Instituto Nacional de Matemática Pura e Aplicada

Doctoral Thesis

From Reaction-Diffusion Models to the study of Stochastic Differential Equations

Conrado Freitas Paulo da Costa

Rio de Janeiro

2016

Instituto Nacional de Matemática Pura e Aplicada

Conrado Freitas Paulo da Costa

FROM REACTION DIFFUSION MODELS TO THE STUDY OF STOCHASTIC DIFFERENTIAL EQUATIONS

Thesis presented to the Post-graduate Program in Mathematics at Instituto de Matemática Pura e Aplicada as partial fulfillment of the requirements for the degree of Doctor in Philosophy in Mathematics.

Advisor: Milton Jara

Rio de Janeiro
March 16, 2016

Aos que vierem depois de nós, que sejam livres
para a amizade e para o pensamento.

Agradecimentos

Gostaria de agradecer a todos que me ajudaram nessa trajetória. Em especial, ao Tertuliano Franco, Daniel Valesin, Ivaldo Nunes pelas conversas durante os cursos de verão.

Ao Francisco Ganacim e Marco Mendes, meus amigos e mestres. Vocês me ensinaram muito sobre a vida e a Matemática.

Ao meu orientador Milton Jara que em um momento crucial me motivou a continuar. Obrigado Milton, eu jamais faria o doutorado em Matemática se não tivéssemos conversado.

Aos professores do grupo de Probabilidade do IMPA, Cláudio Landim, Roberto (Imbuzeiro) de Oliveira, Augusto Teixeira pelos incentivos e apoios para as mais diversas dúvidas.

Aos professores Elon Lages Lima, e Emanuel Carneiro e Benar Svaiter pelas lições de matemática.

Aos funcionários do IMPA em especial à divisão de Ensino pelos cuidados e orientações.

Aos meus amigos de faculdade, economistas que sonham e animam o sonho de um Brasil melhor.

Em especial, agradeço ao meu irmão, Bernardo da Costa, que me ajudou mais do que seria possível imaginar, desde a monografia até a tese de doutorado, seja por telefone, email ou pessoalmente.

Agradeço aos esforços cooperativos que construiram a wikipedia.com o math.stackexchange.com, o http://gen.lib.rus.ec e o sci-hub.io.

Agradeço ao Johel Beltran, pela atenção dedicada ao problema da unicidade de soluções para a equação diferencial da tese e por sugerir a leitura do artigo do Yamada e Watanabe de 1971.

Agradeço aos membros da minha banca de tese, Tertuliano Franco, Luca Avena, Bernardo de Lima, Augusto Teixeira, Roberto Oliveira e Milton Jara, pelo comentários e correções indicadas para a versão final da tese.

Gostaria também de agradecer à universidade de Leiden e à hospitalidade do instituto de matemática. Em especial aos professores Luca Avena e Frank den Hollander.

E por fim, ao meu Pai e minha Mãe, que me ajudaram desde os primeiros passos. Não sei o que seria sem vocês, obrigado.

The World is strange, the whole universe is very strange, but see when you look at the details and you find out that the rules are very simple of the game, the mechanical rules by which you can figure out exactly what is going to happen when the situation is simple, is again this chess game... If you were in just a corner where only a few pieces are involved you can work out exactly what will happen. And you can always do that when there are only a few pieces so you know that you are understanding, and yet in the real game is so many pieces that you can't figure out what is going to happen. So there was a kind of hierarchy of different complexities. It's hard to believe it's incredible in fact most people don't believe, that the behaviour of, say me, one jack jack and you, nodding and all, this stuff is the result of a lot and lot of atoms all obeying very simple rules... Come out, that it evolves into such a creature with a billion years of life with its experiences that has produced the things with prawns that stick out like this and so on... the real... there is such a lot in the world there is so much distance between the fundamental rules and the final phenomena that is almost unbelievable that the final variety of phenomena can come from such a steady operation of such simple rules.

But you have to build from this complex scaffolding to find out the simple rules but it is not complicated, it's just a lot of it, and if you start at the beginning which nobody want to do... I mean, you come in to me now in interview and you are asking me about the latest discoveries that have been made, nobody asks about a simple ordinary phenomena in the street: oh like What about those collors or something like that... I would have a nice interview explaining all about the colors, Butterfly wings, whole big deal you don't care about that, you want the big final result. That is going to be complicated because I am at the end of 400 years of very effective method of finding things out about the world.

It has to do with curiosity, it has to do with people wondering what makes something do something and then to discover that if you try to get's answers they are related to each other that the things that make the wind make the waves, and the motion of water is like the motion of air and is like the motion of sand. The fact that things have common features turns out more and more universal. What we are looking for is how every thing works and what makes everything work, but is curiosity, is the way we are, what we are. It is very much more exciting to discover that we are on a ball, heaven sticked upside down spinning arround in space with as a misterious force which hold us on going arround a big globe of gas that is burning by a fuel by a fire that is completely different that any fire we can make, but know we can make that fire nuclear fire, you know.... That is much more exciting story to many people than the tales which other people used to make up, who were worried about the universe that we were living on back of a turtle or something like that they were wonderful stories, but the truth is so much more remarkable and so, what is the pleasure of physics to me is as is revealed that the truth is so remarkable so amazing and I can't.... I have this disease and many other people that have studied far enough to begin to understand a little how things work are fascinated by it and this fascination drives them on to such an extent that they've able to convince governments and so on to keeps supporting them and this investigation that the race is making into it's own environment.

THE FEYNMAN SERIES - Curiosity
https://www.youtube.com/watch?v=lmTmGLzPVyM

Abstract

Fix a finite graph (V, E). In this article we construct a family of Reaction-Diffusion models that converge after scaling to a solution to the following Stochastic Differential Equations (SDE's):

$$\begin{cases} d\zeta_t(x) = \left[\Delta_V \zeta_t(x) - \beta \left(\zeta_t(x)\right)^k\right] dt + \sqrt{\alpha \left(\zeta_t(x)\right)^l}\, dB_t^x \quad \forall x \in V \\ \zeta_0(x) = \rho_0(x) \end{cases}$$

where $\alpha, \beta > 0$ and k, l are positive integers such that $k > l$.

We show that the limiting points of the tight familly of processes are solutions to a well posed martingale problem and that the limiting measure corresponds to the law induced by a solution to the SDE that we began with.

Keywords: Reaction diffusion models, Scaling limit of particle systems, Stochastic Differential Equations (SDE's), Martingale Problems

Resumo

Fixe um grafo (V, E). Nesse trabalho iremos construir uma família de modelos dereação e difusão que após rescalados convergem para a solução da sequinte Equação Diferencial Estocástica (EDE):

$$\begin{cases} d\zeta_t(x) = \left[\Delta_V \zeta_t(x) - \beta\left(\zeta_t(x)\right)^k\right] dt + \sqrt{\alpha\left(\zeta_t(x)\right)^l}\, dB_t^x \quad \forall x \in V \\ \zeta_0(x) = \rho_0(x) \end{cases} \tag{1}$$

onde $\alpha, \beta > 0$ e k, l são numeros naturais tais que $k > l$, existe uma família de modelos de reação e difusão que converge depois de escalados para a solução da equação diferencial dada.

Mostraremos que os limites de escala dos processoos acima considerados correspondem à solução de um problema martingal bem posto e que a lei induzida por ela trata-se da lei induzida pela solução da equação diferencial que tomamos como ponto de partida.

Palavras-chave: Modelos de reação e difusão, Limites de escala de sistemas de partículas, Equações Diferenciais Estocásticas (EDE's), problemas martingais

Contents

Chapter 1

Introduction

> Philosophy is written in that great book which ever lies before our eyes — I mean the universe — but we cannot understand it if we do not first learn the language and grasp the symbols, in which it is written. This book is written in the mathematical language, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth.
>
> Galileo–"Il Saggiatore" 1623

Somewhat surprisingly we have been discovering patterns in Nature and translating them into mathematical language. For this translation, differential equations play a key role. To describe the evolution of a system with many variables one often makes use of the theory of (partial) differential equations. Differential equations are the common denominator of several modern exact sciences such as physics, chemistry, biology or engineering. Since the central aspect of various applications is the `intrinsic randomness` of those systems, one needs to take this into account. Probability theory has achieved important positive results in this direction: first it provided a theory for the Brownian motion[1], then it opened the path for the theory of Stochastic differential equations[2], Finally it has develloped systematic thecniques to obtain family of processes that converge to solutions of an Stochastic Differential Equation (SDE).

[1] notably with the works of Einstein [Ein56] and Wiener [Wie21]

[2] for instance with the works of Kolmogorov [KF31], Itô [Ito51] and Stroock-Varadhan [SV69a, SV69b]

1.1 Motivations

At this point of our inquiry we look, from a distance, for what might be the possible guiding lines for this work. It's possible to single out a few perspectives:

• To study reaction-diffusion models from the perspective of particle systems

Reaction-diffusion models have been interesting the scientific community in general because of their wide range of applications. Indeed, reaction-diffusion models have been used to model a large class of phenomena in several fields of knowledge, for instance:

- Chemical Physics: Heat explosion, Chemical Kinetics, Polymerization,
- Biology: Morphogenesis, Epidemics, Population dynamics, and
- Physiology: Growth of tissues, Coagulation, Atherosclerosis

According to Volpert [Vol14], reaction-diffusion models originated with the works of Semenov on thermal explosion in the 1930's. Later, in 1952, came a notable work of Alan Turing: "The Chemical Basis of Morphogenesis" [Tur52]. The idea was to capture a physical phenomenon through a suitable differential equation and then use the knowledge from its solutions to understand the relevant features of the phenomenon.

More recent studies of reaction-diffusion models through Particle Systems appear in [AT80], [Blo92], [Kot86] and [FG12]. In this context, we try to understand a physical phenomenon as a consequence of the interactions of its underlying characters, the particles. To do so we construct a family of probabilistic models that try to capture basic evolution patterns of the system. For Reaction-Diffusion models, we consider: motion, birth and death of particles that occur at random times in a given environment.

The models of this family are meant to describe the same physical reality with a different degree of precision. We hope that, as we increase precision, we will be able to extract some knowledge about this phenomenon. So if we prove convergence of a family of particle systems to the solution of an SDE we can use our knowledge about the particle systems to learn about the behaviour of the solution of the SDE.

• To classify the behavior of fluctuations with respect to a constant solution of an SDE

As we deal with Reaction-Diffusion models in particle systems, we learn that they give rise in the limit to solutions to equations of the following kind:

$$d\zeta_t(x) = (\Delta_V \zeta_t(x) + F(\zeta_t(x)))\, dt + \sqrt{G(\zeta_t(x))}\, dB_t^x. \tag{1.1}$$

We are interested in working with functions F and G that behave locally as polynomials. One reason for this is that, in the context of chemical reactions, the rate equation [Vol14, p.14] states that the rate of reaction is proportional to integer powers of the relevant densities (depending on the stoichiometric coefficients of the reaction under consideration). However, it is not true that the same conditions hold independently of the density, this means that reactions might have a polynomial behavior near a given density point but for high densities this polynomial rule might no longer hold due to saturation and interactions. so one expect to be dealing with polynomial rates at a local scale and correcting terms for extreme densities.

Another reason for dealing with F and G as polynomials near a given densitity is because this gives rise to non linear SDE's that have not been studied so far and we hope to gain some knowledge about them using reaction-diffusion models in the particle systems approach.

To explain the square root on the term $\sqrt{G(\zeta(x))}$ we turn to Stroock-Varadhan theory of martingale problems (see [KS91] pgs 311 - 327). For every SDE there corresponds a second order differential operator. In our case for $f \in C^2(\mathbb{R}^V)$ and $\zeta \in \mathbb{R}^V$, we have $L : C^2(\mathbb{R}^V) \to C(\mathbb{R}^V)$ defined by:

$$Lf(\zeta) = \sum_{x \in V} G(\zeta(x))\partial_x\partial_x f(\zeta) + \sum_{x \in V} \left(\Delta_V \zeta_t(x) + F(\zeta(x))\right) \partial_x f(\zeta)$$

The SDE corresponding to L is (1.1).

If F and G are smooth functions and $F(0) = G(0) = 0$ we have that $\zeta_t(x) = 0$ for every $x \in V$ and $t \geq 0$ is a solution to (1.1). We would like to see what happens in a vicinity of 0. The behaviour of the system will depend on the local behavior of the functions F and G. Our first interest will be to study the relation between solutions of the SDE and the derivatives of F and G. The idea is that the first non-null derivative will determine the local behaviour of the system, that is, we consider that there are integers k, l such that:

- $F^{(k)} \neq 0$ and $F^{(j)} = 0$ for every $j < k$ and
- $G^{(l)} \neq 0$ and $G^{(j)} = 0$ for every $j < l$.

In order to obtain solutions that remain in the vicinity of 0 we assume that $F^{(k)} < 0$. Let's say that $F^{(k)}(0) = -\beta < 0$ and $G^{(l)}(0) = \alpha > 0$. then we would like to find solutions to the following SDE:

$$\begin{aligned} d\zeta_t(x) = {} & \left(\Delta_V \zeta_t(x) - \beta(\zeta_t(x))^k (1 + o(1))\right) dt \\ & + \sqrt{\alpha(\zeta_t(x))^l (1 + o(1))}\, dB_t^x \end{aligned} \tag{1.2}$$

To remove this undesired $o(1)$ terms of the equation, we consider the simpler equations:

$$d\zeta_t(x) = \left(\Delta_V \zeta_t(x) - \beta(\zeta_t(x))^k\right) dt + \sqrt{\alpha\,(\zeta_t(x))^l}\, dB_t^x \tag{1.3}$$

It turns out that for this equations we allow for arbitrary initial condition. So we don't need to worry about being in a vicinity of 0.

• **To study nonlinear SPDE's from the perspective of particle systems**

The study of differential equations is at the core of the major advances of the sciences of the modern period. Mathematicians and physicists of that time shared the idea that in some sense natural phenomena could be interpreted by mathematical equations. The differential equations could be understood in some vague sense as the ultimate laws of nature governing a large class of phenomena. This motivation thrived and remains behind the study of differential equations. Indeed, in the words of Richard Feynmann one can hear the echoes of this conceptions (see epigraph or [Gow11]). In this spirit when we derive new equations we learn more about the world we live in, we uncover new laws of nature.

One way to study SDE's is via scaling limits of particle systems (see [KL99]). In the same spirit as before when we described physical phenomena via particle systems, now we describe with increasing precision some idealized physical reality and then find convenient scales to fit together those descriptions into a macroscopic object. When scales are well chosen, the family of rescaled processes converge to a limiting object that solves an SDE. This provides a way to use Reaction-Diffusion models to derive solutions to SDE's.

In our case we consider Reaction-Diffusion models taking values on a finite undirected graph (V, E). Typically one can think of (V, E) as a discrete torus on $\mathbb{R}^3$. This graph is a model for the space. As we obtain convergence of a family of particle systems to the solution of an SDE on the graph, the idea is to make the distance between points smaller and smaller, by increasing the number of vertices and edges and ajusting the scales to preserve the description of a macroscopic object, to obtain a solution in the continuous space, that is, a solution for an SPDE.

Establishing the existence and uniqueness of solutions for SPDE's, such as the KPZ [KPZ86], given formally in [HS15] by:

$$\partial_t h = \partial_x^2 h + \lambda\,(\partial_x h)^2 + \xi \tag{1.4}$$

as well as interpreting their meaning has challenged the community for over 30 years. It was not until recently that major progresses have been made. Notable among those are: Martin Hairer's theory of regularity structures [Hai14], the notion of energy solutions introduced by Gonçalves and Jara [GJ13], and the proof by Gubinelli and Perkowski that those solutions are unique [GP15].

An also interesting SPDE is the Parabolic Anderson Model:

$$\partial_t u = \Delta u + u \cdot \xi \tag{1.5}$$

Where u is a function of $t \geq 0$, $x \in \mathbb{R}^2$ and ξ is a white noise on $\mathbb{R}^2$ see [HL15].

The study of the PAM equation has also benefited from the theories develloped to solve the KPZ. The interesting (mathematically challenging) term in the KPZ equation is $(\partial_x h)^2$ and in the PAM equation is $u \cdot \xi$. Those equations are ill posed because in the continuous setting solutions are distributions and the product of distributions is not in general a well defined object. The recent efforts and advances towards solving those equations have provided grounds for making sense of those products.

One might combine these interesting features in a new equation:

$$\partial_t u = \Delta u - u^2 + u \cdot \xi. \tag{1.6}$$

It is worth noting that this equation is very different from both the KPZ and the PAM. Since for instance we have the term u^2 in the above equation instead of $(\partial_x u)^2$. However, from the our point of view, this equation should challenge us in simmilar ways as the two above have. So there is an interest in solving it. One way to approach the problem is to consider a graph (V, E) as a discrete model for the space and deal with the spatially discrete analog of this equation:

$$du = \left(\Delta_V u - u^2\right)\, dt + u\, dB_t \tag{1.7}$$

In this setting the Laplacian is no longer continuous but discrete:

$$\Delta_V u(x) = \sum_{y: y \sim x} u(y) - u(x)$$

and the noise term is driven by a vector valued Brownian motion. So, in this case, instead of an SPDE we have an ordinary Stochastic Differential Equation (an SDE). The idea is to learn from the discrete setting and then to devise a strategy to move to the continuous case.

If for a moment we ignore the noise term, we get:

$$du = \left(\Delta_V u - u^2\right) dt$$

Which is reminiscent of the family of Reaction-Diffusion differential equations (for an introduction see [Vol14])

This leads us to use Reaction-Diffusion models in particle systems to study solutions to equations such as (1.7).

1.2 Solving a family of SDE's

We fix $k, l \in \mathbb{N}$, $k > l$ and $\alpha, \beta > 0$, and consider the following SDE:

$$\begin{cases} d\zeta_t(x) = \left[\Delta_V \zeta_t(x) - \beta\left(\zeta_t(x)\right)^k\right] dt + \sqrt{\alpha\left(\zeta_t(x)\right)^l}\, dB_t^x \quad \forall x \in V \\ \zeta_0(x) = \rho_0(x) \end{cases} \tag{1.8}$$

where $(B^x)_{x \in V}$ is a $|V|$-dimensional Brownian motion and ρ_0 is an arbitrary initial condition.

The main new result in this work is the explicit construction for each of these SDE's of a family of stochastic processes $(\eta_\cdot^n)_n = (\{\eta_t^n; t \geq 0\})_n$ issued from Particle Systems. Under suitable initial conditions, this family converge after scaling to the solution of the given SDE with respect to a fixed initial condition. Convergence here is meant as convergence in law of the probability measures with respect to the J_1-Skorohod topology [Bil99, chapter 3] and is denoted by $\mathbb{P}^n \xrightarrow[n\to\infty]{J_1} \mathbb{P}^*$ where $\mathbb{P}^n$ and $\mathbb{P}^*$ are probability measures on $D = \big(D([0,\infty), \mathbb{R}^V), J_1\big)$.

We consider the scaling $\zeta_\cdot^n = \frac{\eta_\cdot^n}{n}$, and note that they correspond to probability measures on $\big(D([0,\infty), \mathbb{R}^V), J_1\big)$. We can then state the main theorem of this work:

Theorem 1. *Let $k, l \in \mathbb{N}$, $k > l$, let $\alpha, \beta > 0$ $(B^x)_{x \in V}$ be a $|V|$-dimensional Brownian motion and let $\rho_0 \in \mathbb{R}^V$. There is a family of stochastic processes $(\eta_\cdot^n)_n$ induced by reaction-diffusion models such that for every $x \in V$ $\zeta_0^n(x) \to \rho_0(x)$ and $\zeta_\cdot^n(x) \xrightarrow[n\to\infty]{J_1} \zeta_\cdot^*(x)$ where $\zeta_\cdot^* = (\zeta_\cdot^*(x))_{x \in V}$ is the (unique) solution of the stochastic differential equation* (1.8) *with $\zeta_0^* = \rho_0$.*

An interesting consequence of our proof is the existence of solutions to a class of non-linear SDE's that do not admit solutions via the Picard method. From a technical point of view, the estimate we used for the uniform bounds used in the proof of the Theorem is a usefull tool and we hope it can be used in other contexts.

In a sense, what is really new is the idea to use Reaction-Diffusion models in particle systems to this kind of problems. Not only they allow us to solve differential equations that have reaction and diffusion terms but also provide a framework to deal with the fluctuations terms encoded in the function that multiplies the noise. Traditionnaly, the fluctuation terms were interpreted as the result of random catalytic activity. Our models are thus called *auto-catalytic*, since their randomness stems from the dynamic of the particle system instead of an external source.

One interesting property left to understand in this model is whether the solutions to the SDE's will eventually die. More deeply, we could study the distribution of the time it takes for the process to die. For this, one might hope to get some insight from the discrete models and the simulations they allow us to do. One last related question is to study the implementation of those simulations and compare them with other discretization techniques.

Theorem 1 also suggests investigating its generalizations. A first direction is to include the case $k \leq l$. Another, is the case of infinite sites, $|V| = \infty$. Besides that, one can treat more general neighboring relations, modeling asymetrical environments.

In more broad directions, we could also study the case where the reaction term is positive, which requires studying finite explosion times. Open still is the case in which F has multiple zeros; this might lead to the study of multiple equilibria and metastability. Finally, the ultimate goal is to adapt our procedure to general functions F and G, and general graphs V in such a way that it becomes a framework to obtain solutions to SPDE's.

The proof of this theorem is now the subject of our concern.

1.3 Strategy of proof

To prove the theorem, for each given SDE (1.8) we have to build a particle system and then prove tightness for the scaled processes derived from it, according to conveniently chosen initial conditions, and finally characterize the limit points as the solution of a well posed martingale problem. Before moving on, it is important to discuss somewhat vaguely the relation between a few major ideas that will make the scheme of the proof more sensible. Those are:

- the relation between SDE's and martingale problems
- the relation between scaling limits of particle systems and martingale problems
- a heuristic argument for how scaling limits of particle systems might give rise to solutions of an SDE.

1.3.1 SDE's and martingale problems

Let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^{d\times r} \to \mathbb{R}$ be measurable functions. Let $B_\cdot = (B^1_\cdot, \ldots, B^r_\cdot)$ be an r-dimensional Brownian motion. Consider the following abstract d-dimensional SDE

$$\begin{cases} dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t \\ X_0 = x_0 \end{cases} \tag{1.9}$$

or in coordinates

$$\begin{cases} dX_t^i = b_i(X_t)\,dt + \sum_j \sigma_{i,j}(X_t)\,dB_t^j \\ X_0^i = x_0^i \end{cases}$$

A weak solution [KS91, p. 300] to this equation with initial condition x_0 is a triple $[(X, B), (\Omega, \mathcal{F}, \mathbb{P}), \mathcal{F}_t]$ where

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{\mathcal{F}_t\}_t$ is a filtration satisfying the usual conditions, [KS91, p.10]

(ii) $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a continuous adapted $\mathbb{R}^d$-valued process, $B = \{B_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is an r-dimensional Brownian motion,

(iii) Almost surely $\int_0^t |b_i(X_s)| + \sigma_{i,k}(X_s)\,ds < \infty$ for all $1 \leq i \leq d$, $1 \leq k \leq r$, $0 \leq t < \infty$, and,

(iv) Almost surely

$$X_t^i = x_0^i + \int_0^t b_i(X_s)\,ds + \sum_{k=1}^r \int_0^t \sigma_{i,k}(X_s)\,dB_s^j; \quad \forall t \geq 0,\ \forall\, 1 \leq i \leq d$$

The probability law induced by the weak solution $[(X, W), (\Omega, \mathcal{F}, \mathbb{P}), \mathcal{F}_t]$ is the probability measure $\mathbb{P}^*$ on $\left(C[0,\infty)^d, \mathcal{B}\left(C[0,\infty)^d\right)\right)$, obtained by

$$\mathbb{P}^*(A) = \mathbb{P}\left[X_\cdot \in A\right].$$

Itô's calculus provides a closed formula for the value of $f(X_t)$ for every $f \in C^2(\mathbb{R}^d)$. The following equality holds almost surely:

$$\begin{aligned} f(X_t) = f(X_0) + \sum_{i=1}^d \Bigg(\int_0^t \partial_i f(X_s) b_i(X_s)\,ds + \sum_{k=1}^r \int_0^t \partial_i f(X_s) \sigma_{i,k}(X_s)\,dB_s^k \\ + \frac{1}{2} \sum_{j=1}^d \int_0^t \partial_i \partial_j f(X_s) a_{i,j}(X_s)\,ds \Bigg) \end{aligned}$$

where $a_{i,j}(x) = \sum_{k=1}^r \sigma_{i,k}(x)\sigma_{j,k}(x)$ for $1 \leq i, j \leq d$. Rearranging the terms we obtain

$$f(X_t) - f(X_0) - \sum_{i=1}^{d} \int_0^t \partial_i f(X_s) b_i(X_s)\, ds - \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \partial_i \partial_j f(X_s) a_{i,j}(X_s)\, ds$$
$$= \sum_{i=1}^{d} \sum_{k=1}^{r} \int_0^t \partial_i f(X_s) \sigma_{i,k}(X_s)\, dB_s^k$$

which is a continuous local martingale.

Based on this we define the second order differential operator L by

$$Lf(x) = \sum_{i=1}^{d} b_i(x) \partial_i f(x) + \frac{1}{2} \sum_{i,j=1}^{d} a_{i,j}(x) \partial_i \partial_j f(x)$$

and the function of trajectories

$$M_t^{f,L}(X.) = f(X_t) - f(X_0) - \int_0^t Lf(X_s)\, ds$$

and we see that for each solution $[(X, W), (\Omega, \mathcal{F}, \mathbb{P}), \mathcal{F}_t]$ we obtain a family of continuous local martingales $\left\{ M_t^{f,L}(X.), \mathcal{F}_t; t > 0 \right\}$.

This illuminates the following definition:

Definition 1. *A probability measure $\mathbb{P}$ on $\left(C[0,\infty)^d, \mathcal{B}\left(C[0,\infty)^d\right)\right)$ under which $\left\{ M_t^{f,L}, \mathcal{F}_t; t \geq 0 \right\}$ is a continuous local martingale for every $f \in C^2(\mathbb{R}^d)$ is called a* ***solution to the martingale problem associated with** L**.***

Using the notation defined up to now, the next theorem [KS91, p. 315] completes the relation between Martingale problems and solutions to an SDE.

Theorem. *Let $\mathbb{P}^*$ be a probability measure on $\left(C[0,\infty)^d, \mathcal{B}\left(C[0,\infty)^d\right)\right)$ under which $M_.^{f,L}$ is a continuous local martingale for the choices of $f(x) = x_i$ and $f(x) = x_i x_j$, $1 \leq i,j \leq d$ then there is an r-dimensional Brownian motion W on $(\Omega, \mathcal{F}, \mathbb{P})$ an extension of $\left(C[0,\infty)^d, \mathcal{B}\left(C[0,\infty)^d\right)\right)$ such that $[(X, W), (\Omega, \mathcal{F}, \mathbb{P}), \mathcal{F}_t]$ is a weak solution of*

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

and $\mathbb{P}^*$ is the law induced by this weak solution.

So we see that to each solution to a martingale problem, there corresponds a solution to an SDE. This theorem is interesting not only because it completes the relation between solutions of an SDE and solutions to a martingale problem but also because it gives a simpler criteria to check whether or not, a given measure is a solution to a martingale problem. We don't need to verify for every $f \in C^2(\mathbb{R}^d)$ that the function $M_\cdot^{L,f}$ is a continuous local martingale, now we only need to verify this for the coordinate functions $f(x) = x_i$ and the product of two coordinates $f(x) = x_i x_j$.

We conclude this section with a final remark:

Remark. *The martingale problem that corresponds to a weak solution to* (1.9) *is the martingale problem associated with L given by*

$$Lf(X_s) = \sum_{i=1}^{d} \partial_i f(X_s) b_i(X_s) + \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j f(X_s) a_{i,j}(X_s)$$

where $a = \sigma\sigma^T$ or in coordinates, $a_{i,j}(x) = \sum_{k=1}^{r} \sigma_{i,k}(x)\sigma_{j,k}(x)$.

1.3.2 From Scaling limits of particle systems to solutions of Martingale problems

Let's assume for simplicity that we have a family of particle systems $\zeta_\cdot^n$, associated with the infinitesimal generator L_n. That $\zeta_\cdot^n$ converges to $\zeta_\cdot^*$ and that it happens that for every $f \in C^2(\mathbb{R}^d)$ and $\zeta \in \mathbb{R}^d$

$$L_n f(\zeta) \to L_* f(\zeta)$$

uniformly in compact sets, where L_* is a second order differential operator such that $\displaystyle\sum_{i,j=1}^{d} z_i a^*_{i,j}(\zeta^*) z_j \geq 0\,.$

Consider for each $f \in C^2(\mathbb{R}^d) \cap_n \mathcal{D}(L_n)$ the Dynkin martingales [Lig99, Theorem 3.32]

$$M_t^{f,L_n} = f(\zeta_t^n) - f(\zeta_0^n) - \int_0^t L_n f(\zeta_s^n)\, ds.$$

From the convergence of $\zeta_\cdot^n$ to $\zeta_\cdot^*$ we obtain

$$\begin{aligned} M_t^{f,L_n} &= f(\zeta_t^n) - f(\zeta_0^n) - \int_0^t L_n f(\zeta_s^n)\, ds \\ &\downarrow \\ M_t^{f,L_*} &= f(\zeta_t^*) - f(\zeta_0^*) - \int_0^t L_* f(\zeta_s^*)\, ds. \end{aligned}$$

Here convergence follows from the continuity of the function $M^{f,L_*}(\zeta^n)$ with respect to the Skorohod topology, plus the fact that uniform convergence in compact sets and tightness give us that

$$\mathbb{P}_n\left[\sup_{t\leq T}\left|M_t^{f,L_n}(\zeta)-M_t^{f,L_*}(\zeta)\right|>\epsilon\right]\xrightarrow[n\to\infty]{}0.$$

If we prove that $M_\cdot^{f,L_n}$ are uniformly integrable then we can conclude that the limit $M_\cdot^{f,L_*}$ is also a martingale [Dur10, p. 259 Theorem 5.5.2]. This means that $\zeta_\cdot^*$ is a solution to the martingale problem associated with L_*

1.3.3 From scaling limits of particle systems to solutions of SDE's

Consider the functions $f_i(\zeta) = \zeta(i)$. Rearranging the expressions of the Dynkin Martingales, we get:

$$\zeta_t^n(i) = \zeta_0^n(i) + \int_0^t b_i(\zeta_s^n)\, ds + M^{f_i,L_n}.$$

Using the continuity of the above functions and the convergence of $\zeta_\cdot^n$ to $\zeta_\cdot^*$, we obtain

$$\begin{array}{c} \zeta_t^n(i) = \zeta_0^n(i) + \int_0^t (L_n f_i)(\zeta_s^n)\, ds + M_t^{f_i,L_n} \\ \downarrow \\ \zeta_t^*(i) = \zeta_0^*(i) + \int_0^t (L_* f_i)(\zeta_s^*)\, ds + M_t^{f_i,L_*} \end{array}$$

In the case our family of martingales is uniformly integrable and the limit $\zeta_\cdot^*$ is almost surely continuous, then $M_\cdot^{f_i,L_*}$ is a continuous local martingale. If we do some more calculations which involve the product of two coordinates $f_{i,j}(\zeta) = \zeta(i)\zeta(j)$ we learn about the covariance of M^{f_i,L_*} and M^{f_j,L_*}. By the representation of martingales theorem [KS91, pp.315-316 pp.170-172] we obtain $[(\zeta^*, B), (\Omega, \mathcal{F}, \mathbb{P}), \mathcal{F}_t]$ such that:

$$M_t^{f_i,L_*} = \sum_{j=1}^{d} \int_0^t \sigma_{i,j}^*(\zeta_s^*)\, dB_s^j$$

with

$$\mathbb{P}\left[\int_0^t \left(\sigma_{i,j}^*(\zeta_s^*)\right)^2 ds < \infty\right] = 1; \quad 1 \leq i,j \leq d; \quad t \geq 0$$

and defining $b_i^*(\zeta_s^*) = L_* f_i(\zeta_s^*)$ since M^{f_i, L_*} is a continuous local martingale

$$\mathbb{P}\left[\int_0^t |b_i(\zeta_s^*)|^2 \, ds < \infty\right] = 1; \quad 1 \leq i, j \leq d; \quad t \geq 0$$

So, to sum up:

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{\mathcal{F}_t\}_t$ is a filtration satisfying the usual conditions,

(ii) $\zeta^* = \{\zeta_t^*, \mathcal{F}_t; 0 \leq t < \infty\}$ is a continuous adapted $\mathbb{R}^d$-valued process, $B = \{B_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is an d-dimensional Brownian motion,

(iii) Almost surely, $\int_0^t |b_i^*(\zeta_s^*)| + \sigma_{i,j}^*(\zeta_s^*) \, ds < \infty$ for every $1 \leq i, j \leq d$ $0 \leq t < \infty$, and,

(iv) Almost surely,

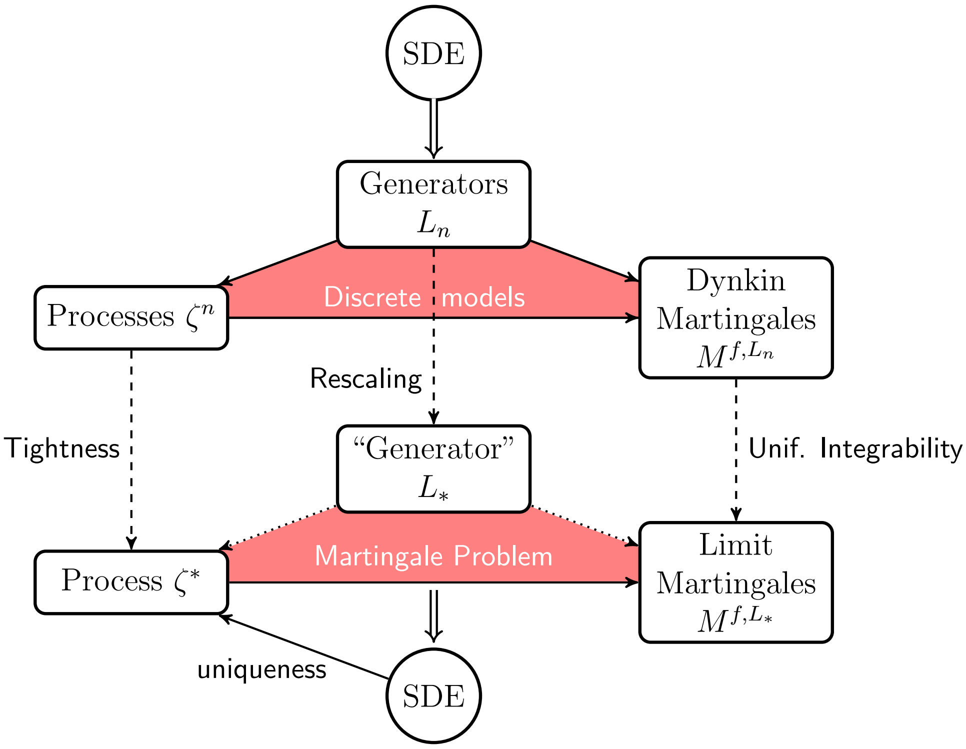
$$\zeta_t^*(i) = \zeta_0^*(i) + \int_0^t b_i(\zeta_s^*) \, ds + \sum_{j=1}^d \int_0^t \sigma_{i,j}^*(\zeta_s^*) \, dB_s^j; \quad \forall t \geq 0 \, \forall 1 \leq i \leq d$$

Once we have this and convergence on the initial condition $\zeta_0^n \to x_0$ we have a weak solution to the SDE

$$\begin{cases} d\zeta_t^* = b^*(\zeta_t^*) \, dt + \sigma^*(\zeta_t^*) \, dB_t \\ \zeta_0^* = x_0 \end{cases} \tag{1.10}$$

1.4 Scheme of the proof

Now that we've put together the relation between the major concepts (SDE's, martingale problems and scaling limit of particle systems) and that we have seen a little bit how tightness and uniform integrability come into play, we can present the scheme of the proof of the theorem and then the steps we will need to complete the proof:



The idea is that given one SDE, we would like to construct a family of particle systems[3], inspired by the relations between martingale problems and SDE's.

Chapter 2 is devoted to the study of the discrete models, from the construction to the analysis of the scaled processes $\zeta^n_\cdot = \frac{\eta^n_\cdot}{n}$ and the Dynkin martingales associated to them. To do this we consider $f_n(\eta) = f\left(\frac{\eta}{n}\right)$ and then:

$$M_t^{f_n,L_n} := f_n(\eta_t^n) - f_n(\eta_0^n) - \int_0^t (L_n f_n)(\eta_s^n)\,ds.$$

We then prove uniform bounds that will be useful in the proof of convergence, since those uniform bounds are fundamental in the proof of tightness on the one hand and uniform integrability on the other.

Chapter 3 deals with the passage to the limit. To do this we prove that:

(a) the family of processes $(\zeta^n_\cdot)_n$ is tight, thus we can consider a subsequence of $\zeta^n_\cdot$ converging to a limit point $\zeta^*_\cdot$;

(b) the martingales M^{f_n,L_n} (for convenient f_n) are uniformly integrable, and the expression of the limit martingales M^{f,L_*} can be written in terms of the limit trajectories $\zeta^*_\cdot$ and a second order operator L_*.

[3]we refer to the family of particle systems as discrete models due to some vague resemblance with Euler discretization scheme used to solve differential equations in the early days.

Then we need to show that the limit is unique. To do so we prove that:

(c) the probability measure corresponding to $\zeta^*_\cdot$ is the unique solution to the martingale problem associated with L^*;

The question of uniqueness deserves some comments. The diffusion coefficient $G(\zeta(x))$ vanishes if $\zeta(x) = 0$, so we don't have uniform ellipticity, which is a usual criterion for uniqueness of martingale problems.

However, since there is a correspondence between martingale problems and SDE's, if we prove uniqueness of solutions to the SDE it will imply uniqueness of solutions to the corresponding martingale problem. We then try to prove pathwise uniqueness for the SDE's. There are two cases to analyse: $l \geq 2$ and $l = 1$. The case $l \geq 2$ is simpler, since the coefficients of the SDE (b, σ) satisfy locally Lipschitz conditions, from which a standard argument shows that pathwise uniqueness follows.

For the case $l = 1$, the dispersion coefficients vanish as $\sqrt{x}$ near the boundary of the positive region, so the Lipschitz condition fails near $x = 0$. Moreover, we know that if the dispersion matrix vanished with rate x^α with $\alpha < \frac{1}{2}$ we would no longer have uniqueness. Therefore, the fact that we can also prove uniqueness here indicates that this a limit situation.

From an intuitive point of view one believes that uniqueness holds since the drift term in the boundary away from zero is a positive vector driving the trajectory towards the interior of the region. So one expects that the local time in the boundary away from is zero is zero and that there is no more than one solution before reaching zero. Since zero is an absorbing state that would give us a global uniqueness result.

We found well established criteria in the papers of Yamada and Watanabe (1971) and in the book of Karatzas that suit the case $l = 1$. So even though we prove uniqueness of solutions in a quite straightforward way, it might be that for similar problems one such criterion is not available. So there is a general interest in understanding more about boundary conditions and other criteria for uniqueness as for instance checking that the limit second order differential operator L_* is a probability generator as in [Lig99, p.312]. This leads us to investigate among other properties, whether $\mathcal{R}(I - \lambda L_*)$ is dense in $C(\mathbb{R}^V)$ for all sufficiently small λ.

Fortunately, for the moment, we don't need to worry about those issues.

Chapter 2

Discrete models and uniform properties

The purpose of this chapter is to complete the first part of the scheme of the proof in (1.4). We first discuss how to build general reaction-diffusion models. Then, inspired by a fixed SDE as in (1.8) we construct a family of particle systems that, after scaling, converges to a solution to the given SDE. Afterwards, we discuss briefly the Dynkin Martingales with respect to the coordinate functions, and introduce the notion of discrete coefficients for L_n. Finally, we derive uniform properties for the family of particle systems.

To sum up, in this chapter we prepare the key ingredients for the passage to the limit that takes place in chapter 3.

2.1 Set up

To construct a particle system, one needs to define its configurations, specify the admissible transitions and the rates at which they occur. This is what we shall do next for a general reaction-diffusion model. In this work, all reaction-diffusion models are going to be particular instances of this general one.

Fix a finite set V. The points of this set shall be denoted by x, y, z, $\ldots$ and are called *sites*. We consider a neighboring relation $\sim$ on V such that $x \nsim x$, $x \sim y \Rightarrow y \sim x$, and we put $E = \{\, \{x, y\} \mid x \sim y \,\}$. The resulting graph $G = (V, E)$ is a model for the space on which our dynamics takes place, where the points in V represent the loci, and the edges of E indicate which loci are close to one another. To keep track of the number of particles at each site, we denote by $\eta(x)$ the number of particles at site x and by $\eta = (\eta(x))_{x \in V}$ the configuration of the system.

The dynamics we consider is of a probabilistic nature. There are 3 types of transition that occur independently of one another and at random times. Particles on site x may jump to each neighboring site $y \sim x$ according to exponential times with rate 1; moreover, at each site x, a particle can be created with rate $F^+(\eta(x))$; and finally at each site x a particle can be annihilated with rate $F^-(\eta(x))$.

We need a notation for the resulting configuration for each transition:

- $\eta^{x,y}$ is the configuration obtained from η by moving one particle from x to y, if possible. That is, if $\eta(x) > 0$:

$$\eta^{x,y}(z) = \begin{cases} \eta(y) + 1 & \text{if } z = y \\ \eta(x) - 1 & \text{if } z = x \\ \eta(z) & \text{otherwise} \end{cases}$$

  while if $\eta(x) = 0$ then $\eta^{x,y} = \eta$.

- $\eta^{x,+}$ corresponds to the creation of one particle at site x:

$$\eta^{x,+}(z) = \begin{cases} \eta(x) + 1 & \text{if } z = x \\ \eta(z) & \text{otherwise} \end{cases}$$

- $\eta^{x,-}$ corresponds to the annihilation of one particle at site x if possible. That is, if $\eta(x) > 0$:

$$\eta^{x,y}(z) = \begin{cases} \eta(x) - 1 & \text{if } z = x \\ \eta(z) & \text{otherwise} \end{cases}$$

  while if $\eta(x) = 0$ then $\eta^{x,-} = \eta$.

Thanks to Hille-Yoshida's Theorem [Lig99, pp.102-103], we can describe this class of models more briefly by encoding all this information on the operator $\mathcal{L}$:

$$\begin{aligned}\mathcal{L}f(\eta) = &\sum_{x \sim y \in V} \eta(x)\left[f(\eta^{x,y}) - f(\eta)\right] + \sum_{x \in V} F^+(\eta(x))\left[f(\eta^{x,+}) - f(\eta)\right] \\ &+ \sum_{x \in V} F^-(\eta(x))\left[f(\eta^{x,-}) - f(\eta)\right].\end{aligned}$$

Transition rates can be read as the factors multiplying the differences between the transitions $f(\eta^\bullet)$ and the original state $f(\eta)$. Note that since all particles at a given site x can move to a neighboring y, the transition rate for jumps from any particle in x to y is $\eta(x)$, and not 1.

Our processes will be well-defined once we rule out explosions. Explosions occur when, in a finite time, an infinite number of transitions occurs. In our case, $F^+(u) \leq C(1+u)$ and there can be no explosions; we postpone the proof of this result to the appendix 4.1.

On these conditions, $\mathcal{L}$ is a probability generator [Lig99, p. 97]. So, we can construct the particle system by first defining a probability semigroup $\left(T^{\mathcal{L}}(t)\right)_{t\geq 0}$ by the following formula:

$$\forall\, f \in C_b(\mathbb{R}^V) : \; T^{\mathcal{L}}(t)f = \lim_n \left(I - \frac{t}{n}\mathcal{L}\right)^{-n} f. \tag{2.1}$$

Then, for each $\xi \in \mathbb{R}^V$, we define $\mathbb{P}^\xi$ as the only measure whose finite-dimensional evaluations are compatible with the given transition semigroup $T^{\mathcal{L}}$. That is, for every $k \in \mathbb{N}$, $t_1 < t_2 < \ldots < t_k$ and $\eta_1, \ldots, \eta_k \in \mathbb{N}^V$:

$$\begin{aligned} &\mathbb{P}^\xi \left[\eta_0 = \eta, \eta_{t_1} = \eta_1 \ldots, \eta_{t_k} = \eta_k\right] = \\ &\qquad \mathbb{1}_{\eta=\xi}\; T^{\mathcal{L}}_{t_1}(\mathbb{1}_{\eta_1})(\eta)\; T^{\mathcal{L}}_{t_2-t_1}(\mathbb{1}_{\eta_2})(\eta_1)\; \cdots\; T^{\mathcal{L}}_{t_k-t_{k-1}}(\mathbb{1}_{\eta_k})(\eta_{k-1}) \end{aligned} \tag{2.2}$$

Finally, we fix μ_0, a measure in $\mathbb{R}^V$ to stand for an initial condition, and define for a bounded and measurable function $f : D([0,\infty), \mathbb{R}^V) \to \mathbb{R}$:

$$\mathbb{E}^{\mu_0}\left[f\right] := \int \mathbb{E}^\xi \left[f\right]\, d\mu_0(\xi). \tag{2.3}$$

This defines the *stochastic process associated with the generator* $\mathcal{L}$ *and the initial condition* μ_0 which is the measure $\mathbb{P}^{\mu_0}$ on $D([0,\infty), \mathbb{R}^V)$ such that $\mathbb{P}^{\mu_0}(A) = \mathbb{E}^{\mu_0}\left[\mathbb{1}_A\right]$. We will denote it by $\left\{\, \eta^{\mathcal{L}}_t; t \geq 0, \eta^{\mathcal{L}}_0 = \mu_0 \,\right\}$ or simply $\eta_\cdot = \left\{\, \eta_t; t \geq 0 \,\right\}$ leaving the generator and the initial condition implicit.

At this point, we can show that

$$\mathcal{L}f(\eta) = \lim_{h\to 0} \frac{1}{h} \mathbb{E}^\eta \left[f(X_h) - f(X_0)\right]$$

for all functions f for which the above limit exists. We say that these functions *belong to the domain of* $\mathcal{L}$.

2.2 Construction of the models

We will consider a family of stochastic processes $(\eta^n_\cdot = \left\{\, \eta^n_t; t \geq 0 \,\right\})_{n\in\mathbb{N}}$ corresponding to reaction-diffusion models on the graph (V, E).

The goal in constructing these models is to find a family of processes converging to the solution of the following SDE:

$$\begin{cases} d\zeta_t(x) = \left[\Delta_V \zeta_t(x) - \beta\left(\zeta_t(x)\right)^k\right]dt + \left(\alpha\zeta_t(x)\right)^{l/2}\, dB_t^x \qquad \forall x \in V \\ \zeta_0(x) = \rho_0(x) \end{cases} \tag{2.4}$$

where $B = (B_\cdot^x)_{x\in V}$ is a vector of independent Brownian motions.

The macroscopic parameter for this family is the "density" of particles $\zeta_t^n(x) = \dfrac{\eta_t^n(x)}{n}$. In the spirit that particle systems describe a same physical reality with different degrees of precision, we can imagine that our model becomes more precise as n gets larger. and by taking $n \to \infty$ we expect that if the initial density of particles converges ($\zeta_0^n(x) \to \rho_0(x)$ for every $x \in V$) then $\zeta_\cdot^n \to \zeta_\cdot^*$ where $\zeta_\cdot^*$ solves (2.4).

The reaction-diffusion models can be described by their infinitesimal generators L_n. In our case they have the following expression:

$$\begin{aligned} L_n f(\eta) = \sum_{x\sim y\in V} \eta(x)\left[f(\eta^{x,y}) - f(\eta)\right] + \sum_{x\in V} F_n^+(\eta(x))\left[f(\eta^{x,+}) - f(\eta)\right] \\ + \sum_{x\in V} F_n^-(\eta(x))\left[f(\eta^{x,-}) - f(\eta)\right] \end{aligned} \tag{2.5}$$

where the rates $F_n^+(\eta(x))$ and $F_n^-(\eta(x))$ now depend on n to account for the macroscopic compatibility after scaling. For this, we consider the simple case where $V = \{\, x \,\}$ as in this case no diffusion occurs and the notation simplifies as $\eta = \eta(x)$.

The relevant quantities observed from these models due to birth and death of particles are the average incremental[1] net result of birth and death in the process ζ_t^n, and the incremental deviation to be expected from this average. These are given by:

$$\begin{cases} F_n(n\zeta^n) &= \frac{1}{n}\left[F_n^+(n\zeta^n) - F_n^-(n\zeta^n)\right] \\ G_n(n\zeta^n) &= \frac{1}{n^2}\left[F_n^+(n\zeta^n) + F_n^-(n\zeta^n)\right] \end{cases} \tag{2.6}$$

After introducing suitable scalings, we want that F_n and G_n converge to the macroscopic rates $-\beta\zeta^k$ and $\alpha\zeta^k$. So we can see how the parameters that appear in our SDE (2.4), the integers (k, l) and the positive reals (α, β), influence the discretized models.

[1] average incremental values are obtained by $\lim_{h\to 0} h^{-1}\mathbb{E}\left[f(X_h) - f(X_0)\right] = Lf(X_0)$ and incremental deviations by $\lim_{h\to 0} h^{-1}\mathbb{E}\left[\left(f(X_h) - f(X_0)\right)^2\right] = Qf(X_0)$

This can be done by defining:

$$\begin{aligned} F(u) &= -\beta u^k + f(u)u^k \\ G(u) &= \ \ \alpha u^l + g(u)u^l. \end{aligned}$$

Here, the functions f and g are correcting factors such that

- $F^+ = \frac{1}{2}(F+G)$ and $F^- = \frac{1}{2}(G-F)$ are positive, since both will be related to rates F_n^+ and F_n^- in our process;
- $F_n^+(u) \leq C_n(1+u)$; and
- $\lim_{u\to 0} f(u) = 0$ and $\lim_{u\to 0} g(u) = 0$.

For concreteness, as one example to have in mind, one can take $f(u) = 0$ and $g(u) = \left(\beta u^{k-l} - \alpha\right) \mathbb{1}_{\beta u^k \geq \alpha u^l}(u)$.

We introduce the scalings

$$F_n^+(u) = n^\lambda F^+\left(\frac{u}{n^\nu}\right) \tag{2.7}$$

$$F_n^-(u) = n^\lambda F^-\left(\frac{u}{n^\nu}\right) \tag{2.8}$$

and go back to the macroscopic quantities (2.6) to obtain

$$\begin{aligned} F_n(n\zeta^n) &= n^{\lambda-1}\left(\zeta^n n^{1-\nu}\right)^k \left(-\beta + f\left(\frac{n\zeta^n}{n^\nu}\right)\right) \\ G_n(n\zeta^n) &= n^{\lambda-2}\left(\zeta^n n^{1-\nu}\right)^l \left(\alpha + g\left(\frac{n\zeta^n}{n^\nu}\right)\right). \end{aligned}$$

Equating the exponents of n to zero, this gives us

$$\begin{cases} \lambda - 1 + k(1-\nu) = 0 \\ \lambda - 2 + l(1-\nu) = 0. \end{cases}$$

This determines the parameters $\nu = 1 + \frac{1}{k-l}$ and $\lambda = 1 + \frac{k}{k-l}$. Since $\nu > 1$ and assuming that ζ^n stays bounded (tightness gives us boundedness with arbitrarily high probability), we have that $\frac{n\zeta^n}{n^\nu} \to 0$. Therefore

$$\begin{aligned} F_n(n\zeta^n) &= (\zeta^n)^k \left(-\beta + f\left(\frac{n\zeta^n}{n^\nu}\right)\right) = (\zeta^n)^k \left(-\beta + f(o(1))\right) \\ G_n(n\zeta^n) &= (\zeta^n)^l \left(\alpha + g\left(\frac{n\zeta^n}{n^\nu}\right)\right) = (\zeta^n)^l \left(\alpha + g(o(1))\right). \end{aligned}$$

This shows that F_n and G_n do converge to the desired macroscopic quantities.

To sum up, $\left((\eta_\cdot^n(x))_{x\in V} = \eta_\cdot^n\right)_{n\in\mathbb{N}}$ will be the family of Markov processes corresponding to the reaction-diffusion process with birth rate F_n^+ as in (2.7) death rate F_n^- as in (2.8) and initial conditions η_0^n satisfying $\zeta_0^n \to \rho_0$.

2.3 Notation

Since the process $\eta_\cdot^n$ corresponds to a probability measure $\mathbb{P}_n$ on the space D it will be convenient to write for an event A:

$$\mathbb{P}\left[\eta_\cdot^n \in A\right] = \mathbb{P}_n\left(A\right).$$

In the same spirit, we let $\mathbb{E}_n$ denote the expectation with respect to $\mathbb{P}_n$ and we write for $f : D \to \mathbb{R}$

$$\mathbb{E}\left[f(\eta_\cdot^n)\right] = \mathbb{E}_n\left[f\right].$$

From the definitions of the previous section, we can write

$$\begin{aligned} F_n^+(\eta^n(x)) &= n^\lambda F^+\left(\frac{\zeta^n(x)}{n^{\nu-1}}\right) = \frac{1}{2}n^\lambda\left[G\left(\frac{\zeta^n(x)}{n^{\nu-1}}\right) + F\left(\frac{\zeta^n(x)}{n^{\nu-1}}\right)\right] \\ &= \frac{1}{2}n^2\zeta^n(x)^l\left[\alpha + g\left(\frac{\zeta^n(x)}{n^{\nu-1}}\right)\right] - n\zeta^n(x)^k\left[\beta - f\left(\frac{\zeta^n(x)}{n^{\nu-1}}\right)\right] \end{aligned}$$

and analogously

$$F_n^-(\eta^n(x)) = \frac{1}{2}n^2\zeta^n(x)^l\left[\alpha + g\left(\frac{\zeta^n(x)}{n^{\nu-1}}\right)\right] + n\zeta^n(x)^k\left[\beta - f\left(\frac{\zeta^n(x)}{n^{\nu-1}}\right)\right].$$

This motivates the following notation:

$$\begin{aligned} g^{n,\alpha}\left(\zeta^n(x)\right) &= \frac{1}{2}\left(\zeta^n(x)\right)^l\left(\alpha + g\left(\frac{\zeta^n(x)}{n^{\nu-1}}\right)\right) \\ f^{n,\beta}\left(\zeta^n(x)\right) &= \frac{1}{2}\left(\zeta^n(x)\right)^k\left(\beta - f\left(\frac{\zeta^n(x)}{n^{\nu-1}}\right)\right) \end{aligned}$$

which simplifies the expressions of F_n^+ and F_n^- to

$$F_n^+(\eta^n(x)) = n^2 g^{n,\alpha}(\zeta^n(x)) - n f^{n,\beta}(\zeta^n(x)) \tag{2.9}$$

$$F_n^-(\eta^n(x)) = n^2 g^{n,\alpha}(\zeta^n(x)) + n f^{n,\beta}(\zeta^n(x)). \tag{2.10}$$

We now define the function $S^n(\eta) := \sum_{x\in V}\frac{\eta(x)}{n}$. When applied to η_t^n we denote it by $S_t^n = \sum_{x\in V}\zeta_t^n(x)$ which is a measure of the total mass of particles in the system for the process $\eta_\cdot^n$ at time t. This allows us to define the following stopping times:

$$\begin{aligned} \tau_K^n &:= \inf\left\{ t > 0; S_t^n > K \right\} \\ \tilde{\tau}^n &:= \inf\left\{ t > 0; S_t^n > w\ n^{\frac{\nu-1}{2}} \right\} \\ \hat{\tau}^n &:= \inf\left\{ t > 0; S_t^n > w\ n^{\nu-1} \right\} \end{aligned}$$

where w is such that $\sup_{|z|\le w}|f(z)| \le \beta$ and $\sup_{|z|\le w}|g(z)| \le \alpha$. So for $S^n(\eta) \le wn^{\nu-1}$:

$$-2f^{n,\beta}(\zeta^n(x)) = (\zeta^n(x))^k \left(-\beta + f\left(\frac{\zeta^n(x)}{n^{\nu-1}}\right)\right) \ge -2\beta\,(\zeta^n(x))^k \tag{2.11}$$

$$2g^{n,\alpha}(\zeta^n(x)) = (\zeta^n(x))^l \left(\alpha + g\left(\frac{\zeta^n(x)}{n^{\nu-1}}\right)\right) \le 2\alpha\,(\zeta^n(x))^l \tag{2.12}$$

which means that for $t < \hat{\tau}^n$ (and consequently for $t < \tilde{\tau}^n$) we have a good estimate on the increment rate of the relevant macroscopic quantities of the process.

2.4 Dynkin Martingales and useful computations

Let $f : \mathbb{N}^V \to \mathbb{R}$ be a function in the domain of L_n in the sense of Section 2.2. We omit the dependence on ω and denote by M_t^{f,L_n} the Dynkin martingale associated with L_n and f:

$$M_t^{f,L_n} = f(\eta_t^n) - f(\eta_0^n) - \int_0^t L_n f(\eta_s^n)\,ds. \tag{2.13}$$

It basically says that the Dynkin martingale is an error term with respect to the prediction $f(\eta_0^n) + \int_0^t (L_n f)(\eta_s^n)\,ds$.

The variance at time t of the Dynkin martingale is

$$\mathbb{E}\left[\left(M_t^{f,n}\right)^2\right] = \int_0^t (L_n f^2)(\eta_s^n) - 2f(\eta_s^n)L_n f(\eta_s^n)\,ds = \int_0^t (Q_n f)(\eta_s^n)\,ds$$

where we define $Q_n f(\eta) := (L_n f^2)(\eta) - 2f(\eta)L_n f(\eta)$.

In the proof of the main theorem, we will need to consider the expression of the Dynkin martingales associated to the functions $f_{x,n}(\eta) = \frac{\eta(x)}{n}$ and $f_{x,y,n} = (f_{x,n} \cdot f_{y,n})(\eta)$. We give here the corresponding values of $L_n f$ and $Q_n f$ to have them all ready when needed.

For f in the domain of L_n and Q_n, $L_n f$ is given by (2.5) and $Q_n f$ is given by:

$$Q_n f(\eta) = \sum_{x\sim y\in V} \eta(x)\left[f(\eta^{x,y}) - f(\eta)\right]^2 + \sum_{x\in V} F_n^+(\eta(x))\left[f(\eta^{x,+}) - f(\eta)\right]^2 + \sum_{x\in V} F_n^-(\eta(x))\left[f(\eta^{x,-}) - f(\eta)\right]^2 \tag{2.14}$$

The Lipschitz function $f_{x,n}$ belongs to the domain of L_n and Q_n; the function $f_{x,y,n}$ is the product of Lipschitz functions and therefore also belongs to the domain of L_n and Q_n (see 4.2). From 4.3, we get

$$L_n(f_{x,n})(\eta_s^n) = \Delta_V \zeta_s^n(x) + (\zeta_s^n(x))^k \left(-\beta + f\left(\frac{\zeta_s^n(x)}{n^{\nu-1}}\right)\right) \tag{2.15}$$

$$Q_n(f_{x,n})(\eta_s^n) = \sum_{y\sim x} \frac{\zeta_s^n(y)+\zeta_s^n(x)}{n} + (\zeta_s^n(x))^l \left(\alpha + g\left(\frac{\zeta_s^n(x)}{n^{\nu-1}}\right)\right). \tag{2.16}$$

Also from 4.3, we see that $L_n f_{x,y,n}(\eta_s^n)$ can be written as:

$$L_n f_{x,n}(\eta_s^n)\cdot f_{y,n}(\eta_s^n) + L_n f_{y,n}(\eta_s^n)\cdot f_{x,n}(\eta_s^n) - \mathbb{1}_{x\sim y}\left(\frac{\zeta_s^n(x)+\zeta_s^n(y)}{n}\right).$$

As for $Q_n f_{x,y,n}(\eta_s^n)$, we only need to know it is polynomialy bounded with respect to the total mass of the system.

2.5 Discrete analogues of a and b for L_n

We would like to show that the limit process is a solution to the martingale problem associated with a second order differential operator. These operators have the general form:

$$(L_* f)(\zeta) = \sum_{x,y\in V} a^*_{x,y}(\zeta)\partial_{x,y} f(\zeta) + \sum_{x\in V} b^*_x(\zeta)\partial_x f(\zeta).$$

To recover a^* and b^*, we use the coordinate functions $f_x(\zeta) = \zeta(x)$:

$$\begin{aligned} b^*_x(\zeta) &= L_* f_x(\zeta) \\ a^*_{x,y}(\zeta) &= L_* f_x \cdot f_y(\zeta) - f_x(\zeta) L_* f_y(\zeta) - f_y(\zeta) L_* f_x(\zeta) \\ a^*_{x,x}(\zeta) &= L_*(\zeta(x)\zeta(y)) - \zeta(x) L_* \zeta(y) - \zeta(y) L_* \zeta(x) \end{aligned}$$

By analogy, we define the coefficients for L_n:

$$\begin{aligned} b^n_x(\zeta^n_s) &:= L^n f_{x,n}(\eta^n_s) \\ a^n_{x,y}(\zeta^n_s) &:= L^n(f_{x,n}\cdot f_{y,n}(\eta^n_s)) - f_{x,n}(\eta^n_s) L^n f_{y,n}(\eta^n_s) - f_{y,n}(\eta^n_s) L^n f_{x,n}(\eta^n_s) \\ a^n_{x,x}(\zeta^n_s) &:= L^n(f_{x,n}\cdot f_{x,n}(\eta^n_s)) - 2 f_{x,n}(\eta^n_s) L^n f_{x,n}(\eta^n_s) = Q^n(f_{x,n}(\eta^n)) \end{aligned}$$

In light of our computations above we have that:

$$\begin{aligned}
b^n_x(\zeta) &= \Delta_V \zeta^n(x) - (\zeta^n_s(x))^k \left(-\beta + f\left(\frac{\zeta^n_s(x)}{n^{\nu-1}}\right)\right) \\
a^n_{x,y}(\zeta) &= \mathbb{1}_{x\sim y}\left[-\frac{\zeta^n(x)+\zeta^n(y)}{n}\right] \\
a^n_{x,x}(\zeta) &= \sum_{y\sim x}\frac{\zeta^n(x)+\zeta^n(y)}{n} + (\zeta^n_s(x))^l\left(\alpha + g\left(\frac{\zeta^n_s(x)}{n^{\nu-1}}\right)\right)
\end{aligned}$$

2.6 Uniform results for the family $\{\eta^n_\cdot\}_n$

Now we can state the first technical results that will be crucial to obtain tightness and uniform integrability.

Proposition 1 (uniform bound on moments)**.** *For every $m \in \mathbb{N}$, there is a constant $C(m)$, independent of n, such that:*

$$\mathbb{E}\left[(S^n_{t\wedge\hat{\tau}^n})^m\right] \le C(m)(1+t).$$

Remark. *We have omitted the dependency of C on the terms k, l, α and β because they are fixed parameters of our SDE. The really important part is that C does not depend on n. Moreover, we have made explicit the dependency of C on m to emphasize this dependence when the estimate is used.*

Proof. Define $h^m_n(\eta) = \min\left\{(S^n(\eta))^m, (wn^{\nu-1}+1)^m\right\}$. We have:

$$h^m_n(\eta^n_t) = h^m_n(\eta^n_0) + \int_0^t (L_n h^m_n)(\eta^n_s)\,ds + M^{h^m_n,n}_t,$$

We note that while $S^n(\eta) \le wn^{\nu-1}$ we have $L_n h^m_n(\eta) \le C_0(m)$ (see 4.4). Therefore, using the stopping time $\hat{\tau}^n := \inf\{t>0; S^n_t > w\,n^{\nu-1}\}$ on the above equation we obtain:

$$(S^n_{t\wedge\hat{\tau}^n})^m = (S^n_0)^m + \int_0^{t\wedge\hat{\tau}^n} (L_n h^m_n)(\eta^n_s)\,ds + M^{h^m_n,n}_{t\wedge\hat{\tau}^n},$$

and taking expectations on both sides suffices to make the martingale term disappear and we can find the desired bound:

$$\mathbb{E}\left[(S^n_{t\wedge\hat{\tau}^n})^m\right] \le (S^n_0)^m + tC_0(m) < C(m)(1+t).$$

□

The final result we will prove in this section is the following

Proposition 2 (Uniform non-explosion)**.** *For all $T > 0$,*

$$\lim_{A\to\infty} \sup_n \mathbb{P}\left[\sup_{s\le T} S^n(\eta_s^n) > A\right] = 0.$$

Proof. If $S^n < wn^{\nu-1}$ then $L_n S^n(\eta) < 0$, indeed:

$$L_n S^n(\eta) = \sum_{x\in V} (\zeta_s^n(x))^k \left(-\beta + f\left(\frac{\zeta_s^n(x)}{n^{\nu-1}}\right)\right) < 0.$$

This yields

$$S^n(\eta^n_{t\wedge\hat{\tau}^n}) \le S^n(\eta^n_0) + M^{S^n,n}_{t\wedge\hat{\tau}^n}.$$

For $wn^{\nu-1} > A$ we have that $\mathbb{P}\left[\sup_{t\le T} S_t^n > A\right] = \mathbb{P}\left[\sup_{t\le T} S^n_{t\wedge\hat{\tau}^n} > A\right]$ and so:

$$\mathbb{P}\left[\sup_{t\le T} S_t^n > A\right] \le \mathbb{P}\left[S_0^n > \frac{A}{2}\right] + \mathbb{P}\left[\sup_{t\le T} M^{S^n,n}_{t\wedge\hat{\tau}^n} > \frac{A}{2}\right].$$

Since $\sup_n \mathbb{P}\left[S_0^n > \frac{A}{2}\right] \xrightarrow[A\to\infty]{} 0$ we need to estimate the second term. To do so we use Chebyshev inequality and Doob's inequality along with the estimates used in (2.12) for the g function plus the fact that $\sum_{x\in V} \zeta_s(x)^l \le \left(S_s^N\right)^l$

$$\begin{aligned}
\mathbb{P}\left[\sup_{t\le T} M^{S^n,n}_{t\wedge\hat{\tau}^n} > \frac{A}{2}\right] &\le \frac{4}{A^2}\mathbb{E}\left[\left(\sup_{t\le T} M^{S^n,n}_{t\wedge\hat{\tau}^n}\right)^2\right] \le \frac{16}{A^2}\mathbb{E}\left[\left(M^{S^n,n}_{T\wedge\hat{\tau}^n}\right)^2\right] \\
&\le \frac{16}{A^2}\mathbb{E}\left[\int_0^{T\wedge\hat{\tau}^n} \sum_{x\in V} (\zeta_s^n(x))^l \left(\alpha^2 + g\left(\frac{\zeta_s^n(x)}{n^{\beta-1}}\right)\right) ds\right] \\
\le \frac{16}{A^2}\mathbb{E}\left[\int_0^{T\wedge\hat{\tau}^n} 2\alpha^2 \left(S_s^n\right)^l \, ds\right] &\le \frac{16}{A^2} 2\alpha^2 C(l)(1+T)T \xrightarrow[A\to\infty]{} 0
\end{aligned}$$

Which gives us that $\lim_{A\to\infty} \sup_n \mathbb{P}\left[\sup_{s\le T} S^n(\eta^n_{t\wedge\hat{\tau}^n}) > A\right] = 0.$ □

Chapter 3

The limit process

After having established the first results in the previous chapter, the final arguments in the proof are of a qualitative nature and involve much less computation. In this chapter, we prove that:

(a) the sequence $(\zeta_\cdot^n)_{n\in\mathbb{N}}$ is tight,

(b) limit points of $(\zeta_\cdot^n)_{n\in\mathbb{N}}$ solve a martingale problem, and that

(c) this martingale problem is well-posed.

Throughout this chapter, we use a notation in the same spirit as in the previous one. We write $\mathbb{P}^n$ to refer to the probability measure induced by $\zeta_\cdot^n$ and similarly we will denote (for A an event in D):

$$\mathbb{P}^n(A) = \mathbb{P}[\zeta_\cdot^n \in A]$$

Analogously, we make $\mathbb{E}^n$ denote the expectation with respect to the probability law $\mathbb{P}^n$, so that

$$\mathbb{E}^n[f(\omega)] = \mathbb{E}[f(\zeta_\cdot^n)].$$

3.1 Tightness

In our context, a tight family of processes $(\zeta_\cdot^n)_{n\in\mathbb{N}}$ is pre-compact by Prohorov's Theorem [Bil99, p. 51 and 138]. This means that every sequence of processes from this family admits a limit point $\zeta_\cdot^*$. That is, a subsequence of $(\zeta_\cdot^n)_n$ converges to some $\zeta_\cdot^*$, which means that the corresponding probabilities $(\mathbb{P}^n)_n$ converge weakly (as measures over D with the Skorohod topology) to some probability law $\mathbb{P}^*$.

By the Portemanteau theorem [Bil99, p. 16], this is equivalent to say that for every (Skorohod) open set $A \subset D\left([0,\infty),\mathbb{R}^V\right)$ we have

$$\mathbb{P}^*(A) \leq \liminf_n \mathbb{P}^n(A).$$

Remember a family $\{\,\mathbb{P}^n\,\}_n$ of probability measures is tight when for every $\epsilon > 0$, there is a (Skorohod) compact set $K(\epsilon)$ such that

$$\inf_n \mathbb{P}^n[K(\epsilon)] \geq 1 - \epsilon.$$

To prove tightness for the vector-valued process $\{\,\zeta^n_\cdot\,\}_{n\in\mathbb{N}}$ it suffices to verify that for every $x \in V$ the sequence of paths $\{\zeta^n_t(x)\,;\, t \in [0,\infty)\}_{n\in\mathbb{N}}$ in $\mathcal{D}([0,\infty);\mathbb{R})$ is tight.

Indeed, if each $\zeta^n_t(x)$ is tight, denote by $K(x,\epsilon)$ the compact set for which

$$\inf_n \mathbb{P}^n\left[\zeta^n_\cdot(x) \in K(x,\epsilon)\right] \geq 1 - \epsilon.$$

Now take $K(\epsilon) = \cup_{x\in V} K(x, \frac{\epsilon}{|V|})$. This gives us, for all n:

$$\mathbb{P}[\zeta^n_\cdot \notin K(\epsilon)] \leq \sum_{x\in V} \mathbb{P}\left[\zeta^n_\cdot(x) \notin K\left(x, \frac{\epsilon}{|V|}\right)\right] \leq \epsilon.$$

So

$$\inf_n \mathbb{P}\left[\zeta^n_\cdot \in K(\epsilon)\right] \geq 1 - \epsilon.$$

Proposition 3. *The sequence $\{\zeta^n_t(x)\,;\, t \in [0,\infty)\}_{n\in\mathbb{N}}$ satisfies Aldous's Criterion [Bil99, p. 178], and therefore it is tight. More precisely, given any $T > 0$, it satisfies the following two conditions:*

i)

$$\lim_{A\to\infty} \sup_n \mathbb{P}\left[\sup_{t\leq T} |\zeta^n_t(x)| > A\right] = 0$$

ii) $\forall\, \epsilon > 0$:

$$\lim_{\delta_0\to 0} \sup_{n\in\mathbb{N}} \sup_{\delta\leq\delta_0} \sup_{\tau\in\mathcal{T}_T} \mathbb{P}\left(\left|\zeta^n_{\tau+\delta}(x) - \zeta^n_\tau(x)\right| > \epsilon\right) = 0$$

where $\mathcal{T}_T = \{\,\tau \mid \tau$ is a stopping time bounded by $T\,\}$[1]

[1] The abuse notation $\tau + \delta = \min\{\,\tau + \delta, T\,\}$ for convenience.

Proof. Condition $i)$ follows from Proposition 2:

$$\lim_{A\to\infty}\sup_n \mathbb{P}\left[\sup_{t\le T}|\zeta^n_t(x)|>A\right]\le \lim_{A\to\infty}\sup_n \mathbb{P}\left[\sup_{t\le T}S^n_t>A\right]=0$$

To see condition $ii)$ let $M^{x,n}$ denote the Dynkin martingale associated with the function $f_{x,n}(\eta)=\frac{\eta(x)}{n}$. Upon rewriting its defining equation (2.13), we get:

$$\zeta^n_t(x)=\zeta^n_0(x)+\int_0^t \Delta_V\zeta^n_s(x)-f^{\beta,n}\left(\zeta^n_s(x)\right)\,ds+M^{x,n}_t.$$

So, for $\delta\le\delta_0$:

$$\zeta^n_{\tau+\delta}(x)-\zeta^n_\tau(x)=\left(\int_\tau^{\tau+\delta}\Delta\zeta^n_s(x)-f^{\beta,n}\left(\zeta^n_s(x)\right)\,ds\right)+\left(M^{x,n}_{\tau+\delta}-M^{x,n}_\tau\right).$$

Since we want a bound on the left-hand side, we will bound each term on the right by $\epsilon/2$. We start with an estimate of the integral term.

Observe that

$$\begin{aligned}\mathbb{P}&\left[\left|\int_\tau^{\tau+\delta}\Delta\zeta^n_s(x)-f^{\beta,n}(\zeta^n(x))\,ds\right|>\frac{\epsilon}{2}\right]\\&\le\mathbb{P}\left[\left|\int_{\tau\wedge\tau^n_A}^{(\tau+\delta)\wedge\tau^n_A}\Delta\zeta^n_s(x)-f^{\beta,n}(\zeta^n(x))\,ds\right|>\frac{\epsilon}{2}\right]+\mathbb{P}\left[\tau^n_A<T\right].\end{aligned}$$

Using again Proposition 2, it follows that

$$\sup_n\mathbb{P}\left[\tau^n_A<T\right]=\sup_n\mathbb{P}\left[\sup_{t\le T}S^n_t(x)>A\right]\xrightarrow[A\to\infty]{}0.$$

The stopping time τ^n_A ensures that the absolute value of the integrated term remains bounded, since

$$\left|\Delta\zeta^n_s(x)-f^{\beta,n}(\zeta^n(x))\right|\le\sum_{y\sim x}\left(\zeta^n_s(y)+\zeta^n_s(x)\right)+2\beta\left(\zeta^n_s(x)\right)^k\le 2\left|V\right|A+2\beta A^k.$$

Coming back to the stopped integral, by Markov's inequality:

$$\begin{aligned}\mathbb{P}&\left[\left|\int_{\tau\wedge\tau^n_A}^{(\tau+\delta)\wedge\tau^n_A}\Delta\zeta^n_s(x)-f^{\beta,n}(\zeta^n(x))\,ds\right|>\frac{\epsilon}{2}\right]\\&\le\frac{2}{\epsilon}\mathbb{E}\left[\int_{\tau\wedge\tau^n_A}^{(\tau+\delta)\wedge\tau^n_A}\left|\Delta\zeta^n_s(x)-f^{\beta,n}(\zeta^n(x))\right|\,ds\right]\\&\le\frac{2}{\epsilon}\delta\left(2\left|V\right|A+2\beta A^k\right)\le\frac{2}{\epsilon}\delta_0\left(2\left|V\right|A+\beta A^k\right)\xrightarrow[\delta_0\to0]{}0.\end{aligned}$$

We estimate $\mathbb{P}\left[\,\left|M^{x,n}_{\tau+\delta} - M^{x,n}_{\tau}\right| > \frac{\epsilon}{2}\,\right]$ by a similar stopping-time argument, so the remaining term is

$$\mathbb{P}\left[\,\left|M^{x,n}_{(\tau+\delta)\wedge\tau^n_A} - M^{x,n}_{\tau\wedge\tau^n_A}\right| > \frac{\epsilon}{2}\right].$$

By Chebychev's inequality and the martingale property of $M^{x,n}_{\cdot}$,

$$\begin{aligned}
&\mathbb{P}\left[\,\left|M^{x,n}_{(\tau+\delta)\wedge\tau^n_A} - M^{x,n}_{\tau\wedge\tau^n_A}\right| > \frac{\epsilon}{2}\right]\\
&\quad\le \frac{4}{\epsilon^2}\mathbb{E}\left[\left(M^{x,n}_{(\tau+\delta)\wedge\tau^n_A}\right)^2 - \left(M^{x,n}_{\tau\wedge\tau^n_A}\right)^2\right]\\
&\quad= \frac{4}{\epsilon^2}\mathbb{E}\left[\int_{\tau\wedge\tau^n_A}^{(\tau+\delta)\wedge\tau^n_A} Q_n(f_{x,n})(\eta^n_s)\right]
\end{aligned}$$

and if $n^{\nu-1}w > A$, then we can again bound the integrated term by:

$$\begin{aligned}
&\frac{4}{\epsilon^2}\mathbb{E}\left[\int_0^t \mathbb{1}_{[\tau\wedge\tau^n_A,\,(\tau+\delta)\wedge\tau^n_A]}(t)\left(\frac{2\,|V|\,A}{n} + 2\alpha A^l\right)\,ds\right]\\
&\quad\le \frac{4}{\epsilon^2}\delta\left(2\,|V|\,A + 2\alpha A^l\right) \le \frac{4}{\epsilon^2}\delta_0\left(2\,|V|\,A + 2\alpha A^l\right) \xrightarrow[\delta_0\to 0]{} 0.
\end{aligned}$$

This shows that the family $\{\,\zeta^n_{\cdot}(x)\,\}_n$ satisfies condition $ii)$ of Aldous's Criterion, and concludes the proof of its tightness. □

3.2 Characterization of the limit

Now that we proved that the sequence $(\zeta^n_{\cdot})_n$ is tight, it remains to characterize the limit points of this sequence. For this, we will show that:

- the limit points are continuous,
- the limit points solve the martingale problem for some second-order elliptic differential operator L_*, and that
- there is a unique solution to the martingale problem for L_*

3.2.1 Continuity of paths of the limit process

We claim that any limit measure of the tight family $(\mathbb{P}^n)_{n\in\mathbb{N}}$ gives measure 1 to continuous trajectories. This follows from an analysis of the jump function:

$$\begin{aligned}
J : D([0,T],E) &\to \mathbb{R}\\
x &\mapsto \sup_{t\in(0,T]} d_E(x(t-),x(t)).
\end{aligned}$$

Since every $x \in D$ is right continuous, the condition that $J(x) = 0$ implies that x is a left-continuous path in $[0, T]$ and therefore it implies that x is a continuous path in $[0, T]$.

Let $\mathbb{P}^*$ be a limit point of the sequence $\mathbb{P}^n$. That is, for some subsequence $\mathbb{P}^{n'}$, we have $\mathbb{P}^{n'} \xrightarrow[n' \to \infty]{J_1} \mathbb{P}^*$. We claim that $\mathbb{P}^*[J(x) = 0] = 1$. The key observation to proof this is to note that J is continuous with respect to the J_1-Skorohod topology [Bil99, p. 125]. This means that

$$[J(x) > a] = \{\, x \mid J(x) > a \,\} = J^{-1}((a, \infty))$$

is an open set. Then, since for $n > K$, $\mathbb{P}_n\left[J(x) > \frac{1}{K}\right] = 0$ (because all jumps of ζ^n have magnitude $1/n$), by the Portemanteau Theorem [Bil99, p. 16]:

$$\mathbb{P}^*\left[J(x) > \frac{1}{K}\right] \leq \liminf_{n'} \mathbb{P}^{n'}\left[J(x) > \frac{1}{K}\right] = 0.$$

This implies that

$$\mathbb{P}^*[J(x) > 0] = \mathbb{P}\left(\cup_{K \in \mathbb{N}}\left[J(x) > \frac{1}{K}\right]\right) \leq \sum_{K \in \mathbb{N}} \mathbb{P}^*\left[J(x) > \frac{1}{K}\right] = 0$$

or in other words, that $\mathbb{P}^*[J(x) = 0] = 1$.

3.2.2 Martingale problem

In this section, we construct a second-order differential operator L_* following the heuristics outlined in the Introduction (more precisely, in subsection 1.3.2), and prove that any limit point of $\mathbb{P}^n$ is a solution to the martingale problem associated with it.

The candidate for L_* is obtained by analysing the expressions of the (discrete analogues of the) coefficients a and b of L_n:

$$\begin{aligned}
b_x^n(\zeta) &= \Delta_V \zeta^n(x) - f^{\beta,n}(\zeta^n(x)) \\
a_{x,y}^n(\zeta) &= \mathbb{1}_{x \sim y}\left[-\frac{\zeta^n(x) + \zeta^n(y)}{n}\right] \\
a_{x,x}^n(\zeta) &= \sum_{y \sim x} \frac{\zeta^n(x) + \zeta^n(y)}{n} + g^{\alpha,n}(\zeta^n(x)).
\end{aligned}$$

Since ζ^n is tight, we can assume that $\zeta^n \xrightarrow[n \to \infty]{J_1} \zeta^*$. So we define:

$$\begin{aligned}
b_x^*(\zeta) &= \Delta_V \zeta(x) - \beta\,(\zeta(x))^k \\
a_{x,y}^*(\zeta) &= 0 \\
a_{x,x}^*(\zeta) &= \alpha\,(\zeta(x))^l,
\end{aligned}$$

which determine the coefficients of the differential operator L_*.

Remember that we don't need to verify that $\left\{ M_t^{f,L_*}, \mathcal{F}_t; t \geq 0 \right\}$ is a continuous local martingale for every $f \in C^2(\mathbb{R}^d)$ [KS91, p. 318]. From the discussion in Subsection 1.3.1, it suffices to verify this property for the coordinate functions $f_x(\zeta) = \zeta(x)$ and $f_{x,y}(\zeta) = \zeta(x)\zeta(y)$. So we only need to show that

Proposition 4.

$$\begin{aligned} M_t^{x,L_*} &:= \zeta_t^*(x) - \zeta_0^*(x) - \int_0^t b_x^*(\zeta_s^*)\, ds \\ M_t^{x,y,L_*} &:= \zeta_t^*(x)\zeta_t^*(y) - \zeta_0^*(x)\zeta_0^*(y) \\ &\qquad - \int_0^t b_x^*(\zeta_s^*)\zeta_s^*(y) + b_y^*(\zeta_s^*)\zeta_s^*(x) + a_{x,y}^*(\zeta_s^*)\, ds \end{aligned}$$

are continuous local martingales.

Proof. This will follow from the study of the family of stopped martingales $M_{t\wedge\tilde{\tau}^n}^{x,L_n}$ and $M_{t\wedge\tilde{\tau}^n}^{x,y,L_n}$, where

$$\tilde{\tau}^n := \inf\left\{ t > 0; S_t^n > w\, n^{\frac{\nu-1}{2}} \right\}. \tag{3.1}$$

We will first prove that these are uniformly integrable martingales, which implies that their limits are martingales as well. Then, we show that the limits are the martingales M_t^{x,L_*} and M_t^{x,y,L_*}.

Let's see the first martingale. It suffices to show that, for every $t \leq T$,

$$\sup_n \mathbb{E}\left[\left(M_{t\wedge\tilde{\tau}^n}^{x,L_n}\right)^2\right] < \infty.$$

The quadratic variation of the stopped Dynkin martingales can be calculated with the operator Q_n. Using the expansion given in equation (2.16), and that before $\tilde{\tau}$ the function g is small by (2.12), we have:

$$\begin{aligned} \sup_n \mathbb{E}\left[\left(M_{t\wedge\tilde{\tau}^n}^{x,L_n}\right)^2\right] &\leq \sup_n \mathbb{E}\left[\int_0^{t\wedge\tilde{\tau}^n} \sum_{y:y\sim x} \frac{\zeta_s^n(y) + \zeta_s^n(x)}{n} + 2\alpha\left(\zeta_s^n(x)\right)^l\, ds\right] \\ &\leq T\left(2\,|V|\,C(1)(1+T) + 2\alpha C(l)(1+T)\right) < \infty \end{aligned}$$

where we bounded $|\zeta|$ and $|\zeta|^l$ by the uniform estimates of Proposition 1.

Now, we claim that $M^{x,L_n}_{t\wedge\tilde{\tau}^n} \xrightarrow[n\to\infty]{J_1} M^{x,L_*}_t$. Indeed, define by analogy with

$$M^{x,L_n}_{t\wedge\tilde{\tau}^n} = \zeta^n_{t\wedge\tilde{\tau}^n}(x) - \zeta^n_0(x) - \int_0^{t\wedge\tilde{\tau}^n} b^n_x(\zeta^n_s)\,ds$$

the processes (not stopped, and with the limit function b^* in place of b^n):

$$\tilde{M}^{x,n}_t = \zeta^n_t(x) - \zeta^n_0(x) - \int_0^t b^*_x(\zeta^n_s)\,ds$$

Note that since $\zeta^n_\cdot \xrightarrow[n\to\infty]{J_1} \zeta^*_\cdot$, the continuity of the functions $\tilde{M}^{x,n}_t$ implies that $\tilde{M}^{x,n}_t \xrightarrow[n\to\infty]{J_1} M^{x,L_*}_t$.

Moreover, we have

$$\lim_n \mathbb{P}\left(\sup_{t\le T}\left|M^{x,L_n}_{t\wedge\tilde{\tau}^n} - \tilde{M}^{x,n}_t\right| > \epsilon\right) = 0,$$

Indeed, because $\tilde{\tau}^n > \tau_A$ for n large enough:

$$\begin{aligned}\lim_n \mathbb{P}\left(\sup_{t\le T}\left|M^{x,L_n}_{t\wedge\tilde{\tau}^n} - \tilde{M}^{x,n}_t\right| > \epsilon\right) \le \limsup_n \mathbb{P}\left(\sup_{t\le T}\left|M^{x,L_n}_{t\wedge\tau^n_A} - \tilde{M}^{x,n}_{t\wedge\tau^n_A}\right| > \epsilon\right)\\ + \limsup_n \mathbb{P}\left[\tau^n_A < T\right].\end{aligned}$$

Define $\epsilon_n(A) = \sup_{z\le\frac{A}{n^{\nu-1}}} |f(z)|$ and note that $\epsilon_n \xrightarrow[n\to\infty]{} 0$, so:

$$\begin{aligned}\left|M^{x,L_n}_{t\wedge\tau^n_A} - \tilde{M}^{x,n}_{t\wedge\tau_A}\right| &\le \int_0^{t\wedge\tau^n_A} \left|\beta\left(\zeta^n_s\right)^k - f^{n,\beta}(\zeta^n_s)\right|\,ds\\ &= \int_0^{t\wedge\tau^n_A} \left|f\left(\frac{\zeta^n_s}{n^{\nu-1}}\right)(\zeta^n_s)^k\right|\,ds \le \int_0^{t\wedge\tau^n_A} \epsilon_n A^k \le \epsilon_n A^k T \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

We then conclude that:

$$\lim_n \mathbb{P}\left(\sup_{t\le T}\left|M^{x,L_n}_{t\wedge\tilde{\tau}^n} - \tilde{M}^{x,n}_t\right| > \epsilon\right) \le \limsup_n \mathbb{P}\left[\tau^n_A < T\right] \xrightarrow[A\to\infty]{} 0.$$

We have proved that $d(\tilde{M}^{x,n}, M^{x,L_n}_{\cdot\wedge\tilde{\tau}^n}) \xrightarrow[n\to\infty]{} 0$ in probability, and also that $\tilde{M}^{x,n} \to M^{x,L_*}$, so $M^{x,L_n}_{\cdot\wedge\tilde{\tau}^n} \xrightarrow[n\to\infty]{J_1} M^{x,L_*}$ [Bil99, p. 27]. Since the family $M^{x,L_n}_{t\wedge\tilde{\tau}^n}$ is uniformly integrable, we conclude that the limit M^{x,L_*} is a martingale.

We do the same for the martingales $M^{x,y,L_n}_{\cdot\wedge\tilde{\tau}^n}$. For this purpose we remember that the quadratic variations of these martingales are given by

$$\mathbb{E}_n\left[\left(M^{x,y,L_n}_{T\wedge\tilde{\tau}^n}\right)^2\right] = \mathbb{E}_n\left[\int_0^{T\wedge\tilde{\tau}^n} Q_n f_{x,y,n}(\eta^n_s)\,ds\right].$$

Since the expression of $Q_n f_{x,y,n}(\eta^n_s)$ is given by a finite sum of products of $\zeta^n_s(x)$ and $\zeta^n_s(y)$, we can again find a convenient constant $C > 0$ such that:

$$Q_n f_{x,y,n}(\eta^n_s) \leq C\left(1 + (S^n_s)^{l+2}\right)$$

Therefore:

$$\begin{aligned}\mathbb{E}_n\left[\left(M^{x,y,L_n}_{t\wedge\tilde{\tau}^n}\right)^2\right] &= \mathbb{E}_n\left[\int_0^{T\wedge\tilde{\tau}^n} Q_n f_{x,y,n}(\eta^n_s)\,ds\right] \\ &< T(1+T)C\left(1 + (S^n_s)^{l+2}\right) < \infty.\end{aligned}$$

The proof that $M^{x,y,L_n}_{\cdot\wedge\tilde{\tau}^n} \xrightarrow[n\to\infty]{J_1} M^{x,y,L_*}$ follows an analog construction of martingales

$$\tilde{M}^{x,y,n}_t = \zeta^n_t(x)\zeta^n_t(y) - \zeta^n_0(x)\zeta^n_0(y) - \int_0^t \zeta^n_s(x)b^*_x(\zeta^n_s) + \zeta^n_s(y)b^*_x(\zeta^n_s) + a^*_{x,y}(\zeta^n_s)\,ds.$$

□

3.2.3 Uniqueness

To conclude that the tight sequence of processes $\{\,\zeta^n_\cdot\,\}$ converges, it suffices to prove that it has a unique limit point. This is the final step needed for the characterization of the limits of the particle systems associated with L_n. Previously we have shown that the limit measures are concentrated on continuous paths, then we have shown that every limit point is a solution to the martingale problem associated with L_* and now we shall show that there is only one such solution.

The correspondence between solutions to martingale problems and solution to SDE's is given in [KS91, Corollaries 4.8 and 4.9, p. 317]. Existence and uniqueness of solutions $[(X, W), (\Omega, \mathcal{F}, \mathbb{P}), \mathcal{F}_t]$ in the sense of probability law to an SDE with a fixed but arbitrary initial distribution

$$\mathbb{P}\left[X_0 \in \Gamma\right] = \mu(\Gamma)$$

is equivalent to existence and uniqueness of solutions P to the corresponding martingale problem with the initial condition

$$P\left[y \in C([0,\infty),\mathbb{R}^d \mid y(0) \in \Gamma\right] = \mu(\Gamma)\ \Gamma \in \mathcal{B}(\mathbb{R}^d).$$

The two solutions are related by $\mathbb{P}(X \in A) = P(A)$, that is, the solution to the martingale problem is the law induced by the weak solution to the SDE.

Also, uniqueness in the sense of probability law follows from pathwise uniqueness [YW71, KS91, p. 301 and 331]. So, we only need to verify that pathwise uniqueness holds for our equations. Remember from the discussion at the end of Section 1.4 that we need to treat two cases separately.

For the case $l \geq 2$, since both b and σ are C^1, we can apply:

Theorem ([KS91, p.287]). *Suppose that the coefficients $b(t,\zeta), \sigma(t,\zeta)$ are locally Lipschitz continuous in the space variable; i.e., for every $n \geq 1$ there exists a constant $K_n > 0$ such that for every $t \geq 0$, $\|\zeta\| \leq n$ and $\left\|\tilde{\zeta}\right\| \leq n$:*

$$\left\|b(t,\zeta) - b(t,\tilde{\zeta})\right\| + \left\|\sigma(t,\zeta) - \sigma(t,\tilde{\zeta})\right\| \leq K_n \left\|\zeta - \tilde{\zeta}\right\|.$$

Then pathwise uniqueness holds for (b,σ), that is, for the vector valued SDE

$$d\zeta_t = b(t,\zeta)dt + \sigma(t,\zeta)\,dB_t.$$

For the case $l = 1$ we note that the local Lipschitz condition is not true for σ, since it behaves as a square-root near the boundary $\zeta(x) = 0$. At this point, the first two criteria we could use for uniqueness have failed, namely uniform elipticity of the diffusion matrix and local Lipschitz condition for the coefficients of the corresponding SDE.

Before looking for alternative methods to prove uniqueness, one might suspect that this case $l = 1$ does not admit a unique solution. Let's examine some cases where uniqueness fails due to multiple behaviour at singular points.

First, in the context of martingale problems, multiplicity might be a consequence of incomplete definition of the domain of the operator. For instance, suppose we consider $L = \Delta$ in the half-line $(0,\infty)$. If we specify the domain of L as

$$D_\rho = \left\{ f \in C_0^2(0,+\infty) \,\middle|\, \frac{f'(0+)}{\rho} = \lim_{x\to 0} \frac{1}{2} f''(x) \right\}$$

the resulting solution of the martingale problem is the *Sticky* Brownian motion with parameter $\rho > 0$. This shows that the resulting random walk depends not only on the differential operator formula, but also on the choice of the functions it's applied to.

Another way to prove uniqueness to the martingale problem is to consider a suitable Green function that helps showing $R(I - \lambda L_*) = C_0(\mathbb{R}^V)$ for all small $\lambda > 0$. Finding a Green function is not an easy task but this method proved succesfull in the case of the reflected Brownian motion on a wedge [Wil83].

From the point of view of SDE's, there are also examples where uniqueness fails because trajectories can remain at the singular point $X = 0$ for an arbitrary time [KS91, pp.292]. First, from the family

$$dX_t = |X_t^\alpha|\, dB_t, \qquad \alpha < \frac{1}{2}$$

it becomes clear that our case $\alpha = \frac{1}{2}$ is a limit situation. Moreover, the SDE

$$dX_t = 3X_t^{1/3}\, dt + 3X^{2/3}\, dB_t.$$

shows that it is not enough to have regularity in the noise term (note that $2/3 > 1/2$) but we also need a good behaviour in the drift term.

In fact, in our case, both drift and dispersion terms have the regularities needed to assure uniqueness. Coming back to our problem, labeling the sites V of our graph by $\{\, x_1, \ldots, x_n \,\}$, we observe that the associated dispersion matrix σ fits the particular form required by the following criterion, which will again give pathwise uniqueness in this case, concluding the last case for our theorem.

Theorem ([YW71, Theorem 1])**.** *Let*

$$d\zeta_t = \sigma(\zeta_t)\, dB_t + b(\zeta_t)\, dt \tag{3.2}$$

where

$$\sigma(\zeta) = \begin{bmatrix} \sigma_1(\zeta(x_1)) & & 0 & \\ & \sigma_2(\zeta(x_2)) & & \\ & & \ddots & \\ & 0 & & \sigma_n(\zeta(x_n)) \end{bmatrix}, b(\zeta) = \big(b^1(\zeta), b^2(\zeta), \ldots, b^n(\zeta)\big)$$

such that

(i) *there exists a positive increasing function $\rho(u), u \in (0, \infty)$ such that $\int_0^\epsilon \frac{1}{\rho^2(u)}\, du = +\infty$ for every $\epsilon > 0$*

$$|\sigma_i(u) - \sigma_i(v)| \le \rho(|u - v|), \quad \forall\, u, v \in \mathbb{R}, \quad , 1 \le i \le n$$

(ii) *for every* $n \geq 1$ *there exists a constant* $K_n > 0$ *such that for every* $t \geq 0, \|\zeta\| \leq n$ *and* $\left\|\tilde{\zeta}\right\| \leq n$*:*

$$\left\|b_i(\zeta) - b_i(\tilde{\zeta})\right\| \leq K_n \left\|\zeta - \tilde{\zeta}\right\|$$

Then pathwise uniqueness holds.

While condition *ii*) is again satisfied, since b remains polynomial, using $\rho(u) = \sqrt{u}$ gives us condition *i*).

And this concludes the proof of the uniqueness. So for every SDE we considered there is a family of Reaction-Diffusion models that converge after scaling to the law induced by unique solution of the given SDE.

Chapter 4

Technical issues and computations

In this chapter we discuss some results that we believe are better treated here than elsewhere. To deal with them in the middle of the text would interrupt the flow of ideas and furthermore would obscure the general understanding of the proof.

The first two sections deal with standard results of the construction of particle systems. In Section 4.1 we rule out explosions and in Section 4.2 we prove that polynomial functions of the number of particles in each site are in the domain of the generator.

Section 4.3 is mostly computations that will be used throughout the text a couple of times. We encourage the reader to calculte these quantities on his own but if trouble arises we hope he finds help in the explicit computations.

Section 4.4 is the gem of this Chapter. If we should single out one result that is key and original in the proof of tightness it is the fact that we can uniformly bound $L_n \left(S^n_{s\wedge\hat{\tau^n}}\right)^m$. That is, that we can bound uniformly the infinitesimal increase of polynomials of the total mass of the system. These uniform estimates are used indirectly for the proof of tightness since they are required to prove Proposition 2. This result will be used once again for the proof of uniform integrability of the Martingales in Proposition 4.

4.1 Non explosion

Let's consider the particle system defined in the set up 2.1. Since we allow for an infinite number of configurations $\eta \in \mathbb{N}^V$, to complete the definition of the process we need to rule out explosions, that is, an infinite number of transitions in a finite time. We will see that it suffices to require that

$F^+(u) \leq C(1+u)$.

Indeed, let $S_t = \sum_{x \in V} \eta_t(x)$ be the total number of particles in the system at time t. Summing over all rates $F^+(\eta_t(x))$, we see that the rate at which a particle is born at time t is less than $\Lambda_t = C(|V| + S_t)$. Thus, the expected time for one such transition is greater than $1/\Lambda_t$. Let T_n be the time it takes for the n-th particle to be born, therefore by independence of those transitions we have that

$$\begin{aligned}
\mathbb{E}\left[T_0\right] &\geq 1/\left(C(|V| + S_0)\right) \\
\mathbb{E}\left[T_1\right] &\geq 1/\left(C(|V| + S_0 + 1)\right) \\
&\vdots \qquad\qquad \vdots \\
\mathbb{E}\left[T_n\right] &\geq 1/\left(C(|V| + S_0 + n)\right).
\end{aligned}$$

Therefore

$$\mathbb{E}\left[\sum_{n=0}^{\infty} T_n\right] \geq \sum_{n=0}^{\infty} \left(C(|V| + S(\eta_0) + n)\right)^{-1} = \infty.$$

This implies that $\sum_{k=1}^{\infty} T_n = \infty$ almost surely [Nor97, theorem 2.3.2]. Since

$$\mathbb{P}\left[\text{infinite birth transitions before time } T\right] = \mathbb{P}\left[\sum_{n=1}^{\infty} T_n < T\right] = 0,$$

we conclude that almost surely there can't be an infinte number of particles before any arbitrary time T.

This means that the number of transitions (birth, death or jump) is almost surely finite as well. Indeed, define $\lambda(N)$ to be supremum of all transitions rates admissible between configurations with at most N particles.

$$\lambda(N) = \sup\left\{ F^+(\eta(x)), F^-(\eta(x)), \eta(x) \,\middle|\, \sum_{x \in V} \eta(x) \leq N \right\} < \infty. \tag{4.1}$$

Consider the event that the total number of particle births up to time T stay bounded by n and that the number of particles at the begining is less then n. We need to rule out an infinite number of jumps or deaths of particles before time T as well. Since those transitions are independent of the birth transitions, the time it takes for each occurence depends on exponencial times $\hat{T}_i$ with rate less that $\lambda(2n)$. By the same argument as before,

$$\mathbb{P}\left[\text{infinite jumps or deaths before time } T\right] = \left[\sum_{i=1}^{\infty} T_i < T\right] = 0$$

and this rules out explosions.

4.2 Lipschitz functions are in the domain of L_n and Q_n

A function $f : \mathbb{N}^V \to \mathbb{R}$ is Lipschitz if

$$\sup_{x\in V} \sup_{\eta\in\mathbb{N}^V} |f(\eta + \delta_x) - f(\eta)| = C_f < \infty.$$

We claim that a function satisfying the Lipschitz condition is on the domain of L_n. Indeed, we compute

$$\begin{aligned} L_n f(\eta) &= \lim_{h\to 0} \frac{1}{h}\mathbb{E}\left[f(\eta_h^n) - f(\eta)\right] \\ &= \lim_{h\to 0} \frac{1}{h}\Bigg(\mathbb{E}\left[f(\eta_h^n) - f(\eta) \;\middle|\; \text{at most 1 birth before time } h\right] \\ &\qquad + \mathbb{E}\left[f(\eta_h^n) - f(\eta) \;\middle|\; \text{2 births before time } h\right] + \dots \\ &\qquad + \mathbb{E}\left[f(\eta_h^n) - f(\eta) \;\middle|\; k \text{ births before time } h\right] + \dots\Bigg) \end{aligned}$$

We claim that

$$\begin{aligned} \lim_{h\to 0} \frac{1}{h}\Bigg(&\mathbb{E}\left[f(\eta_h^n) - f(\eta) \;\middle|\; \text{2 births before time } h\right] + \dots \\ &+ \mathbb{E}\left[f(\eta_h^n) - f(\eta) \;\middle|\; k \text{ births before time } h\right] + \dots\Bigg) = 0 \end{aligned}$$

As f is Lipschitz, $|f(\eta_h^n) - f(\eta)|\, \mathbb{1}_{\{\,k \text{ births before time } h\,\}} \le C_f k \mathbb{1}_{\{\,k \text{ births before time } h\,\}}$, and then the limit above is less than

$$\lim_{h} \frac{1}{h} C_f \sum_{k\ge 2} \mathbb{P}\left(k \text{ or more birth transitions before time } h\right).$$

Now, $p_k(h) = \mathbb{P}(k \text{ births or more before time } h)$ is given by

$$\int \dots \int \lambda_1 e^{-\lambda_1 x_1} \dots \lambda_k e^{-\lambda_k x_k} \mathbb{1}_{\{\,x_1+\dots+x_k\le h\,\}}\, dx_1 \dots dx_k.$$

Bounding all exponentials by 1 and observing that we're integrating over the k-dimensional simplex with side h, whose volume is $h^k (k!)^{-1}$, we see that

$$p_k(h) \le \frac{\prod_{i=1}^k \lambda_i}{\prod_{i=1}^k i} h^k.$$

Now, since $\lambda_k \leq C(|V|+S_0+k)$ and $\sup_k \frac{C(|V|+S_0+k)}{k} = \hat{C} < \infty$ we have:

$$p_k(h) \leq \prod_{i=1}^{k} \frac{\lambda_i}{i} h^k \leq \hat{C}^k h^k.$$

So for $\hat{C}h \leq \frac{1}{2}$:

$$\frac{C_f}{h} \sum_{k\geq 2} p_k(h) \leq \frac{C_f}{h} \sum_{k\geq 2} (\hat{C}h)^k \leq \frac{C_f}{h} 2(\hat{C}h)^2 \xrightarrow[h\to 0]{} 0$$

as we claimed.

Therefore

$$\begin{aligned} L_n f(\eta) &= \lim_{h\to 0} \frac{1}{h} \mathbb{E}\left[f(\eta_h^n) - f(\eta)\right] \\ &= \lim_{h\to 0} \frac{1}{h} \left(\mathbb{E}\left[f(\eta_h^n) - f(\eta) \mid \text{at most 1 birth before time } h \right] \right) \\ &= \sum_{x\sim y\in V} \eta(x) \left[f(\eta^{x,y}) - f(\eta)\right] + \sum_{x\in V} F_n^+(\eta(x)) \left[f(\eta^{x,+}) - f(\eta)\right] \\ &\qquad + \sum_{x\in V} F_n^-(\eta(x)) \left[f(\eta^{x,-}) - f(\eta)\right] \end{aligned}$$

so we see that f is in the domain of L_n.

To see that f is in the domain of Q_n it suffices to prove that

$$\lim_{h\to 0} \frac{1}{h} E_n\left[(f(X_h) - f(X_0))\right]$$

exists pointwisely. In this case we set $Q_n f = \lim_{h\to 0} \frac{1}{h} \mathbb{E}_n\left[(f(X_h) - f(X_0))^2\right]$. To prove this we follow analogous steps as above. We just note that if f is Lipschitz, $\left(f(\eta_h^n) - f(\eta)\right)^2 \mathbb{1}_{\{k \text{ births before time } h\}} \leq (C_f k)^2 \mathbb{1}_{\{k \text{ births before time } h\}}$, and then

$$\begin{aligned} &\lim_{h\to 0} \frac{1}{h} \Bigg(\mathbb{E}\left[(f(\eta_h^n) - f(\eta))^2 \mid 2 \text{ births before time } h \right] + \dots \\ &\qquad + \mathbb{E}\left[(f(\eta_h^n) - f(\eta))^2 \mid k \text{ births before time } h \right] + \dots \Bigg) \\ &\qquad \leq \frac{1}{h} \sum_{k\geq 2} (C_f k)^2 p_k(h) \end{aligned} \tag{4.2}$$

so for $\hat{C}h < 1/2$ we have

$$\frac{C_f}{h} \sum_{k\geq 2} k^2 p_k(h) \leq \frac{C_f}{h} \sum_{k\geq 2} k^2 (\hat{C}h)^k \leq \frac{C_f}{h} (\hat{C}h)^2 \sum_{k\geq 2} k^2 (\hat{C}h)^{k-2} \xrightarrow[h\to 0]{} 0 \tag{4.3}$$

since $\sum_{k\geq 2} k^2(\hat{C}h)^{k-2}$ is uniformly bounded. This concludes the proof that f is in the domain of Q_n.

An analogous argument holds for functions that are products of Lipschitz functions such as $H = f \cdot g$. Let $\xi \in \mathbb{N}^V$ and let $\|\xi\|_1 = \sum_{x\in V} \xi(x)$ and use the triangle inequality to obtain

$$|H(\eta+\xi) - H(\eta)| \leq C_f \|\xi\|_1 (g(\eta) + C_g \|\xi\|_1) + f(\eta) C_g \|\xi\|_1$$

We want to see now that H is in the domain of L_n. The key estimate now is $|H(\eta_h^n) - H(\eta)| \mathbb{1}_{\{k \text{ births before time } h\}} \leq C_H(\eta)(k)^2 \mathbb{1}_{\{k \text{ births before time } h\}}$ where $C_H(\eta)$ is some constant that depends on C_f, C_g, $f(\eta)$ and $g(\eta)$.

With this we can follow the same arguments as in (4.2) and (4.3) to conclude that H is in the domain of L_n. Analogously we can prove that H is in the domain of Q_n.

4.3 Explicit formulae for L_n and Q_n over coordinate functions

We put $f_{x,n}(\eta) = \frac{\eta(x)}{n} = \zeta^n(x)$ for the normalized projection on the coordinate x of the process η^n. Then, we have:

$$\begin{aligned} L_n(f_{x,n})(\eta) = &\sum_{w\sim z\in V} \eta(w) \left[f_{x,n}(\eta^{w,z}) - f_{x,n}(\eta)\right] \\ &+ \sum_{x\in V} F_n^{+,k,l}(\eta(x)) \left[f_{x,n}(\eta^{x,+}) - f_{x,n}(\eta)\right] \\ &+ \sum_{x\in V} F_n^{-,k,l}(\eta(x)) \left[f_{x,n}(\eta^{x,-}) - f_{x,n}(\eta)\right] \end{aligned}$$

and summing the terms with F^+ together with F^-, we can replace them with the net result F and obtain:

$$= \left(\sum_{y\sim x} \frac{\eta(y) - \eta(x)}{n}\right) + \left(\frac{\eta(x)}{n}\right)^k \left(-\beta + f\left(\frac{\eta(x)}{n^\nu}\right)\right).$$

Passing to the macroscopic quantities ζ^n, and using the discrete laplacian Δ_V, this becomes

$$\begin{aligned} L_n(f_{x,n})(\eta_s^n) &= \sum_{y\sim x} (\zeta_s^n(y) - \zeta_s^n(x)) + (\zeta_s^n(x))^k \left(-\beta + f\left(\frac{\zeta_s^n(x)}{n^{\nu-1}}\right)\right) \\ &= \Delta_V \zeta_s^n(x) + (\zeta_s^n(x))^k \left(-\beta + f\left(\frac{\zeta_s^n(x)}{n^{\nu-1}}\right)\right). \end{aligned}$$

Analogously, for the quadratic variation operator:

$$Q_n(f_{x,n})(\eta) = \sum_{y\sim x} \frac{\eta(y)+\eta(x)}{n^2} + \left(\frac{\eta(x)}{n}\right)^l \left(\alpha + g\left(\frac{\eta(x)}{n^\nu}\right)\right)$$
$$Q_n(f_{x,n})(\eta^n_s) = \sum_{y\sim x} \frac{\zeta^n_s(y)+\zeta^n_s(x)}{n} + \left(\zeta^n_s(x)\right)^l \left(\alpha + g\left(\frac{\zeta^n_s(x)}{n^{\nu-1}}\right)\right)$$

Now, we move to the functions $f_{x,y,n}(\eta) = \frac{\eta(x)}{n}\frac{\eta(y)}{n} = \zeta^n(x)\zeta^n(y)$. Proceeding in the same way as above, we now must take care of the case when x and y are neighbors, so we split the sum comming from the jumps in three pieces: one to x, one to y, and one to handle the case when $x \sim y$.

$$\begin{aligned}
&L_n f_{x,y,n}(\eta)\\
&= \sum_{\substack{w\sim x\\ w\neq y}} \eta(w)\left[f_{x,y,n}(\eta^{w,x}) - f_{x,y,n}(\eta)\right] + \eta(x)\left[f_{x,y,n}(\eta^{x,w}) - f_{x,y,n}(\eta)\right]\\
&+ \sum_{\substack{w\sim y\\ w\neq x}} \eta(w)\left[f_{x,y,n}(\eta^{w,y}) - f_{x,y,n}(\eta)\right] + \eta(y)\left[f_{x,y,n}(\eta^{y,w}) - f_{x,y,n}(\eta)\right]\\
&+ \mathbb{1}_{x\sim y}\left(\eta(x)\left[f_{x,y,n}(\eta^{x,y}) - f_{x,y,n}(\eta)\right] + \eta(y)\left[f_{x,y,n}(\eta^{y,x}) - f_{x,y,n}(\eta)\right]\right)\\
&+ F^+_n(\eta(x))\left[f_{x,y,n}(\eta^{x,+}) - f_{x,y,n}(\eta)\right] + F^-_n(\eta(x))\left[f_{x,y,n}(\eta^{x,-}) - f_{x,y,n}(\eta)\right]\\
&+ F^+_n(\eta(y))\left[f_{x,y,n}(\eta^{y,+}) - f_{x,y,n}(\eta)\right] + F^-_n(\eta(y))\left[f_{x,y,n}(\eta^{y,-}) - f_{x,y,n}(\eta)\right]
\end{aligned}$$

Now we compute the values of the differences inside brackets:

$$\begin{aligned}
&= \sum_{\substack{w\sim x\\ w\neq y}} \frac{\eta(w)-\eta(x)}{n}\left(\frac{\eta(y)}{n}\right) + \sum_{\substack{w\sim y\\ w\neq y}} \frac{\eta(w)-\eta(y)}{n}\left(\frac{\eta(x)}{n}\right)\\
&+ \mathbb{1}_{x\sim y}\left(\eta(x)\left[\frac{\eta(x)}{n^2} - \frac{\eta(y)}{n^2} - \frac{1}{n^2}\right] + \eta(y)\left[\frac{-\eta(x)}{n^2} + \frac{\eta(y)}{n^2} - \frac{1}{n^2}\right]\right)\\
&- 2f^{\beta,n}\left(\frac{\eta(x)}{n}\right)\left(\frac{\eta(y)}{n}\right) - 2f^{\beta,n}\left(\frac{\eta(y)}{n}\right)\left(\frac{\eta(x)}{n}\right)
\end{aligned}$$

Finally, we recombine the terms from the different cases we end up with a simpler expression:

$$
\begin{aligned}
&= \sum_{w\sim x} \frac{\eta(w)-\eta(x)}{n}\left(\frac{\eta(y)}{n}\right) + \sum_{w\sim y} \frac{\eta(w)-\eta(y)}{n}\left(\frac{\eta(x)}{n}\right) \\
&- \mathbb{1}_{x\sim y}\left(\frac{\eta(x)}{n^2}+\frac{\eta(y)}{n^2}\right) \\
&- 2f^{\beta,n}\left(\frac{\eta(x)}{n}\right)\left(\frac{\eta(y)}{n}\right) - 2f^{\beta,n}\left(\frac{\eta(y)}{n}\right)\left(\frac{\eta(x)}{n}\right).
\end{aligned}
$$

Again we translate in terms of the ζ^n:

$$
\begin{aligned}
L_n f_{x,y,n}(\eta_s^n) &= L_n f_{x,n}(\eta_s^n)\cdot f_{y,n}(\eta_s^n) + L_n f_{y,n}(\eta_s^n)\cdot f_{x,n}(\eta_s^n) \\
&- \mathbb{1}_{x\sim y}\left(\frac{\zeta_s^n(x)+\zeta_s^n(y)}{n}\right)
\end{aligned}
$$

To conclude, we procede in the same manner when it commes to the term $Q_n f_{x,y,n}(\eta)$ this ammounts first to split in cases the jump terms:

$$
\begin{aligned}
&Q_n f_{x,y,n}(\eta) \\
&= \sum_{\substack{w\sim x\\ w\neq y}} \eta(w)\left[f_{x,y,n}(\eta^{w,x})-f_{x,y,n}(\eta)\right]^2 + \eta(x)\left[f_{x,y,n}(\eta^{x,w})-f_{x,y,n}(\eta)\right]^2 + \\
&+ \sum_{\substack{w\sim y\\ w\neq x}} \eta(w)\left[f_{x,y,n}(\eta^{w,y})-f_{x,y,n}(\eta)\right]^2 + \eta(y)\left[f_{x,y,n}(\eta^{y,w})-f_{x,y,n}(\eta)\right]^2 \\
&+ \mathbb{1}_{x\sim y}\left(\eta(x)\left[f_{x,y,n}(\eta^{x,y})-f_{x,y,n}(\eta)\right]^2 + \eta(y)\left[f_{x,y,n}(\eta^{y,x})-f_{x,y,n}(\eta)\right]^2\right) \\
&+ F_n^+(\eta(x))\left[f_{x,y,n}(\eta^{x,+})-f_{x,y,n}(\eta)\right]^2 + F_n^-(\eta(x))\left[f_{x,y,n}(\eta^{x,-})-f_{x,y,n}(\eta)\right]^2 \\
&+ F_n^+(\eta(y))\left[f_{x,y,n}(\eta^{y,+})-f_{x,y,n}(\eta)\right]^2 + F_n^-(\eta(y))\left[f_{x,y,n}(\eta^{y,-})-f_{x,y,n}(\eta)\right]^2
\end{aligned}
$$

then we compute the differences inside brackets:

$$
\begin{aligned}
&= \sum_{\substack{w\sim x\\ w\neq y}} \frac{\eta(w)+\eta(x)}{n^2}\left(\frac{\eta(y)}{n}\right)^2 + \sum_{\substack{w\sim y\\ w\neq y}} \frac{\eta(w)+\eta(y)}{n^2}\left(\frac{\eta(x)}{n}\right)^2 \\
&+ \mathbb{1}_{x\sim y}\left(\eta(x)\left[\frac{\eta(x)}{n^2}-\frac{\eta(y)}{n^2}-\frac{1}{n^2}\right]^2 + \eta(y)\left[-\frac{\eta(x)}{n^2}+\frac{\eta(y)}{n^2}-\frac{1}{n^2}\right]^2\right) \\
&+ 2g^{\alpha,n}\left(\frac{\eta(x)}{n}\right)\left(\frac{\eta(y)}{n}\right)^2 + 2g^{\alpha,n}\left(\frac{\eta(y)}{n}\right)\left(\frac{\eta(x)}{n}\right)^2
\end{aligned}
$$

and finally we recombine the terms from the different cases to obtain a simpler expression:

$$
\begin{aligned}
&= \sum_{w\sim x} \frac{\eta(w)+\eta(x)}{n^2}\left(\frac{\eta(y)}{n}\right)^2 + \sum_{w\sim y} \frac{\eta(w)+\eta(y)}{n^2}\left(\frac{\eta(x)}{n}\right)^2 \\
&+ \mathbb{1}_{x\sim y}\Bigg(\left[-2\eta(x)\frac{\eta(x)}{n^2}\frac{\eta(y)}{n^2} + 2\eta(x)\left(\frac{\eta(y)}{n^2} - \frac{\eta(x)}{n^2}\right)\frac{1}{n^2} + \eta(x)\frac{1}{n^4}\right] \\
&\left[-2\eta(y)\frac{\eta(y)}{n^2}\frac{\eta(x)}{n^2} + 2\eta(y)\left(\frac{\eta(x)}{n^2} - \frac{\eta(y)}{n^2}\right)\frac{1}{n^2} + \eta(y)\frac{1}{n^4}\right]\Bigg) \\
&+ 2g^{\alpha,n}\left(\frac{\eta(x)}{n}\right)\left(\frac{\eta(y)}{n}\right)^2 + 2g^{\alpha,n}\left(\frac{\eta(y)}{n}\right)\left(\frac{\eta(x)}{n}\right)^2
\end{aligned}
$$

Again we translate in terms of the ζ^n:

$$
\begin{aligned}
&Q_n f_{x,y,n}(\eta^n_s) \\
&= \sum_{w\sim x} \frac{\zeta^n_s(w)+\zeta^n_s(x)}{n}\left(\zeta^n_s(y)\right)^2 + \sum_{w\sim y} \frac{\zeta^n_s(w)+\zeta^n_s(y)}{n}\left(\zeta^n_s(x)\right)^2 \\
&+ \mathbb{1}_{x\sim y}\Bigg(\left[-2\zeta^n_s(x)\zeta^n_s(x)\frac{\zeta^n_s(y)}{n} + 2\zeta^n_s(x)\left(\frac{\zeta^n_s(y)}{n} - \frac{\zeta^n_s(x)}{n}\right)\frac{1}{n} + \zeta^n_s(x)\frac{1}{n^3}\right] \\
&\left[-2\zeta^n_s(y)\zeta^n_s(y)\frac{\zeta^n_s(x)}{n} + 2\zeta^n_s(y)\left(\frac{\zeta^n_s(x)}{n} - \frac{\zeta^n_s(y)}{n}\right)\frac{1}{n} + \zeta^n_s(y)\frac{1}{n^3}\right]\Bigg) \\
&+ 2g^{\alpha,n}\left(\zeta^n_s(x)\right)\left(\zeta^n_s(y)\right)^2 + 2g^{\alpha,n}\left(\zeta^n_s(y)\right)\left(\zeta^n_s(x)\right)^2
\end{aligned}
$$

and these expressions written in terms the macroscopic quantities ζ^n_s will be used throughout the thesis.

4.4 Uniform bounds on $L_n h^m_n$

From Proposition 1, we recall that $h^m_n(\eta) = \min\left\{ \left(S^n(\eta)\right)^m, \left(wn^{\nu-1}+1\right)^m \right\}$. We needed to show that while $S^n(\eta) \le wn^{\nu-1}$, $L_n h^m_n(\eta) \le C_0(m)$. This requires a rather long computation, so in order to make it more direct we single out the following inequalities:

$$\sum_{x\in V} f^{n,\beta}\left(\zeta^n(x)\right) \ge \sum_{x\in V} \frac{\beta}{2}\left(\zeta^n(x)\right)^k \ge \frac{\beta}{2} c\left(S^n(\eta^n)\right)^k \tag{4.4}$$

$$\sum_{x\in V} g^{n,\alpha}\left(\zeta^n(x)\right) \le \sum_{x\in V} 2\alpha\left(\zeta^n(x)\right)^l \le 2\alpha\left(S^n(\eta^n)\right)^l \tag{4.5}$$

where c is some constant that depends on $|V|$ and m.

Since a jump transition conserves the total number of particles of the system, in the expression of $L_n\left(S^n(\eta)\right)^m$ we only need to take into account the birth/death transitions at each site, so for $S^n(\eta) \le wn^{\nu-1}$:

$$\begin{aligned} L_n\left(S^n(\eta)\right)^m = &\sum_{x\in V} F_n^+(\eta(x))\left[\left(S^n(\eta^{x,+})\right)^m - \left(S^n(\eta)\right)^m\right] \\ &\sum_{x\in V} F_n^-(\eta(x))\left[\left(S^n(\eta^{x,-})\right)^m - \left(S^n(\eta)\right)^m\right] \end{aligned}$$

Then expanding the expressions for F_n^+ and F_n^- according to (2.9) (2.10) and using the binomial expression for the terms in brackets, we get:

$$\begin{aligned} L_n\left(S^n(\eta)\right)^m = &\sum_{x\in V}\left(n^2 g^{n,\alpha}\left(\zeta^n(x)\right) - nf^{n,\beta}\left(\zeta^n(x)\right)\right)\left[\sum_{j\ge 1}\binom{m}{j}\left(S^n\right)^{m-j}\left(\frac{1}{n}\right)^j\right] \\ &+\left(n^2 g^{n,\alpha}\left(\zeta^n(x)\right) + nf^{n,\beta}\left(\zeta^n(x)\right)\right)\left[\sum_{j\ge 1}\binom{m}{j}\left(S^n\right)^{m-j}\left(\frac{-1}{n}\right)^j\right] \end{aligned}$$

Now note that, depending if j is either odd or even, the terms in the above expression simplify:

$$\begin{aligned} L_n\left(S^n(\eta)\right)^m = &\sum_{x\in V} -2nf^{n,\beta}\left(\left(\zeta^n(x)\right)\right)\left[\sum_{1\le j \text{ odd}}\binom{m}{j}\left(S^n\right)^{m-j}\frac{1}{n^j}\right] \\ &\sum_{x\in V}\left(2n^2 g^{n,\alpha}\left(\zeta^n(x)\right)\right)\left[\sum_{1\le j \text{ even}}\binom{m}{j}\left(S^n\right)^{m-j}\frac{1}{n^j}\right] \end{aligned}$$

So writing all in terms of even numbers we obtain for the right-hand side:

$$2\sum_{1\le j \text{ odd}}\frac{\left(S^n\right)^{m-j-1}}{n^{j-1}}\left[\sum_{x\in V}\left(-f_{n,\beta}\left(\left(\zeta^n(x)\right)\right)\binom{m}{j}S^n\right) + g^{n,\alpha}\left(\zeta^n(x)\right)\binom{m}{j+1}\right]$$

Since $S^n(\eta) \le wn^{\nu-1}$ we obtain estimates (2.12) for $f^{n,\beta}$ and $g^{n,\alpha}$:

$$\le 2\sum_{1\le j \text{ odd}}\frac{\left(S^n\right)^{m-j-1}}{n^{j-1}}\left[\sum_{x\in V}\left(2\alpha\left(\zeta^n(x)\right)\right)^l\binom{m}{j+1} - 2\beta\left(\zeta^n(x)\right)^k\binom{m}{j}S^n\right]$$

From the inequalities (4.4) (4.5) we obtain:

$$\leq 2 \sum_{1 \leq j \text{ odd}} \frac{(S^n)^{m-j-1}}{n^{j-1}} \left[\left(2\alpha^2 (S^n)\right)^l \binom{m}{j+1} - 2\beta c (S^n)^k \binom{m}{j} S^n \right]$$

$$\leq 2 \sum_{1 \leq j \text{ odd}} \frac{(S^n)^{m+l-j-1}}{n^{j-1}} \left[2\alpha^2 \binom{m}{j+1} - 2\beta c (S^n)^{k+1-l} \binom{m}{j} \right]$$

$$\leq C_0(m)$$

Where the last inequality follows from the fact that the terms

$$\left[2\alpha^2 \binom{m}{j+1} - 2\beta c (S^n)^{k+1-l} \binom{m}{j} \right]$$

are decreasing in S^n

Bibliography

[AT80] Ludwig Arnold and M. Theodosopulu. deterministic limit of the stochastic model of chemical reactions with diffusion. *Advances in Applied Probability*, 12(2):367–379, 1980.

[Bil99] Patrick Billingsley. *Convergence of probability measures.* Wiley series in probability and statistics. Wiley-Interscience, 2ed edition, 1999.

[Blo92] Douglas Blount. Law of large numbers in the supremum norm for a chemical reaction with diffusion. *The annals of applied probability*, 2(1):131–141, 1992.

[Dur10] Rick Durrett. *Probability Theory and Examples.* Cambridge University press, 2010.

[Ein56] Albert Einstein. *INVESTIGATIONS ON THE THEORY OF THE BROWNIAN MOVEMENT*, volume d. Dover, New York, 1956.

[FG12] Tertuliano Franco and Pablo Groisman. A particle system with explosions: law of large numbers for the density of particles and the blow-up time. *Journal of Statistical Physics*, 4:629–642, November 2012.

[GJ13] Patrícia Gonçalves and Milton Jara. Nonlinear fluctuations of weakly asymmetric interacting particle systems. *Archive for Rational Mechanics and Analysis*, 212(2):597–644, Dec 2013.

[Gow11] Reid Gower. THE FEYNMAN SERIES - Curiosity. `https://www.youtube.com/watch?v=lmTmGLzPVyM`, 2011. [Online; accessed 27-Jan-2016].

[GP15] Massimiliano Gubinelli and Nicholas Perkowski. Energy solutions of KPZ are unique. *arXiv preprint: 1508.07764*, Aug 2015.

[Hai14] Martin Hairer. A theory of regularity structures. *Inventionnes Mathematica*, 198(2):269–504, 2014.

[HL15] Martin Hairer and Cyril Labbé. A simple construction of the continuum parabolic anderson model on $\mathbb{R}^2$. *arXiv:1501.00692*, Jan 2015.

[HS15] Martin Hairer and Hao Shen. A central limit theorem for the KPZ equation. *arXiv:1507.01237*, Jul 2015.

[Ito51] Kiyoshi Ito. On a formula concerning stochastic differentials. *Nagoya Mathematical Journal*, 3(1):55–65, 1951.

[KF31] Andrey Nikolaevich Kolmogorov and Sergei Vasilyevich Fomin. On analytical methods in the theory of probability. *Mathematische Annalen*, 104(1):415–458, 1931.

[KL99] Claude Kipnis and Claudio Landim. *Scaling Limits of Interacting Particle Systems*. Springer, 1999.

[Kot86] Peter Kotelenez. Law of large numbers and central limit theorem for linear chemical reactions with diffusion. *The Annals of Probability*, 14(1):173–193, 1986.

[KPZ86] Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang. Dynamic scaling of growing interfaces. *Physics Reviews Letters*, 56(9):889–892, Mar 1986.

[KS91] Ioannis Karatzas and Steven E. Shreve. *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics. Springer Verlag, 2ed edition, 1991.

[Lig99] Thomas Milton Liggett. *Continuous Time Markov Processes*, volume 113 of *Graduate Texts in Mathematics*. The American Mathematical Society, 1999.

[Nor97] J.R. Norris. *Markov Chains*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University press, 1997.

[SV69a] Daniel Wyler Stroock and S. R. I. S. Varadhan. Diffusion processes with continuous coefficients, i. *Communications on Pure and Applied Mathematics*, 22(3):345–400, 1969.

[SV69b] Daniel Wyler Stroock and S. R. I. S. Varadhan. Diffusion processes with continuous coefficients, ii. *Communications on Pure and Applied Mathematics*, 22(3):479–530, 1969.

[SV71] Daniel Wyler Stroock and S. R. I. S. Varadhan. Diffusion processes with boundary conditions. *Communications on Pure and Applied Mathematics*, 24:147–225, 1971.

[Tur52] Alan Turing. The chemical basis of morphogenesis. *Philosophical Transactions of the Royal Society of London*, 237(641):37–72, 1952.

[Vol14] Vitaly Volpert. *Elliptic Partial Differential Equations : Volume 2: Reaction-Diffusion Equations.* Monographs in Mathematics. Springer Basel, 2014.

[Wie21] Norbert Wiener. The average of an analytic functional and the brownian movement. *Proceedings of the National Academy of Sciences of the United States of America*, 7(10):294–298, 1921.

[Wil83] Ruth Jeannette Williams. *Brownian Motion in a Wedge with Oblique Reflection at the Boundary.* Stanford University, 1983.

[YW71] Toshio Yamada and Shinzo Watanabe. On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.*, 11(1):155–167, 1971.