Elements of Analytic Hypoellipticity

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1. Preface

These notes form the basis of a five lecture mini–course given at the *II Escola Brasileira de Equações Diferenciais*, January 23–27, 2006. This was held at IMPA, *Instituto Nacional de Matemática Pura e Aplicada*, Rio de Janeiro, Brazil.

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2. Introduction

These notes are an introduction to the general problem of analytic regularity for linear partial differential operators with analytic coefficients. We assume that the reader is familiar with elementary real and complex analysis, together with the rudiments of functional analysis and distribution theory.

Definition 1. Let $\Omega \subset \mathbb{R}^n$ be open. If $f : \Omega \to \mathbb{C}$, we say the f is real analytic on Ω if $f \in C^\infty(\Omega)$ and furthermore, for each compact $K \subset \Omega$, there exists $C = C(K) > 0$ such that

$$|\partial_x^\alpha f(x)| \leq C^{1+|\alpha|}\alpha!$$

for all $x \in K$ and for all multi-indices α.

Note that if f is real analytic on Ω, then the Taylor series for f converges to f uniformly on a small neighborhood of each point of Ω. Hence it follows that there is an open subset $\tilde{\Omega}$, such that $\Omega \subset \tilde{\Omega} \subset \mathbb{C}^n$ and f extends to a holomorphic function on $\tilde{\Omega}$.

Note that the constant C in the Definition gives a rough indication of "how far" f can be extended into the complex domain.

We begin with some examples of smooth functions that are not real analytic.

Examples

(1) Define $f : \mathbb{R} \to \mathbb{R}$ by the following rule: $f(t) = 0, t \le 0$ and $f(t) = e^{-1/t}, t > 0$. Observe that f is smooth everywhere. Note that f is analytic on any open set that does not contain the point $t = 0$. However, f is not analytic on any open set that contains the point $t = 0$. Indeed, this follows from the principle of analytic continuation.

Let $r \in \mathbb{R}, r > 0$. If we define, for $x \in \mathbb{R}^n$, $g(x) = f(r^2 - |x|^2)$, then g is smooth, with support equal to the compact set $\{|x| \le r\}$. Note that g is analytic, as long as $|x| \ne r$. However, g is not analytic near any point where $|x| = r$.

(2) Define $h : \mathbb{R} \to \mathbb{R}$ by the following formula, assuming that $k \ge 2$ is an integer,

$$h(t) = \int_0^\infty e^{i\rho^k t} e^{-\rho} d\rho. \tag{1}$$

Observe that h is smooth everywhere. Note that h is analytic on any open set that does not contain the point $t = 0$. However, h is not analytic on any open set that contains the point $t = 0$. Indeed, this follows from the fact that

$$\partial_t{}^j h(0) = i^j \int_0^\infty \rho^{kj} e^{-\rho} d\rho = i^j (kj)! \tag{2}$$

and hence h is not analytic near 0, since $k \ge 2$. Note that h belongs to the Gevrey class k, but to no better Gevrey class.

We are interested in partial differential operators with analytic coefficients defined on open subsets $\Omega \subset \mathbb{R}^n$. Such operators can be written as follows:

$$P = \sum_{|\alpha| \le m} a_\alpha(x) \partial_x^\alpha,$$

where each a_α is real analytic on Ω and m is a positive integer.

We will use the following definition.

DEFINITION 2. Let P be a linear partial differential operator with analytic coefficients defined on an open set $\Omega \subset \mathbb{R}^n$. We say that P is *analytic hypoelliptic* on Ω if for every open $O \subset \Omega$ we have the following: Pu analytic on O implies that u is analytic on O. Here u is a distribution on O.

Sometimes this is referred to as analytic hypoellipticity in the strong sense. Several other definitions are used in the literature. The reader should be careful on this point. We will have more to say about this later.

If, in Definition 2, the term *analytic* is replaced by C^∞, then we say that P is *hypoelliptic* on Ω. It is important to compare these two ideas, for a given operator with analytic coefficients. The literature contains many interesting examples which challenge the intuition.

3. Operators with constant coefficients

In this section we will discuss analytic hypoellipticity for operators with constant coefficients. The theory is well understood for this class of operators. An excellent reference for this section is the text of Treves [**27**].

Let $(x_1, \ldots, x_n)$ denote natural coordinates in $\mathbb{R}^n$. We say that the partial differential operator P has constant coefficients if P can be written as follows,

$$P = \sum_{|\alpha| \leq m} a_\alpha \partial_x^\alpha,$$

where $a_\alpha \in \mathbb{C}$.

We recall the following

DEFINITION 3. Let P be a partial differential operator with constant coefficients defined on $\mathbb{R}^n$. Let $E \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution. We say that E is a fundamental solution for P if

$$PE = \delta.$$

Here δ denotes the Dirac delta function concentrated at $0 \in \mathbb{R}^n$.

The reader should keep in mind that fundamental solutions are not unique. Indeed, if E_1, E_2 are fundamental solutions for P, then

$$P(E_1 - E_2) = 0.$$

That is, $E_1 - E_2$ is in the kernel of P.

One important consequence of the existence of a fundamental solution is local solvability. Indeed, if P has a fundamental solution E and $f \in \mathcal{E}'(\mathbb{R}^n)$ is arbitrary (i.e. f a distribution with compact support), it follows that we have

$$P(E * f) = f.$$

Note that $E * f$ denotes the convolution of distributions. We have the following classical result

THEOREM 1. *Let P be a partial differential operator with constant coefficients defined on $\mathbb{R}^n$. Assume that P is not identically 0. Then P has a fundamental solution.*

For a proof of this, we refer the reader, for example, to the book of Hörmander [**14**], Theorem 7.3.10.

So we see that all operators with constant coefficients are locally solvable. What can the fundamental solution tell us about regularity for such operators ? The reader should notice that if a constant coefficient operator P is analytic hypoelliptic on $\mathbb{R}^n$, then any fudamental solution E for P must be itself be an analytic function away from 0. Indeed, this follows since $PE = 0$ in the complement of the origin. We have the following result.

THEOREM 2. *Let P be a partial differential operator with constant coefficients defined on $\mathbb{R}^n$. Then P is analytic hypoelliptic on $\mathbb{R}^n$ if and only if there exists a fundamental solution for P that is real analytic away from the origin.*

PROOF. We have already observed that the condition is necessary for analytic hypoellipticity. So we will assume that we have an open set $U \subset \mathbb{R}^n$, $u \in \mathcal{D}'(U)$ such that Pu is analytic on U. Let $x_0 \in U$ be arbirary. We must show that u is analytic near x_0.

By the Cauchy – Kovalevska Theorem, we may assume that

$$Pu = 0$$

on U. We will discuss this important theorem (and similar reductions) in the next section.

Let $\varphi \in C_0^\infty(U)$ such that $\varphi = 1$ near x_0. Then we have $v \in \mathcal{E}'(U)$ such that

$$P(\varphi u) = v.$$

Note that v vanishes near x_0. Hence, we may assume that there exists $\epsilon > 0$ such that

$$\text{supp}(v) \subset \{x : \epsilon \leq |x - x_0| \leq 2\epsilon\}. \tag{3}$$

Note that supp(v) denotes the support of v.

Let E be a fundamental solution for P that is analytic away from the origin. Observe that

$$E * v = E * P(\varphi u) = PE * (\varphi u) = \delta * (\varphi u) = \varphi u.$$

We now must only show that $E * v$ is analytic near x_0.

We introduce another cut–off function. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\psi(x) = 1, |x| \leq \delta$$

and

$$\text{supp}(\psi) \subset \{x : |x| \leq 2\delta\}. \tag{4}$$

We choose $\delta > 0$ such that

$$0 < \delta < \epsilon/2. \tag{5}$$

We return to $E * v$. We write

$$E * v = (\psi E) * v + [(1 - \psi)E] * v.$$

Observe that $(\psi E) * v$ vanishes in a neighborhood of x_0. This follows from the classical formula

$$\operatorname{supp}((\psi E) * v) \subset \operatorname{supp}(\psi E) + \operatorname{supp}(v) \subset \operatorname{supp}(\psi) + \operatorname{supp}(v).$$

Indeed, if $x_0 \in \operatorname{supp}((\psi E)*v)$, then $x_0 = a+b$, with $a \in \operatorname{supp}(v)$ and $b \in \operatorname{supp}(\psi)$. Hence, using (3) and (4), we see that

$$\epsilon \leq |x_0 - a| = |b| \leq 2\delta.$$

But this contradicts (5). Hence $x_0 \notin \operatorname{supp}((\psi E)*v)$, i.e. $(\psi E)* v$ vanishes near x_0.

In order to show that $E * v$ is analytic near x_0, we must only show that $[(1-\psi)E] * v$ is analytic near x_0. First observe that $(1-\psi)E \in C^\infty(\mathbb{R}^n)$. Hence it follows that $[(1-\psi)E] * v$ is smooth everywhere. It remains to estimate the derivatives in order to show analyticity.

Since v is a distribution with compact support, it follows that there exists $C > 0$, an integer k and positive real numbers p, q such that

$$(6) \qquad |< v, f >| \leq C \sup_{y \in K, |\beta| \leq k} |\partial^\beta f(y)|$$

for all $f \in C^\infty(\mathbb{R}^n)$. Here K is the compact neighborhood of $\operatorname{supp}(v)$ given by

$$K = \{y : p \leq |y - x_0| \leq q\},$$

where $p < \epsilon$ and $q > 2\epsilon$.

We must estimate $\partial^\alpha([(1-\psi)E] * v)=(\partial^\alpha[(1-\psi)E]) * v$. Using Leibniz formula we see that

$$\partial^\alpha[(1-\psi)E] = \sum \binom{\alpha}{\gamma} \partial^{\alpha-\gamma}(1-\psi)\partial^\gamma E.$$

Notice that, when $\gamma \neq \alpha$ we have

$$\partial^{\alpha-\gamma}(1-\psi) = -\partial^{\alpha-\gamma}\psi.$$

Note that the function on the right has support in the ball $\{x : |x| \leq 2\delta\}$, because of (4). Hence it follows, by using our previous argument, that when $\gamma \neq \alpha$, the function

$$(\partial^{\alpha-\gamma}(1-\psi)\partial^{\gamma}E) * v$$

vanishes in a neighborhood of x_0.

Hence it follows that

$$\partial^{\alpha}([(1-\psi)E] * v) = [(1-\psi)\partial^{\alpha}E] * v + R,$$

where R vanishes near x_0.

It remains to estimate $[(1-\psi)\partial^{\alpha}E] * v$. Note that

$$|([(1-\psi)\partial^{\alpha}E] * v)(x)| = | < v, [(1-\psi)\partial_x^{\alpha}E](x-\cdot) > |$$
$$\leq C \sup |\partial_y^{\beta}[(1-\psi(x-y))\partial_x^{\alpha}E(x-y)]|.$$

Here the supremun is taken for $|\beta| \leq k$, and $p \leq |y-x_0| \leq q$.

Note that the right hand side vanishes for $|x-y| \leq \delta$. So we may assume $|x-y| \geq \delta$. On this set $E(x-y)$ is analytic by assumption.

If we assume that $|x-x_0| \leq p/2$, it follows then that $|x-y| \geq p/2$ in the above estimate. In other words, the quantity $|x-y|$ varies in a compact set which is bounded away from 0. $E(x-y)$ is analytic in a neighborhood of this compact set. Hence it follows that there exists $C_1 > 0$ such that

$$|([(1-\psi)\partial^{\alpha}E] * v)(x)| \leq C_1^{|\alpha|+k+1}(\alpha! + k!)$$

for all α.

We finally have $C_2 > 0$ such that

$$|([(1-\psi)\partial^{\alpha}E] * v)(x)| \leq C_2^{|\alpha|+1}(\alpha)!$$

for all α and all x such that $|x-x_0| \leq p/2$. Thus it follows that $[(1-\psi)E] * v$ is analytic near x_0. The proof is complete. □

The reader should note that the C^{∞} analog of Theorem 2 is also true. That is we have

THEOREM 3. *Let P be a partial differential operator with constant coefficients defined on $\mathbb{R}^n$. Then P is hypoelliptic on $\mathbb{R}^n$ if and only if there exists a fundamental solution for P that is C^{∞} away from the origin.*

For a proof of this, we refer the reader to [**27**]. Next, we discuss some examples.

(1) First we consider the operator $\frac{d}{dx}$ on the real line $\mathbb{R}$. A fundamental solution for $\frac{d}{dx}$ is given by the Heaviside function H. Note that H is the locally integrable function defined by: $H(x) = 1, x > 0$ and $H(x) = 0, x < 0$.

If $\varphi \in C_0^\infty(\mathbb{R})$, it follows that

$$< \frac{dH}{dx}, \varphi >= - < H, \frac{d\varphi}{dx} >= - \int_0^\infty \frac{d\varphi}{dx}(x)dx = \varphi(0),$$

by the Fundamental Theorem of Calculus. Thus,

$$\frac{dH}{dx} = \delta.$$

Since H is analytic away from 0, we see that $\frac{d}{dx}$ is analytic hypoelliptic and hypoelliptic on $\mathbb{R}$. Also note that all fundamental solutions for $\frac{d}{dx}$ are of the form $H + C$, where C is an arbitrary constant.

(2) Next we consider the Cauchy – Riemann operator

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$$

in the plane $\mathbb{R}^2$, with coordinates (x, y). We will construct a fundamental solution E for $\partial_{\bar{z}}$ by using the partial Fourier transform.

If $u \in \mathcal{S}(\mathbb{R}^2)$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^2$, we define the partial Fourier transform $\tilde{u}$ as follows:

$$\tilde{u}(x, \eta) = \int e^{-iy\eta} u(x, y)dy.$$

If E is a tempered distribution on $\mathbb{R}^2$, we define $\tilde{E}$ by duality. That is

$$< \tilde{E}, u >=< E, \tilde{u} >$$

for $u \in \mathcal{S}(\mathbb{R}^2)$.

If E is a tempered distribution, which is a fundamental solution for $\partial_{\bar{z}}$, we see that we must have

$$\partial_x \tilde{E} - \eta \tilde{E} = 2\delta(x) \otimes 1_\eta,$$

where 1_η denotes the constant function 1 in the η variable. If we write

$$\tilde{E}(x,\eta) = v(x,\eta)e^{x\eta}$$

we see that

$$\partial_x v = 2e^{-x\eta}\delta(x) = 2\delta(x).$$

Hence

$$\tilde{E}(x,\eta) = 2(H(x) + C(\eta))e^{x\eta},$$

where $C(\eta)$ is an arbitrary function of η. We must choose $C(\eta)$ so that $\tilde{E}(x,\eta)$ is tempered. Notice that $e^{x\eta}$ is bounded as long as the product $x\eta < 0$. This leads us to define $C(\eta) = -1, \eta > 0$ and $C(\eta) = 0, \eta < 0$.

It follows that

$$\tilde{E}(x,\eta) = -2H(-x)e^{\eta x}, \eta > 0$$

and

$$\tilde{E}(x,\eta) = 2H(x)e^{\eta x}, \eta < 0.$$

Using the Fourier inversion formula, we have for $x \neq 0$,

$$E(x,y) = \frac{1}{2\pi}\int e^{iy\eta}\tilde{E}(x,\eta)d\eta$$

$$= \frac{-H(-x)}{\pi}\int_0^\infty e^{iy\eta}e^{x\eta}d\eta + \frac{H(x)}{\pi}\int_{-\infty}^0 e^{iy\eta}e^{x\eta}d\eta$$

$$= \frac{1}{\pi z}(H(-x) + H(x)) = \frac{1}{\pi z},$$

where $z = x + iy$. So we see that when $x \neq 0$, $E(x,y) = \frac{1}{\pi z}$. However, using polar coordinates, one sees easily that $\frac{1}{\pi z}$ is a locally integrable function in the plane. Hence $\frac{1}{\pi z}$ is a tempered distribution in the plane, whose partial Fourier transform equals $\tilde{E}$. We have reached the conclusion that $\frac{1}{\pi z}$ is a fundamental solution for the Cauchy – Riemann operator $\partial_{\bar{z}}$.

We observe that $\frac{1}{\pi z}$ is real analytic away from 0. Hence $\partial_{\bar{z}}$ is both analytic hypoelliptic and hypoelliptic in the plane. Also note that if E is any fundamental solution of $\partial_{\bar{z}}$, it follows that $E(x,y) = \frac{1}{\pi z} + h(z)$, where h is an entire function.

Next we discuss how to establish the fact that $\frac{1}{\pi z}$ is a fundamental solution for the Cauchy – Riemann operator $\partial_{\bar{z}}$ in a direct way, without using the Fourier transform.

Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ and consider

$$< \partial_{\bar{z}}(\frac{1}{\pi z}), \varphi >= - < \frac{1}{\pi z}, \partial_{\bar{z}}\varphi >$$
$$= - \int\int \frac{1}{\pi z}\partial_{\bar{z}}\varphi(x,y)\frac{d\bar{z}\wedge dz}{2i}$$
$$= - \lim_{\epsilon\to 0}\int\int_{|z|\geq\epsilon} \frac{1}{\pi z}\partial_{\bar{z}}\varphi(x,y)\frac{d\bar{z}\wedge dz}{2i}.$$

We observe that

$$\frac{1}{\pi z}\partial_{\bar{z}}\varphi(x,y)\frac{d\bar{z}\wedge dz}{2i} = d(\frac{1}{\pi z}\varphi(x,y)\frac{dz}{2i}),$$

when $|z| > \epsilon$, for any $\epsilon > 0$.

It follows by Stokes formula that

$$< \partial_{\bar{z}}(\frac{1}{\pi z}), \varphi >= - \lim_{\epsilon\to 0}\int_{|z|=\epsilon} \frac{1}{\pi z}\varphi(x,y)\frac{dz}{2i},$$

where the circle $|z| = \epsilon$ is oriented *clockwise*. We have

$$< \partial_{\bar{z}}(\frac{1}{\pi z}), \varphi >= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_0^{2\pi} \varphi(\epsilon\cos(\theta), \epsilon\sin(\theta))d\theta = \varphi(0,0).$$

So we see again that $\partial_{\bar{z}}(\frac{1}{\pi z}) = \delta$.

(3) Next we discuss Laplace's operator. Let $(x_1, \ldots, x_n)$ denote coordinates in $\mathbb{R}^n$, for $n \geq 2$. Laplace's operator is given by

$$\triangle = \frac{\partial^2}{\partial {x_1}^2} + \cdots + \frac{\partial^2}{\partial {x_n}^2}.$$

This operator has rotationally symmetric fundamental solutions given by the following:

$$\frac{1}{2\pi} \log |x|, \text{ when } n = 2,$$

and

$$\frac{-1}{(n-2)|S^{n-1}|} \frac{1}{|x|^{n-2}}, \text{ when } n > 2.$$

Here $|x|$ denotes the Euclidean norm of x and $|S^{n-1}|$ the area of the unit sphere in $\mathbb{R}^n$. Note that $|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, where Γ denotes the Euler gamma function.

It is clear that these fundamental solutions are locally integrable functions that are analytic when $x \neq 0$, for all $n \geq 2$. Hence $\triangle$ is analytic hypoelliptic and hypoelliptic for all $n \geq 2$. For a detailed discussion of these results, we refer the reader to [**27**].

We have seen that the three operators $\frac{d}{dx}, \partial_{\bar{z}}, \triangle$ are all analytic hypoelliptic and hypoelliptic. These operators are all examples of *elliptic* partial differential operators. This is the class we will discuss next.

In order to define elliptic operators, we return to the Fourier transform. If $u \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decresing functions, we define the Fourier transform $\hat{u}$ by

$$\hat{u}(\xi) = \int e^{-i<x,\xi>} u(x) dx.$$

If we define

$$D_j = \frac{1}{i} \frac{\partial}{\partial x_j},$$

then we have the formula

$$\widehat{D_j u}(\xi) = \xi_j \hat{u}(\xi), j = 1, \ldots, n.$$

Thus, the operator D_j has as "symbol" the function ξ_j.

Every partial differential operator with analytic coefficients on an open set $\Omega \subset \mathbb{R}^n$ can be written as

$$P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$$

with a_α analytic on Ω, and $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$.

DEFINITION 4. The *symbol* of $P(x, D)$ is the function

$$P(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha,$$

where $\xi \in \mathbb{R}^n$ and $x \in \Omega$. Note that for each $x \in \Omega$, $P(x, \xi)$ is a polynomial in ξ of degree m.

The *principal symbol* of $P(x, D)$ is the function

$$p_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha,$$

which is a homogeneous polynomial in ξ of degree m, for each $x \in \Omega$.

We say that the operator $P(x, D)$ is *elliptic* on Ω if

$$p_m(x, \xi) \neq 0$$

for all $x \in \Omega$ and $\xi \neq 0$, $\xi \in \mathbb{R}^n$.

We have the following

THEOREM 4. *Let*

$$P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$$

be an operator with constant coefficients. If P is elliptic, then P is analytic hypoelliptic and hypoelliptic on $\mathbb{R}^n$.

PROOF. If $\xi \neq 0$, then there exist $C_1 > 0$ such that

$$\frac{1}{|\xi|^m}|p_m(\xi)| = |p_m(\frac{\xi}{|\xi|})| \geq C_1,$$

since P is elliptic, the unit sphere is compact, and p_m is homogeneous of degree m. Note that $P - p_m$ is a polynomial of degree $m-1$. Hence there exists $C_2 > 0$ such that

$$|P(\xi) - p_m(\xi)| \leq C_2(1+|\xi|)^{m-1}, \xi \in \mathbb{R}^n.$$

It follows that there exists $C > 0$ and $R > 0$ such that

$$|P(\xi)| \geq C|\xi|^m, |\xi| \geq R.$$

Let φ be a smooth function with compact support, such that φ is equal to 1 in a neighborhood of the set

$$\{|\xi| \leq R\}.$$

Note that the function $(1-\varphi(\xi))/P(\xi)$ is a smooth, bounded function on all of $\mathbb{R}^n$; and hence, a tempered distribution. Thus there exists a tempered distribution E on $\mathbb{R}^n$ such that

$$\hat{E}(\xi) = (1-\varphi(\xi))/P(\xi).$$

In other words,

$$PE = \delta - \psi, \tag{7}$$

where $\hat{\psi} = \varphi$.

Since φ has compact support, it follows that ψ can be extended to all of $\mathbb{C}^n$ as an entire function. In particular, ψ is real analytic. We will now show that E is analytic away from the origin.

First observe that any derivative of order k of $1/P$ is of the form Q/P^{k+1}, where Q is a polynomial of degree $\leq (m-1)k$. This follows by mathematical induction. It follows that when $|\xi| > R$, there exists for each α, β a constant $C_{\alpha,\beta} > 0$ such that

$$|\xi^\beta D^\alpha(1/P(\xi))| \leq C_{\alpha,\beta}|\xi|^{|\beta|-|\alpha|-m}.$$

From this it follows that $D^\beta x^\alpha E$ is continuous when

$$|\beta| - |\alpha| - m < -n.$$

Hence E is smooth away from $x = 0$. Next to prove the analyticity.

Note that if $-|\alpha| - m \leq -n - 1$, then we may write

$$x^\alpha E(x) = \int e^{i<x,\xi>}(-1)^{|\alpha|} D^\alpha((1-\varphi)/P)(\xi) \frac{d\xi}{(2\pi)^n},$$

where the integral is absolutely convergent. It follows that

$$D^\beta(x^\alpha E)(x)$$

$$= \lim_{\epsilon \to 0} \int e^{i<x,\xi>-\epsilon|\xi|} \xi^\beta (-1)^{|\alpha|} D^\alpha((1-\varphi)/P)(\xi) \frac{d\xi}{(2\pi)^n}.$$

Indeed, we perform the change of contour, (see also [**29**], pages 244–245)

$$\xi \to \xi + i\delta x |\xi| \chi(\xi).$$

Here $\delta > 0$ and χ is a smooth function such that $\chi(\xi) = 0$ for $|\xi| < 2R$ and $\chi(\xi) = 1$ for $|\xi| > 3R$. It follows that if x varies in a compact set K which does not contain 0, and $\delta > 0$ is small enough, that we have $C_1, C_2 > 0$ such that

$$|D^\beta(x^\alpha E)(x)| \leq C_1 \int e^{-C_2|\xi|} |\xi|^{|\beta|-n-1} d\xi.$$

Hence there exists $C > 0$, such that for $x \in K$ we have

$$|D^\beta(x^\alpha E)(x)| \leq C^{|\beta|+1} |\beta|!.$$

It follows that E is analytic for all $x \neq 0$.

The fact that P is analytic hypoelliptic now follows from (7). First observe that the convolution $\psi * u$ is analytic everywhere, for any distribution u with compact support. Now the proof follows the lines of Theorem 2. We leave the details to the reader. □

Next we discuss the one dimensional heat equation. Let (x, t) be coordinates in $\mathbb{R}^2$. The heat operator is the operator P

$$P = \partial_t - \partial_x^2.$$

We will construct a fundamental solution E for P by using the partial Fourier transform, in a similar way which we used to study the Cauchy – Riemann operator.

The reader should note that the symbol of P is given by $P(\xi, \tau) = i\tau - \xi^2$, while the principal symbol p_2 is given by $p_2(\xi, \tau) = -\xi^2$. Note that $p_2(0, \tau) = 0$ for all $\tau \neq 0$. Hence P is *not* elliptic.

If E is a tempered distribution, which is a fundamental solution for P, we see that we must have

$$\partial_t \tilde{E} + \xi^2 \tilde{E} = \delta(t) \otimes 1_\xi,$$

where 1_ξ denotes the constant function 1 in the ξ variable. If we write

$$\tilde{E}(\xi, t) = v(\xi, t) e^{-t\xi^2}$$

we see that

$$\partial_t v = e^{t\xi^2} \delta(t) = \delta(t).$$

Hence we may choose $v(\xi, t) = H(t)$, the Heaviside function. We have

$$\tilde{E}(\xi, t) = H(t) e^{-t\xi^2}.$$

Using the Fourier inversion formula we have,

$$E(x, t) = H(t) \int e^{ix\xi - t\xi^2} \frac{d\xi}{2\pi}.$$

By completing the square we observe that

$$ix\xi - t\xi^2 = -t(\xi - ix/2t)^2 - x^2/4t.$$

Hence we have

$$E(x, t) = H(t) e^{-x^2/4t} \int e^{-t(\xi - ix/2t)^2} \frac{d\xi}{2\pi}. \tag{8}$$

To complete our calculation we consider the following integral

$$I_y = \int_{-\infty}^{+\infty} e^{-tz^2} dx,$$

where $z = x + iy$ is a complex variable and $t > 0$. First we know, from mathcalculus, that $I_0 = \sqrt{\frac{\pi}{t}}$. Now consider the contour integral

$$\int_{C_R} e^{-tz^2} dz = 0,$$

where C_R is the rectangle in the plane with vertices

$$-R, R, R + iy, -R + iy$$

and $R > 0$ is arbitrary. It follows, by letting $R \to \infty$, that $I_y = I_0 = \sqrt{\frac{\pi}{t}}$ for all real y.

Returning to (8) we have

$$E(x,t) = \frac{1}{2\sqrt{\pi t}} H(t) e^{-x^2/4t}. \tag{9}$$

So E is a fundamental solution for P. Clearly, E is analytic as long as $t \neq 0$. Also, at points where $t = 0$ and $x \neq 0$ we see that E is C^∞. So E is smooth away from the origin, and P is hypoelliptic. However, since E vanishes for all $t < 0$, we see that E cannot be analytic at any point where $t = 0$. Thus we see that P is *not* analytic hypoelliptic.

In summary we see that the heat operator P is hypoelliptic, but neither elliptic, nor analytic hypoelliptic.

We now show that P is not analytic hypoelliptic in a more direct way. We search for a solution to the equation $Pu = 0$, where u has the special form

$$u(x,t) = \sum_{j \geq 0} a_j(t) x^j.$$

Substituting into the equation, we see that the a_j must satisfy

$$\partial_t a_j = (j+1)(j+2) a_{j+2}, j \geq 0.$$

We choose $a_j = 0$ for j odd, and

$$a_{2j} = \frac{\partial_t^j a_0}{(2j)!}, j \geq 0.$$

With these choices we see that u will solve the equation $Pu = 0$ with the correct choice of a_0. We choose $a_0(t) = h(t)$, where h is defined in (1). We choose $k = 2$. We see that

$$|\partial_t^j h(t)| \leq (2j)!, t \in \mathbb{R}.$$

Hence the series defining u converges to a smooth solution for $|x| < 1$. However, since we also have

$$|\partial_t^j h(0)| = (2j)!,$$

it follows that u cannot be analytic.

The fact that the heat opearator is not elliptic and also not analytic hypoelliptic is no coincidence. We will see that all non–elliptic operators with constant coefficients are not analytic hypoelliptic.

To begin the discussion, we consider operators with constant coefficients on $\mathbb{R}^n$ whose *full symbol* is a homogeneous polynomial of degree m. That is, we now assume that

$$P(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha.$$

We also assume that P is not elliptic. That is, we assume that there exists $N \in \mathbb{R}^n, N \neq 0$ such that

$$p_m(N) = P(N) = 0.$$

We now proceed to construct a particular solution u to the equation $Pu = 0$. First we define f, a real valued function of a real variable. Let $f(t) = 0, t \leq 0$ and $f(t) = t^{m+1}, t > 0$. Note that f is analytic for $t \neq 0$ and that $f \in C^m(\mathbb{R})$, but $f \notin C^{m+1}(\mathbb{R})$. Let $x \in \mathbb{R}^n$. As usual $< x, N >= x_1 N_1 + \cdots + x_n N_n$. Now define $u(x) = f(< x, N >)$.

We observe that

$$(Pu)(x) = (-i)^m \sum_{|\alpha|=m} a_\alpha {\partial_x}^\alpha (f(< x, N >))$$

$$= f^{(m)}(< x, N >) \sum_{|\alpha|=m} a_\alpha N^\alpha = f^{(m)}(< x, N >) p_m(N) = 0.$$

Hence $Pu = 0$ and P is neither hypoelliptic nor analytic hypoelliptic.

Note that u can be constructed to be smooth, and not analytic. Indeed, f can be chosen smooth, with $f(t) = 0, t \leq 0$ and $f(t) > 0, t > 0$.

An important example that fits the above situation is the n dimensional wave operator. That is the operator on $\mathbb{R}^{n+1}$ given by

$$\partial_t^2 - \sum_{j=1}^{n} \partial_j^2.$$

The above discussion shows that the wave operator is neither hypoelliptic nor analytic hypoelliptic.

We have seen that when the full symbol is homogeneous, the study is rather easy. However, a more serious result is the following one, which completes the study of analytic hypoellipticity for operators with constant coefficients.

THEOREM 5. *Let $P(D)$ be an operator with constant coefficients on $\mathbb{R}^n$. Assume that there exists $N \in \mathbb{R}^n, N \neq 0$ such that $p_m(N) = 0$. Then there exists $u \in C^\infty(\mathbb{R}^n)$ such that $Pu = 0$ and supp $u = \{x :< x, N >\geq 0\}$.*

A proof of this result can be found in [**14**], Theorem 8.6.7, page 310.

We see then that all non–elliptic operators with constant coefficients are not analytic hypoelliptic. This follows from Theorem 5 by using the principle of analytic continuation. We can summarize our work in this section with the following beautiful theorem :

THEOREM 6. *Let $P(D)$ be an operator with constant coefficients on $\mathbb{R}^n$. Then $P(D)$ is analytic hypoelliptic on $\mathbb{R}^n$ if and only if $P(D)$ is elliptic.*

4. The Cauchy – Kovalevska Theorem

We begin our study of operators with variable, real analytic coefficients, with a brief discussion of the Cauchy – Kovalevska Theorem. We refer the reader to [**27**] for a detailed study.

We are interested in partial differential operators P with analytic coefficients, defined on an open subset $\Omega \subset \mathbb{R}^n$. Such an operator can be written as follows:

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha,$$

where each a_α is real analytic on Ω and m is a positive integer.

We introduce the following definition:

DEFINITION 5. We say that $S \subset \Omega$ is an analytic hypersurface if every $y \in \Omega$ has an open neighborhood U such that there is a real analytic, real valued function φ defined on U with the following properties: grad $\varphi \neq 0$ on U; $S \cap U$ is exactly the set of points $y \in U$ such that $\varphi(y) = 0$.

DEFINITION 6. Let P and S be as above, with $y \in S$. We say that S is characteristic for P at y if

$$p_m(y, \text{grad } \varphi(y)) = 0.$$

Recall that p_m denotes the principal symbol of P.

We can now state the well known Cauchy – Kovalevska Theorem.

THEOREM 7. *Let P be defined on Ω as above. Let $S \subset \Omega$ be a real analytic hypersurface which is not characteristic for P at each of its points. Also assume that S has a well defined exterior normal at each of its points. Then given functions f analytic on Ω and $u_j, j = 0, \dots, m-1$ analytic on S, there* *exists a neighborhood $V \subset U$ of S and a function u defined on V such that*

$$Pu = f, \text{ on } V$$

and

$$\partial_\nu^j u = u_j, \text{ on } S, j = 0, \ldots, m-1.$$

We denote the derivative in the direction of the exterior normal by ∂_ν.

When studying analytic hypoellipticity, the following situation commonly occurs. One has a distribution u such that $Pu = f$, near a point p in Euclidean space. P has analytic coefficients and the function f is known to be analytic near p. If there is one direction which is not characteristic for P at the point p, then the Cauchy – Kovalevska Theorem tells us that there exists a real analytic v defined near p such that $Pv = f$ there. Hence we have $P(u - v) = 0$ near p. Furthermore $u - v$ is analytic near p if and only if u is analytic near p. Hence we may reduce the study to a homogeneous equation. The reduction is quite common. We have already used it in the proof of Theorem 2.

5. Elliptic regularity and the analytic wave front set

Our goal in this section is to give a brief introduction to microlocal analysis. We will define the analytic wave front set and indicate how analytic hypoellipticity can be derived for elliptic operators using microlocal methods.

Let $\Omega \subset \mathbb{R}^n$ be open and let $u \in \mathcal{D}'(\Omega)$ be a distribution on Ω. If $x \in \Omega$, we say that u is analytic at x if u is analytic in a neighborhood of x. Hence the set of points at which u is analytic is an open subset of Ω. The complement of this open set (in Ω) is called the *analytic singular support* of u, which we denote by

$$SS_A(u).$$

Note that $SS_A(u)$ is always a relatively closed subset of Ω.

For example, if $\delta(x - x_0)$ denotes the Dirac mass concentrated at the point x_0, then $SS_A(\delta(x - x_0)) = \{x_0\}$. Also, if f denotes the smooth function discussed in Example 1 of the Introduction, we see that $SS_A(f) = \{0\}$.

Next we introduce the analytic wave front set, which is a generalization of the analytic singular support. Let $(x, \xi) \in \Omega \times \mathbb{R}^n$ be natural coordinates. If $V \subset \Omega \times \mathbb{R}^n$, we say that V is *conic* if whenever we have $(x, \xi) \in V$, it follows that $(x, \lambda\xi) \in V$, for all $\lambda > 0$.

We are now ready to define the analytic wave front set for $u \in \mathcal{D}'(\Omega)$. In analogy with $SS_A(u)$, we define it by its complement.

Definition 7. Let $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus 0)$ and let $u \in \mathcal{D}'(\Omega)$. We say that $(x_0, \xi_0) \notin WF_A(u)$ if there exists $\chi \in C_0^\infty(\Omega)$, with $\chi = 1$ near x_0 and a conic neighborhood of (x_0, ξ_0), $V \subset \Omega \times (\mathbb{R}^n \setminus 0)$, and constants $C > 0, \epsilon > 0$, such that

$$\left| \int_{\mathbb{R}^n} e^{i<x-y,\xi>-|\xi||x-y|^2/2} (\chi u)(y) dy \right| \leq C e^{-\epsilon|\xi|}$$

for all $(x, \xi) \in V$.

So we see that $WF_A(u)$ is always a closed, conic subset of $\Omega \times (\mathbb{R}^n \setminus 0)$. The reader should also be aware that a much more general and flexible definition can be given. Indeed there are many choices for the "phase"

$$i\langle x - y, \xi\rangle - |\xi||x - y|^2/2$$

and "elliptic symbol" χ. This freedom is guaranteed by an analytic version of the stationary phase formula. Furthermore, the correct, invariant definition demands that we view $\Omega \times (\mathbb{R}^n \setminus 0)$ as $T^*(\Omega) \setminus 0$, which denotes the real cotangent bundle of Ω, minus the zero section. But these ideas take us far from the task at hand. We refer to the work of Sjöstrand [**22**] for this.

We denote by $\pi : \Omega \times (\mathbb{R}^n \setminus 0) \to \Omega$ the natural projection. That is, $\pi(x, \xi) = x$. Our next result relates $WF_A(u)$ and $SS_A(u)$.

THEOREM 8. *Let $\Omega \subset \mathbb{R}^n$ be open and let $u \in \mathcal{D}'(\Omega)$. Then*

$$\pi(WF_A(u)) = SS_A(u).$$

PROOF. To begin, assume that $x_0 \in \Omega \setminus SS_A(u)$. Then u is analytic near x_0. Choose $\psi \in C_0^\infty(\Omega)$ with support near x_0, contained in the set where u is analytic. Also assume that $\psi = 1$ near x_0. We make the change of contour

$$y \to y - i\epsilon\psi(y)\xi/|\xi|$$

in the integral of the definition. It follows that we have exponential decay near x_0, for all choices of $\xi \neq 0$. Hence $(x_0, \xi) \notin WF_A(u)$ for all $\xi \neq 0$. We have shown that $\pi(WF_A(u)) \subset SS_A(u)$.

Now to prove the other inclusion. Assume that $(x_0, \xi) \notin WF_A(u)$ for all $\xi \neq 0$. Note that

$$\delta(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i<x,\xi>} d\xi,$$

where we interpret the integral on the right as the following oscillatory integral

$$\lim_{\epsilon \to 0} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i<x,\xi> - \epsilon|\xi|^2/2} d\xi.$$

We deform the contour as follows:

$$\xi \to \xi + i|\xi|x/2,$$

to obtain

$$\delta(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i<x,\xi> - |\xi||x|^2/2} a(x, \xi) d\xi,$$

where a is a function determined by the change of contour. Note that a is an "elliptic symbol of order 0" in the sense of Sjöstrand [**22**]. It follows from our assumption, using stationary phase methods, that

$$\left| \int_{\mathbb{R}^n} e^{i<x-y,\xi> - |\xi||x-y|^2/2} a(x - y, \xi)(\chi u)(y) dy \right| \leq Ce^{-\epsilon|\xi|},$$

for x near x_0 and all $\xi \neq 0$. Now using our characterization of δ we see that

$$(\chi u)(x) = \int\int e^{i<x-y,\xi>-|\xi||x-y|^2/2} a(x-y,\xi)(\chi u)(y)dyd\xi.$$

Now it follows that u is analytic near x_0 by estimating the derivatives directly, using the exponential decay in ξ. □

Suppose now that P is a partial differential operator on Ω of order m, with analytic coefficients. Let p_m denotes its principal symbol. We denote by Σ the *characteristic set* of P, which is defined as follows:

$$\Sigma = \{(x,\xi) \in \Omega \times (\mathbb{R}^n \setminus 0) : p_m(x,\xi) = 0\}.$$

We have the following classical result, originally due to Sato, within the framework of hyperfunctions. The version we state is due to Hörmander [**13**].

THEOREM 9. *If $u \in \mathcal{D}'(\Omega)$, then*

$$WF_A(u) \subset \Sigma \cup WF_A(Pu).$$

We refer the reader to [**22**] for a proof. Note that Theorem 9 immediately gives us a result on analytic hypoellipticity.

THEOREM 10. *Let P be a partial differential operator with analytic coefficients on the open subset $\Omega \subset \mathbb{R}^n$. If P is elliptic on Ω, then P is analytic hypoelliptic on Ω.*

PROOF. Let $O \subset \Omega$ be open, and let $u \in \mathcal{D}'(O)$ such that Pu is analytic on O. Hence $WF_A(Pu) = \emptyset$. But P elliptic is equivalent to $\Sigma = \emptyset$. Hence $WF_A(u) = \emptyset$, and u is analytic. □

6. Operators of real principal type

We have seen in the last section that elliptic operators with analytic coefficients are always analytic hypoelliptic. Furthermore, we know that when non–elliptic operators have solutions with singularities, these singularities must be contained in the characteristic set. We now discuss a class of operators where the characteristic set has the simplest possible geometry.

Let P be a partial differential operator with analytic coefficients defined on the open set $\Omega \subset \mathbb{R}^n$. Let p denote its principal symbol. We say that P is of *real principal type* if p is real valued, and whenever $p(x,\xi) = 0$, it follows that $\partial_\xi p(x,\xi) \neq 0$. That is, the gradient of p with respect to ξ is not zero at each point of the characteristic set Σ. Note that Σ is then a real analytic hypersurface of $\Omega \times (\mathbb{R}^n \setminus 0)$.

We associate to p its *Hamilton field* which is by definition the vector field on $\Omega \times (\mathbb{R}^n \setminus 0)$ given by

$$\mathcal{H}_p = \sum_{j=1}^{n} \Big(\frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}\Big).$$

Note that $\mathcal{H}_p p = 0$ always. When P is of real principal type, it follows that $\mathcal{H}_p$ is never zero and always tangent to Σ. Indeed, through each point of Σ there passes a unique integral curve of $\mathcal{H}_p$. Such a curve is called a *bicharacteristic strip* of p. The projection of a bicharacteristic strip to the base Ω is called a *bicharacteristic curve* for p.

Simple examples of operators of real principal type are $\frac{\partial}{\partial x_1}$ on $\mathbb{R}^n, n \geq 2$ or the wave operator. Both of these operators are neither hypoelliptic nor analytic hypoelliptic. Indeed, we have the following classical result of Zerner [**32**],

THEOREM 11. *Let P be an operator of real principal type with analytic coefficients on the open subset $\Omega \subset \mathbb{R}^n$. Let $x \in \Omega$ and let Γ be a bicharacteristic curve for P that contains x. Then there exists an open neighborhood O of x and $u \in \mathcal{D}'(O)$ such that $Pu = 0$ and $SS(u) = \Gamma$.*

Note that $SS(u)$ denotes the *singular support* of u, which is the complement of the largest open set in O where u is C^∞.

We see that a consequence of Theorem 11 is that P is neither hypoelliptic nor analytic hypoelliptic. Indeed, the correct theorem concerning operators of real principal type is the next result of Hörmander [**13**], on the propagation of singularities.

THEOREM 12. *Let P be an operator of real principal type with analytic coefficients on the open subset $\Omega \subset \mathbb{R}^n$. If $u \in \mathcal{D}'(\Omega)$ and Pu is analytic, then $WF_A(u)$ is invariant under the Hamilton flow.*

The phrase "$WF_A(u)$ is invariant under the Hamilton flow" means the following: If $p \in WF_A(u)$, then the entire bicharacteristic strip that contains p is contained in $WF_A(u)$.

7. Operators of the form "Sum of Squares"

We have seen that when the principal symbol is real and has simple zeroes, then there are no hypoellipticity results of any kind. We next study a class of operators with real principal symbol, which vanishes at least to order two on its characteristic set. These are the "Sum of Squares" operators, studied by Hörmander in [**12**].

Let $\Omega \subset \mathbb{R}^n$ be an open set, with coordinates $x_1, \ldots, x_n$. Consider $m+1$ real, C^∞ vectorfields X_j of the form

$$X_j = \sum_{k=1}^{n} a_{jk}(x) \frac{\partial}{\partial x_k}, j = 0, \ldots, m.$$

Here each $a_{jk} \in C^\infty(\Omega)$. We will study operators of the form

$$P = \sum_{j=1}^{m} X_j^2 + X_0 + c(x), \tag{10}$$

where c is a complex valued C^∞ function on Ω.

We are interested in the hypoellipticity of P. Clearly we must make some assumptions on the vector fields X_j.

If X and Y are smooth vector fields on Ω, then we denote the commutator bracket by $[X, Y] = XY - YX$. Note that $[X, Y]$ is also a smooth vector field. The collection of all smooth vector fields on Ω forms a Lie algebra. We denote by $\mathfrak{g}$ the smallest Lie subalgebra that contains $X_j, j = 0, \ldots, m$. Note that $\mathfrak{g}$ is the real vector space spanned by all the successive brackets of the X_j.

If we "freeze" $\mathfrak{g}$ at a point $x \in \Omega$, we get a vector space. The dimension of this space is called the rank of $\mathfrak{g}$ at x. In general the rank may vary from point to point.

We are now ready to state the result of Hörmander [**12**].

THEOREM 13. *Suppose that the rank of $\mathfrak{g}$ is a nonzero constant on Ω. Then the operator P in* (10) *is hypoelliptic on Ω if and only if the rank of $\mathfrak{g}$ is equal to n.*

Besides the original proof [**12**], the reader should also consult the book of Treves [**29**].

Observe that the heat operator is in the form (10) and so Theorem 13 applies. Recall that the heat operator is hypoelliptic but not analytic hypoelliptic. We pose the following question:

When P has analytic coefficients and has the form (10)*, what is the correct condition that guarantees analytic hypoellipticity for P ?*

This question has motivated much of the research on analytic hypoellipticity in recent years.

Before we continue our survey, we pause to discuss some classical results from the theory of ordinary differential equations.

8. The Hermite operator

The Hermite operator and its spectral theory arise in a natural way in the study of several classical examples in the theory of analytic hypoellipticity. We pause to recall some essential facts.

First, by definition, the Hermite operator is the ordinary differential operator H given by

$$H = -\frac{d^2}{dx^2} + x^2.$$

The eigenfunctions and eigenvalues of H are well known. By eigenfunction, with eigenvalue λ, we mean a measurable function u, defined on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |u|^2 < \infty$$

and

$$Hu = \lambda u.$$

The eigenvalues of H are given by

$$\lambda = \lambda_k = 2k + 1, k = 0, 1, \dots$$

with corresponding eigenspaces of dimension 1, each generated by

$$u_k(x) = e^{-x^2/2} H_k(x).$$

Here H_k denotes the Hermite polynomial

$$H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k}(e^{-x^2}).$$

Clearly each u_k is a real analytic function of Schwartz class.

9. The Baouendi–Goulaouic Example

At the time of Hörmander's "sum of squares" theorem, the question of analytic hypoellipticity for such operators was asked. Of course, it was well known that the heat operator was an immediate counterexample. But, the question became, suppose that P was a "pure" sum of squares, i.e. suppose that P has the form

$$P = \sum_{j=1}^{m} X_j^2, \tag{11}$$

where the X_j were analytic vector fields, satisfying Hörmander's bracket condition. Could it be that such operators were always analytic hypoelliptic ?

At the time this was a reasonable question. Consider the following example on $\mathbb{R}^2$ with coordinates (x, t). Let $X_1 = \frac{\partial}{\partial t}$ and $X_2 = t\frac{\partial}{\partial x}$. The operator $X_1^2 + X_2^2$ was known to be analytic hypoelliptic back then. This example was typical of what was known. Hence it came as a surprise that Baouendi and Goulaouic [**1**] were able to show that the following operator is *not* analytic hypoelliptic on $\mathbb{R}^3$, with coordinates (x, y, t):

$$B = \frac{\partial^2}{\partial t^2} + t^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{12}$$

We will now give a proof of this fact by constructing a smooth function u, such that $Bu = 0$, but u is not analytic. The proof that we present is different from the one in [**1**] and has at its roots an argument of Oleinik [**21**]. A similar integral formula for u can be found in [**14**].

We see that the operator B is elliptic on any open set not intersecting the hyperplane $t = 0$. So we will concentrate our work near a fixed point $(x_0, y_0, 0)$. We seek a solution to the equation $Bu = 0$. We take partial Fourier transform with respect to x, and obtain

$$\hat{u}_{tt} - t^2\xi^2\hat{u} + \hat{u}_{yy} = 0.$$

We define v by

$$\hat{u}(\xi, y, t) = v(\xi, y, \xi^{1/2}t).$$

Letting $s = \xi^{1/2}t$, we obtain the following equation for v:

$$v_{ss} - s^2 v + \frac{1}{\xi}v_{yy} = 0.$$

We now separate variables; that is, we assume that $v(\xi, y, s) = A(s)B(\xi, y)$. This yields the following equations for A and B:

$$-A_{ss} + s^2 A = \lambda A$$

and

$$B_{yy} = \lambda \xi B,$$

where λ is a constant to be chosen.

We choose $\lambda = \lambda_k = 2k + 1$ for some fixed $k \geq 0$ and $A(s) = u_k(s)$, the corresponding eigenfunction, as discussed in the previous section.

Then B must have the form

$$B(\xi, y) = f(\xi) e^{y\sqrt{\lambda_k \xi}},$$

where f is an arbitrary function of ξ. We choose f as follows:

$$H(\xi) e^{-ix_0\xi - y_0\sqrt{\lambda_k \xi} - \sqrt{\xi}},$$

where $H(\xi)$ denotes the Heaviside function. It follows that

$$v(\xi, y, s) = H(\xi) e^{-ix_0\xi - (1-(y-y_0)\sqrt{\lambda_k})\sqrt{\xi}} u_k(s).$$

Finally we see that our solution is given by

$$u(x, y, t) = \int_0^\infty e^{i(x-x_0)\xi - (1-(y-y_0)\sqrt{\lambda_k})\sqrt{\xi}} u_k(\xi^{1/2} t) \frac{d\xi}{2\pi}.$$

Note that as long as y stays near y_0, we have

$$1 - (y - y_0)\sqrt{\lambda_k} \geq 1/2.$$

Hence u will be C^∞ in such a neighborhood. One also checks that $Bu = 0$.

Now we show that u cannot be analytic near the point $(x_0, y_0, 0)$. We see that

$$(\partial_x^j u)(x_0, y_0, 0) = i^j u_k(0) \int_0^\infty \xi^j e^{-\sqrt{\xi}} \frac{d\xi}{2\pi}.$$

It follows that

$$(\partial_x^j u)(x_0, y_0, 0) = \frac{i^j u_k(0)}{\pi} (2j + 1)!.$$

Thus u cannot be analytic, as long as $u_k(0) \neq 0$. If we have $u_k(0) = 0$, then it must follow that $u_k'(0) \neq 0$, since u satisfies a second order ode. To obtain the nonanalyticity in this case, it suffices to calculate the derivatives

$$(\partial_x^j u_t)(x_0, y_0, 0).$$

We leave this calculation to the reader.

10. An Example with complex first order terms

In this section we study a class of operators of the form (10), where the vector field X_0 is complex valued.

Let (x,t) be coordinates on $\mathbb{R}^2$. First we introduce the complex vector field L given by

$$L = \partial_t - it\partial_x.$$

We denote its complex conjugate by $\bar{L}$, ie

$$\bar{L} = \partial_t + it\partial_x.$$

We will study the operator P_λ which we define to be

$$P_\lambda = \bar{L}L + \lambda[\bar{L}, L],$$

where λ is a parameter. Note that the commutator is given by

$$[\bar{L}, L] = -2i\partial_x,$$

and we have

$$P_\lambda = \partial_t^2 + t^2\partial_x^2 - i(1+2\lambda)\partial_x.$$

When the quantity $-i(1+2\lambda)$ is real, then Hörmander's Theorem applies, and P_λ is hypoelliptic. (In this case, we also get analytic hypoellipticity. More about this later.) Our interest here is when $-i(1+2\lambda)$ may not be real.

We will now show that when $\lambda = 0, 1, 2, \ldots$ P_λ is neither hypoelliptic nor analytic hypoelliptic.

Consider the equation $P_\lambda u = 0$. Perform a partial Fourier transform with respect to x to obtain

$$\hat{u}_{tt} - t^2\xi^2\hat{u} + (1+2\lambda)\xi\hat{u} = 0.$$

We define v by

$$\hat{u}(\xi, t) = v(\xi, \xi^{1/2}t).$$

Letting $s = \xi^{1/2}t$, we obtain the following equation for v:

$$v_{ss} - s^2 v + (1 + 2\lambda)v = 0.$$

We may choose

$$v(s) = f(\xi)u_\lambda(s),$$

where u_λ is the appropriate eigenfunction of the Hermite operator, and f is arbirary. We choose

$$f(\xi) = H(\xi)(1 + \xi^2)^{-N},$$

where N is a fixed positive integer and H denotes the Heaviside function. We obtain

$$u(x,t) = \int_0^{+\infty} e^{ix\xi}(1 + \xi^2)^{-N} u_\lambda(\xi^{1/2}t)\frac{d\xi}{2\pi}.$$

One checks that $P_\lambda u = 0$. But u has only finite differentiability, depending on the choice of the integer N. Hence both hypoellipticity and analytic hypoellipticity fail for P_λ, when $\lambda = 0, 1, 2, \ldots$

When λ avoids these "bad" values, P_λ is both hypoelliptic and analytic hypoelliptic. We will discuss this later on.

11. Concatenations

In this section we will briefly discuss the concatenation method of Treves. To illustrate the method, we will discuss again the results of the previous section.

Let L and P_λ be defined as before. Observe that if we define $z = x + it^2/2$, then

$$Lz = 0.$$

If h is any holomorphic function defined on the image of z, we see that also

$$L(h \circ z) = 0.$$

If we define $w = z^{3/2}$, then $Lw = 0$. Hence L is neither hypoelliptic nor analytic hypoelliptic. But

$$P_0 w = \bar{L}Lw = 0,$$

and so we see (again) that P_0 is neither hypoelliptic nor analytic hypoelliptic.

One checks easily, that for any constant λ we have

$$P_\lambda \bar{L} = \bar{L} P_{\lambda-1}.$$

Indeed we have, for any $k = 0, 1, 2, \ldots$, the following *concatenation formula* of Treves

$$P_\lambda \bar{L}^k = \bar{L}^k P_{\lambda-k}.$$

Note that when $\lambda = k$, we obtain

$$P_k \bar{L}^k = \bar{L}^k P_0.$$

Since $P_0 w = 0$, we see that P_k is not hypoelliptic, nor analytic hypoelliptic for each $k = 0, 1, 2, \ldots$. This reestablishes the result of the previous section.

12. A Little Symplectic Geometry

In this section we give a quick introduction to symplectic geometry. For a more systematic and invariant treatment see, for example, [**14**].

Let $\Omega \subset \mathbb{R}^n$ be open. Let (x, ξ) denote coordinates in $\Omega \times \mathbb{R}^n$. We introduce the symplectic form

$$\omega = \sum_{j=1}^{n} d\xi_j \wedge dx_j.$$

We see that ω is an antisymmetric bilinear form on the tangent space of $\Omega \times \mathbb{R}^n$. By this we mean that if X and Y are tangent vectors on $\Omega \times \mathbb{R}^n$, we have

$$\omega(X, Y) = -\omega(Y, X).$$

A curious consequence of this is the fact that $\omega(X, X) = 0$ for all X.

Furthermore, ω is non–degenerate. By this we mean that if

$$\omega(X, Y) = 0$$

for all tangent vectors Y, then it follows that $X = 0$.

If we let

$$X = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} + b_j \frac{\partial}{\partial \xi_j}$$

and

$$X' = \sum_{j=1}^{n} a'_j \frac{\partial}{\partial x_j} + b'_j \frac{\partial}{\partial \xi_j},$$

then we see that

$$\omega(X, X') = \sum_{j=1}^{n} b_j a'_j - a_j b'_j.$$

If p is a smooth function on $\Omega \times \mathbb{R}^n$, recall the its Hamilton field is defined by

$$\mathcal{H}_p = \sum_{j=1}^{n} (\frac{\partial p}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j} \frac{\partial}{\partial \xi_j}).$$

If q is another smooth function on $\Omega \times \mathbb{R}^n$, we see that

$$\omega(\mathcal{H}_p, \mathcal{H}_q) = \mathcal{H}_p q.$$

The Poisson bracket of p and q is defined as

$$\{p, q\} = \mathcal{H}_p q.$$

Suppose now that $\Sigma \subset \Omega \times \mathbb{R}^n$ is a submanifold of codimension d. If $\rho \in \Sigma$, we will be interested in the space $T_\rho^\perp(\Sigma)$, the space orthogonal (with respect to the symplectic form) to the tangent space of Σ at the point ρ. By definition this is given by

$$T_\rho^\perp(\Sigma) = \{X \in T_\rho(\Omega \times \mathbb{R}^n) : \omega(X, Y) = 0 \text{ for all } Y \in T_\rho(\Sigma)\},$$

where $T_\rho(\Sigma)$ is the tangent space to Σ at the point ρ.

If the codimension of Σ is equal to d and $\rho \in \Sigma$, then near ρ we can find d smooth functions $f_j, j = 1, \ldots, d$ with linearly independent differentials, such that near ρ we have

$$\Sigma = \{(x, \xi) \in \Omega \times \mathbb{R}^n : f_j(x, \xi) = 0, j = 1, \ldots, d\}.$$

In this situation it is not hard to see that

$$T_\rho^\perp(\Sigma) = \text{ linear span } \{\mathcal{H}_{f_j}, j = 1, \dots, d\}.$$

For example suppose that Σ is a hypersurface (i.e. codimension 1), with $\rho \in \Sigma$. Then near ρ we have a smooth function f, with $df \neq 0$ such that $\Sigma = \{f = 0\}$. We see that $T_\rho^\perp(\Sigma) =$ linear span $\{\mathcal{H}_f\}$. But trivially we have $\mathcal{H}_f f = 0$, so that $\mathcal{H}_f$ is also tangent to Σ. Hence, in this case,

$$T_\rho^\perp(\Sigma) \cap T_\rho(\Sigma) = \text{ linear span } \{\mathcal{H}_f\} \neq 0.$$

Recall that this is the geometric situation encountered when studying operators of real principal type.

Consider another example. Suppose we define

$$\Sigma = \{(x, \xi) \in \Omega \times \mathbb{R}^n : x_1 = 0 = \xi_1\},$$

where we assume that $n \geq 2$. We see that $\mathcal{H}_{\xi_1} = \frac{\partial}{\partial x_1}$ and $\mathcal{H}_{x_1} = -\frac{\partial}{\partial \xi_1}$. Neither of these vector fields is tangent to Σ. Indeed, we have in this case

$$T_\rho^\perp(\Sigma) \cap T_\rho(\Sigma) = 0, \tag{13}$$

for all $\rho \in \Sigma$. Note that this is the geometric situation encountered when studying the characteristic set for the operators P_λ of Section 10.

Observe that (13) guarantees that the restriction of ω to $T_\rho(\Sigma)$ is non–degenerate, making Σ itself into a symplectic manifold.

Indeed, if Σ is any submanifold of $\Omega \times \mathbb{R}^n$ which satisfies (13) for all $\rho \in \Sigma$, we call such Σ a *symplectic submanifold* of $\Omega \times \mathbb{R}^n$.

13. The Symplectic Case

In this section we discuss the case of a "sum of squares" operator with symplectic characteristic set.

Let $\Omega \subset \mathbb{R}^n$ be an open set, with coordinates $x_1, \ldots, x_n$. Consider $m+1$ real valued, real analytic vectorfields X_j of the form

$$X_j = \sum_{k=1}^{n} a_{jk}(x) \frac{\partial}{\partial x_k}, j = 0, \ldots, m.$$

Here each a_{jk} is analytic on Ω. We will study an operator of the form

$$P = \sum_{j=1}^{m} X_j^2 + X_0 + c(x), \tag{14}$$

where c is a complex valued, analytic function on Ω.

We will assume that Σ, the characteristic set of P, is a symplectic submanifold of $\Omega \times \mathbb{R}^n$. We also must assume that p, the principal symbol of P, *vanishes precisely to second order on* Σ. By this, we mean the following: For every compact $K \subset\subset \Omega$ there exists $C > 0$ such that

$$C^{-1}|\xi|^2(d(x,\xi))^2 \leq -p(x,\xi) \leq C|\xi|^2(d(x,\xi))^2,$$

for all $(x,\xi) \in K \times \mathbb{R}^n, |\xi| > 1$. Here

$$d(x,\xi) = \inf_{(y,\eta)\in\Sigma} \{|x-y| + |\frac{\xi}{|\xi|} - \eta|\}$$

is the distance from the point $(x, \frac{\xi}{|\xi|})$ to Σ. Also, the reader should note that $p \leq 0$, so $|p| = -p$.

We have the following Theorem, which is a consequence of a more general result proved independently by Treves [**28**] and Tartakoff [**25**].

THEOREM 14. *Let P be defined on an open subset $\Omega \subset \mathbb{R}^n$. Assume that P has the form* (14). *Also assume that Σ, the characteristic set of P, is a symplectic submanifold of $\Omega \times \mathbb{R}^n$. Furthermore, assume that p, the principal symbol of P, vanishes precisely to second order on Σ. Then P is analytic hypoelliptic on Ω.*

One important example of the situation described above is the operator P_λ of Section 10, when the quantity

$$-i(1+2\lambda)$$

is real.

On the other hand, by a result of Matsuzawa [**16**], the operator on $\mathbb{R}^2$

$$\partial_t^2 + t^{2k}\partial_x^2$$

is analytic hypoelliptic on $\mathbb{R}^2$ for all integers $k \geq 1$. Observe that Theorem 14 does not apply when $k \geq 2$.

14. Treves' Original Conjecture

At this point we know that analytic hypoellipticity holds in the symplectic case (with uniform vanishing to second order of the principal symbol). We also know that the Baouendi–Goulaouic operator is a counterexample. These facts were part of the motivation that led Treves to make a conjecture that is still open today. First some terminology.

Let $\Omega \subset \mathbb{R}^n$ and let $(0,1) \subset \mathbb{R}$ denote the open unit interval. We have the following :

DEFINITION 8. Let $\Sigma \subset \Omega \times \mathbb{R}^n$ be an analytic submanifold and let $\gamma : (0,1) \to \Sigma$ be a non-constant analytic curve. We call γ a *bicharacteristic strip* for Σ if

$$\frac{d\gamma}{dt}(t) \in (T_{\gamma(t)}\Sigma)^{\perp} \tag{15}$$

for all $t \in (0,1)$.

Note that this definition generalizes the terminology originally reserved for operators of real principal type.

In [**28**] Treves conjectured that when the characteristic set Σ is a manifold, and contains such curves, then the associated operator is not analytic hypoelliptic. Later, in [**30**], Treves extended his conjecture. We will discuss this later on.

Certainly the Baouendi–Goulaouic example is consistent with the conjecture. Indeed, the operator is given by

$$B = {\partial_t}^2 + t^2 {\partial_x}^2 + {\partial_y}^2, \tag{16}$$

with characteristic set Σ given by $\tau = t = \eta = 0$. We see that $T\Sigma$, the tangent bundle of Σ, is spanned by the vector fields

$$\partial_x, \partial_y, \partial_\xi.$$

Also $T\Sigma^\perp$, the orthogonal bundle with respect to the symplectic form, is spanned by

$$\partial_t, \partial_y, \partial_\tau.$$

Note that

$$\partial_y \in T\Sigma \cap T\Sigma^\perp.$$

Hence Σ is not symplectic. Indeed if we let γ be a particular y line, then we have

$$T_p\gamma = T_p\Sigma \cap T_p\Sigma^\perp$$

for each $p \in \gamma$. These y lines are bicharacteristic strips for Σ.

More evidence is supplied by the example of Metivier [**18**]:

$$M = {\partial_x}^2 + (x^2 + y^2){\partial_y}^2. \tag{17}$$

M is not analytic hypoelliptic on any open set containing the origin. (M is elliptic away from the origin.) Note that the characteristic set Σ is given by

$$\Sigma = \{\xi = x = y = 0\},$$

and hence Σ itself is a bicharacteristic strip for Σ.

On the other hand, operators with nonsymplectic characteristic set Σ may still be analytic hypoelliptic. Consider the operator

$$P = {\partial_x}^2 + (\partial_y + (\frac{4}{3}x^3 + 4xy^2)\partial_t)^2.$$

This is the principal part of the Kohn Laplacian for the domain

$$\{(z, w) \in \mathbb{C}^2 : \Im w > |z|^4\}.$$

In this case, $\Sigma = \{\xi = 0 = q\}$, where we define $q = \eta + (\frac{4}{3}x^3 + 4xy^2)\tau$. Note that Σ is symplectic as long as $x^2 + y^2 \neq 0$. However, when $x^2 + y^2 = 0$, we see that $\mathcal{H}_\xi = \frac{\partial}{\partial x}$ and $\mathcal{H}_q$ are tangent to Σ. Hence Σ is not a symplectic manifold when $x = y = \xi = \eta = 0$. Observe that this set where the symplectic form degenerates is itself a symplectic manifold of codimension 4. Certainly Σ has no bicharacteristic strips. P is analytic hypoelliptic by work of Sjöstrand [**23**].

Compare this with the following example of Oleinik [**21**]. Let p, r be integers ≥ 1 and consider

$$P = {\partial_t}^2 + t^{2p}{\partial_x}^2 + t^{2r}{\partial_y}^2.$$

Oleinik proved that P is analytic hypoelliptic if and only if $p = r$. Note that the characteristic set is given by $\tau = 0 = t$, for any choice of $p \geq 1, r \geq 1$. This is always a symplectic manifold.

Hence we see that analytic hypoellipticity can fail in the symplectic case, when the order of vanishing is not uniform.

At this point, we should point out that Christ [**4**] has proved a very special, 3 dimensional version of the Treves conjecture.

15. The Poisson Stratification of Σ

We introduce the Poisson stratification of the characteristic set. This is an important idea and the basis for Treves next conjecture.

Let $\Omega \subset \mathbb{R}^m$ be open and let $X_0, \ldots, X_\nu$ be real analytic vector fields on Ω. Let P have the form

$$P = {X_0}^2 + \cdots + {X_\nu}^2.$$

We assume P satisfies Hörmander's condition. Let $f_j, j = 0, \ldots, \nu$ denote the symbols of the X_j. The characteristic set of P is defined as

$$\Sigma = \{p \in \Omega \times (\mathbb{R}^n \setminus 0) : f_j(p) = 0, j = 0, \ldots, \nu\}.$$

It is a theorem of Treves [**30**] that Σ can be decomposed in the following way.

- There exist connected, pairwise disjoint analytic submanifolds $\Sigma_j \subset \Sigma$ such that
$$\Sigma = \cup \Sigma_j.$$
Furthermore, the union is locally finite.
- For each j we have
$$T_p\Sigma_j \cap T_p{\Sigma_j}^{\perp}$$
has constant dimension at each $p \in \Sigma_j$.
- There exists, for each j, an integer n_j such that f_I vanishes on Σ_j for all $|I| < n_j$, but for each $p \in \Sigma_j$, there exists I with $|I| = n_j$ such that $f_I(p) \neq 0$. Note that if $I = (i_1, \ldots, i_q)$, then $f_I = \{f_{i_1}, \ldots, \{f_{i_{q-1}}, f_{i_q}\}, \ldots\}$.
- Σ_j is maximal for the above three properties.

16. Examples

We study the stratification for
$$P = {\partial_x}^2 + (\partial_y + (\frac{4}{3}x^3 + 4xy^2)\partial_t)^2.$$
We see that the characteristic set is stratified in the following way:
$$\Sigma_1 = \{\xi = 0 = \eta + (\frac{4}{3}x^3 + 4xy^2)\tau, x^2 + y^2 \neq 0\}$$
$$\Sigma_2^j = \{x = \xi = 0 = \eta = y, (-1)^j\tau > 0\},$$
for $j = 1, 2$. All strata are symplectic and the operator is analytic hypoelliptic.

Now consider the Oleinik operator.
$$P = {\partial_t}^2 + t^{2p}{\partial_x}^2 + t^{2r}{\partial_y}^2,$$
with p, r both ≥ 1. In all cases,
$$\Sigma = \{t = 0 = \tau\}.$$

If $p = r$, Σ is the only stratum, which is symplectic. P is analytic hypoelliptic.

However when $p < r$ we have, for $j = 1, 2$

$$\Sigma_1^j = \{t = 0 = \tau, (-1)^j \xi_2 > 0\}$$

$$\Sigma_2^j = \{t = \tau = \xi_2 = 0, (-1)^j \xi_1 > 0\}.$$

The Σ_1^j are symplectic, while the Σ_2^j are not. The operator is not analytic hypoelliptic.

17. Treves' Conjecture

Let $U \subset \mathbb{R}^m$ be open and let $X_0, \ldots, X_\nu$ be real analytic vector fields on U. Let P have the form

$$P = {X_0}^2 + \cdots + {X_\nu}^2.$$

We assume P satisfies Hörmander's condition. Let Σ denote the characteristic set of P. Treves' conjecture asserts that the following statement is true:

For P to be analytic hypoelliptic on U it is necessary and sufficient that every Poisson stratum of Σ be symplectic.

Treves' conjecture is consistent with all known results. However, the analog of the conjecture is not true in the global sense or in the sense of germs. See Cordaro–Himonas [**6**] and Hanges [**10**]. Also, the contribution of Bove, Derridj, Tartakoff [**3**] is a very interesting generalization of [**10**]. Indeed, these papers have motivated Treves to give a more generalized conjecture, see [**31**].

18. Symplectic Strata of Codimension Two

We have the following recent result of Cordaro and Hanges [**5**], which establishes Treves' conjecture (in the positive direction) in the codimension two case. [1]

THEOREM 15. *Let $U \subset \mathbb{R}^m$ be open and let $X_0, \dots, X_\nu$ be real analytic vector fields on U which satisfy Hörmander's condition. Let P have the form*

$$P = {X_0}^2 + \cdots + {X_\nu}^2,$$

with Σ the characteristic set of P. Let $q \in \Sigma$. We assume that near q, Σ is a symplecitc Poisson stratum of codimension 2. Then P is analytic hypoelliptic at q.

This means that whenever $u \in \mathcal{D}'(U)$, with $q \notin WF_A(Pu)$, it follows that $q \notin WF_A(u)$.

[1] Recently it has come to our attention that Theorem 15 follows from the results of Okaji [**20**]. The methods of [**20**] are quite different from ours [**5**], which we believe are applicable to more general situations.

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