+\vec{e}_j) - A_{r_{\vec{n}+\vec{e}_j}+1} f(\vec{n}).
\end{aligned}$$

Fixed a point $\vec{p} = (p_1, p_2, \ldots, p_d) \in \mathbb{Z}^d$, we want to evaluate the maximal contribution of $f(\vec{p})$ to the right-hand side of (3.3.1).

3.3.2 Auxiliary results

We now prove the following lemma of arithmetic character, which will be particularly useful in the rest of the proof.

Lemma 15. *If $d \geq 1$, then*

$$N_{1,d}(k)^2 > N_{1,d}(k+1)N_{1,d}(k-1) \quad \forall\ k \geq 1. \qquad (3.3.2)$$

Proof. We prove this via induction. For $d = 1$ we have that $N_{1,1}(k) = 2k+1$, therefore

$$N_{1,1}(k)^2 = 4k^2 + 4k + 1 > (2k+3)(2k-1) = N_{1,1}(k+1)N_{1,1}(k-1).$$

Since $N_{1,d}(k) = \left|\{(x_1, \ldots, x_d) \in \mathbb{Z}^d; |x_1| + \cdots + |x_d| \leq k\}\right|$, fixing the value of the last variable, we can verify that

$$N_{1,d}(k) = N_{1,d-1}(k) + 2\sum_{j=0}^{k-1} N_{1,d-1}(j). \qquad (3.3.3)$$

Now, let us assume that the result is true for d, i.e.

$$N_{1,d}(k)^2 > N_{1,d}(k+1)N_{1,d}(k-1) \quad \forall\ k \geq 1. \tag{3.3.4}$$

We want to prove that this implies that the result is also true for $d+1$. For simplicity we denote $g(k) := N_{1,d}(k)$ and $f(k) := N_{1,d+1}(k)$ for all $k \geq 0$. Thus by (3.3.4) we have that

$$\frac{g(1)}{g(0)} > \frac{g(2)}{g(1)} > \cdots > \frac{g(k)}{g(k-1)} > \frac{g(k+1)}{g(k)} > \ldots \tag{3.3.5}$$

and by (3.3.3) we have that

$$f(k) = g(k) + 2\sum_{j=0}^{k-1} g(j) \qquad \forall\ k \geq 0.$$

The latter implies that

$$f(k+1) - f(k) = g(k+1) + g(k) \qquad \forall\ k \geq 0.$$

Therefore, by (3.3.5), we obtain that

$$\frac{g(k+1)}{g(k)} > \frac{g(k+2)+g(k+1)}{g(k+1)+g(k)}$$

and

$$\frac{g(k+1) + 2\sum_{j=1}^{k} g(j)}{g(k) + 2\sum_{j=1}^{k} g(j-1)} > \frac{g(k+1)}{g(k)}.$$

Combining these inequalities we arrive at

$$\begin{aligned} \frac{f(k+1)}{f(k)} &\geq \frac{g(k+1) + 2\sum_{j=1}^{k} g(j)}{g(k) + 2\sum_{j=1}^{k} g(j-1)} > \frac{g(k+1)}{g(k)} \\ &> \frac{g(k+2)+g(k+1)}{g(k+1)+g(k)} = \frac{f(k+2)-f(k+1)}{f(k+1)-f(k)}, \end{aligned}$$

and hence

$$\frac{f(k+1)-f(k)}{f(k)} > \frac{f(k+2)-f(k+1)}{f(k+1)}.$$

This implies that

$$\frac{f(k+1)}{f(k)} > \frac{f(k+2)}{f(k+1)} \ \ \forall\, k \geq 0,$$

which establishes the desired result. □

Corollary 16. *If $d \geq 1$, we have that*

$$\frac{1}{N_{1,d}(k)} - \frac{1}{N_{1,d}(k+1)} > \frac{1}{N_{1,d}(k+1)} - \frac{1}{N_{1,d}(k+2)} \ \ \forall\, k \geq 0. \tag{3.3.6}$$

Proof. We notice that (3.3.6) is equivalent to

$$\frac{N_{1,d}(k+1)}{N_{1,d}(k)} + \frac{N_{1,d}(k+1)}{N_{1,d}(k+2)} > 2.$$

This follows from Lemma 15 and the arithmetic mean - geometric mean inequality because

$$\frac{N_{1,d}(k+1)}{N_{1,d}(k)} + \frac{N_{1,d}(k+1)}{N_{1,d}(k+2)} > \frac{N_{1,d}(k+2)}{N_{1,d}(k+1)} + \frac{N_{1,d}(k+1)}{N_{1,d}(k+2)} \geq 2.$$

□

3.3.3 Proof of Theorem 13

Let us simplify notation by writing $N_1(k) := N_{1,d}(k)$. Given $1 \leq j \leq d$, using Corollary 16 we make the following observations.

Case 1: If $\vec{n} \in X_j^-$ and $n_j \geq p_j$. In this situation we have that the contribution of $f(\vec{p})$ to $A_{r_{\vec{n}}} f(\vec{n}) - A_{r_{\vec{n}}+1} f(\vec{n}+\vec{e}_j)$ is non-positive (if $|\vec{n}-\vec{p}|_1 > r_{\vec{n}}$) or $\frac{1}{N_1(r_{\vec{n}})} - \frac{1}{N_1(r_{\vec{n}}+1)}$ (if $|\vec{n}-\vec{p}|_1 \leq r_{\vec{n}}$). In the second case we have

$$\begin{aligned} \frac{1}{N_1(r_{\vec{n}})} - \frac{1}{N_1(r_{\vec{n}}+1)} &\leq \frac{1}{N_1(|\vec{n}-\vec{p}|_1)} - \frac{1}{N_1(|\vec{n}-\vec{p}|_1+1)} \\ &= \frac{1}{N_1(|\vec{n}-\vec{p}|_1)} - \frac{1}{N(|\vec{n}+\vec{e}_j-\vec{p}|_1)}. \end{aligned}$$

The equality is attained if and only if $r_{\vec{n}} = |\vec{n}-\vec{p}|_1$.

Case 2: If $\vec{n} \in X_j^+$ and $n_j \geq p_j$. Now we have that the contribution of $f(\vec{p})$ to $A_{r_{\vec{n}+\vec{e}_j}} f(\vec{n} + \vec{e}_j) - A_{r_{\vec{n}+\vec{e}_j}+1} f(\vec{n})$ is non-positive (if $|\vec{n}+\vec{e}_j-\vec{p}|_1 > r_{\vec{n}+\vec{e}_j}$) or $\frac{1}{N_1(r_{\vec{n}+\vec{e}_j})} - \frac{1}{N_1(r_{\vec{n}+\vec{e}_j}+1)}$ (if

$|\vec{n}+\vec{e}_j-\vec{p}|_1 \le r_{\vec{n}+\vec{e}_j})$. In the second case we have

$$\begin{aligned}\frac{1}{N_1(r_{\vec{n}+\vec{e}_j})}-\frac{1}{N_1(r_{\vec{n}+\vec{e}_j}+1)} &\le \frac{1}{N_1(|\vec{n}+\vec{e}_j-\vec{p}|_1)}-\frac{1}{N_1(|\vec{n}+\vec{e}_j-\vec{p}|_1+1)}\\ &= \frac{1}{N_1(|\vec{n}-\vec{p}|_1+1)}-\frac{1}{N_1(|\vec{n}-\vec{p}|_1+2)}\\ &< \frac{1}{N_1(|\vec{n}-\vec{p}|_1)}-\frac{1}{N_1(|\vec{n}-\vec{p}|_1+1)}\\ &= \frac{1}{N_1(|\vec{n}-\vec{p}|_1)}-\frac{1}{N(|\vec{n}+\vec{e}_j-\vec{p}|_1)}.\end{aligned}$$

Case 3: If $\vec{n}\in X_j^-$ and $n_j<p_j$. In this situation we have that the contribution of $f(\vec{p})$ to $A_{r_{\vec{n}}}f(\vec{n})-A_{r_{\vec{n}}+1}f(\vec{n}+\vec{e}_j)$ is non-positive (if $|\vec{n}-\vec{p}|_1>r_{\vec{n}}$) or $\frac{1}{N_1(r_{\vec{n}})}-\frac{1}{N_1(r_{\vec{n}}+1)}$ (if $|\vec{n}-\vec{p}|_1\le r_{\vec{n}}$). In the second case we have

$$\begin{aligned}\frac{1}{N_1(r_{\vec{n}})}-\frac{1}{N_1(r_{\vec{n}}+1)} &\le \frac{1}{N_1(|\vec{p}-\vec{n}|_1)}-\frac{1}{N_1(|\vec{p}-\vec{n}|_1+1)}\\ &< \frac{1}{N_1(|\vec{p}-\vec{n}-\vec{e}_j|_1)}-\frac{1}{N_1(|\vec{p}-\vec{n}|_1)}.\end{aligned}$$

Case 4: If $\vec{n}\in X_j^+$ and $n_j<p_j$. Now we have that the contribution of $f(\vec{p})$ to $A_{r_{\vec{n}+\vec{e}_j}}f(\vec{n}+\vec{e}_j)-A_{r_{\vec{n}+\vec{e}_j}+1}f(\vec{n})$ is non-positive (if $|\vec{p}-\vec{n}-\vec{e}_j|_1>r_{\vec{n}+\vec{e}_j}$) or $\frac{1}{N_1(r_{\vec{n}+\vec{e}_j})}-\frac{1}{N_1(r_{\vec{n}+\vec{e}_j}+1)}$ (if $|\vec{p}-\vec{n}-\vec{e}_j|_1\le r_{\vec{n}+\vec{e}_j})$. In the second case we have

$$\begin{aligned}\frac{1}{N_1(r_{\vec{n}+\vec{e}_j})}-\frac{1}{N_1(r_{\vec{n}+\vec{e}_j}+1)} &\le \frac{1}{N_1(|\vec{p}-\vec{n}-\vec{e}_j|_1)}-\frac{1}{N_1(\vec{p}-\vec{n}-\vec{e}_j|_1+1)}\\ &= \frac{1}{N_1(|\vec{p}-\vec{n}-\vec{e}_j|_1)}-\frac{1}{N_1(|\vec{p}-\vec{n}|_1)}.\end{aligned}$$

The equality is achieved if and only if $r_{\vec{n}+\vec{e}_j}=|\vec{p}-\vec{n}-\vec{e}_j|_1$.

Conclusion: Given a line l in the lattice, we define the distance from $\vec{p}$ to l by

$$d(l,\vec{p})=\min\{|\vec{m}-\vec{p}|_1;\ \vec{m}\in l\}.$$

If the direction of l is the same as the direction of $\vec{e}_j$, by intersecting l with the hyperplane $H_j=\{\vec{z}\in\mathbb{Z}^d; z_j=p_j\}$ we obtain the point that realizes the distance from p to l. By the previous analysis we have that the contribution of $f(\vec{p})$ to

$$\sum_{\vec{n}\in l\cap X_j^-} A_{r_{\vec{n}}} f(\vec{n}) - A_{r_{\vec{n}}+1} f(\vec{n}+\vec{e}_j) + \sum_{\vec{n}\in l\cap X_j^+} A_{r_{\vec{n}+\vec{e}_j}} f(\vec{n}+\vec{e}_j) - A_{r_{\vec{n}+\vec{e}_j}+1} f(\vec{n})$$

is less than or equal to

$$\frac{2}{N_{1,d}(d(l,\vec{p}))}. \tag{3.3.7}$$

As p belongs to d lines of the lattice, given $k \in \mathbb{N}$ there exist $d(N_{1,d-1}(k) - N_{1,d-1}(k-1))$ lines such that $d(l,\vec{p}) = k$. Thus the contribution of $f(\vec{p})$ to the right-hand side of (3.3.1) is less than or equal to

$$\left(2d + \sum_{k\geq 1} \frac{2d(N_{1,d-1}(k) - N_{1,d-1}(k-1))}{N_{1,d}(k)}\right),$$

and as a consequence of this we obtain the desired inequality.

If f is a delta function, then there exist $\vec{y} \in \mathbb{Z}^d$ and $k \in \mathbb{R}$ such that

$$f(\vec{y}) = k \quad \text{and} \quad f(\vec{x}) = 0 \;\; \forall\, \vec{x} \in \mathbb{Z}^d \setminus \{y\}.$$

Considering the contribution of $|f(\vec{y})|$ to a line l in the lattice $\mathbb{Z}^d$ we have equality in (3.3.7), and hence in (3.1.9). On the other hand, let us assume that $f : \mathbb{Z}^d \to \mathbb{R}$ is a nonnegative function that verifies the equality in (3.1.9). We define $P = \{\vec{t} \in \mathbb{Z}^d; f(\vec{t}) \neq 0\}$ and then

$$\operatorname{Var} M_1 f = \left(2d + \sum_{k\geq 1} \frac{2d(N_{1,d-1}(k) - N_{1,d-1}(k-1))}{N_{1,d}(k)}\right) \sum_{\vec{t}\in P} f(\vec{t}).$$

Therefore, given $\vec{s} = (s_1, s_2, \ldots, s_d) \in P$ and a line l in the lattice, the contribution of $f(\vec{s})$ to l in (3.3.7) must be $\frac{2}{N_{1,d}(d(l,\vec{s}))}$ by the previous analysis. Then, if there exists $\vec{u} \in P \setminus \{\vec{s}\}$, the contribution of $f(\vec{u})$ to l in (3.3.1) must also be $\frac{2}{N_{1,d}(d(l,\vec{u}))}$. Assume without loss of generality that $s_d > u_d$ and consider the line $l = \{(s_1, s_2, \ldots, s_{n-1}, x); x \in \mathbb{Z}\}$. As we have equality in (3.1.9), given $\vec{n} \in l$ such that $n_d \geq s_d$, we need to have that $\vec{n} \in X_j^-$ and $|\vec{n} - \vec{s}|_1 = r_{\vec{n}} = |\vec{n} - \vec{u}|_1$, which gives us a contradiction. Thus f must be a delta function.

3.4 Proof of Theorem 14

3.4.1 Preliminaries

As before we start noticing that, since $f \in \ell^1(\mathbb{Z}^d)$, for each $\vec{n} \in \mathbb{Z}^d$ there exist $r_{\vec{n}} \in \mathbb{R}^+$ and $c_{\vec{n}} \in \mathbb{R}^d$ such that $\vec{n} \in c_{\vec{n}} + Q(r_{\vec{n}})$ and $\widetilde{M}f(\vec{n}) = A_{r_{\vec{n}}}f(c_{\vec{n}})$, where $Q_{r_{\vec{n}}} = \{m \in \mathbb{Z}^d; |m|_\infty \leq r_{\vec{n}}\} = \{m \in \mathbb{Z}^d, \max\{|m_1|, \ldots, |m_d|\} \leq r_{\vec{n}}\}$. We now introduce the local maxima and minima of a discrete function $g : \mathbb{Z} \to \mathbb{R}$.[1] We say that an interval $[n, m]$ is a *string of local maxima* of g if

$$g(n-1) < g(n) = \ldots = g(m) > g(m+1).$$

If $n = -\infty$ or $m = \infty$ (but not both simultaneously) we modify the definition accordingly, eliminating one of the inequalities. The rightmost point m of such a string is a *right local maximum* of g, while the leftmost point n is a *left local maximum* of g. We define *string of local minima*, *right local minimum* and *left local minimum* analogously.

Given a line l in the lattice $\mathbb{Z}^d$ parallel to $\vec{e}_d$ there exists $n' \in \mathbb{Z}^{d-1}$ such that $l = \{(n', m); m \in \mathbb{Z}\}$. Let us assume that $\widetilde{M}f(n', x)$ is not constant as function of x (otherwise the variation of the maximal function over this line will be zero). Let $\{[a_j^-, a_j^+]\}_{j\in\mathbb{Z}}$ and $\{[b_j^-, b_j^+]\}_{j\in\mathbb{Z}}$ be the ordered strings of local maxima and local minima of $\widetilde{M}f(n', x)$ (we allow the possibilities of a_j^- or $b_j^- = -\infty$ and a_j^+ or $b_j^+ = \infty$), i.e.

$$\ldots < a_{-1}^- \leq a_{-1}^+ < b_{-1}^- \leq b_{-1}^+ < a_0^- \leq a_0^+ < b_0^- \leq b_0^+ < a_1^- \leq a_1^+ < b_1^- \leq b_1^+ < \ldots \qquad (3.4.1)$$

This sequence may terminate in one or both sides and we adjust the notation and the proof below accordingly. Note that we have at least one string of local maxima since $\widetilde{M}f(\vec{n}) \to 0$ as $|\vec{n}|_\infty \to \infty$, therefore, if the sequence terminates in one or both sides, it must terminate in a string of local maxima. The variation of the maximal function in l is given by

$$2\sum_{j\in\mathbb{Z}} \widetilde{M}f(n', a_j^+) - \widetilde{M}f(n', b_j^-) \leq 2\sum_{j\in\mathbb{Z}} A_{r_{(n',a_j^+)}} f(c_{(n',a_j^+)}) - A_{r_{(n',a_j^+)}+|a_j^+ - b_j^-|} f(c_{(n',a_j^+)}). \quad (3.4.2)$$

We now prove an auxiliary lemma.

Lemma 17. *Given $\vec{q} \in \mathbb{Z}^d$ and a line l in the lattice $\mathbb{Z}^d$. There exists at most one string of local maxima of $\widetilde{M}f$ in l such that there exists $\vec{n}$ in the string whose contribution of $f(\vec{q})$ to $A_{r_{\vec{n}}}f(c_{\vec{n}})$ is positive.*

[1] The local extrema are defined slightly differently in [8, 16], but used with the meaning stated here.

Proof. Assume without loss of generality that $l = \{(m_1, m_2, \ldots, m_{d-1}, x);\ x \in \mathbb{Z}\}$ $= \{(m', x);\ x \in \mathbb{Z}\}$. Consider a string of local maxima of $\widetilde{M}f$ in l

$$\widetilde{M}f(m', a-1) < \widetilde{M}f(m', a) = \ldots = \widetilde{M}f(m', a+n) > \widetilde{M}f(m', a+n+1). \qquad (3.4.3)$$

Let

$$\widetilde{M}f(m', a+i) = A_{r_{(m',a+i)}} f(c_{(m',a+i)}) \ \ \forall\ 0 \le i \le n.$$

Given $\vec{q} = (q_1, q_2, \ldots, q_d) \in \mathbb{Z}^d$, a necessary condition for the contribution of $f(\vec{q})$ to $A_{r_{(m',a+i)}} f(c_{(m',a+i)})$ to be positive for some i is that $a - 1 < q_d < a + n + 1$ (otherwise this would violate one of the endpoint inequalities in (3.4.3)). The result follows from this observation. $\square$

3.4.2 Proof of Theorem 14

Given $\vec{p} \in \mathbb{Z}^d$ and a line l in the lattice $\mathbb{Z}^d$, we define $d(l, \vec{p}) = \min\{|\vec{p} - \vec{m}|_\infty;\ \vec{m} \in l\}$ and $d(l, \vec{p})_+ = \max\{1, d(l, \vec{p})\}$. As consequence of Lemma 17, given $\vec{p} = (p_1, p_2, \ldots, p_{d-1}, p_d) \in \mathbb{Z}^d$ and a line $l = \{(n_1, n_2, \ldots, n_{d-1}, x) \in \mathbb{Z}^d;\ x \in \mathbb{Z}\}$ such that $\big|\{i \in \{1, 2, \ldots, d-1\}; |n_i - p_i| = d(l, \vec{p})\}\big| = j$, the contribution of $f(\vec{p})$ to the right-hand side of (3.4.2) is less than or equal to

$$\frac{2}{(d(l, \vec{p}) + 1)^j (d(l, \vec{p}))_+^{d-j}}. \qquad (3.4.4)$$

In fact, if an ℓ^∞-cube contains $\vec{p}$ and a point in l then it must have side at least $d(l, \vec{p})$, and it must contain $(d(l, \vec{p}) + 1)$ lattice points in each direction $\vec{e_i}$ for i such that $|n_i - p_i| = d(l, \vec{p})$. In the other $d - j$ directions the cube contains at least $d(l, \vec{p})$ lattice points. This leads to (3.4.4).

If equality in (3.4.4) is attained for a point $\vec{p}$ and a line l, then there is a point $\vec{q} \in l$ that realizes the distance to $\vec{p}$, belongs to a string of local maxima of l, and such that $\vec{p} \in c_{\vec{q}} + Q(r_{\vec{q}})$. Moreover, this string of local maxima must be unique, otherwise $f(\vec{p})$ would also have a negative contribution coming from a string of minimum in (3.4.2). In particular this implies that $\widetilde{M}f(\vec{p}) \ge \widetilde{M}f(\vec{n})$ for all $\vec{n} \in l$. If we fix a point $\vec{p}$ and assume that equality in (3.4.4) is attained *for all lines* l in our lattice, then $\widetilde{M}f(\vec{p}) \ge \widetilde{M}f(\vec{n})$ for all $\vec{n} \in \mathbb{Z}^d$.

Therefore, as $\vec{p}$ belong to d lines of the lattice $\mathbb{Z}^d$, and given $k \in \mathbb{N}$ and $j \in \{1, 2, \ldots, d-1\}$ there exist $2^j \binom{d-1}{j} (2(k-1) + 1)^{d-1-j}$ lines $l = \{(n_1, n_2, \ldots, n_{d-1}, x);\ x \in \mathbb{Z}\}$ such that $d(l, \vec{p}) = k$ and $\big|\{i \in \{1, 2, d \ldots, d-1\}; |n_i - p_i| = k\}\big| = j$, the contribution of $f(\vec{p})$ to the variation of the maximal function in $\mathbb{Z}^d$ is less than or equal to

$$\begin{aligned}
&2d + d\sum_{k\geq 1}\sum_{j=1}^{d-1} 2^j \binom{d-1}{j}(2k-1)^{d-1-j}\frac{2}{(k+1)^j\, k^{d-j}} \\
&= 2d + \sum_{k\geq 1}\frac{2d}{k}\sum_{j=1}^{d-1}\binom{d-1}{j}\left(\frac{2}{k+1}\right)^j\left(\frac{2k-1}{k}\right)^{d-1-j} \\
&= 2d + \sum_{k\geq 1}\frac{2d}{k}\left(\left(\frac{2}{k+1}+\frac{2k-1}{k}\right)^{d-1} - \left(\frac{2k-1}{k}\right)^{d-1}\right).
\end{aligned}$$

This concludes the proof of (3.1.10).

If f is a delta function, with $f(\vec{n}) = 0$ for all $n \in \mathbb{Z}^d \setminus \{\vec{p}\}$ for some $p \in \mathbb{Z}^d$, it is easy to see that we have equality in (3.4.4) for the contribution of $|f(\vec{p})|$ to all lines l, which implies equality in (3.1.10). On the other hand, let us assume that $f : \mathbb{Z}^d \to \mathbb{R}$ is a nonnegative function that verifies the equality in (3.1.10). We define $P = \{\vec{t} \in \mathbb{Z}^d; f(\vec{t}) \neq 0\}$ and thus

$$\operatorname{Var} \widetilde{M}f = \left(2d + \sum_{k\geq 1}\frac{2d}{k}\left(\left(\frac{2}{k+1}+\frac{2k-1}{k}\right)^{d-1} - \left(\frac{2k-1}{k}\right)^{d-1}\right)\right)\sum_{t\in P} f(t).$$

Then, given $\vec{s} \in P$, if there exists $\vec{u} \in P \setminus \{\vec{s}\}$, we consider a line l in the lattice $\mathbb{Z}^d$ such that $\vec{s} \in l$ and $\vec{u} \notin l$. The contribution of $f(\vec{s})$ to l must be 2, $\widetilde{M}f(\vec{s}) = f(\vec{s})$ belongs to the unique string of local maxima of $\widetilde{M}f$ in l and the right-hand side of (3.4.2) must be $2f(\vec{s})$, by the previous analysis. Therefore the contribution of $f(\vec{u})$ to the line l is 0 and then $f(\vec{u})$ does not provide the maximum contribution as predicted in (3.4.4), hence (3.1.10) cannot attained. We conclude that f must be a delta function.

3.5 Another explicit bound

The same strategy of the proof of Theorem 13 allows us to prove that:

Proposition 18. *Let $d \geq 2$ and $f : \mathbb{Z}^d \to \mathbb{R}$ be a function in $\ell^1(\mathbb{Z}^d)$. Then*

$$\begin{aligned}
\operatorname{Var} Mf &\leq 2d\left(1 + \sum_{k\geq 1}((2k+1)^{d-1} - (2k-1)^{d-1})\left(\frac{1+k}{(2k+1)^d} - \frac{k}{(2k+3)^d}\right)\right)\|f\|_{l^1(\mathbb{Z}^d)} \\
&= : C(d)\|f\|_{l^1(\mathbb{Z}^d)}.
\end{aligned}$$

Remark: This result is not optimal, the main reason why this happens is that given a point $\vec{p}$ and a line l we can have several points that realizes the distance from $\vec{p}$ to l. The problem of finding the sharp constant for this inequality is an interesting problem.

Proof. Since $f \in \ell^1(\mathbb{Z}^d)$, given $\vec{n} \in \mathbb{Z}^d$ we have that there exists $r_{\vec{n}} \in \mathbb{Z}$ such that $Mf(\vec{n}) = A_{r_{\vec{n}}} f(\vec{n})$. For all $\vec{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$ we define

$$|\vec{m}|_\infty = \max\{|m_i|; 1 \leq i \leq d\},$$

and for $1 \leq i \leq d$, we define

$$Y_i = \{\vec{n} \in \mathbb{Z}^d; |\vec{n}|_\infty = |n_i|\},$$

$$L_i = \{l \subset \mathbb{Z}^d;\ l \ \text{ is a line parallel to the vector } \ \vec{e_i}\},$$

$$X_i^- = \{\vec{n} \in \mathbb{Z}^d; Mf(\vec{n}) \geq Mf(\vec{n}+\vec{e_i})\} \ \text{ and } \ X_i^+ = \{\vec{n} \in \mathbb{Z}^d; Mf(\vec{n}+\vec{e_i}) > Mf(\vec{n})\}.$$

We then have

$$\begin{aligned}
\operatorname{Var} Mf &= \sum_{\vec{n}\in\mathbb{Z}^d} \sum_{j=1}^{d} |Mf(\vec{n}) - Mf(\vec{n}+\vec{e_j})| \\
&= \sum_{j=1}^{d} \sum_{l\in L_j} \sum_{\vec{n}\in l\cap X_j^-} Mf(\vec{n}) - Mf(\vec{n}+\vec{e_j}) + \sum_{j=1}^{d} \sum_{l\in L_j} \sum_{\vec{n}\in l\cap X_j^+} Mf(\vec{n}+\vec{e_j}) - Mf(\vec{n}) \\
&\leq \sum_{j=1}^{d} \sum_{l\in L_j} \sum_{\vec{n}\in l\cap X_j^-} A_{r_{\vec{n}}} f(\vec{n}) - A_{r_{\vec{n}}+1} f(\vec{n}+\vec{e_j}) \qquad (3.5.1) \\
&\quad + \sum_{j=1}^{d} \sum_{l\in L_j} \sum_{\vec{n}\in l\cap X_j^+} A_{r_{\vec{n}+\vec{e_j}}} f(\vec{n}+\vec{e_j}) - A_{r_{\vec{n}+\vec{e_j}}+1} f(\vec{n}).
\end{aligned}$$

Given $\vec{p} \in \mathbb{Z}^d$ and a line l in the lattice $\mathbb{Z}^d$, we define $d(l,\vec{p}) = \min\{|\vec{p}-\vec{m}|_\infty; \vec{m} \in l\}$ as in §3.4.2. If the direction of l is the same that the direction of e_j, intersecting l with the hiperplane $H_j = \{\vec{z} \in \mathbb{Z}^d; z_j = p_j\}$ we obtain a point $\vec{p_l}$ that realizes the distance from $\vec{p}$ to l, we consider

$$l_1 = \{\vec{n} \in l; n_j \geq p_j + d(l,\vec{p})\},\ l_2 = \{\vec{n} \in l; p_j - d(l,\vec{p}) < n_j < p_j + d(l,\vec{p})\} \ \text{ and }$$
$$l_3 = \{\vec{n} \in l; n_j \leq p_j - d(l,\vec{p})\}.$$

Since $\vec{n}-\vec{p}\in Y_j$ for all $\vec{n}\in l_1\cup l_3$, then if we do an analysis by cases as in the proof of the Theorem 13 we have that the contribution of $f(\vec{p})$ to the variation of the maximal operator in $l_1\cup l_3$ is less than or equal to

$$\frac{2}{(2d(l,\vec{p})+1)^d},$$

moreover, since $|\vec{m}-\vec{p}|_\infty = d(l,\vec{p})$ for all $m\in l_2$ we obtain that the contribution of $f(\vec{p})$ to the variation of the maximal operator in l_2 is bounded by

$$2d(l,\vec{p})\left(\frac{1}{(2d(l,\vec{p})+1)^d}-\frac{1}{(2d(l,\vec{p})+3)^d}\right).$$

Since p belong to d lines of the lattice and given $k\in\mathbb{N}\setminus\{0\}$ there exist $d((2k+1)^{d-1}-(2k-1)^{d-1})$ lines such that $d(l,\vec{p})=k$. Thus the contribution of $f(\vec{p})$ to the right hand side of (3.5.1) is less than or equal to

$$\left(2d+\sum_{k\geq 1} d((2k+1)^{d-1}-(2k-1)^{d-1})\left(\frac{2+2k}{(2k+1)^d}-\frac{2k}{(2k+3)^d}\right)\right).$$

This implies the result.

□

Chapter 4

Extremal functions of exponential type

4.1 Introduction

In this chapter we address a class of problems that have come to be known as *Beurling-Selberg extremal problems*. One of the most well-known examples of such a problem is due to Selberg himself [60, 68, 76]. Given an interval $I \subset \mathbb{R}$ and $\delta > 0$, Selberg constructed an integrable function $M(x)$ that satisfies

(i) $\hat{M}(\xi) = 0$ if $|\xi| > \delta$ where $\hat{M}(\xi)$ is the Fourier transform (see §4.2) of $M(x)$,

(ii) $M(x) \geq \chi_I(x)$ for each $x \in \mathbb{R}$, and

(iii) $M(x)$ has the smallest integral [1] among all functions satisfying (i) and (ii).

The key constraint (which is common in Beurling-Selberg problems) is condition (ii), that $M(x)$ *majorizes* the characteristic function of I. Such problems are sometimes called *one-sided* approximation problems.

Most work on these sorts of problems have been focused on solving Selberg's problem but with $\chi_I(x)$ replaced by a different single variable function, such as $e^{-\lambda|x|}$. Such problems are considered in [17, 18, 15, 20, 24, 25, 52, 53, 54, 77]. Some work has also been done in the several variables setting. Shortly after his construction of the function $M(x)$, Selberg was able to construct majorants and minorants of the characteristic function of a box whose

[1] This was shown by Selberg in the case when $\delta\mathrm{Length}(I) \in \mathbb{Z}$. If $\delta\mathrm{Length}(I) \notin \mathbb{Z}$, then the minimal integral was found by B. F. Logan ([55], unpublished) and Littmann [54].

Fourier transforms are supported in a (possibly different) box. His majorant can be shown to be extremal for certain configurations of boxes, and it is unknown if the minorant is ever extremal for any given configuration of boxes. Nevertheless, such approximations have proven to be useful in applications [3, 4, 28, 31, 39, 40, 41, 46] because the best known approximations are *asymptotically extremal* as the Fourier support becomes uniformly large. In what follows we will use Selberg's method of constructing minorants (see Proposition 22).

We first study the Beurling-Selberg problem for multivariate Gaussian functions.[2] We then generalize the *Gaussian subordination* and *distribution method*, originally developed in [20], to higher dimensions and apply the method to study the Beurling-Selberg problem for a class of radial functions. We conclude our investigations with some applications to Hilbert-type inequalities and adapt the construction to periodic functions.

To state the first of our main results we shall use the following notation (see §4.2 for additional information). For $c > 0$, let $g_c(t) = e^{-c\pi t^2}$ and for $a, \lambda \in \Lambda$ (where $\Lambda = (0,\infty)^d$) let

$$x \in \mathbb{R}^d \mapsto G_\lambda(x) = \prod_{j=1}^{d} g_{\lambda_j}(x_j) = \exp\left\{ -\sum_{j=1}^{d} \lambda_j \pi x_j^2 \right\}.$$

and let

$$Q(a) = \prod_{j=1}^{d} [-a_j, a_j].$$

Our first result is a solution to the Beurling-Selberg extremal problem of determining optimal *majorants* of the Gaussian function $G_\lambda(x)$ that have Fourier transform supported in $Q(a)$.

Theorem 19. *Let $a, \lambda \in \Lambda$. If $F : \mathbb{R}^d \to \mathbb{R}$ is an integrable function that satisfies*

(i) $F(x) \geq G_\lambda(x)$ for each $x \in \mathbb{R}^d$, and

1. $\widehat{F}(\xi) = 0$ for each $\xi \notin Q(a)$,

then

$$\int_{\mathbb{R}^d} F(x)dx \geq \prod_{j=1}^{d} \lambda_j^{-\frac{1}{2}} \Theta(0; ia_j^2/\lambda_j). \tag{4.1.1}$$

where $\Theta(v;\tau)$ is Jacobi's theta function (see §4.2). Moreover, equality holds if $F(z) = M_{\lambda,a}(z)$ where $M_{\lambda,a}(z)$ is defined by (4.3.3).

[2] This problem, for a single variable, was solved in [20].

This theorem is essentially a corollary of Theorem 3 of [20]. The proof simply uses the product structure and positivity of $G_\lambda(x)$ in conjunction with Theorem 3 of [20]. It would be interesting to determine the analogue of the above theorem for *minorants* of $G_\lambda(x)$ (i.e. the high dimensional analogue of Theorem 2 of [20]) where the extremal functions cannot be obtained by a tensor product of lower dimensional extremal functions.

In our second result we address this problem by constructing minorants of the Gaussian function $G_\lambda(x)$ that have Fourier transform supported in $Q(a)$ and that are asymptotically extremal as a becomes uniformly large in each coordinate.

Theorem 20. *Let $a, \lambda \in \Lambda$. If $F : \mathbb{R}^d \to \mathbb{R}$ is an integrable function that satisfies*

(i) $F(x) \le G_\lambda(x)$ for each $x \in \mathbb{R}^d$, and

1. $\hat{F}(\xi) = 0$ for each $\xi \notin Q(a)$,

then

$$\int_{\mathbb{R}^d} F(x)dx \le \prod_{j=1}^{d} \lambda_j^{-\frac{1}{2}} \Theta(\tfrac{1}{2}; ia_j^2/\lambda_j). \tag{4.1.2}$$

Furthermore, there exist a positive constant $\gamma_0 = \gamma_0(d)$ such that if $\gamma := \min\{a_j^2/\lambda_j : 1 \le j \le d\} \ge \gamma_0$, then

$$\prod_{j=1}^{d} \lambda_j^{-\frac{1}{2}} \Theta(\tfrac{1}{2}; ia_j^2/\lambda_j) \le (1 + 5de^{-\pi\gamma}) \int_{\mathbb{R}^d} L_{\lambda,a}(x)dx \tag{4.1.3}$$

where $L_{\lambda,a}(x)$ is defined by (4.3.2).

Remark: For a fixed a, if λ is large enough then the right hand side (RHS) of (4.1.3) would be negative and the inequality would not hold, but this is not true for large values of γ. In fact, this happens because the zero function would be a better minorant.
Remark: Inequality (4.1.3) implies that if λ is fixed and a is large, then $\int_{\mathbb{R}^d} L_{\lambda,a}(x)dx$ approaches exponentially fast to the optimal answer. In this sense, we say that $L_{\lambda,a}(x)$ is asymptotically optimal with respect to the type.

Our next set of results (which are too lengthy to state here) are Theorems 25, 26, and 27. See §4.4 for the statements of the theorems. These theorems generalize the so-called *distribution* and *Gaussian subordination* methods of [20]. The main idea behind these methods goes back to the paper of Graham and Vaaler [36]. We will describe a "watered down" version of the approach here. Let us begin with the inequality

$$G_\lambda(x) \leq F_\lambda(x)$$

where $F_\lambda(x)$ is defined by (4.3.3). The idea is to *integrate the free parameter* λ in the function $G_\lambda(x)$ with respect to a (positive) measure ν on $\Lambda = (0,\infty)^d$ to obtain a pair of new functions of x:

$$g(x) = \int_\Lambda G_\lambda(x) d\nu(\lambda) \leq \int_\Lambda F_\lambda(x) d\nu(\lambda) = f(x).$$

The process simultaneously produces a function $g(x)$ and a majorant $f(x)$ having $\hat{f}(\xi)$ supported in $Q(a)$. The difference of the functions in L^1-norm is similarly obtained by integrating against ν. The method that we present allows us to produce majorants and minorants for $g(x)$ equal to the one of the following functions (among others):

$$\begin{aligned}
g(x) &= e^{-\alpha|x|^r}, \text{ for } \alpha > 0 \ \ \textit{and} \ \ 0 < r < 2, \\
g(x) &= (|x|^2 + \alpha^2)^{-\beta}, \text{for } \alpha > 0 \ \textit{and} \ \beta > 0, \\
g(x) &= -\log\left(\frac{|x|^2 + \alpha^2}{|x|^2 + \beta^2}\right), \text{for } 0 < \alpha < \beta, \text{ and} \\
g(x) &= |x|^\sigma, \text{for } \sigma \in (0,\infty) \setminus 2\mathbb{Z}_+
\end{aligned}$$

where $|x|$ is the Euclidean norm of x (see §4.2). In §4.5.2, we will discuss the full class of functions for which our method produces majorants and minorants.

One dimensional extremal functions have proven to be useful in several problems in analytic number theory [11, 12, 13, 19, 20, 26, 32, 33, 36, 40, 59, 60, 67, 68, 77]. The most well known examples make use of Selberg's extremal functions for the characteristic function of the interval. These include sharp forms of the large sieve and the Erdős-Turán inequality. Recent activity include estimates of quantities related to the Riemann zeta function. These include estimates of the zeta function on vertical lines in the critical strip and estimates of the pair correlation of the zeros of the zeta function. It would be interesting to see similar applications in the multidimensional setting.

Organization of the chapter

In §4.2 we introduce notation and gather the necessary background material for the remainder of the chapter. Then in §4.3 we will discuss and prove Theorems 19 and 20. Next, in §4.4 we present and prove a generalization of the *distribution method* introduced in [20] as well as a generalization of their Gaussian subordination result. In §4.5.1 we study periodic analogues of Theorems 19, 20, 25, 26, and 27. We conclude with some applications to Hilbert-type inequalities in §4.5.3 and some final remarks in §4.6

4.2 Preliminaries

Let us first have a word about our notation. Throughout this chapter vectors in $\mathbb{R}^d$ will be denoted by lowercase letters such as $x = (x_1, ..., x_d)$, and the Euclidean norm of x is given by

$$|x| = \left\{x_1^2 + \cdots + x_d^2\right\}^{1/2}.$$

We will write $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, $\Lambda = (0, \infty)^d$ and for each $a \in \Lambda$ we let

$$Q(a) = \{x \in \mathbb{R}^d : |x_j| \leq a_j\}. \tag{4.2.1}$$

The vector $u = (1, ..., 1) \in \Lambda$ will be called the *unitary* vector and we will use the notation $Q(R) = Q(Ru)$ whenever $R > 0$ is a positive real number.

Now we will introduce some notation that is not standard, but convenient for our purposes. Given vectors $x, y \in \mathbb{R}^d$ we will write $xy = (x_1y_1, ..., x_dy_d)$ and if $y_j \neq 0$ for each j, then we will write $x/y = (x_1/y_1, ..., x_d/y_d)$. We say that $x < y$ ($x \leq y$) if $x_j < y_j$ $\forall j$ ($x_j \leq y_j$ $\forall j$). We will always denote the inner product of $x, y \in \mathbb{R}^d$ by a central dot, that is, $x \cdot y$.

One of the main objects of study in this chapter is the *Fourier transform*. Given an integrable function $F(x)$ on $\mathbb{R}^d$, we define the Fourier transform of $F(x)$ by

$$\hat{F}(\xi) = \int_{\mathbb{R}^d} e(x \cdot \xi) F(x) dx$$

where $\xi \in \mathbb{R}^d$ and $e(\theta) = e^{-2\pi i\theta}$. We extend the definition in the usual way to tempered distributions (see for instance [72]). We will mainly be considering functions whose Fourier transforms are supported in a bounded subset of $\mathbb{R}^d$. Such functions are called *band-limited*. It is well-known that band-limited functions can be extended to an entire functions on $\mathbb{C}^d$ satisfying an exponential growth condition. An entire function $F : \mathbb{C}^d \to \mathbb{C}$ is said to be of *exponential type* if there exists a number $\tau \geq 0$ such that for every $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

$$|F(z)| \leq C_\epsilon e^{(\tau+\epsilon)|z|}.$$

If $F(z)$ satisfies such a growth estimate, then $F(z)$ is said to be of exponential type at most τ.

There is a refinement of the definition of exponential type due to Stein [69, 72]. Given an origin symmetric convex body K with supporting function

$$H(z) = H_K(z) = \sup_{\xi \in K} z \cdot \xi, \tag{4.2.2}$$

an entire function $F : \mathbb{C}^d \to \mathbb{C}$ is said to be of exponential type with respect to K if

$$|F(z)| \leq C_\epsilon e^{(1+\epsilon)H(z)}.$$

This is the natural generalization of exponential type used by Stein in his generalization of the Paley-Wiener theorem [69, 72]. The Paley-Wiener theorem has another generalization to higher dimensions that is formulated for certain tempered distributions. In this formulation, it is not necessary that the body K be symmetric. We will now state this generalization of the Paley-Wiener which can be found in [43], Theorem 7.3.1.

Theorem 21 (Paley–Wiener–Schwartz)**.** *Let K be a convex compact subset of $\mathbb{R}^d$ with supporting function $H(x)$ given by (4.2.2). If F is a tempered distribution such that the support of $\widehat{F}$ is contained in K, then $F : \mathbb{C}^d \to \mathbb{C}$ is an entire function and exist $N, C > 0$ such that*

$$|F(x+iy)| \leq C(1+|x+iy|)^N e^{2\pi H(y)}.$$

for every $x + iy \in \mathbb{C}^d$.

Conversely, every entire function $F : \mathbb{C}^d \to \mathbb{C}$ satisfying an estimate of this form defines a tempered distribution with Fourier transform supported on K.

We will now define and compile some results about Gaussians and theta functions that we will need in the sequel. Given a positive real number $\delta > 0$ the Gaussian $g_\delta : \mathbb{R} \to \mathbb{R}$ is defined by

$$g_\delta(t) = e^{-\delta\pi t^2},$$

and its Fourier transform is given by $\hat{g}_\delta(\xi) = \delta^{-1/2} g_{1/\delta}(\xi)$.

For a $\tau = \sigma + it$ with $t > 0$, if $q = e^{\pi i \tau}$, then Jacobi's theta function (see [27]) is defined by

$$\Theta(v;\tau) = \sum_{n\in\mathbb{Z}} e(nv)q^{n^2}. \tag{4.2.3}$$

These functions are related through the Poisson summation formula by

$$\sum_{m\in\mathbb{Z}} g_\delta(v+m) = \sum_{n\in\mathbb{Z}} e(nv)\hat{g}_\delta(n) = \delta^{-1/2}\Theta(v; i\delta^{-1}). \tag{4.2.4}$$

The one dimensional case of Theorems 19 and 20 are proven in [20]. There it is proved that the functions

$$l_\delta(z) = \Big(\frac{\cos \pi z}{\pi}\Big)^2 \Big\{ \sum_{k\in\mathbb{Z}} \frac{g_\delta(k+\frac{1}{2})}{(z-k-\frac{1}{2})^2} + \sum_{k\in\mathbb{Z}} \frac{g'_\delta(k+\frac{1}{2})}{(z-k-\frac{1}{2})} \Big\} \tag{4.2.5}$$

and

$$m_\delta(z) = \Big(\frac{\sin \pi z}{\pi}\Big)^2 \Big\{ \sum_{k\in\mathbb{Z}} \frac{g_\delta(k)}{(z-k)^2} + \sum_{k\in\mathbb{Z}} \frac{g'_\delta(k)}{(z-k)} \Big\} \tag{4.2.6}$$

are entire functions of exponential type at most 2π and they satisfy

$$l_\delta(x) \le g_\delta(x) \le m_\delta(x) \tag{4.2.7}$$

for all real x. Moreover,

$$\int_{-\infty}^{\infty} m_\delta(x)dx = \delta^{-\frac{1}{2}}\Theta(0; i/\delta) \tag{4.2.8}$$

and

$$\int_{-\infty}^{\infty} l_\delta(x)dx = \delta^{-\frac{1}{2}}\Theta(\tfrac{1}{2}; i/\delta). \tag{4.2.9}$$

In view of (4.2.4), (4.2.8), and (4.2.9), the functions $l_\delta(z)$ and $m_\delta(z)$ are the best one-sided L^1-approximations of g_δ having exponential type at most 2π.

4.3 The multidimensional Gaussian function

In this section we will prove Theorem 19 and 20. To construct a minorant of the Gaussian, we begin with the functions $m_\delta(z)$ and $l_\delta(z)$ defined by (4.2.5) and (4.2.6) and use Selberg's bootstrapping technique to obtain multidimensional minorants. The majorant is constructed by taking $m_\delta(z)$ tensored with itself d-times.

For every $\lambda = (\lambda_1, ..., \lambda_n) \in \Lambda$ we define the function $G_\lambda : \mathbb{R}^d \to \mathbb{R}$ by

$$G_\lambda(x) = \prod_{j=1}^{d} g_{\lambda_j}(x_j) = \prod_{j=1}^{d} e^{-\lambda_j \pi x_j^2}. \tag{4.3.1}$$

The following proposition is due to Selberg, but it was never published [76]. We called it *Selberg's bootstrapping method* because it enables us to construct a minorant for a tensor product of functions provided that we have majorants and minorants of each component at our disposal. The method has been used in one form or the other in [4, 31, 39, 40, 41].

Proposition 22. *Let $d > 0$ be natural number and $f_j : \mathbb{R} \to (0, \infty)$ be functions for every $j = 1, ..., d$. Let $l_j, m_j : \mathbb{R} \to \mathbb{R}$ be real-valued functions such that*

$$l_j(x) \le f_j(x) \le m_j(x)$$

for every x and j. Then

$$-(d-1)\prod_{k=1}^{d} m_k(x_k) + \sum_{k=1}^{d} l_k(x_k) \prod_{\substack{j=1 \\ j\neq k}}^{d} m_j(x_j) \le \prod_{k=1}^{d} f_k(x_k).$$

This proposition is easily deduced from the following inequality.

Lemma 23. *If $\beta_1, ..., \beta_d \ge 1$, then*

$$\sum_{k=1}^{d} \prod_{\substack{j=1 \\ j\neq k}}^{d} \beta_j \le 1 + (d-1)\prod_{k=1}^{d} \beta_k.$$

Proof. We give a proof by induction, starting with the inductive step since the base case is simple.

Suppose that the claim is true for $d = 1, ..., L$. Let $\beta_1, ..., \beta_L, \beta_{L+1}$ be a sequence of real numbers not less than one and write $\beta_j = 1 + \epsilon_j$. We obtain

$$\begin{aligned}
\sum_{k=1}^{L+1} \prod_{\substack{j=1 \\ j\neq k}}^{L+1} \beta_j &= \prod_{j=1}^{L} \beta_j + (1+\epsilon_{L+1}) \sum_{k=1}^{L} \prod_{\substack{j=1 \\ j\neq k}}^{L} \beta_j \\
&\leq \prod_{j=1}^{L} \beta_j + (1+\epsilon_{L+1}) \left\{ 1 + (L-1) \prod_{j=1}^{L} \beta_j \right\} \\
&= 1 + \epsilon_{L+1} + \prod_{j=1}^{L} \beta_j + (L-1) \prod_{j=1}^{L+1} \beta_j \\
&\leq 1 + \epsilon_{L+1} \prod_{j=1}^{L} \beta_j + \prod_{j=1}^{L} \beta_j + (L-1) \prod_{j=1}^{L+1} \beta_j \\
&= 1 + L \prod_{j=1}^{L+1} \beta_j
\end{aligned}$$

□

Now we can define our candidates for majorant and minorant of $G_\lambda(x)$. For a given $\lambda \in \Lambda$ define the functions

$$z \in \mathbb{C}^d \mapsto L_\lambda(z) = -(d-1) \prod_{j=1}^{d} m_{\lambda_j}(z_j) + \sum_{k=1}^{d} l_{\lambda_k}(z_k) \prod_{\substack{j=1 \\ j\neq k}}^{d} m_{\lambda_j}(z_j) \tag{4.3.2}$$

and

$$z \in \mathbb{C}^d \mapsto M_\lambda(z) = \prod_{j=1}^{d} m_{\lambda_j}(z_j). \tag{4.3.3}$$

It follows from Proposition 22 and (4.2.7) that

$$L_\lambda(x) \leq G_\lambda(x) \leq M_\lambda(x) \text{ for all } x \in \mathbb{R}^d. \tag{4.3.4}$$

Moreover, since $l_\delta(x)$ and $m_\delta(x)$ have exponential type at most 2π, we conclude that the Fourier transforms of $L_\lambda(x)$ and $M_\lambda(x)$ are supported on Q. We modify $L_\lambda(z)$ and $M_\lambda(z)$ to have exponential type with respect to $Q(a)$ in the following way. Given $a, \lambda \in \Lambda$ we define the functions

$$L_{\lambda,a}(z) = L_{\lambda/a^2}(az) \tag{4.3.5}$$

and

$$M_{\lambda,a}(z) = M_{\lambda/a^2}(az). \tag{4.3.6}$$

By (4.3.4) we obtain

$$L_{\lambda,a}(x) \le G_\lambda(x) \le M_{\lambda,a}(x) \text{ for all } x \in \mathbb{R}^d \tag{4.3.7}$$

and using the scaling properties of the Fourier transform, we conclude that $L_{\lambda,a}(x)$ and $M_{\lambda,a}(x)$ have exponential type with respect to $Q(a)$. By formula (4.2.6), we have $m_\delta(k) = g_\delta(k)$ for all integers k, hence we obtain

$$M_{\lambda,a}(k/a) = G_\lambda(k/a) \tag{4.3.8}$$

for all $k \in \mathbb{Z}^d$ (recall that $k/a = (k_1/a_1, ..., k_d/a_d)$).

We are now in a position to prove Theorems 19 and 20.

Proof of Theorem 19. It follows from (4.3.4) and (4.3.6) that the function $M_{\lambda,a}(x)$ is majorant of $G_\lambda(x)$ of exponential type with respect to $Q(a)$. Define $\alpha = a_1 \cdots a_d$. Using definition (4.3.3) and (4.3.6) we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} M_{\lambda,a}(x)\mathrm{d}x \;=\; \alpha^{-1}\int_{\mathbb{R}^d} M_{\lambda/a^2}(x)\mathrm{d}x \;&=\; \alpha^{-1}\prod_{j=1}^{d}\int_{\mathbb{R}^d} m_{\lambda_j/a_j^2}(x_j)dx_j \\ &=\; \prod_{j=1}^{d}\lambda_j^{-\frac{1}{2}}\Theta(0; ia_j^2/\lambda_j), \end{aligned}$$

where the first equality is due to a change of variables, the second one due to the product structure and the third one due to (4.2.8).

Now we will prove that (4.3.6) is extremal. Suppose that $F(z)$ is an entire majorant of $G_\lambda(x)$ of exponential type with respect to $Q(a)$ and integrable on $\mathbb{R}^d$ (and therefore absolutely integrable on $\mathbb{R}^d$). We then have

$$\int_{\mathbb{R}^d} F(x)dx = \alpha^{-1}\sum_{k\in\mathbb{Z}^d} F(k/a) \ge \alpha^{-1}\sum_{k\in\mathbb{Z}^d} G_\lambda(k/a) = \prod_{j=1}^{d}\lambda_j^{-\frac{1}{2}}\Theta(0; ia_j^2/\lambda_j) \tag{4.3.9}$$

because $M_{\lambda,a}(x)$ majorizes G_λ, and the rightmost equality is given by (4.2.4).

□

To prove Theorem 20 we will need the following lemma.

Lemma 24. *For all $t > 0$ we have*

$$1 - 4q/(1-q)^2 < \frac{\Theta(\frac{1}{2}; it)}{\Theta(0; it)} < e^{-2q}, \tag{4.3.10}$$

where $q = e^{-\pi t}$.

Proof of Theorem 20. Suppose that $F(z)$ is an entire minorant of exponential type with respect to $Q(a)$ and absolutely integrable on $\mathbb{R}^d$. Recalling that $u = (1, ..., 1)$ and applying the Poisson summation formula, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} F(x)dx = \alpha^{-1} \sum_{k\in\mathbb{Z}^d} F(k/a + u/2a) &\leq \alpha^{-1} \sum_{k\in\mathbb{Z}^d} G_\lambda(k/a + u/2a) \\ &= \prod_{j=1}^{d} \lambda_j^{-\frac{1}{2}} \Theta(\tfrac{1}{2}; ia_j^2/\lambda_j).\end{aligned}$$

where the last equality is given by (4.2.4). This proves (4.1.2). By construction, $L_{\lambda,a}(z)$ is an entire minorant of exponential type with respect to $Q(a)$. Using definitions (4.3.2) and (4.3.5) we conclude that

$$\int_{\mathbb{R}^d} L_{\lambda,a}(x)dx = \left\{ \sum_{j=1}^{d} \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} - (d-1) \right\} \prod_{j=1}^{d} \Theta(0; ia_j^2/\lambda_j)\lambda_j^{-\frac{1}{2}}. \tag{4.3.11}$$

Thus, to deduce (4.1.3), we only need to prove that

$$(1 + 5de^{-\pi\gamma})\left\{ \sum_{j=1}^{d} \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} - (d-1) \right\} \geq \prod_{j=1}^{d} \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)}. \tag{4.3.12}$$

for large γ (recall that $\gamma = \min\{\lambda_j/a_j^2\}$). If we let $q_j = e^{-\pi a_j^2/\lambda_j}$ and γ sufficient large such that $(1 - e^{-\pi\gamma})^2 > 4/5$, we can use Lemma 24 to obtain

$$\sum_{j=1}^{d} \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} - (d-1) \geq 1 - 4\sum_{j=1}^{d} q_j/(1-q_j)^2 \geq 1 - 5\sum_{j=1}^{d} q_j. \tag{4.3.13}$$

Applying Lemma 24 for a sufficient large γ we obtain

$$\prod_{j=1}^{d} \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} \leq \exp\Big\{ -2\sum_{j=1}^{d} q_j \Big\} \leq 1 - \sum_{j=1}^{d} q_j. \tag{4.3.14}$$

where the last inequality holds, say, if $\sum_{j=1}^{d} q_j < \log 2$. Write $\beta = \sum_{j=1}^{d} q_j$ and note that

$$1 - \beta \leq (1 - 5\beta)(1 + 5\beta)$$

if β is sufficiently small, say, if $\beta \in [0, 1/25)$. If γ is sufficiently large such that $\sum_{j=1}^{d} q_j < 1/25$ we obtain

$$1 - \sum_{j=1}^{d} q_j \leq \big(1 - 5\sum_{j=1}^{d} q_j\big)\big(1 + 5\sum_{j=1}^{d} q_j\big) < \big(1 - 5\sum_{j=1}^{d} q_j\big)(1 + 5de^{-\pi\gamma}). \tag{4.3.15}$$

By (4.3.13), (4.3.14) and (4.3.15) we conclude that there exists an $\gamma_0 = \gamma_0(d) > 0$ such that if $\gamma \geq \gamma_0$, then (4.3.12) holds.

□

Proof of Lemma 24. Recall that $e(v) = e^{2\pi i v}$ and $q = e^{\pi i \tau}$. By Theorem 1.3 of chapter 10 of [71] the Theta function has the following product representation (known as the Jacobi's triple product identity)

$$\Theta(v;\tau) = \prod_{n=1}^{\infty} \big(1 - q^{2n}\big)\big(1 + q^{2n-1}e(v)\big)\big(1 + q^{2n-1}e(-v)\big). \tag{4.3.16}$$

It follows from (4.3.16) that

$$\frac{\Theta(\frac{1}{2}; it)}{\Theta(0; it)} = \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n-1}}{1 + q^{2n-1}}\right)^2 = \exp\left\{2\sum_{n=0}^{\infty} \log\left(1 - \frac{2q^{2n+1}}{1 + q^{2n+1}}\right)\right\}.$$

Using the inequality $\log(1-x) \geq -x/(1-x)$ for all $x \in [0,1)$ we obtain

$$\begin{aligned}
\frac{\Theta(\frac{1}{2}; it)}{\Theta(0; it)} \geq \exp\left\{ -4\sum_{n=0}^{\infty} \frac{q^{2n+1}}{1-q^{2n+1}} \right\} &\geq \exp\left\{ -\frac{4}{1-q}\sum_{n=0}^{\infty} q^{2n+1} \right\} \\
&= \exp\left\{ -\frac{4q}{(1-q)(1-q^2)} \right\} \\
&> e^{-4q/(1-q)^2} \\
&> 1 - 4q/(1-q)^2
\end{aligned}$$

and this proves the left hand side (LHS) inequality in (4.3.10). The (RHS) inequality in (4.3.10) is deduced by a similar argument using the inequality $\log(1-x) \leq -x$ for all $x \in [0,1)$.

4.4 Gaussian subordination method

In this section we adapt the distribution method developed in [20] to the several variables setting, and we apply the Gaussian subordination method to the majorant and minorant for the multidimensional Gaussian defined in the previous section. This method allow us to extend the class of function $g(x)$ for which we can solve the corresponding Beurling–Selberg extremal problems. The central idea is to integrate the functions $G_\lambda(x)$, $L_{\lambda,a}(z)$ and $M_{\lambda,a}(z)$, in a distributional sense, with respect to a non–negative measure ν defined on $\Lambda = (0,\infty)^d$.

We begin with a generalization of the distribution method developed in [20] for existence of majorants and minorants, we will deal with extremality later.

Theorem 25 (Distribution method – (Existence)). *Let $K \subset \mathbb{R}^d$ be a compact convex set, Λ be a measurable space of parameters, and for each $\lambda \in \Lambda$ let $G(x;\lambda) \in L^1(\mathbb{R}^d)$ be a real-valued function. For each λ let $F(z;\lambda)$ be an entire function defined for $z \in \mathbb{C}^d$, of exponential type with respect to K. Let ν be a non–negative measure on Λ that satisfies*

$$\int_\Lambda \int_{\mathbb{R}^d} |F(x;\lambda) - G(x;\lambda)| \mathrm{d}x \mathrm{d}\nu(\lambda) < \infty. \tag{4.4.1}$$

and

$$\int_\Lambda \int_{\mathbb{R}^d} |\widehat{G}(x;\lambda)\varphi(x)| \mathrm{d}x \mathrm{d}\nu(\lambda) < \infty \tag{4.4.2}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ *supported in* K^c.

Let $\mathcal{G} \in \mathcal{S}'(\mathbb{R}^d)$ *be a real-valued continuous function and*

$$\widehat{\mathcal{G}}(\varphi) = \int_{\mathbb{R}^d} \int_{\Lambda} \widehat{G}(x;\lambda) d\nu(\lambda) \varphi(x) dx \tag{4.4.3}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ *supported in* K^c.

(i) If $G(x;\lambda) \le F(x;\lambda)$ *for each* $x \in \mathbb{R}^d$ *and* $\lambda \in \Lambda$, *then there exists a real entire majorant* $\mathcal{M}(x)$ *for* $\mathcal{G}(x)$ *of exponential type with respect to* K *and*

$$\int_{\mathbb{R}^d} \{\mathcal{M}(x) - \mathcal{G}(x)\} dx$$

is equal to the quantity at (4.4.1).

(ii) If $F(x;\lambda) \le G(x;\lambda)$ *for each* $x \in \mathbb{R}^d$ *and* $\lambda \in \Lambda$, *then there exists a real entire minorant* $\mathcal{L}(x)$ *for* $\mathcal{G}(x)$ *of exponential type with respect to* K *and*

$$\int_{\mathbb{R}^d} \{\mathcal{G}(x) - \mathcal{L}(x)\} dx$$

is equal to the quantity at (4.4.1).

With the exception of Theorem 25, Λ will always stand for $(0,\infty)^d$. For a given $a \in \Lambda$ define $\mathfrak{G}_+^d(a)$ as the set of ordered pairs $(\mathcal{G}, \nu)$ where $\mathcal{G} : \mathbb{R}^d \to \mathbb{R}$ is a continuous function and ν is a non-negative Borel measure in Λ such that

(C1) $\mathcal{G} \in \mathcal{S}'(\mathbb{R}^d)$ is a tempered distribution.

(C2) For all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ supported in $Q(a)^c$ we have

$$\int_{\mathbb{R}^d} \int_{\Lambda} |\widehat{G}_\lambda(x) \varphi(x)| d\nu(\lambda) dx < \infty.$$

(C3) For all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ supported in $Q(a)^c$ we have

$$\int_{\mathbb{R}^d} \mathcal{G}(x) \widehat{\varphi}(x) = \int_{\mathbb{R}^d} \int_{\Lambda} \widehat{G}_\lambda(x) \mathrm{d}\nu(\lambda) \varphi(x) dx. \tag{4.4.4}$$

(C4+) The following integrability condition holds

$$\int_{\Lambda} \prod_{k=1}^{d} \lambda_k^{-\frac{1}{2}} \left\{ \prod_{j=1}^{d} \Theta(0; ia_j^2/\lambda_j) - 1 \right\} d\nu(\lambda) < \infty. \tag{4.4.5}$$

In a analogous way, we define the class $\mathfrak{G}_{-}^{d}(a)$ replacing condition **(C4+)** by

(C4-) The following integrability condition holds

$$\int_{\Lambda} \left\{ 1 - \left(\sum_{j=1}^{d} \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} - (d-1) \right) \prod_{j=1}^{d} \Theta(0; ia_j^2/\lambda_j) \right\} \prod_{j=1}^{d} \lambda_j^{-\frac{1}{2}} \mathrm{d}\nu(\lambda) < \infty. \tag{4.4.6}$$

Theorem 26 offers a optimal resolution of the Majorization Problem for the class functions $\mathfrak{G}_{+}^{d}(a)$ and the Theorem 27 offers asymptotically optimal resolution of the Minorization Problem for the class of functions $\mathfrak{G}_{-}^{d}(a)$.

Theorem 26 (Gaussian subordination – Majorant)**.** *For a given $a \in \Lambda$, let $(\mathcal{G}, \nu) \in \mathfrak{G}_{+}^{d}(a)$. Then there exists an extremal majorant $\mathcal{M}_a(z)$ of exponential type with respect to $Q(a)$ for $\mathcal{G}(x)$. Furthermore, $\mathcal{M}_a(x)$ interpolates $\mathcal{G}(x)$ on $\mathbb{Z}^d/a$ and satisfies*

$$\int_{\mathbb{R}^d} \mathcal{M}(x) - \mathcal{G}(x) \mathrm{d}x = \int_{\Lambda} \prod_{k=1}^{d} \lambda_k^{-\frac{1}{2}} \left\{ \prod_{j=1}^{d} \Theta(0; ia_j^2/\lambda_j) - 1 \right\} d\nu(\lambda). \tag{4.4.7}$$

Theorem 27 (Gaussian subordination – Minorant)**.** *For a given $a \in \Lambda$, let $(\mathcal{G}, \nu) \in \mathfrak{G}_{-}^{d}(a)$. Then, if $\mathcal{F}(z)$ is a real entire minorant of $\mathcal{G}(x)$ of exponential type with respect to $Q(a)$, we have*

$$\int_{\mathbb{R}^d} \mathcal{G}(x) - \mathcal{F}(x) dx \geq \int_{\Lambda} \prod_{j=1}^{d} \lambda_k^{-\frac{1}{2}} \left\{ 1 - \prod_{j=1}^{d} \Theta(\tfrac{1}{2}; ia_j^2/\lambda_j) \right\} d\nu(\lambda). \tag{4.4.8}$$

Furthermore, there exists a family of minorants $\{\mathcal{L}_a(z) : a \in \Lambda\}$ where $\mathcal{L}_a(z)$ is of exponential type with respect to $Q(a)$ such that

$$\int_{\mathbb{R}^d} \mathcal{G}(x) - \mathcal{L}_a(x) \mathrm{d}x$$

is equal to the (LHS) of (4.4.6). Also

$$\lim_{a\uparrow\infty}\int_{\mathbb{R}^d}\mathcal{G}(x)-\mathcal{L}_a(x)\mathrm{d}x=0, \tag{4.4.9}$$

where $a\uparrow\infty$ means $a_j\uparrow\infty$ for each j.

Corollary 28. *Under the hypothesis of Theorem 27, suppose also that exists an $R>0$ such that $\mathrm{supp}(\nu)\subset\Lambda\cap Q(R)$, $\mathcal{G}\in L^1(\mathbb{R}^d)$ and*

$$\int_{\mathbb{R}^d}\mathcal{G}(x)\mathrm{d}x=\int_\Lambda\prod_{j=1}^d\lambda_j^{-\frac{1}{2}}\mathrm{d}\nu(\lambda)<\infty.$$

Then, there exist a constant $\alpha_0>0$ such that, if $\alpha:=\min\{a_j\}\geq\alpha_0$ and if $\mathcal{F}(x)$ is a real entire minorant of $\mathcal{G}(x)$ of exponential type with respect to $Q(a)$, then

$$\int_{\mathbb{R}^d}\mathcal{F}(x)\mathrm{d}x\leq(1+5de^{-\pi\alpha^2/R})\int_{\mathbb{R}^d}\mathcal{L}_a(x)\mathrm{d}x.$$

Proof of Corollary 28. By a direct application of Theorem 20 we obtain

$$\int_{\mathbb{R}^d}\mathcal{F}(x)\mathrm{d}x\leq\int_{\Lambda_R}\prod_{j=1}^d\Theta(\tfrac{1}{2};ia_j^2/\lambda_j)\lambda_j^{-\frac{1}{2}}\mathrm{d}\nu(\lambda).$$

If we choose $\gamma_0>0$, as in Theorem 20 and define $\alpha_0=R\gamma_0$, we can use inequality (4.1.3) to conclude that

$$\prod_{j=1}^d\Theta(\tfrac{1}{2};ia_j^2/\lambda_j)\lambda_j^{-\frac{1}{2}}\leq(1+5de^{-\pi\alpha^2/R})\int_{\mathbb{R}^d}L_{\lambda,a}(x)\mathrm{d}x.$$

for all $\lambda\in\Lambda\cap Q(R)$, if each $a_j\geq\alpha_0$. If we integrate this last inequality with respect to $\mathrm{d}\nu(\lambda)$ we obtain the desired result.

□

4.4.1 Proofs of Theorems 25, 26 and 27

Proof of Theorem 25. We follow the proof of Theorem 14 from [20] proving only the majorant case, since the minorant case is nearly identical. Let

$$D(x;\lambda) = F(x;\lambda) - G(x;\lambda) \geq 0.$$

By condition (4.4.1) and Fubini's theorem the function

$$\mathcal{D}(x) = \int_\Lambda D(x;\lambda)\mathrm{d}\nu(\lambda) \geq 0,$$

is defined for almost all $x \in \mathbb{R}^d$ and $\mathcal{D} \in L^1(\mathbb{R}^d)$. The Fourier transform of $\mathcal{D}(x)$ is a continuous function given by

$$\widehat{\mathcal{D}}(\xi) = \int_\Lambda \widehat{D}(\xi;\lambda)\mathrm{d}\nu(\lambda), \tag{4.4.10}$$

and, due to (4.4.2), for almost every $\xi \notin K$ we have the alternative representation

$$\widehat{\mathcal{D}}(\xi) = -\int_\Lambda \widehat{G}(\xi;\lambda)\mathrm{d}\nu(\lambda). \tag{4.4.11}$$

Let $\mathcal{M}$ be the tempered distribution given by

$$\mathcal{M}(\varphi) = \int_{\mathbb{R}^d} \{\mathcal{D}(x) + \mathcal{G}(x)\}\varphi(x)dx.$$

Now for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ supported in K^c, we have by combining (4.4.3) and (4.4.11)

$$\widehat{\mathcal{M}}(\varphi) = \widehat{\mathcal{D}}(\varphi) + \widehat{\mathcal{G}}(\varphi) = 0. \tag{4.4.12}$$

Hence $\widehat{\mathcal{M}}$ is supported on K, in the distributional sense. By the Theorem 21, it follows that the distribution $\mathcal{M}$ is identified with an analytic function $\mathcal{M} : \mathbb{C}^d \to \mathbb{C}$ of exponential type with respect to K and that

$$\mathcal{M}(\varphi) = \int_{\mathbb{R}^d} \mathcal{M}(x)\varphi(x)dx \tag{4.4.13}$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$. It then follows from the definition of $\mathcal{M}$ and (4.4.13) that for almost every $x \in \mathbb{R}^d$

$$\mathcal{M}(x) = \mathcal{D}(x) + \mathcal{G}(x),$$

which implies $\mathcal{M}(x) \geq \mathcal{G}(x)$ for all $x \in \mathbb{R}^d$ since $\mathcal{G}(x)$ is continuous, and

$$\int_{\mathbb{R}^d} \{\mathcal{M}(x) - \mathcal{G}(x)\} dx = \int_{\Lambda} \int_{\mathbb{R}^d} \{F(x;\lambda) - G(x;\lambda)\} dx d\nu(\lambda) < \infty.$$

□

Now we turn to the proof of Theorem 26.

Proof of Theorem 26. We follow the proof of Theorem 14 from [20], skipping some parts but including changes needed for higher dimensions.

By conditions (C1),(C2),(C3) and (C4+) we are at the position of applying Theorem 25 for the functions $M_{\lambda,a}(z)$ defined at the previous section. Let $\mathcal{M}_a(z)$ be the majorant given by Theorem 25 part (i), for $K = Q(a)$, $G_\lambda(x) = G(x;\lambda)$ and $F(z;\lambda) = M_{\lambda,a}(z)$. First, we show that $\mathcal{M}_a(n/a) = \mathcal{G}(n/a)$ for each $n \in \mathbb{Z}^d$ and then we will conclude that $\mathcal{M}_a(z)$ is extremal. Let $\mathcal{D}(x) :=: \mathcal{M}_a(x) - \mathcal{G}(x)$. By Theorem 25 we know that $\mathcal{D} \in L^1(\mathbb{R}^d)$. Define $\alpha = a_1 a_2 \ldots a_d$ and

$$P(x) = \frac{1}{\alpha} \sum_{n \in \mathbb{Z}^d} \mathcal{D}((x+n)/a).$$

It follows from Fubini's theorem that $P(x)$ is defined almost everywhere, is integrable on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and $\widehat{P}(k) = \widehat{\mathcal{D}}(ak)$ for all $k \in \mathbb{Z}^d$. Therefore, we have the following identity

$$P * F_R(x) = \sum_{\substack{n \in \mathbb{Z}^d \\ |n_j| \le R}} \prod_{j=1}^{d} \left(1 - \frac{|n_j|}{R+1}\right) \widehat{\mathcal{D}}(an) e(x \cdot n) \tag{4.4.14}$$

for each positive integer R and $x \in \mathbb{R}^d$, where

$$F_R(x) = \prod_{j=1}^{d} \frac{1}{R+1} \left(\frac{\sin \pi (R+1) x_j}{\sin \pi x_j}\right)^2 = \sum_{\substack{n \in \mathbb{Z}^d \\ |n_j| \le R}} \prod_{j=1}^{d} \left(1 - \frac{|n_j|}{R+1}\right) e(x \cdot n) \tag{4.4.15}$$

is the product of one–dimensional Fejér kernels . From (4.4.11) and (4.4.14) we have

$$\begin{aligned} P * F_R(0) &= \widehat{\mathcal{D}}(0) - \sum_{\substack{n\neq 0 \\ |n_j|\leq R}} \prod_{j=1}^{d} \left(1 - \frac{|n_j|}{R+1}\right) \int_{\Lambda} \widehat{G}_{\lambda}(an) d\nu(\lambda) \\ &= \widehat{\mathcal{D}}(0) - \int_{\Lambda} \Bigg\{ \sum_{\substack{n\neq 0 \\ |n_j|\leq R}} \prod_{j=1}^{d} \left(1 - \frac{|n_j|}{R+1}\right) \widehat{G}_{\lambda}(an) \Bigg\} d\nu(\lambda). \end{aligned}$$

Since $P * F_R$ is a non–negative function and the term in the brackets above is positive we may apply Fatou's lemma to obtain

$$\begin{aligned} \widehat{\mathcal{D}}(0) &\geq \liminf_{R\to\infty} P * F_R(0) + \int_{\Lambda} \liminf_{R\to\infty} \Bigg\{ \sum_{\substack{n\neq 0 \\ |n_j|\leq R}} \prod_{j=1}^{d} \left(1 - \frac{|n_i|}{R+1}\right) \widehat{G}_{\lambda}(an) \Bigg\} d\nu(\lambda) \\ &= \liminf_{R\to\infty} P * F_R(0) + \int_{\Lambda} \sum_{n\neq 0} \widehat{G}_{\lambda}(an) d\nu(\lambda). \end{aligned} \tag{4.4.16}$$

From the Poisson Summation formula and the fact that $M_{\lambda,a}(n/a) = G_{\lambda}(n/a)$ for all $n \in \mathbb{Z}^d$, we have

$$\widehat{M}_{\lambda,a}(0) - \widehat{G}_{\lambda}(0) = \sum_{\substack{n\in\mathbb{Z}^d \\ n\neq 0}} \widehat{G}_{\lambda}(an).$$

By (4.4.10) we conclude that second term on the (RHS) of (4.4.16) is equal to $\widehat{\mathcal{D}}(0)$. This implies that

$$\liminf_{R\to\infty} P * F_R(0) \leq 0 \implies \liminf_{R\to\infty} P * F_R(0) = 0.$$

Using this fact with the definition of $P(x)$, we have

$$\begin{aligned} \liminf_{R\to\infty} P * F_R(0) &= \liminf_{R\to\infty} \int_{\frac{1}{2}Q} P(-y) F_R(y) dy \\ &= \liminf_{R\to\infty} \int_{\frac{1}{2}Q} \sum_{n\in\mathbb{Z}^d/a} \frac{1}{\alpha} \mathcal{D}((n-y)/a) F_R(y) dy \\ &= \liminf_{R\to\infty} \sum_{n\in\mathbb{Z}^d/a} \int_{\frac{1}{2}Q} \frac{1}{\alpha} \mathcal{D}((n-y)/a) F_R(y) dy \end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{n\in\mathbb{Z}^d} \liminf_{R\to\infty} \int_{\frac{1}{2}Q} \frac{1}{\alpha}\mathcal{D}((n-y)/a)F_R(y)dy \\
&= \sum_{n\in\mathbb{Z}^d} \frac{1}{\alpha}\mathcal{D}(n/a),
\end{aligned}
$$

where we have used the positivity of $\mathcal{D}(x)$ and $F_R(x)$, Fubini's theorem and Fatou's lemma. The last equality is due to Fejér's theorem for the continuity of $\mathcal{D}(x)$ at the lattice points $\mathbb{Z}^d/a$ (see section 3.3 of [35]). Recalling that $\mathcal{D}(x) \geq 0$ for each $x \in \mathbb{R}^d$, it implies that $\mathcal{D}(n/a) = 0$ which in turn implies $\mathcal{M}_a(n/a) = \mathcal{G}(n/a)$ for each $n \in \mathbb{Z}^d$.

To conclude, note that if $\mathcal{F}(z)$ is an real entire majorant of exponential type with respect to $Q(a)$ such that $\mathcal{F} - \mathcal{G} \in L^1(\mathbb{R}^d)$ then $\mathcal{F} - \mathcal{M}_a \in L^1(\mathbb{R}^d)$. Thus, the following Poisson summation formula holds pointwise

$$
\int_{\mathbb{R}^d} \{\mathcal{F}(x) - \mathcal{M}_a(x)\}dx = \frac{1}{\alpha}\sum_{n\in\mathbb{Z}^d} \{\mathcal{F}((n+y)/a) - \mathcal{M}_a((n+y)/a)\}
$$

for every $y \in \mathbb{R}^d$. If we take $y = 0$ and use that $\mathcal{M}_a(z)$ interpolates $\mathcal{G}(x)$ at the lattice $\mathbb{Z}^d/a$ we conclude that $\mathcal{M}_a(z)$ is extremal, and this concludes the theorem.

□

Before we turn to the proof of Theorem 27, we need a technical lemma, we present the proof of that later.

Lemma 29. *The functions*

$$
\phi_a : \lambda \in \Lambda \mapsto \left\{ \sum_{j=1}^{d} \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} - (d-1) \right\} \prod_{j=1}^{d} \Theta(0; ia_j^2/\lambda_j)
$$

indexed by $a \in \Lambda$ satisfy

(i) $\phi_a(\lambda) \leq \phi_b(\lambda)$ if $a_j \leq b_j$ for all $j \in \{1, ..., d\}$

(ii) For all $\lambda \in \Lambda$, we have

$$
\lim_{a\uparrow\infty} \phi_a(\lambda) = 1
$$

where $a \uparrow \infty$ means that $a_j \uparrow \infty$ for all j.

Proof of Theorem 27. First we prove (4.4.8) and then we conclude with a proof of (4.4.9).

Let $\mathcal{F}(z)$ be a real entire minorant of $\mathcal{G}(x)$ with exponential type with respect to $Q(a)$. We can assume that $\mathcal{D}(x) = \mathcal{G}(x) - \mathcal{F}(x) \in L^1(\mathbb{R}^d)$, otherwise (4.4.8) is trivial. Define $\alpha = a_1 ... a_d$. By Fubini's theorem the function

$$P(x) = \frac{1}{\alpha} \sum_{n \in \mathbb{Z}^d} \mathcal{D}((x+n)/a)$$

is defined almost everywhere, integrable over $\frac{1}{2}Q$ and $\widehat{P}(n) = \widehat{\mathcal{D}}(an)$ for all $n \in \mathbb{Z}^d$. Let $F_R(x)$ be the multidimensional Fejér kernel as defined in (4.4.15), hence we have the following equality

$$P * F_R(x) = \sum_{\substack{n \in \mathbb{Z}^d \\ |n_j| \leq R}} \prod_{j=1}^{d} \left(1 - \frac{|n_j|}{R}\right) \widehat{\mathcal{D}}(an) e(x \cdot n). \tag{4.4.17}$$

By condition (C3) we obtain

$$\begin{aligned} \widehat{\mathcal{D}}(0) &= P * F_R(u/2) - \sum_{\substack{n \neq 0 \\ |n_j| \leq R}} \prod_{j=1}^{d} \left(1 - \frac{|n_j|}{R+1}\right) \int_{\Lambda} \widehat{G}_\lambda(an) d\nu(\lambda) (-1)^{u \cdot n} \\ &= P * F_R(u/2) + \int_{\Lambda} \left\{ - \sum_{\substack{n \neq 0 \\ |n_j| \leq R}} \prod_{j=1}^{d} \left(1 - \frac{|n_j|}{R+1}\right) \widehat{G}_\lambda(an)(-1)^{u \cdot n} \right\} d\nu(\lambda). \end{aligned}$$

Since $x \mapsto \widehat{G}_\lambda(x)$ is a product of radially decreasing functions we easily see that the term in the brackets is positive, thus we can apply Fatou's lemma to obtain

$$\begin{aligned} \widehat{\mathcal{D}}(0) &\geq \liminf_{R \to \infty} P * F_R(u/2) + \int_{\Lambda} \liminf_{R \to \infty} \left\{ - \sum_{\substack{n \neq 0 \\ |n_j| \leq R}} \prod_{j=1}^{d} \left(1 - \frac{|n_i|}{R+1}\right) \widehat{G}_\lambda(an)(-1)^{u \cdot n} \right\} d\nu(\lambda) \\ &= \liminf_{R \to \infty} P * F_R(u/2) + \int_{\Lambda} - \sum_{n \neq 0} \widehat{G}_\lambda(an)(-1)^{u \cdot n} d\nu(\lambda). \end{aligned} \tag{4.4.18}$$

Using the properties of the Fourier transform of $G_\lambda(x)$ and the definition of the Theta function (4.2.3) , we find that the term inside the integral in (4.4.18) is equal to

$$\prod_{j=1}^{d} \lambda_j^{-\frac{1}{2}} - \prod_{j=1}^{d} \Theta(\tfrac{1}{2}; i a_j^2/\lambda_j) \lambda_j^{-\frac{1}{2}}.$$

This proves the lower bound estimate (4.4.8), since $P * F_R$ is a non-negative function. Now, we turn to the proof of the L^1 convergence (4.4.9). By conditions (C1),(C2),(C3) and (C4-) we are at the position of applying Theorem 25 for the functions $L_{\lambda,a}(z)$ defined at the previous section. Let $\mathcal{L}_a(z)$ be the minorant given by Theorem 25 part (ii), for $K = Q(a)$, $G_\lambda(x) = G(x;\lambda)$ and $F(z;\lambda) = L_{\lambda,a}(z)$. We obtain

$$\int_{\mathbb{R}^d} \mathcal{G}_\lambda(x) - \mathcal{L}_a(x)dx = \int_\Lambda \int_{\mathbb{R}^d} G_\lambda(x) - L_{\lambda,a}(x)dxd\nu(\lambda) =$$
$$\int_\Lambda \left\{1 - \left\{\sum_{j=1}^d \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} - (d-1)\right\} \prod_{j=1}^d \Theta(0; ia_j^2/\lambda_j)\right\} \prod_{j=1}^d \lambda_j^{-\frac{1}{2}} \mathrm{d}\nu(\lambda). \quad (4.4.19)$$

Using Lemma 29 we see that the functions inside the integral form a decreasing sequence (indexed by $a \in \Lambda$) converging to zero as $a \uparrow \infty$. Therefore, by the monotone convergence theorem , the integral goes to 0 and the proof is complete. □

Proof of Lemma 29. To see that the functions

$$\phi_a : \lambda \in \Lambda \mapsto \left\{\sum_{j=1}^d \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} - (d-1)\right\} \prod_{j=1}^d \Theta(0; ia_j^2/\lambda_j) \quad (4.4.20)$$

form an increasing sequence of functions indexed by $a \in \Lambda$ as each $a_j \uparrow \infty$, is enough to prove that for every $\lambda \in \Lambda$ and $k \in \{1, ..., d\}$

$$\frac{\partial}{\partial a_k} \phi_a(\lambda) \geq 0.$$

For all $t > 0$ denote

$$u_j(t) = \Theta(0; it^2/\lambda_j) \quad \text{and} \quad v_j(t) = \Theta(\tfrac{1}{2}; it^2/\lambda_j),$$

we can use the Leibniz rule to obtain

$$\frac{\partial}{\partial a_k} \phi_a(\lambda) = \left(v_k'(a_k) + \left\{\sum_{\substack{j=1 \\ j \neq k}}^d \frac{v_j(a_j)}{u_j(a_j)} - (d-1)\right\} u_k'(a_k)\right) \prod_{\substack{j=1 \\ j \neq k}}^d u_j(a_j). \quad (4.4.21)$$

Using the summation formula (4.2.3), we see that the functions $t \mapsto (u_j(t) - 1)$ is a sum of positive decreasing functions that decreases to 0 as $t \uparrow \infty$, thus $u_j(t)$ is a decreasing function

that decreases to 1 as $t \uparrow \infty$. Analogously, using the product formula (4.3.16), each $v_j(t)$ is a product of positive increasing functions that increases to 1 as $t \uparrow \infty$, thus $v_j(t)$ is a positive increasing function that increases to 1 as $t \uparrow \infty$. Therefore, the term inside the parenthesis in (4.4.21) is positive, which implies that $\frac{\partial}{\partial a_k}\phi_a(\lambda) > 0$, and this proves item (i).
Since

$$\phi_a(\lambda) = \left\{ \sum_{j=1}^{d} \frac{v_j(a_j)}{u_j(a_j)} - (d-1) \right\} \prod_{j=1}^{d} u_j(a_j),$$

we see that $\phi_a(\lambda)$ converges to 1, for every $\lambda \in \Lambda$, as each $a_j \uparrow \infty$ and this proofs item (ii).

□

4.5 Further results

In this section we will use the machinery of the previous section to construct one-sided approximations by trigonometric polynomials. After that, as in Part III of [20], we will give some examples of functions that our method is applicable and then we present some Hilbert-type inequalities that arise from the constructions of section §4.4.

4.5.1 Periodic analogues

In this subsection we find the best approximations by trigonometric polynomials for functions that are, in some sense, subordinate to Theta functions. The proofs of the theorems in this section are almost identical to the proofs of the previous sections, and thus we state the theorems without proof.

Definition 30. Let $a = (a_1, a_2, ..., a_d) \in \mathbb{Z}_+^d$ (i.e $a_j \geq 1 \; \forall j$) , we will say that the degree of a trigonometric Polynomial $P(x)$ is less than a (degree $P < a$) if

$$P(x) = \sum_{-a<n<a} \widehat{P}(n) e(n \cdot x).$$

The problems we are interested to solve have the following general form

Periodic majorization problem. Fix an $a \in \mathbb{Z}_+^d$ (called the *degree*) and a Lebesgue measurable real periodic function $g : \mathbb{T}^d \to \mathbb{R}$. Determine the value of

$$\inf \int_{\mathbb{T}^d} |F(x) - g(x)| dx, \tag{4.5.1}$$

where the infimum is taken over functions $F : \mathbb{T}^d \to \mathbb{R}$ satisfing

(i) $F(x)$ is a real trigonometric polynomial.

(ii) Degree of $F(x)$ is less than a.

(iii) $F(x) \geq g(x)$ for every $x \in \mathbb{T}^d$.

If the infimum is achieved, then identify the extremal polynomial $F(z)$. Similarly, there exist the minorant problem

Periodic minorization problem. Solve the previous problem with condition (iii) replaced by the condition

$$\text{(iv) } F(x) \leq g(x) \text{ for every } x \in \mathbb{T}^d.$$

Now we define for every $\lambda \in \Lambda$, the periodization of the Gaussian function $G_\lambda(x)$ by

$$f_\lambda(x) := \sum_{n \in \mathbb{Z}^d} G_\lambda(x+n) = \sum_{n \in \mathbb{Z}^d} \prod_{j=1}^{d} e^{-\lambda_j \pi (x_j + n_j)^2} = \prod_{j=1}^{d} \Theta(x_j; i/\lambda_j) \lambda_j^{-\frac{1}{2}}.$$

Theorem 31 (Existence)**.** *For a given $a = (a_1, a_2, ..., a_d) \in \mathbb{Z}_+^d$, let Λ be a measurable space of parameters, and for each $\lambda \in \Lambda$, let $R_\lambda(x)$ be a real trigonometric polynomial with degree less than a. Let ν be a non–negative measure in Λ that satisfies*

$$\int_\Lambda \int_{\mathbb{T}^d} |R_\lambda(x) - f_\lambda(x)| dx d\nu(\lambda) < \infty. \tag{4.5.2}$$

Suppose that $g : \mathbb{T}^d \to \mathbb{R}$ is a continuous periodic function such that

$$\widehat{g}(k) = \int_\Lambda \widehat{G}_\lambda(k) d\nu(\lambda), \tag{4.5.3}$$

for all $k \in \mathbb{Z}^d$ such that $|k_j| \geq a_j$ for some $j \in \{1, 2, ..., d\}$. Then

(i) if $f_\lambda(x) \leq R_\lambda(x)$ for each $x \in \mathbb{T}^d$ and $\lambda \in \Lambda$, then there exist a trigonometric polynomial $m_a(x)$ with degree $m_a < a$, such that of $m_a(x) \geq g(x)$ for all $x \in \mathbb{T}^d$ and

$$\int_{\mathbb{T}^d} m_a(x) - g(x) dx$$

is equal to the (LHS) of (4.5.2).

(ii) if $R_\lambda(x) \leq f_\lambda(x)$ for each $x \in \mathbb{T}^d$ and $\lambda \in \Lambda$, then there exist a trigonometric polynomial $l_a(x)$ with degree $l_a < a$, such that $l_a(x) \leq g(x)$ in $\mathbb{T}^d$, and

$$\int_{\mathbb{T}^d} g(x) - l_a(x) dx,$$

is equal to (LHS) of (4.5.2).

Before we state the main theorems of this section we need some definitions. The functions $M_{\lambda,a}(x)$ and $L_{\lambda,a}(x)$, defined in (4.3.3) and (4.3.2), belong to $L^1(\mathbb{R}^d)$, thus, by the Plancharel-Pólya theorem (see [64]) and the periodic Fourier inversion formula, their respective periodizations are trigonometric polynomials of degree less than a, that is

$$m_{\lambda,a}(x) := \sum_{n \in \mathbb{Z}^d} M_{\lambda,a}(x+n) = \sum_{-a<n<a} \widehat{M}_{\lambda,a}(n) e^{2\pi i n \cdot x} \tag{4.5.4}$$

and

$$l_{\lambda,a}(x) := \sum_{n \in \mathbb{Z}^d} L_{\lambda,a}(x+n) = \sum_{-a<n<a} \widehat{L}_{\lambda,a}(n) e^{2\pi i n \cdot x} \tag{4.5.5}$$

holds for each $x \in \mathbb{T}^d$. The following theorem offers a resolution to the majorization problem for a specific class of functions.

Theorem 32 (Gaussian subordination – Periodic majorant). *Let $a \in \mathbb{Z}_+^d$ and ν be non-negative Borel measure on Λ that satisfies*

$$\int_\Lambda \prod_{j=1}^d \lambda_k^{-\frac{1}{2}} \left\{ \prod_{j=1}^d \Theta(0; i a_j^2/\lambda_j) - 1 \right\} \mathrm{d}\nu(\lambda) < \infty. \tag{4.5.6}$$

Let $g : \mathbb{T}^d \to \mathbb{R}$ be a continuous periodic function such that

$$\widehat{g}(k) = \int_\Lambda \widehat{G}_\lambda(k) d\nu(\lambda), \tag{4.5.7}$$

for all $k \in \mathbb{Z}^d$ such that $|k_j| \geq a_j$ for some $j \in \{1, 2, ..., d\}$. Then for every real trigonometric polynomial $P(x)$, with degree $P < a$ and $P(x) \geq g(x)$ for all $x \in \mathbb{T}^d$, we have

$$\int_{\mathbb{T}^d} P(x) - g(x) dx \geq \int_\Lambda \prod_{j=1}^d \lambda_k^{-\frac{1}{2}} \left\{ \prod_{j=1}^d \Theta(0; i a_j^2/\lambda_j) - 1 \right\} \mathrm{d}\nu(\lambda). \tag{4.5.8}$$

Moreover, there exists a real trigonometric polynomial m_a, with degree $m_a < a$, such that $m_a(x)$ is a majorant of $g(x)$ that interpolates $g(x)$ on the lattice $\mathbb{Z}^d/a$ and equality at (4.5.8) holds.

Theorem 33 (Gaussian subordination – Periodic minorant)**.** *Let $a \in \mathbb{Z}^d_+$ and ν be non-negative Borel measure on Λ such that*

$$\int_\Lambda \left\{1 - \left\{\sum_{j=1}^d \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} - (d-1)\right\} \prod_{j=1}^d \Theta(0; ia_j^2/\lambda_j)\right\} \prod_{j=1}^d \lambda_j^{-\frac{1}{2}} \mathrm{d}\nu(\lambda) < \infty. \tag{4.5.9}$$

Let $g : \mathbb{T}^d \to \mathbb{R}$ be continuous periodic function such that

$$\widehat{g}(k) = \int_\Lambda \widehat{G}_\lambda(k) d\nu(\lambda) \tag{4.5.10}$$

for all $k \in \mathbb{Z}^d$ such that $|k_j| \geq a_j$ for some $j \in \{1, 2, ..., d\}$. Then, if $P(z)$ is a real trigonometric polynomial with degree less than a that minorizes $g(x)$, we have

$$\int_{\mathbb{T}^d} g(x) - P(x) dx \geq \int_\Lambda \prod_{j=1}^d \lambda_k^{-\frac{1}{2}} \left\{1 - \prod_{j=1}^d \Theta(\tfrac{1}{2}; ia_j^2/\lambda_j)\right\} d\nu(\lambda). \tag{4.5.11}$$

Furthermore, there exists a family of trigonometric polynomial minorants $\{l_a(x) : a \in \mathbb{Z}^d_+\}$ with degree $l_a < a$, such that the integral

$$\int_{\mathbb{T}^d} g(x) - l_a(x) dx$$

is equal to the quantity in (4.5.9), and

$$\lim_{a \uparrow \infty} \int_{\mathbb{T}^d} g(x) - l_a(x) dx = 0, \tag{4.5.12}$$

where $a \uparrow \infty$ means $a_j \uparrow \infty$ for every j.

Corollary 34. *Under the hypothesis of Theorem 33, suppose also that there exist $R > 0$*

such that $\operatorname{supp}(\nu) \subset \Lambda \cap Q(R)$, *and*

$$\int_{\mathbb{T}^d} g(x)dx = \int_{\Lambda} \prod_{j=1}^{d} \lambda_j^{-\frac{1}{2}} \mathrm{d}\nu(\lambda) < \infty.$$

Then, there exist a constant $\alpha_0 > 0$*, such that if* $\alpha := \min\{a_j\} \geq \alpha_0$ *and if* $P(x)$ *is a trigonometric polynomial with degree* $P < a$ *that minorizes* $g(x)$*, we have*

$$\int_{\mathbb{T}^d} P(x)dx \leq (1 + 5de^{-\alpha^2/R}) \int_{\mathbb{T}^d} l_a(x)dx.$$

4.5.2 The class of admissible functions

We define the class

$$\mathfrak{G}^d = \bigcap_{a\in\Lambda} \mathfrak{G}^d_-(a) \cap \mathfrak{G}^d_+(a).$$

This is the class of pairs such that Theorems 26 and 27 are applicable for every $a \in \Lambda$. In this subsection we present conditions for a pair $(\mathcal{G}, \nu)$ belong to this class.

Some interesting properties arise when ν is concentrated in the diagonal. For every $\eta \in [0,1]$ we define $\Lambda_\eta = \{\lambda \in \Lambda : \eta\lambda_j \leq \lambda_k \forall j,k\}$ and we note that $\Lambda_0 = \Lambda$ and $\Lambda_1 = \{\lambda \in \Lambda : \lambda_j = \lambda_k \forall j,k\}$ is the diagonal.

Proposition 35. *Let* $(\mathcal{G}, \nu)$ *be a pair that satisfies conditions* **(C1)**,**(C2)** *and* **(C3)** *for every* $a \in \Lambda$*. Suppose that* $\operatorname{supp}(\nu) \subset \Lambda_\eta$ *for some* $\eta \in (0,1]$ *and* $\nu(\Lambda_\eta \setminus Q(R)) < \infty$ *for every* $R > 0$*. Then* $(\mathcal{G}, \nu) \in \mathfrak{G}^d$*.*

Proof. We only prove that condition **(C4-)** holds, the condition **(C4+)** is analogous. Given an $a \in \Lambda$, define the function

$$\phi_a : \lambda \in \Lambda \mapsto \left(\sum_{j=1}^{d} \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} - (d-1) \right) \prod_{j=1}^{d} \Theta(0; ia_j^2/\lambda_j).$$

By (4.2.3) and the Poisson summation formula , we have the following estimates

$$1 - \Theta(\tfrac{1}{2}; i/t) \sim 2e^{-\pi/t} \quad \text{as } t \to 0, \tag{4.5.13}$$

$$\Theta(\tfrac{1}{2}; i/t) \sim 2t^{\frac{1}{2}} e^{-\pi t/4} \quad \text{as } t \to \infty \tag{4.5.14}$$

and

$$\Theta(0;i/t)-1\sim 2e^{-\pi/t}\quad \text{as } t\to 0, \tag{4.5.15}$$
$$\Theta(0;i/t)\sim t^{\frac{1}{2}}\quad \text{as } t\to \infty \tag{4.5.16}$$

where the symbol $\sim$ means that the quotient converges to 1. Using the (LHS) inequality of Lemma 24 we conclude that exists an $R>0$ and a $C>0$ such that

$$\phi_a(\lambda)\geq 1-C\sum_{j=1}^{d} e^{-\pi a_j^2/\lambda_j}$$

for every $\lambda\in\Lambda\cap Q(R)$. Choose $l\in\{1,...,d\}$ such that $a_l\leq a_j$ for every j. If $\lambda\in\Lambda_\eta\cap Q(R)$ we have

$$\begin{aligned}\big\{1-\phi_a(\lambda)\big\}\prod_{j=1}^{d}\lambda_j^{-\frac{1}{2}} &\leq C\bigg(\sum_{j=1}^{d} e^{-\pi a_j^2/\lambda_j}\bigg)\prod_{j=1}^{d}\lambda_j^{-\frac{1}{2}}\leq dCe^{-\pi a_l^2\eta/\lambda_l}\prod_{j=1}^{d}\lambda_j^{-\frac{1}{2}}\\ &\leq dC\prod_{j=1}^{d} e^{-\pi a_l^2\eta^2/(d\lambda_j)}\prod_{j=1}^{d}\lambda_j^{-\frac{1}{2}}=dC\widehat{G}_\lambda(\beta u),\end{aligned}$$

where $\beta=a_l\eta d^{-\frac{1}{2}}$ and $u=(1,...,1)$. By estimates (4.5.13)-(4.5.16) we see that the functions $\Theta(\frac{1}{2};i/t)t^{-\frac{1}{2}}$ and $\Theta(0;i/t)t^{-\frac{1}{2}}$ are bounded for $t\in[\eta R,\infty)$, and thus, we conclude that the function

$$\lambda\in\Lambda\mapsto\phi_a(\lambda)\prod_{j=1}^{d}\lambda_j^{-\frac{1}{2}}$$

is bounded on $\Lambda_\eta\setminus Q(R)$, since it is a finite sum of products of these theta functions. Since $\eta>0$, we obtain that the function

$$\lambda\in\Lambda\mapsto\big\{1-\phi_a(\lambda)\big\}\prod_{j=1}^{d}\lambda_j^{-\frac{1}{2}}$$

is bounded in $\Lambda_\eta\setminus Q(R)$, let say by C'. Therefore, we have

$$\int_{\Lambda_\eta} \{1-\phi_a(\lambda)\} \prod_{j=1}^{d} \lambda_j^{-\frac{1}{2}} \mathrm{d}\nu(\lambda) \le dC \int_{\Lambda_\eta \cap Q(b)} \widehat{\mathcal{G}}_\lambda(\beta u)\mathrm{d}\nu(\lambda) + C'\nu(\Lambda_\eta \setminus Q(b)) < \infty,$$

which is finite by condition **(C3)** and the hypotheses of this lemma. Thus ν satisfies condition **(C4-)** and this concludes the proof.

□

Proposition 36. *Let ν be a probability measure on Λ with* $\mathrm{supp}(\nu) \subset \Lambda_\eta$ *for some* $\eta \in (0,1]$. *Define the function*

$$\mathcal{G}(x) = \int_\Lambda G_\lambda(x)\mathrm{d}\nu(\lambda) \tag{4.5.17}$$

for all $x \in \mathbb{R}^d$. *Then* $(\mathcal{G}, \nu) \in \mathfrak{G}^d$.

Proof. Is easy to see that $\mathcal{G}$ is a bounded, continuous and radially decreasing function, thus $\mathcal{G}$ satisfies conditions **(C1)**. Note that for every $x \neq 0$, the function

$$\lambda \in \Lambda \mapsto \widehat{G}_\lambda(x)$$

is bounded on Λ_η and this bound can be taken uniform for x outside any neighborhood of the origin. Thus condition **(C2)** holds, and using Fubini's theorem, condition **(C3)** also holds. We conclude, by Proposition 35, that $(\mathcal{G}, \nu) \in \mathfrak{G}^d$.

□

Due to a clasical result of Schoenberg (see [66]), a radial function $\mathcal{G}(x) = \mathcal{G}(|x|)$ admits the representation (4.5.17) for a probability ν supported on the diagonal Λ_1 if and only if the radial extension to $\mathbb{R}^n$ of $\mathcal{G}(r)$ is positive definite, for all $n > 0$. And this occurs if and only if the function $\mathcal{G}(r^{\frac{1}{2}})$ is completely monotone. As consequence of this fact and Proposition 36 the following multidimensional versions of the functions in Section 11 of [20] are admissible

Example 1.

$$g(x) = e^{-\alpha|x|^r} \in \mathfrak{G}^d, \quad \alpha > 0 \quad \textit{and} \quad 0 < r < 2.$$

Example 2.

$$g(x) = (|x|^2 + \alpha^2)^{-\beta} \in \mathfrak{G}^d, \ \alpha > 0 \ \textit{and} \ \beta > 0.$$

Example 3.

$$g(x) = -\log\left(\frac{|x|^2+\alpha^2}{|x|^2+\beta^2}\right) \in \mathfrak{G}^d,\ \ for\ 0<\alpha<\beta.$$

The following example is a high dimensional analogue of Corollary 21 [20].

Example 4. Given a $\sigma \in (0,\infty)\setminus 2\mathbb{Z}_+$ consider the measure ν_σ, that is supported on the diagonal Λ_1 and defined on a Borel set $E \subset \Lambda$ by

$$\nu_\sigma(E) = C(\sigma)\int_{P(E\cap\Lambda_1)} t^{-\frac{\sigma}{2}-1}\mathrm{d}t \tag{4.5.18}$$

where $P : \Lambda_1 \to \mathbb{R}$ is the projection $P(tu) = t$ for all $t \geq 0$ and

$$C(\sigma) = \pi^{-\frac{\sigma}{2}}/\Gamma(-\frac{\sigma}{2}).$$

where $\Gamma(z)$ is the classical Gamma function. Define on $\mathbb{R}^d$ the function

$$\mathcal{G}_\sigma(x) = |x|^\sigma \ \text{ for } \ \sigma \in (0,\infty)\setminus 2\mathbb{Z}_+$$

we claim that $(\mathcal{G}_\sigma, \nu_\sigma) \in \mathfrak{G}^d$. To see this, first we note that Lemma 18 of [20] has a trivial generalization to the several variable setting, that is, for every function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ that is zero on a neighborhood of the origin, we have

$$\int_{\mathbb{R}^d}\widehat{\varphi}(x)|x|^\sigma\mathrm{d}x = A(d,\sigma)\int_{\mathbb{R}^d}\varphi(x)|x|^{-d-\sigma}\mathrm{d}x,$$

where

$$A(d,\sigma) = \pi^{-\sigma-d/2}\Gamma(\frac{d+\sigma}{2})/\Gamma(-\frac{\sigma}{2}).$$

Secondly, note that

$$\frac{A(d,\sigma)}{|x|^{d+\sigma}} = \int_\Lambda \widehat{G}_\lambda(x)\mathrm{d}\nu_\sigma(\lambda).$$

Hence, by Proposition 35 the pair $(\mathcal{G}_\sigma, \nu_\sigma)$ belongs to $\mathfrak{G}^d$.

The periodic case

Given a pair $(\mathcal{G}, \nu) \in \mathfrak{G}^d$, suppose that $\mathcal{G} \in L^1(\mathbb{R}^d)$ and the periodization

$$x \in \mathbb{T}^d \mapsto \sum_{n \in \mathbb{Z}} \mathcal{G}(n + x)$$

is equal almost everywhere to a continuous function $g(x)$. We easily see that the pair (g, ν) is admissible by the Theorems 32 and 33 for every degree $a \in \mathbb{Z}_+^d$. Thus, the period method contemplates the following functions

Example 5.

$$g(x) = \prod_{j=1}^{d} \Theta(x_j; i/\lambda_j) = \prod_{j=1}^{d} \lambda_j^{\frac{1}{2}} \sum_{n \in \mathbb{Z}^d} G_\lambda(n + x), \text{ for all } \lambda \in \Lambda. \tag{4.5.19}$$

Example 6.

$$g(x) = \sum_{n \in \mathbb{Z}^d} e^{-\alpha |n+x|^r}, \text{ for all } \alpha > 0 \text{ and } 0 < r < 2. \tag{4.5.20}$$

However, we cannot use this construction for the case of the functions $\mathcal{G}_\sigma(x) = |x|^\sigma$ of example 4. The next proposition tell us that if the Fourier coefficients of $\mathcal{G}$ decay sufficiently fast, then the periodization of $\mathcal{G}$ via Poisson summation formula is admissible by the periodic method.

Proposition 37. *Let $(\mathcal{G}, \nu) \in \mathfrak{G}^d$. Suppose that exist constants $C > 0$ and $\delta > d/2$ such that*

$$\int_\Lambda \widehat{G}_\lambda(x) \mathrm{d}\nu(\lambda) \leq C|x|^{-\delta}$$

if $|x| \geq 1$. Define the function

$$x \mapsto \lim_{N \to \infty} \sum_{\substack{n \in \mathbb{Z}^d \\ 0 < |n| < N}} \int_\Lambda \widehat{G}_\lambda(n) \mathrm{d}\nu(\lambda) e(n \cdot x),$$

where the limit is taken in $L^2(\mathbb{T}^d)$, and suppose this can be identified with a continuous function $g(x)$. Then the pair (g, ν) satisfies all the conditions of Theorems 32 and 33 for every $a \in \mathbb{Z}_+^d$.

With this last proposition we see that the following example is admissible by the periodic method of subsection 4.5.1.

Example 7.

$$g_\sigma(x) = \sum_{\substack{n\in\mathbb{Z}^d \\ n\neq 0}} |n|^{-d-\sigma} e(n\cdot x), \text{ for all } \sigma > 0. \tag{4.5.21}$$

4.5.3 Hilbert-type inequalities

Given an $a \in \Lambda$ and $x \in \mathbb{R}^d$ define the norm

$$|x|_a = \max\{|x_j/a_j| : j = 1, ..., d\}.$$

Given an $a \in \Lambda$ we say that a sequence $\{\xi_k\}_{k\in\mathbb{Z}}$ of vectors in $\mathbb{R}^d$ is a-separated if $|\xi_k - \xi_l|_a \geq 1$ for all $l \neq k$. We have the following proposition

Proposition 38. *Let $(\mathcal{G}, \nu) \in \mathfrak{G}^d$, $a \in \Lambda$ and $\{\xi_k\}_{k\in\mathbb{Z}}$ an a-separated sequence of vectors. Then for every finite sequence of complex numbers $\{w_{-N}, ..., w_0, ..., w_N\}$, we have*

$$-A(a,d,\nu)\sum_{n=-N}^{N}|w_n|^2 \leq \sum_{\substack{n,m=-N \\ n\neq m}}^{N} w_n\overline{w}_m\widehat{\mathcal{G}}(\xi_n - \xi_m) \leq B(a,d,\nu)\sum_{n=-N}^{N}|w_n|^2 \tag{4.5.22}$$

where $A(a,d,\nu)$ is equal to the quantity (4.4.6) and $B(a,d,\nu)$ is equal to the quantity (4.4.5). Furthermore the constant $B(a,d,\nu)$ is sharp.

Remark: For $\xi \neq 0$ we write

$$\widehat{\mathcal{G}}(\xi) = \int_\Lambda \widehat{G}_\lambda(\xi)\mathrm{d}\nu(\lambda).$$

Proof. To prove inequality in (4.5.22), define the function $D = \mathcal{M}_a - \mathcal{G}$ where $\mathcal{M}_a$ is given by Theorem 26. Since $\mathcal{M}_a$ if of exponential type with respect to $Q(a)$ we obtain

$$\sum_{n,m=-N}^{N} w_n\overline{w}_m\widehat{\mathcal{D}}(\xi_n - \xi_m) = B(a,d,\nu)\sum_{n=-N}^{N}|w_n|^2 - \sum_{\substack{n,m=-N \\ n\neq m}}^{N} w_n\overline{w}_m\widehat{\mathcal{G}}(\xi_n - \xi_m).$$

But the (LHS) of this last equality is positive since $\mathcal{D}(x)$ is a non-negative function. In an analogous way we can prove the (LHS) inequality of (4.5.22).

For the sharpness of the (RHS) inequality in (4.5.22) suppose that we could change $B(a,d,\nu)$ by some other constant B'. Consider the sequence $w_n = 1$ for all $n \in \mathbb{Z}$ and let the sequence $\{\xi_n\}$ be an enumeration of the points in the set $J(R) = Q(Ra) \cap a\mathbb{Z}^d$ where $R \in \mathbb{Z}_+$. We obtain

$$-\sum_{\xi,\xi' \in J(R)} \widehat{\mathcal{D}}(\xi - \xi') = \sum_{\substack{\xi,\xi' \in J(R) \\ \xi \neq \xi'}} \widehat{\mathcal{G}}(\xi - \xi') - (2R+1)^d \widehat{\mathcal{D}}(0) \leq (2R+1)^d (B' - \widehat{D}(0)).$$

On the other hand we have

$$\sum_{\xi,\xi' \in J(R)} \widehat{\mathcal{D}}(\xi - \xi') = \sum_{|n_j| \leq 2R} \left\{ \prod_{j=1}^{d} (2R + 1 - |n_j|) \right\} \widehat{\mathcal{D}}(an).$$

Hence, we conclude that

$$B' - \widehat{\mathcal{D}}(0) \geq -\sum_{|n_j| \leq 2R} \prod_{j=1}^{d} \left(1 - \frac{|n_j|}{2R+1}\right) \widehat{\mathcal{D}}(an) - = F_{2R} * P(0),$$

where $F_{2R}(x)$ is the Fejér's Kernel defined in (4.4.15) and

$$P(x) = \frac{1}{\alpha} \sum_{n \in \mathbb{Z}^d} \mathcal{D}((n+x)/a),$$

with $\alpha = a_1 ... a_d$. Applying again the arguments of the Theorem 26 we would conclude that

$$\liminf_{R \to \infty} F_{2R} * P(0) = 0$$

and this implies that $B' \geq \widehat{\mathcal{D}}(0)$, since $\widehat{\mathcal{D}}(0) = B(a,d,\nu)$ this concludes the proof.

□

The next corollary is a generalization of Corollary 22 of [20] in the multidimensional setting and is a direct application of Proposition 38 for example 4. Below $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^d$.

Corollary 39. *Let $\sigma > 0$, $a \in \Lambda$ and $\{\xi_k\}_{k \in \mathbb{Z}}$ an a-separated sequence of vectors. Then for* *every finite sequence of complex numbers* $\{w_{-N}, ..., w_0, ..., w_N\}$*, we have*

$$-A(a,d,\sigma)\sum_{n=-N}^{N}|w_n|^2 \le \sum_{\substack{n,m=-N\\ n\neq m}}^{N}\frac{w_n\overline{w}_m}{|\xi_n-\xi_m|^{d+\sigma}} \le B(a,d,\sigma)\sum_{n=-N}^{N}|w_n|^2, \tag{4.5.23}$$

where

$$A(a,d,\sigma) = -\sum_{j=1}^{d}\sum_{n\in\mathbb{Z}^d\setminus 0}\frac{(-1)^{n_j}}{|an|^{d+\sigma}} + (d-1)\sum_{n\in\mathbb{Z}^d\setminus 0}\frac{1}{|an|^{d+\sigma}}$$

and

$$B(a,d,\sigma) = \sum_{n\in\mathbb{Z}^d\setminus 0}\frac{1}{|an|^{d+\sigma}}.$$

Furthermore, the constant $B(a,d,\sigma)$ *is sharp.*

4.6 Concluding remarks

We mentioned in the introduction that Selberg generalized his construction of majorizing and minorizing the characteristic function of an interval, to majorizing and minorizing the characteristic function of a box by functions whose Fourier transforms are supported in a (possibly different) box. Another way to generalize Selberg's original construction to the several variables setting is to consider majorizing and minorizing the characteristic function of a ball by functions whose Fourier transforms are supported in a (possibly different) ball. This problem was considered by Holt and Vaaler [42] and their methods were recently extended by Carneiro and Littmann [19].

As far as we know, almost nothing is known about the Beurling-Selberg extremal problem in higher dimensions when the Fourier transform is supported on a fixed symmetric convex body K. The following question (perhaps the simplest Beurling-Selberg extremal problem in higher dimensions) is open:

> Let K be a symmetric convex body in $\mathbb{R}^d$. Determine the value of
>
> $$\eta(K) = \inf \int_K F(x)dx$$
>
> where the infimum is taken over continuous integrable functions $F : \mathbb{R}^d \to \mathbb{R}$ that satisfy (i) $F(0) \ge 1$, (ii) $F(x) \ge 0$ for all $x \in \mathbb{R}^d$, and (iii) $\hat{F}(\xi) = 0$ if $\xi \notin K$.

In one dimension, the solution to this problem is the Fejér kernel for $\mathbb{R}$. So the solution to the above problem can be thought of as an analogue of the Fejér kernel for K.

It is conjectured [6] by Michael Kelly and Gabriele Bianchi, that $\eta(K) = 2^d/\mathrm{vol}_d(K)$. Vaaler [76] has shown that this conjecture is true if K is an *extremal body.* That is, if K achieves equality in Minkowski's convex body theorem, then the conjecture in [6] is true. For example, the regular hexagon in $\mathbb{R}^2$ is an extremal body. The only other body K for which the conjecture is known to hold is the Euclidean ball.

$\square$

Bibliography

[1] J. M. Aldaz, L Colzani and J. Pérez Lázaro, Optimal bounds on the modulus of continuity of the uncentered Hardy-Littlewood maximal function, J. Geom. Anal. 22 (2012), 132–167.

[2] J. M. Aldaz and J. Pérez Lázaro, Functions of bounded variation, the derivative of the one dimensional maximal function, and applications to inequalities, Trans. Amer. Math. Soc. 359 (2007), no. 5, 2443–2461.

[3] R. C. Baker, Sequences that omit a box (modulo 1), Adv. Math. 227 (2011), no. 5, 1757–1771.

[4] J. T. Barton, H. L. Montgomery, and J. D. Vaaler, Note on a Diophantine inequality in several variables, Proc. Amer. Math. Soc. 129 (2001), no. 2, 337–345.

[5] S. Barza and M. Lind, A new variational characterization of Sobolev spaces, J. Geom. Anal. 25 (2015), 2185–2195

[6] G. Bianchi and M. Kelly, A Fourier Analytic Proof of the Blaschke-Santal Inequality, Proc. Amer. Math. Soc. 143 (2015), 4901–4912 .

[7] R. P. Boas, Jr, *Entire functions*, Academic Press Inc. (1954), New York.

[8] J. Bober, E. Carneiro, K. Hughes and L. B. Pierce, On a discrete version of Tanaka's theorem for maximal functions, Proc. Amer. Math. Soc. 140 (2012), 1669–1680.

[9] H. Brezis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (1983), 486–490.

[10] E. Carneiro, Sharp approximations to the Bernoulli periodic functions by trigonometric polynomials, J. Approx. Theory, 154, no. 2 (2008), 90–104.

[11] E. Carneiro and V. Chandee, Bounding $\zeta(s)$ in the critical strip, J. Number Theory, 131, no. 3 (2011), 363–384.

[12] E Carneiro, V Chandee, F. Littmann and M. Milinovich, Hilbert spaces and the pair correlation of zeros of the riemann zeta-function, J. Reine Angew. Math. (to appear).

[13] E. Carneiro, V. Chandee and M. Milinovich, Bounding $S(t)$ and $S_1(t)$ on the Riemann hypothesis, Math. Ann. 356 (2013), 939–968.

[14] E. Carneiro, R. Finder and M. Sousa, On the variation of maximal operators of convolution type II, preprint at http://arxiv.org/abs/1512.02715v1.

[15] E. Carneiro and F.F Gonçalves, Extremal problems in de Branges spaces: The case of truncated and odd functions. Mathematische Zeitschrift 280 (2015), 17–45.

[16] E. Carneiro and K. Hughes, On the endpoint regularity of discrete maximal operators, Math. Res. Lett. 19, no. 6 (2012), 1245–1262.

[17] E. Carneiro and F. Littmann, Extremal functions in de Branges and Euclidean spaces, Adv. Math. 260 (2014), 281–349.

[18] E. Carneiro and F. Littmann, Bandlimited approximations to the truncated Gaussian and applications, Constr. Approx. 38 (2013), 19–57.

[19] E. Carneiro and F. Littmann, Entire approximations for a class of truncated and odd functions, J. Fourier Anal. Appl. 19 (2013), 967–996.

[20] E. Carneiro, F. Littmann, and J. D. Vaaler, Gaussian subordination for the Beurling-Selberg extremal problem, Trans. Amer. Math. Soc. 365 (2013), 3493–3534.

[21] E. Carneiro and J. Madrid, Derivative bounds for fractional maximal functions, Trans. Amer. Math. Soc., to appear.

[22] E. Carneiro and D. Moreira, On the regularity of maximal operators, Proc. Amer. Math. Soc. 136 (2008), no. 12, 4395–4404.

[23] E. Carneiro and B. F. Svaiter, On the variation of maximal operators of convolution type, J. Funct. Anal. 265 (2013), 837–865.

[24] E. Carneiro and J. D. Vaaler, Some extremal functions in Fourier analysis, II, Trans. Amer. Math. Soc. 362 (2010), 5803–5843.

[25] E. Carneiro and J. D. Vaaler, Some extremal functions in Fourier analysis, III, Constr. Approx. 31, no. 2 (2010), 259–288.

[26] V. Chandee and K. Soundararajan, Bounding $|\zeta(1/2+it)|$ on the Riemann hypothesis, Bull. Lond. Math. Soc. 43, no. 2 (2011), 243250.

[27] K. Chandrasekharan, Elliptic functions, volume 281 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin (1985).

[28] T. Cochrane, Trigonometric approximation and uniform distribution modulo one, Proc. Amer. Math. Soc. 103, no. 3 (1988), 695–702.

[29] L. de Branges, Homogeneous and periodic spaces of entire functions, Duke Math. J. 29 (1962), 203–224.

[30] L. de Branges, *Hilbert spaces of entire functions*. Prentice-Hall Inc., Englewood Cliffs, N.J. (1968).

[31] M. Drmota and R. F. Tichy, Sequences, discrepancies and applications, volume 1651 of Lecture Notes in Mathematics. Springer-Verlag, Berlin (1997).

[32] P. X. Gallagher, Pair correlation of zeros of the zeta function, J. Reine Angew. Math. 362 (1985), 72–86.

[33] D. A. Goldston and S. M. Gonek, A note on S(t) and the zeros of the Riemann zeta-function, Bull. London Math. Soc. 39 (2007), 482–486.

[34] F.F. Gonçalves, M. Kelly and J. Madrid, One-sided band-limited approximations of some radial functions, Bulletin of the Brazilian Mathematical Society 46, no. 4 (2015), 563–599 .

[35] L. Grafakos, *Classical Fourier analysis*, volume 249 of Graduate Texts in Mathematics. Springer, New York, second edition (2008).

[36] S. W. Graham and J. D. Vaaler, A class of extremal functions for the Fourier transform, Trans. Amer. Math. Soc. 265 (1981), 283–382.

[37] P. Hajłasz and J. Maly, On approximate differentiability of the maximal function, Proc. Amer. Math. Soc. 138 (2010), 165–174.

[38] P. Hajłasz and J. Onninen, On boundedness of maximal functions in Sobolev spaces, Ann. Acad. Sci. Fenn. Math. 29 (2004), no. 1, 167–176.

[39] G. Harman, Small fractional parts of additive forms, Philos. Trans. Roy. Soc. London Ser. A, 345 (1676) (1993), 327–338.

[40] G. Harman, Metric number theory, volume 18 of London Mathematical Society Monographs. New Series. The Clarendon Press Oxford University Press, New York (1998).

[41] A. Haynes, M. Kelly and B. Weiss, Equivalence relations on separated nets arising from linear toral flows, Proc. London Math. Soc., to appear.

[42] J. J. Holt and J. D. Vaaler, The Beurling-Selberg extremal functions for a ball in Euclidean space, Duke Math. J. 83, no. 1 (1996), 202–248.

[43] L. Hörmander, *The analysis of linear partial differential operators.*. I, volume 256 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1983. Distribution theory and Fourier analysis.

[44] M. Kelly and T. H. Lê, Uniform dilations in higher dimensions, Journal of the London Mathematical Society, 88, no. 3 (2013), 925–940.

[45] J. Kinnunen, The Hardy-Littlewood maximal function of a Sobolev function, Israel J. Math. 100 (1997), 117–124.

[46] J. Kinnunen and P. Lindqvist, The derivative of the maximal function, J. Reine Angew. Math. 503 (1998), 161–167.

[47] J. Kinnunen and E. Saksman, Regularity of the fractional maximal function, Bull. London Math. Soc. 35 (2003), no. 4, 529–535.

[48] M.G. Krein, On the best approximation of continuous differentiable functions on the whole real axis, Dokl. Akad. Nauk SSSR (Russian), 18 (1938), 615–624.

[49] O. Kurka, On the variation of the Hardy-Littlewood maximal function, Ann. Acad. Sci. Fenn. Math. 40 (2015), 109–133.

[50] S. Lang, *Algebraic number theory*, Addison-Wesley Publishing Co. Inc. (1970).

[51] X. Li and J. D. Vaaler, Some trigonometric extremal functions and the Erdős-Turán type inequalities, Indiana Univ. Math. J. 48, no. 1 (1999), 183–236.

[52] F. Littmann, Entire approximations to the truncated powers, Constr. Approx. 22, no. 2 (2005), 273–295.

[53] F. Littmann, One-sided approximation by entire functions, J. Approx. Theory, 141, no. 1 (2006), 1–7.

[54] F. Littmann, Quadrature and extremal bandlimited functions, SIAM J. Math. Anal. 45, no. 2 (2013), 732–747.

[55] B. F. Logan, Bandlimited functions bounded below over an interval, Notices of the Amer. Math. Soc. 24 (1977).

[56] H. Luiro, Continuity of the maximal operator in Sobolev spaces, Proc. Amer. Math. Soc. 135 (2007), no. 1, 243–251.

[57] H. Luiro, On the regularity of the Hardy-Littlewood maximal operator on subdomains of $\mathbb{R}^d$, Proc. Edinburgh Math. Soc. 53 (2010), no 1, 211–237.

[58] J. Madrid, Sharp inequalities for the variation of the discrete maximal function, preprint.

[59] H. L. Montgomery, *Ten lectures on the interface between analytic number the- ory and harmonic analysis*, volume 84 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI (1994).

[60] H. L. Montgomery, The analytic principle of the large sieve, Bull. Amer. Math. Soc. 84, no. 4 (1978), 547–567.

[61] H.L. Montgomery and R.C. Vaughan, Hilberts inequality, J. London Math. Soc. (2), 8 (1974), 73–82.

[62] I. P. Natanson, *Theory of Functions of a Real Variable*, Frederick Ungar Publishing Co., New York (1950).

[63] L. Pierce, *Discrete analogues in harmonic analysis*, Ph.D. Thesis (2009), Princeton University.

[64] M. Plancherel and G. Pólya, Fonctions entieres et intégrales de Fourier multiples, Comment. Math. Helv. 10, no. 1 (1937), 110–163.

[65] M. Rosenblum and J. Rovnyak, *Topics in Hardy classes and univalent functions.* Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel (1994).

[66] I.J. Schoenberg, Metric spaces and completely monotone functions, Ann. of Math. (2), 39, no. 4 (1938), 811–841

[67] A. Selberg, *Remarks on sieves.* In Proceedings of the Number Theory Conference (Univ. Colorado, Boulder, Colo., 1972), pages 205–216. Univ. Colorado, Boulder, Colo., (1972).

[68] A. Selberg, *Collected papers. Vol. II.* Springer-Verlag, Berlin, 1991. With a foreword by K. Chandrasekharan.

[69] E. M. Stein, Functions of exponential type, Ann. of Math. (2), 65 (1957), 582–592.

[70] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, 1970.

[71] E. M. Stein and R. Shakarchi, *Complex analysis*, Princeton Lectures in Analysis, II. Princeton University Press, Princeton, N.J., (2003).

[72] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[73] S. Steinerberger, A rigidity phenomenon for the Hardy-Littlewood maximal function, preprint at http://arxiv.org/abs/1410.0588

[74] H. Tanaka, A remark on the derivative of the one-dimensional Hardy-Littlewood maximal function, Bull. Austral. Math. Soc. 65 (2002), no. 2, 253–258.

[75] F. Temur, On regularity of the discrete Hardy-Littlewood maximal function, preprint at http://arxiv.org/abs/1303.3993.

[76] J. Vaaler, Personal communication. (2012).

[77] J. Vaaler, Some extremal functions in Fourier analysis, Bull. Amer. Math. Soc. (N.S.), 12, no. 2 (1985), 183–216.

[78] A. Zygmund, *Trigonometric series. 2nd ed. Vols. I, II*, Cambridge University Press, New York (1959).

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