Canonical Metrics on Compact almost Complex Manifolds

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Canonical Metrics on Compact almost Complex Manifolds

Santiago R. Simanca
State University of New York at Stony Brook

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Preface

This monograph is based on a set of notes written for the minicourse *Geometria Riemanniana de Variedades com Estruturas Quase Complexas*, dictated at IMPA in January of 2003. The changes from the original notes amount to improvements in the exposition, with little mathematics added or subtracted.

When preparing the notes for the minicourse, we set up three goals for ourselves:

a) To provide a quick yet thorough introduction to the subject.
b) To provide illustrations of some fundamental techniques from global analysis that are often used in the proofs of important results in the field.
c) To motivate the importance of open problems stated throughout for the consideration of the reader.

With the particular signature anybody can give to his own work, we feel confident that the notes have accomplished each of these goals.

It will be quite evident to the reader that we have tried to associate canonical Hermitian metrics to a given almost complex structure. The intention is the study of the latter through properties of the firsts, and it is perhaps in order to assert from the outset that this is still work in progress, and that a lot remains to be understood in this area. When all is done, we should at least have a better understanding of particular types of almost Hermitian metrics canonically associated, in some suitable sense, to the given almost complex structure. Samples of these are already present in this monograph.

We would like to express our gratitude to all at IMPA for the opportunity we were given, and for the warm hospitality that we received during our entire visit to the Institute. We would like to extend special thanks to Marcos Dajczer, who was instrumental in arranging our visit, and to Mònica Manjarin, with whom we engaged in very interesting discussions while in Rio de Janeiro, and who made observations about the original notes that led to important improvements.

Contents

1. Almost complex structures

In this chapter we introduce the type of manifolds we shall deal with throughout this minicourse. We present these concepts with a great deal of generality. A substantial part of our efforts will be aimed later on at the geometric understanding of complex manifolds of Kähler type, but will not be restricted to it. Rather, we will present the Kähler case as particular to a more general theory.

1.1. **Almost complex manifolds.** Let M be a Hausdorf topological space. In order to analyze M locally, we use *open charts*, that is to say, pairs of the type (U,φ) where U is an open subset of M, and $\varphi : U \mapsto \varphi(U) \subset \mathbb{R}^k$ is a homeomorphism of U onto an open subset of $\mathbb{R}^k$. A collection of charts $\{(U_\alpha, \varphi_\alpha)_{\alpha\in A}\}$ gives M the structure of a smooth manifold of dimension k if the open sets U_α cover M, and if for all pair of indices α, β, the transition function $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \mapsto \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map. We then say that $\{(U_\alpha, \varphi_\alpha)_{\alpha\in A}\}$ is an *atlas* of M.

Definition 1. A complex structure on a topological space M consists of a family $\{(U_\alpha, \varphi_\alpha)_{\alpha\in A}\}$, where U_α is an open subset of M and $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ is a homeomorphism onto an open subset of $\mathbb{C}^n$, such that

a) $M = \cup_{\alpha\in A} U_\alpha$.

b) For each pair of indices α, β en A, the function
$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \mapsto \varphi_\beta(U_\alpha \cap U_\beta)$$
is holomorphic.

Each pair $(U_\alpha, \varphi_\alpha)$ is called a complex chart, and the whole collection $\{(U_\alpha, \varphi_\alpha)_{\alpha\in A}\}$ is called a complex atlas. The integer n is the complex dimension of M.

Notice that a complex manifold of dimension n is, in a natural way, a real manifold of dimension $2n$. For given a point $p \in M$, let us consider a complex chart (U,φ) with $p \in U$ and $\varphi(q) = (z^1(q), \ldots, z^n(q))$. The complex valued functions z^j can be decomposed in terms of their real and imaginary parts, $z^j(q) = x^j(q) + iy^j(q)$, decomposition that in turn induces a map
$$q \mapsto (x^1(q), y^1(q), \ldots, x^n(q), y^n(q))$$
from U onto an open subset of $\mathbb{R}^{2n}$. This function defines a *real* local chart of M. It is easy to see that the transition functions of these charts on M are smooth functions. Thus, the collection of all such charts defines M as a real differentiable manifold of dimension $2n$.

The set $\{\partial_{x^j}\mid_p, \partial_{y^j}\mid_p\}$ forms a basis of the tangent space T_pM. Using it, we define a linear isomorphism $J = J_p : T_pM \to T_pM$ by

$$J(\partial_{x^j}\mid_p) = \partial_{y^j}\mid_p\,, \quad J(\partial_{y^j}\mid_p) = -\partial_{x^j}\mid_p\ .$$

This map is in effect independent of the choice of coordinates made.

For if $\tilde{\varphi}(q) = (\tilde{z}^1(q), \ldots, \tilde{z}^n(q))$ is another local chart in a neighborhood of p such that $\tilde{z} = v^j(q) + iw^j(q)$, the linear map

$$\tilde{J}(\partial_{v^j}\mid_p) = \partial_{w^j}\mid_p\,, \quad \tilde{J}(\partial_{w^j}\mid_p) = -\partial_{v^j}\mid_p$$

coincides with J. Indeed, we have that

$$\begin{aligned}
\partial_{x^j} &= \sum_k \left(\frac{\partial v^k}{\partial x^j}\frac{\partial}{\partial v^k} + \frac{\partial w^k}{\partial x^j}\frac{\partial}{\partial w^k} \right), \\
\partial_{y^j} &= \sum_k \left(\frac{\partial v^k}{\partial y^j}\frac{\partial}{\partial v^k} + \frac{\partial w^k}{\partial y^j}\frac{\partial}{\partial w^k} \right).
\end{aligned}$$

Since the transition function $\tilde{\varphi} \circ \varphi^{-1}$ is holomorphic, the functions v^j, w^j satisfy the Cauchy-Riemann equations

$$\frac{\partial v^k}{\partial x^j} - \frac{\partial w^k}{\partial y^j} = 0\,, \quad \frac{\partial v^k}{\partial y^j} + \frac{\partial w^k}{\partial x^j} = 0\,.$$

Thus

$$\begin{aligned}
\partial_{x^j} &= \sum_k \left(\frac{\partial v^k}{\partial x^j}\frac{\partial}{\partial v^k} + \frac{\partial w^k}{\partial x^j}\frac{\partial}{\partial w^k} \right) = \sum_k \left(\frac{\partial v^k}{\partial x^j}\frac{\partial}{\partial v^k} - \frac{\partial v^k}{\partial y^j}\frac{\partial}{\partial w^k} \right), \\
\partial_{y^j} &= \sum_k \left(\frac{\partial v^k}{\partial y^j}\frac{\partial}{\partial v^k} + \frac{\partial w^k}{\partial y^j}\frac{\partial}{\partial w^k} \right) = \sum_k \left(\frac{\partial v^k}{\partial y^j}\frac{\partial}{\partial v^k} + \frac{\partial v^k}{\partial x^j}\frac{\partial}{\partial w^k} \right).
\end{aligned}$$

It then follows easily that

$$\tilde{J}(\partial_{x^j}\mid_p) = \partial_{y^j}\mid_p\,, \quad \tilde{J}(\partial_{y^j}\mid_p) = -\partial_{x^j}\mid_p\,,$$

which shows that $\tilde{J}$ agrees with J on the basis elements $\{\partial_{x^j}\mid_p, \partial_{y^j}\mid_p\}$. Thus, $\tilde{J} = J$.

In this way, we obtain a globally defined tensor $p \mapsto J_p : T_pM \mapsto T_pM$, that squares to minus the identity, $J^2 = -1\!\!1$.

Definition 2. Let M be a differentiable manifold. An *almost complex structure* on M is a tensor J of type $(1,1)$ such that $J(J(X)) = -X$ for all vector fields X on M. We say that the pair (M, J) is an almost complex manifold if M is a manifold and J is an almost complex structure on it.

By the observations above, a complex structure on a manifold M induces a *canonical* almost complex structure J. This structure has a very particular property. For we may consider the Lie bracket operation $[\,\cdot\,,\,\cdot\,]$ and extend it by complex linearity in each argument to a bracket operation on sections of the complexified tangent bundle $\mathbb{C}\otimes TM$. The structure J is *integrable* in the sense that the bracket of any two vectors in the i-eigenspace of J produces another vector in this same eigenspace. Indeed, the family of local vector fields $\{\partial_{x^j} - i\partial_{y^j}\}$ forms a local basis for the eigenspace of J associated to the eigenvalue i. But we have that $[\partial_{x^j} - i\partial_{y^j}, \partial_{x^l} - i\partial_{y^l}] = 0$, $1 \leq j, l \leq n$, and the assertion made follows.

The remark above has a converse, a deep and famous theorem of Newlander and Nirenberg that we now proceed to discuss in some detail: an integrable almost complex structure J on a manifold M is the canonical almost complex structure induced by a complex structure on M.

Let J be an almost complex structure on M. At each point p of M, J_p is a linear map $J_p : T_pM \mapsto T_pM$ whose square $-\mathbb{1}$. We may decompose the complexified tangent space $\mathbb{C}\otimes TM$ into $T^{1,0}M \oplus T^{0,1}M$, where the summands are the eigenspaces of J of eigenvalue $+i$ and $-i$, respectively. This in turn induces a decomposition of the complexified tensor algebra, and for example, $\mathbb{C}\otimes \Lambda^r M = \sum_{p+q=r} \Lambda^p(T^{1,0}M)^* \otimes \Lambda^q(T^{0,1}M)^*$. We shall denote $\Lambda^p(T^{1,0}M)^* \otimes \Lambda^q(T^{0,1}M)^*$ as $\Lambda^{q,p}_{\mathbb{C}}M$, and refer to its elements as forms of type (q,p).

The definition of J can be extended to T^*M by

$$J\alpha(X) = -\alpha(JX)\,,\ \alpha \in T^*M,\ X \in TM\,,$$

and to $\Lambda^*_{\mathbb{C}}M$ by complex linearity. Thus, $J\beta = i^{p-q}\beta$ if $\beta \in \Lambda^{p,q}_{\mathbb{C}}M$.

The definition of this extension is forced upon us if we want the identification between TM and T^*M induced by a J-invariant metric to intertwine the action of the almost complex structure on TM and T^*M, respectively. Because of that, the i-eigenspace of J on $\mathbb{C}\otimes T^*M$ is precisely $\Lambda^{1,0}_{\mathbb{C}}M$, the dual of $T^{0,1}M$.

Let us extend the Lie bracket operation by complex linearity in each argument to a bracket operation on $\mathbb{C}\otimes TM$. We then ask under what conditions is $T^{1,0}M$ an integrable distribution, that is to say, conditions such that $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$. But sections of $T^{1,0}M$ are in one-to-one correspondence with complex vector fields of the form $X - iJX$, for X a real vector field on M. Given two such complex fields $X - iJX$ and $Y - iJY$, we have that

$$[X - iJX, Y - iJY] = [X,Y] - [JX,JY] - i([X,JY] + [JX,Y])\,,$$

and this is once again of the form $W - iJW$ iff

$$N(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY]$$

is identically zero.

It is fairly easy to see that if f and g are functions on M, $N(fX,gY) = fgN(X,Y)$. Thus, N defines a tensor, the *Nijenhuis* tensor of the almost complex structure J. This tensor measures the obstruction to J arising from a holomorphic structure on M.

We define an almost complex structure J to be integrable if its Nijenhuis tensor is identically zero. The argument above shows that J is integrable if and only if $T^{1,0}M$ is an integrable distribution of complex fields.

Moreover, using the bigrading that J induces on $\mathbb{C} \otimes \Lambda^* M$, the composition of the exterior differentiation operator d with the corresponding projections defines the operators ∂ and $\overline{\partial}$

(1)
$$\partial : C^\infty(\Lambda^{p,q}M) \to C^\infty(\Lambda^{p+1,q}M)\,, \quad \overline{\partial} : C^\infty(\Lambda^{p,q}M) \to C^\infty(\Lambda^{p,q+1}M)\,.$$

Then we have:

Theorem 3. (Newlander-Nirenberg) *On an almost complex manifold (M,J), the following conditions are equivalent:*

a) *The almost complex structure J is induced by a holomorphic structure on M.*
b) *The exterior differentiation d is equal to $d = \partial + \overline{\partial}$.*
c) *The distribution $T^{1,0}M$ is integrable.*

Therefore, on a almost complex manifold (M,J) with integrable J, we can always find a complex atlas whose canonical almost complex structure is J. In that case, let $U \ni q \mapsto (z^1(q), \ldots, z^n(q))$ be a local chart, with $z^j = x^j + iy^j$ as above. Given a multi-index $I = (i_1, \ldots, i_n)$, we define dz^I as $(dz^1)^{i_1} \wedge \cdots \wedge (dz^n)^{i_n}$, where $dz^j = dx^j + idy^j$. We may similarly define $d\bar{z}^I$, where $d\bar{z}^j = dx^j - idy^j$. Then, $\{dz^I \wedge d\bar{z}^J,\ |I| = p,\ |J| = q\}$ spans $\Lambda^{p,q}M$ locally, and

$$\partial(\sum f_{I,J} dz^I \wedge d\bar{z}^J) = \sum_{j=1}^n \sum \frac{\partial f_{I,J}}{\partial z^j} dz^j \wedge dz^I \wedge d\bar{z}^J\,.$$

Similarly,

$$\overline{\partial}(\sum f_{I,J} dz^I \wedge d\bar{z}^J) = \sum_{j=1}^n \sum \frac{\partial f_{I,J}}{\partial \bar{z}^j} d\bar{z}^j \wedge dz^I \wedge d\bar{z}^J\,,$$

and we clearly have

$$d = \partial + \overline{\partial}\,.$$

Furthermore, $\partial\overline{\partial} + \overline{\partial}\partial = 0$, $\partial^2 = 0$, and $\overline{\partial}^2 = 0$, properties that we shall use repeatedly below.

Let M and N be complex manifolds of the same complex dimension n, and f be a holomorphic function $f : M \mapsto N$ between them (that is to say, a function whose local expression is a holomorphic map between open subsets of $\mathbb{C}^n$). Then, $f_*(T^{1,0}M) \subset T^{1,0}N$. Let us choose local complex charts (U, φ) and (V, ψ) in M and N, respectively, with $\varphi(p) = z^j = x^j + iy^j$ and $\psi(a) = w^k = u^k + iv^k$. In the bases $\{\partial_{x^j}, \partial_{y^j}\}$ and $\{\partial_{u^j}, \partial_{v^j}\}$, the Jacobian of f is given by

$$J_{\mathbb{R}}(f) = \begin{pmatrix} \partial_{x^j} u^\alpha & \partial_{y^j} u^\alpha \\ \partial_{x^j} v^\alpha & \partial_{y^j} v^\alpha \end{pmatrix} ,$$

while in the bases $\{\partial_{z^j}, \partial_{\bar{z}^j}\}$ and $\{\partial_{w^j}, \partial_{\bar{w}^j}\}$, we have that

$$J_{\mathbb{C}}(f) = \begin{pmatrix} J(f) & 0 \\ 0 & \overline{J(f)} \end{pmatrix} .$$

where

$$J(f) = (\partial_{z^j} w^\alpha) \ .$$

Then,

$$\det J_{\mathbb{R}}(f) = \det J(f) \overline{\det J(f)} = |\det J(f)|^2 > 0\,,$$

which shows that holomorphic maps preserve orientation. Thus, a complex manifold carries a canonical orientation.

1.2. **Examples.** Since holomorphic functions on compact manifolds are constants, examples of compact complex manifolds are a bit harder to obtain than those of compact real differentiable manifolds. The main source of examples are submanifolds of *complex projective space* $\mathbb{CP}^n$.

1.2.1. *Complex projective space.* Given points $\zeta = (\zeta^0, \zeta^1, \ldots, \zeta^n)$ and $\tilde{\zeta} = (\tilde{\zeta}^0, \tilde{\zeta}^1, \ldots, \tilde{\zeta}^n)$, we declare them to be equivalent iff $\tilde{\zeta} = \lambda\zeta$ for some $\lambda \neq 0$, and write $\zeta \sim \tilde{\zeta}$. As a topological space, $\mathbb{CP}^n = \frac{\mathbb{C}^{n+1} - \vec{0}}{\sim}$. It is given the quotient topology to make the projection map

$$\pi : \mathbb{C}^{n+1} - \vec{0} \mapsto \mathbb{CP}^n$$

continuous.

Given the point $(\zeta^0, \zeta^1, \ldots, \zeta^n) \in \mathbb{C}^{n+1} - \vec{0}$, we denote by $[\zeta^0 : \zeta^1 : \cdots : \zeta^n]$ the corresponding point p in $\mathbb{CP}^n$, and say that p has homogeneous coordinates $(\zeta^0, \zeta^1, \ldots, \zeta^n)$.

The complex structure on $\mathbb{CP}^n$ is defined as follows. Let $\tilde{U}_j = \{\zeta \in \mathbb{C}^{n+1} : \zeta^j \neq 0\}$. We set $U_j = \pi(\tilde{U}_j)$, an open subset of $\mathbb{CP}^n$. Using homogeneous coordinates on U_j, we define $\varphi_j : U_j \to \mathbb{C}^n$ by

$$[\zeta^0, \zeta^1, \ldots, \zeta^n] \mapsto \left(\frac{\zeta^0}{\zeta^j}, \ldots, \frac{\zeta^{j-1}}{\zeta^j}, \frac{\zeta^{j+1}}{\zeta^j}, \ldots, \frac{\zeta^n}{\zeta^j}\right),$$

an obviously continuous map. That this map turns out to be a homeomorphism can be verified easily.

Thus, we have a collection of $n+1$ complex charts $\{(U_j, \varphi_j)\}_{j=0}^n$. It is clear that $\cup_j U_j = \mathbb{CP}^n$. On the other hand, for convenience let us set

$$z_k^j = \frac{\zeta^k}{\zeta^j},$$

so that we may write φ_j as

$$\varphi_j[\zeta^0, \zeta^1, \ldots, \zeta^n] = (z_0^j, \ldots, \widehat{z_j^j}, \ldots, z_n^j).$$

Here the symbol $\widehat{z_j^j}$ indicates that z_j^j is being omitted from consideration. Then, the k-th component of the function $\varphi_l \circ \varphi_j^{-1}$, defined on $\varphi_j(U_j \cap U_l)$, is given by

$$z_k^l = (z_l^j)^{-1} z_k^j.$$

In other words, the transition function $\varphi_l \circ \varphi_j^{-1}$ is just multiplication by $1/z_l^j$, a holomorphic mapping. Thus, $\{(U_j, \varphi_j)\}_{j=0}^n$ is a complex atlas on $\mathbb{CP}^n$ which, endowed with it, becomes a complex manifold.

1.2.2. *Hypersurfaces of degree d in $\mathbb{CP}^n$.* Let $p(\zeta_0, \ldots, \zeta_n)$ be a homogeneous polynomial of degree d in $n+1$ variables. Hence, $p(\lambda\zeta_0, \ldots, \lambda\zeta_n) = \lambda^d p(\zeta_0, \ldots, \zeta_n)$, and this implies that if $\zeta = (\zeta_0, \ldots, \zeta_n)$ is a zero of p, then the entire class of ζ under the relation $\sim$ that defines $\mathbb{CP}^n$ consists of zeroes of p. Therefore,

$$H_p = \{P \in \mathbb{CP}^n : p(P) = 0\}$$

is a well-defined subset of $\mathbb{CP}^n$. If H_p does not have singularities, it defines a hypersurface of $\mathbb{CP}^n$ that automatically inherits a complex structure from that of the ambient space.

Exercise 1. Let p_1 and p_2 be homogeneous polynomials of the same degree, and assume that $H_i = \{P \in \mathbb{CP}^n : p_i(P) = 0\}$ $(i = 1, 2)$ is a smooth hypersurface in $\mathbb{CP}^n$. Show that H_1 is diffeomorphic to H_2.

Hint. The coefficients of a homogeneous polynomial of degree d define a point in $\mathbb{C}^N$ for some N, and the zero set of this polynomial depends only of the class of this point in $\mathbb{CP}^{N-1}$.

The implicit function theorem for holomorphic functions says that if $f_1, \dots, f_k$ are holomorphic functions in $\mathbb{C}^n$ such that

$$\det \begin{pmatrix} \partial_{z_1} f_1 & \cdots & \partial_{z_k} f_1 \\ \vdots & \ddots & \vdots \\ \partial_{z_1} f_k & \cdots & \partial_{z_k} f_k \end{pmatrix} \neq 0 \,,$$

then the equations

$$f_1(z) = 0 \,, \quad f_2(z) = 0 \,, \quad \dots \,, \; f_k(z) = 0 \,,$$

locally define $(z_1, \dots, z_k)$ as holomorphic functions of $(z_{k+1}, \dots, z_n)$.

Let us then consider the polynomial $p_d(\zeta_0, \dots, \zeta_n) = \zeta_0^d + \cdots + \zeta_n^d$. Applying the implicit function theorem, we see that the set

$$S_d = \{z = [z_0 : \cdots : z_n] \in \mathbb{CP}^n \,:\, p_d(z) = 0\}$$

is a complex hypersurface in $\mathbb{CP}^n$. In light of the result in Exercise 1, this hypersurface is commonly referred to as the hypersurface of degree d in $\mathbb{CP}^n$. Its topology depends strongly upon d, fact that will be described in detail later on for the case of a hypersurface of degree d inside $\mathbb{CP}^3$.

1.2.3. *Calabi-Eckmann manifolds.* We now describe examples consisting of the product of spheres of odd dimension $\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$ endowed with an integrable almost complex structure.

The almost complex structure in question is defined in the following manner: let N_1 and N_2 be the exterior normal of the spheres $\mathbb{S}^{2n+1}$ and $\mathbb{S}^{2m+1}$ as subsets of $\mathbb{C}^{n+1}$ and $\mathbb{C}^{m+1}$, respectively, and let J_1 and J_2 be canonical complex structures on these complex vector spaces, respectively. Since $J_1 N_1$ and $J_2 N_2$ are globally defined vector fields on the corresponding spheres, we may decompose any vector field X in $\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$ as

$$X = X_1 + X_2 + a_1(X) J_1 N_1 + a_2(X) J_2 N_2 \,, \tag{2}$$

where X_1 is tangent to $\mathbb{S}^{2n+1}$ and perpendicular to $J_1 N_1$, while X_2 is tangent to $\mathbb{S}^{2m+1}$ and perpendicular to $J_2 N_2$. The structure J is defined by

$$JX = J_1 X_1 + J_2 X_2 - a_2 J_1 N_1 + a_1 J_2 N_2 \,.$$

Its Nijenhuis tensor is identically zero. Thus, $(\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}, J)$ is a complex manifold. Furthermore, with this J on the total space and the obvious one on the base, we obtain a holomorphic fibration of $\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$ over $\mathbb{CP}^n \times \mathbb{CP}^m$ with two-dimensional tori as fibers. We leave the detailed verification of the two assertions made to the interested reader.

1.2.4. *Even dimensional spheres:* $\mathbb{S}^6$. Of all the even dimensional spheres $\mathbb{S}^{2n} \subset \mathbb{R}^{2n+1}$, the only ones that admit almost complex structures are $\mathbb{S}^2$ and $\mathbb{S}^6$ [5]. The almost complex structure on the first of these is integrable, as is the case of any almost complex structure on a two dimensional manifold. The known almost complex structures on $\mathbb{S}^6$ are not integrable. We describe one of these next. It is yet to be determined if this manifold admits an integrable almost complex structure.

The standard almost complex structure on $\mathbb{S}^6$ is defined using the *vector product* in $\mathbb{R}^7$ induced by the multiplication operation on Cayley numbers $\mathbb{O}$, or *octonions*. Let us recall that the octonions form an algebra over $\mathbb{R}$ generated by $\{1, e_0, e_1, e_2, e_3, e_4, e_5, e_6\}$. Each octonion a can be written as

$$a = a_r \cdot 1 + \sum_{j=0}^{6} a_j \cdot e_j = a_r + \vec{a}\,,$$

where the component $\vec{a}$ is interpreted in the obvious manner as a vector in $\mathbb{R}^7$. The component a_r of a is called the real part of a, while $\vec{a}$ is called its imaginary part. The set of purely imaginary octonions is isomorphic (as a vector space) to $\mathbb{R}^7$.

The product of octonions $a = a_r + \vec{a}$ y $b = b_r + \vec{b}$ is defined by

$$a \cdot b = a_r b_r - \langle \vec{a}, \vec{b} \rangle + a_r \vec{b} + b_r \vec{a} + \sum_{i \neq j} a_i b_j e_i \cdot e_j\,,$$

where $\langle \vec{a}, \vec{b} \rangle \in \mathbb{R}$ is the standard inner product of the vectors $\vec{a}$ and $\vec{b}$, respectively, and the products $e_i \cdot e_j$ are given according to the following multiplication table:

	e_0	e_1	e_2	e_3	e_4	e_5	e_6
e_0	-1	e_2	$-e_1$	e_4	$-e_3$	e_6	$-e_5$
e_1	$-e_2$	-1	e_0	$-e_5$	e_6	e_3	$-e_4$
e_2	e_1	$-e_0$	-1	e_6	e_5	$-e_4$	$-e_3$
e_3	$-e_4$	e_5	$-e_6$	-1	e_0	$-e_1$	e_2
e_4	e_3	$-e_6$	$-e_5$	$-e_0$	-1	e_2	e_1
e_5	$-e_6$	$-e_3$	e_4	e_1	$-e_2$	-1	e_0
e_6	e_5	e_4	e_3	$-e_2$	$-e_1$	$-e_0$	-1

The product of two purely imaginary octonions $a = \vec{a}$ and $b = \vec{b}$ yields

$$\vec{a} \cdot \vec{b} = -\langle \vec{a}, \vec{b} \rangle + \sum_{i \neq j} a_i b_j e_i \cdot e_j \ ,$$

where the expression on the right shows the decomposition of $\vec{a} \cdot \vec{b}$ in its real and imaginary parts, respectively. The imaginary part of $\vec{a} \cdot \vec{b}$ defines the *vector product* operation in $\mathbb{R}^7$. That is to say,

$$\vec{a} \times \vec{b} = \sum_{i,j} a_i b_j e_i \times e_j \ ,$$

where $e_i \times e_j$ is given by the multiplication table above after replacing all its diagonal elements by zero. This operation is antisymmetric in $\vec{a}$ and $\vec{b}$, and we have that $\langle \vec{a}, \vec{a} \times \vec{b} \rangle = 0$. Thus, the mapping $\times$ yields a vector perpendicular to each one of its arguments.

Using the canonical metric $\langle \ \ , \ \ \rangle$ in $\mathbb{R}^7$ and given $x \in \mathbb{S}^6 \subset \mathbb{R}^7$, we may identify the tangent space $T_x\mathbb{S}^6$ at x with a subspace of $\mathbb{R}^7$. Consider the mapping

$$\begin{array}{rcl} J_x : T_x\mathbb{S}^6 & \longrightarrow & T_x\mathbb{S}^6 \\ Y & \mapsto & x \times Y \end{array} .$$

Since $x \times Y$ is an element of $\mathbb{R}^7$ perpendicular to x, this mapping is well-defined, and being linear in Y, it defines an endomorphism of $T_x\mathbb{S}^6$. By the *alternative* property of the Cayley product (that is to say, $X \cdot (X \cdot Y) = (X \cdot X) \cdot Y$ for all $X, Y \in \mathbb{O}$), we have that

$$J_x^2 Y = x \times (x \times Y) = x \cdot (x \cdot Y) = (x \cdot x) \cdot Y = -\langle x, x \rangle \, Y = -Y \ .$$

Therefore, J_x is a complex structure on the vector space $T_x\mathbb{S}^6$.

The endomorphism J_x depends smoothly on x. Thus, $x \mapsto J_x$ defines an almost complex structure on $\mathbb{S}^6$. This structure is not integrable.

Let $y \in \mathbb{R}^6$. Under stereographic projection from the south pole, y corresponds to

$$x = (x^0, x^1, \ldots, x^6) = \frac{1}{1 + |y|^2}(1 - |y|^2, 2y^1, \ldots, 2y^6) \ .$$

Under this mapping, the vector ∂_{y^i} corresponds to

$$V_y = \frac{1}{1 + |y|^2}(-2y^i \partial_{x^0} + 2\partial_{x^i}) - \frac{2y^i}{1 + |y|^2} \sum_{j=0}^{6} x^j \partial_{x^j}$$

in $T_x\mathbb{S}^6$. On the other hand, a tangent vector in $T_x\mathbb{S}^6$ with components $(V_0, V_1, \dots, V_6)$ corresponds to the vector

$$\frac{1}{1+x^0}\sum_{j=1}^{6}(V_j - V_0 y^j)\partial_{y^j}$$

in $T_y\mathbb{R}^6$. With this information, we can write down explicitly the almost complex structure J in the stereographic projection coordinates $y \in \mathbb{R}^6$.

Indeed, in the canonical basis $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6\}$ for $\mathbb{R}^7$, the components of the vector $Z = x \times Y$ are given by

$$Z^i = V^i_j Y^j \,,$$

where, if the upper index indicates the row, we have that

$$(V^i_j) = \begin{pmatrix} 0 & -x_2 & x_1 & -x_4 & x_3 & -x_6 & x_5 \\ x_2 & 0 & -x_0 & x_5 & -x_6 & -x_3 & x_4 \\ -x_1 & x_0 & 0 & -x_6 & -x_5 & x_4 & x_3 \\ x_4 & -x_5 & x_6 & 0 & -x_0 & x_1 & -x_2 \\ -x_3 & x_6 & x_5 & x_0 & 0 & -x_2 & -x_1 \\ x_6 & x_3 & -x_4 & -x_1 & x_2 & 0 & -x_0 \\ -x_5 & -x_4 & -x_3 & x_2 & x_1 & x_0 & 0 \end{pmatrix} .$$

Since the second summand of the vector V_y above that corresponds to ∂_{y^i} is parallel to the radial vector field, $J_y(\partial_{y^i})$ is just the push-forward under stereographic projection of the vector

$$x \times \left[\frac{1}{1+|y|^2}(-2y^i\partial_{x^0} + 2\partial_{x^i})\right] .$$

For convenience, let us set

$$\begin{aligned} S &= -y^2\partial_{y^1} + y^1\partial_{y^2} - y^4\partial_{y^3} + y^3\partial_{y^4} - y^6\partial_{y^5} + y^5\partial_{y^6} \,, \\ R &= \sum_{j=1}^{6} y^i\partial_{y^i} \,, \quad n(y) = \frac{2}{1+|y|^2} \,, \quad c(y) = \frac{1-|y|^2}{2} \,. \end{aligned}$$

Then

$$\begin{aligned} J(\partial_{y^1}) &= n(y)\left(-y^1S - y^2R - c(y)\partial_{y^2} + y^5\partial_{y^3} - y^6\partial_{y^4} - y^3\partial_{y^5} + y^4\partial_{y^6}\right) \,, \\ J(\partial_{y^2}) &= n(y)\left(-y^2S + y^1R + c(y)\partial_{y^1} - y^6\partial_{y^3} - y^5\partial_{y^4} + y^4\partial_{y^5} + y^3\partial_{y^6}\right) \,, \\ J(\partial_{y^3}) &= n(y)\left(-y^3S - y^4R - y^5\partial_{y^1} + y^6\partial_{y^2} - c(y)\partial_{y^4} + y^1\partial_{y^5} - y^2\partial_{y^6}\right) \,, \\ J(\partial_{y^4}) &= n(y)\left(-y^4S + y^3R + y^6\partial_{y^1} + y^5\partial_{y^2} + c(y)\partial_{y^3} - y^2\partial_{y^5} - y^1\partial_{y^6}\right) \,, \\ J(\partial_{y^5}) &= n(y)\left(-y^5S - y^6R + y^3\partial_{y^1} - y^4\partial_{y^2} - y^1\partial_{y^3} + y^2\partial_{y^4} - c(y)\partial_{y^6}\right) \,, \\ J(\partial_{y^6}) &= n(y)\left(-y^6S + y^5R - y^4\partial_{y^1} - y^3\partial_{y^2} + y^2\partial_{y^3} + y^1\partial_{y^4} + c(y)\partial_{y^5}\right) . \end{aligned}$$

Notice that $JR = -S$.

With this explicit description, we may prove that this structure is not integrable. However, there are computationally easier —and conceptually better— ways of obtaining that same conclusion. Later on, we shall outline one such as an exercise.

2. Almost Hermitian manifolds

We now study Riemannian metrics that are adjusted to a given almost complex structure. We describe these metrics in general. Later on, we shall restrict our attention to complex manifolds of Kähler type, and discuss some particular properties of the space of Kähler metrics on it.

Definition 4. Let (M, J) be an almost complex manifold. A Riemannian metric g is said to be J-Hermitian if $g(JX, JY) = g(X, Y)$ for all pair of vector fields X, Y. We say that (J, g) is an *almost Hermitian structure* on M which, when provided with one such structure, will be called an *almost Hermitian manifold*.

On an almost Hermitian manifold (M, J, g), we may defined tensors that tie up properties of J and g. Indeed, let us start by introducing a J-twisted version of the *Ricci tensor*, that we shall call the *J-Ricci tensor* from now on. Let us recall that the usual Ricci tensor $r(X, Y)$ of a Riemannian manifold (M, g) is the trace of the linear map $L \to R(L, X)Y$, where R is the Riemann curvature tensor $R(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z$, ∇ the Levi-Civita connection of g. If g is Hermitian relative to J, we reproduce this concept with a J-twist, and define the J-Ricci tensor by

$$r^J(X, Y) = \operatorname{trace} L \to -J(R(L, X)JY)\,, \tag{3}$$

The tensor defined above is essentially the only new tensor we can obtain by computing the trace of a J-twisting of R in two different positions. Indeed, varying the type of trace we take and using the symmetries of the curvature tensor R, up to a constant factor or a permutation of the arguments, we only obtain the expressions r^J, $J^* r^J$ or $J^* r$, respectively.

Unlike r, r^J does not turn out to be a symmetric tensor in general. We shall see examples of this below.

The usual *scalar curvature* s is the total contraction of the curvature tensor, that is to say, the metric trace of the Ricci tensor r. Analogously, we define the *J-scalar curvature* s^J as the metric trace of r^J.

A straightforward calculation shows that in terms of the components of R and J, we have that

$$s = R^{li}{}_{li}, \qquad s^J = -J^i{}_t R_{ilm}{}^t J^{lm}\,. \tag{4}$$

For an almost Hermitian manifold (M, J, g), consider the tensor

$$\omega(X, Y) = \omega_g^J(X, Y) = g(JX, Y)\,. \tag{5}$$

The invariance of g under J makes this an alternate tensor, which is referred to at the *fundamental form* of (M, J, g). This form is J-invariant, but does not have any other special property unless we impose further conditions on the metric g. On the other hand, despite the fact that generally speaking r^J is neither symmetric nor J-invariant, the tensor

$$\rho^J(X, Y) = -r^J(X, JY)\,,$$

is alternate. This 2-form will be called the *J-Ricci form* of the almost Hermitian structure (J, g).

Exercise 2. Let (M, J, g) be an almost Hermitian manifold and consider the space $\Lambda^2 TM$ of bivectors with Riemannian metric defined by

$$g(X \wedge Y, V \wedge W) = g(X, V)g(Y, W) - g(X, W)g(Y, V)\,.$$

The *curvature operator* $\mathfrak{R} : \Lambda^2 TM \to \Lambda^2 TM$ is defined by the identity

$$g(\mathfrak{R}(X \wedge Y), V \wedge W) = g(R(X, Y)W, V)\,.$$

One may use the metric to view this curvature operator as an endomorphism of $\Lambda^2 T^*M$. Prove that when doing so, $\rho^J = \mathfrak{R}\omega$.

2.1. **Kähler metrics.** We now consider a special but important set of metrics of the type above.

Definition 5. Let (M, J) be a complex manifold and let g be a J-Hermitian metric. We say that g is Kähler if its fundamental form ω_g is closed. If the complex manifold (M, J) admits a Kähler metric g, we say that (M, J) is of Kähler type, and that g is a Kähler structure on it.

We may reformulate the definition above in the following manner: *an almost Hermitian manifold (M, J, g) is Kähler iff the tensor J is covariantly constant with respect to the Levi-Civita connection of the metric g.*

Indeed, let us assume that this alternative condition $\nabla_X J = 0$ holds. Since ∇ is torsion free, we have that

$$\begin{array}{rcl} J[X, JY] & = & -\nabla_X(Y) - J\nabla_{JY}X\,, \\ J[JX, Y] & = & \nabla_Y(X) + J\nabla_{JX}Y\,, \\ [JX, JY] & = & J\nabla_{JX}Y - J\nabla_{JY}X\,, \end{array}$$

and it then follows that the Nijenhuis tensor $N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$ is identically zero. By the Newlander-Nirenberg theorem, (M, J) is a complex manifold.

Moreover,

$$\begin{array}{rcl} (\nabla_X \omega)(Y, Z) & = & Xg(JY, Z) - g(J\nabla_X Y, Z) - g(JY, \nabla_X Z) \\ & = & g(\nabla_X JY - J\nabla_X Y, Z) = 0\,, \end{array}$$

so ω_g is covariantly constant, and therefore, its exterior derivative must vanish because $3d\omega(X, Y, Z) = (\nabla_X\omega)(Y, Z) - (\nabla_Y\omega)(X, Z) + (\nabla_Z\omega)(X, Y)$. Thus, (M, J, g) is a Kähler manifold in the sense of the original definition.

Conversely, suppose we an almost Hermitian manifold (M, J, g) with (M, J) complex and $d\omega_g = 0$. Observe that $(\nabla_X J)Y = \nabla_X(JY) - J\nabla_X Y$. Therefore,

$$g((\nabla_X J)Y, Z) = 3d\omega_g(X, JY, JZ) - 3d\omega_g(X, Y, Z) + g(N(Y, Z), JX)\,.$$

The hypothesis imply that this expression is zero for all Y, Z. Thus, $\nabla_X J = 0$ for all X, as desired.

The point in bringing up this equivalent definition is to exhibit the Kähler condition as one that says that the properties of g and J are quite interrelated with each other.

On a Kähler manifold (M, J, g), the J-Ricci tensor r^J coincides with the usual Ricci tensor r. Indeed, when computing the trace of the map $L \rightarrow -J(R(L, X)JY)$, the inner-most J can be pulled out all the way to the front, and we conclude that trace $L \rightarrow -J(R(L, X)JY =$ trace $L \rightarrow -J^2(R(L, X)Y)$. The same argument shows that the Ricci tensor r is J-invariant, and therefore, the J-Ricci form equals $\rho(X, Y) = r(JX, Y)$, a form of type $(1, 1)$. In the Kähler context, this is called this the *Ricci form*. If we write its exterior derivative in terms of covariant differentiation, we see that $d\rho = 0$, so ρ represents a cohomology class in $H^{1,1}(M, \mathbb{C}) \subset H^2(M, \mathbb{R})$.

Let (M, J) be a complex manifold of Kähler type. If we choose holomorphic coordinates $(z^1, \ldots, z^n)$, we get induced local bases $\{\frac{\partial}{\partial z^j}\}$ and $\{\frac{\partial}{\partial z^{\bar{j}}} := \frac{\partial}{\partial \bar{z}^j}\}$ for $T^{1,0}M$ y $T^{0,1}M$, respectively. Let g be a Kähler metric on (M, J), and extend it by complex-multilinearity to a $(0,2)$-tensor on the complexified tangent space. If we define

$$g_{\mu\nu} := g\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right),$$

where the indices μ, ν run along $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$, the Hermitian condition $g(X, Y) = g(JX, JY)$ implies that $g_{jk} = g_{\bar{j}\bar{k}} = 0$, $j, k = 1, \ldots, n$, and that the only non-zero components are those of the form $g_{j\bar{k}} = \bar{g}_{k\bar{j}}$. Therefore, the fundamental form can be locally expressed as

$$\omega = \omega_{j\bar{k}} dz^j \wedge d\bar{z}^k = i g_{j\bar{k}} dz^j \wedge d\bar{z}^k .$$

If we choose the complex coordinates $z^j = x^j + iy^j$ such that $\{\partial_{x^j}, \partial_{y^j}\}$ are orthonormal at some given arbitrary point, then at this point we have that

$$\omega = \frac{i}{2} \sum_k dz^k \wedge dz^{\bar{k}} ,$$

and a relatively simple argument shows that the volume form of the metric, defined as $d\mu_g = \sqrt{\bar{g}} dx^1 \wedge dy^1 \wedge \cdots dx^n \wedge dy^n$ where $\bar{g}$ is the determinant of the matrix of components of g, is given by

$$d\mu_g = \frac{\omega^n}{n!} . \tag{6}$$

Thus, the volume of a Kähler manifold (M, J, g) is the n-th cup product $\Omega^n/n!$, where Ω is the cohomology class represented by the fundamental form of the Kähler metric g.

The non-zero components of the Levi-Civita connection of the metric are given by

$$\Gamma_{ij}^k = g^{k\bar{t}} \frac{\partial g_{i\bar{t}}}{\partial z^j} ,$$

and

$$\Gamma_{\bar{i}\bar{j}}^{\bar{k}} = \overline{\Gamma_{ij}^k} .$$

From this expression for the Christoffel symbols of the connection, we may easily compute the curvature tensor of the Kähler metric. We have that

$$R_{i\bar{j}k\bar{l}} = -g_{t\bar{j}} \frac{\partial \Gamma_{ik}^t}{\partial z^{\bar{l}}} = -\frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial z^{\bar{j}}} + g^{p\bar{q}} \frac{g_{k\bar{q}}}{\partial z^i} \frac{g_{p\bar{l}}}{\partial z^{\bar{j}}} ,$$

where we made use of the fact that

$$g^{i\bar{j}} g_{i\bar{k}} = \delta_{\bar{k}}^{\bar{j}} .$$

Since the Ricci tensor $r(X,Y)$ of a Riemannian metric is the trace of the linear map $L \to R(L,X)Y$, by the previous expression we obtain that

$$r_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -g^{k\bar{l}} \frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial z^{\bar{j}}} + g^{k\bar{l}} g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{l}}}{\partial z^{\bar{j}}} .$$

On the other hand, a direct computation yields

$$\bar{\partial} \log \det (g_{i\bar{j}}) = g^{i\bar{j}} \bar{\partial} g_{i\bar{j}} ,$$

and therefore,

$$\partial\bar{\partial} \log \det (g_{i\bar{j}}) = g^{i\bar{j}} \partial\bar{\partial} g_{i\bar{j}} - g^{p\bar{j}} g^{i\bar{q}} \partial g_{p\bar{q}} \wedge \bar{\partial} g_{i\bar{j}} .$$

Comparing this expression with the Ricci tensor, we conclude that

(7)
$$r_{i\bar{j}} = -\frac{\partial^2 \log \det (g_{i\bar{j}})}{\partial z^i \partial z^{\bar{j}}} .$$

The scalar curvature is the trace of the Ricci tensor:

$$s = 2g^{i\bar{j}} r_{i\bar{j}} .$$

Usually this tensor is conveniently calculated from the relation

(8)
$$s\omega^n = 2n\rho \wedge \omega^{n-1}$$

between s, the fundamental form ω, and the Ricci form ρ. This relationship can be verified easily by the reader.

2.2. **Chern Classes: the Ricci form.** Let M be a differentiable manifold. A smooth complex vector bundle of rank n over M consists of a topological space E, a continuous surjective map $\pi : E \to M$ and the structure of a complex vector space in the set $\pi^{-1}(p)$, subject to the following trivialization condition: for each point $p \in M$ there exists an open neighborhood $U \subset M$ and a homeomorphism $\varphi_U : U \times \mathbb{C}^n \to \pi^{-1}(U)$ such that, for each $q \in U$, the correspondence $v \to \varphi_U(q,v)$ defines an isomorphism between $\mathbb{C}^n$ and $\pi^{-1}(q)$.

Thus, if $\pi : E \to M$ is a complex vector bundle over M, we may cover the manifold M by a collection of open sets $\{U_\alpha\}$ with the property that for each α there exists a trivializing homeomorphism $\varphi_a : U \times \mathbb{C}^n \to \pi^{-1}(U)$ identifying the fiber $\pi^{-1}(q)$ with $\mathbb{C}^n$. This trivializing homeomorphism allows one to find a local basis of sections $\{s^1_\alpha, \ldots, s^n_\alpha\}$. The matrix changing the local basis defined by the homeomorphism φ_α to the local basis defined by the homeomorphism φ_β produces a map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{GL}(n, \mathbb{C})$, the *transition function* of the bundle relative to the trivializations $\varphi_\alpha, \varphi_\beta$. These transition functions satisfy the identities $g_{\alpha\beta}(x) g_{\beta\alpha}(x) = \mathbb{1}$ for all $x \in U_\alpha \cap U_\beta$, and $g_{\alpha\beta}(x) g_{\beta\gamma}(x) g_{\gamma\alpha} = \mathbb{1}$

for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$, and characterize the bundle. For if there exists a cover $\{U_\alpha\}$ of M and functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{GL}(n, \mathbb{C})$ satisfying these identities, then there exists, up to equivalence, a unique complex vector bundle $E \to M$ whose transition functions are the given ones. Two bundles with transition functions g_{ij} and $\tilde{g}_{ij}$ are said to be equivalent iff there are functions $h_i : U_i \to \mathbb{GL}(n, \mathbb{C})$ such that $\tilde{g}_{ij} = h_i g_{ij} h_j^{-1}$.

A typical example is given by $\pi : L \to \mathbb{CP}^n$, where $L = \{([\zeta], z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : z \in \zeta\}$, and π is the projection onto the first factor. The points in L are pairs of the form $([\zeta], z)$ where z belongs to the complex line defined by $[\zeta] \in \mathbb{CP}^n$. In this case the rank is 1, and the bundle is referred to as the *tautological line bundle* over $\mathbb{CP}^n$.

The Chern classes of a complex vector bundle $E \xrightarrow{\pi} M$ are characterized axiomatically as follows:

(1) For all i, $c_i(E) \in H^{2i}(M, \mathbb{Z})$ and $c_0(E) = 1$.
(2) Let N be any manifold and $f : N \to M$ a continuous function. Then $c(f^*(E)) = f^* c(E)$, where $c(E) = \sum_{i=0}^{\infty} c_i(E)$.
(3) (Whitney's formula) If E and F are complex vector bundles over M, then $c(E \oplus F) = c(E) \cdot c(F)$.
(4) Let $\gamma = \gamma_n$ be the tautological line bundle over $\mathbb{CP}^n$, and let h be the standard generator of $H^2(\mathbb{CP}^n, \mathbb{Z})$. Then $c(\gamma) = 1 - h$.

These classes can be constructed explicitly in the following manner. Pick a connection ∇ on the bundle E. If we choose a local frame of sections $\{s_1, \ldots, s_n\}$, the connection is completely determined locally by the connection 1-forms ω_{ij} defined by

$$\nabla s_i = \sum_j \omega_{ij} \otimes s_j \,.$$

Under a change of local frame $\tilde{s}_i = h_{ij} s_j$, it is fairly easy to see that the connection 1-form transforms as $\tilde{\omega}_{ij} = dh_{ik} h_{kj}^{-1} + h_{ik} \omega_{kl} h_{lj}^{-1}$, which is written succinctly as $\tilde{\omega} = dh \cdot h^{-1} + h \omega h^{-1}$.

Extend the connection to a mapping on $C^\infty(\Lambda T^* M \otimes E)$, by defining it on $C^\infty(\Lambda^p T^* M \otimes E)$ as

$$\nabla(\theta_p \otimes s) = d\theta_p \otimes s + (-1)^p \theta_p \wedge \nabla s \,.$$

Then the curvature mapping $\nabla^2 : C^\infty(E) \to C^\infty(\Lambda^2 T^* M \otimes E)$ acts tensorially on sections of E. For if f is any function and s a section of E, we have that

$$\nabla^2(fs) = \nabla(df \otimes s + f \nabla s) = -df \otimes \nabla s + df \otimes \nabla s + f \nabla^2 s = f \nabla^2 s \,.$$

Thus, there exists a curvature 2-form $\Omega = \Omega_{ij}$ with values in the space of endomorphisms of E, such that

$$\nabla^2 s = \Omega s\,.$$

In terms of the local frame $\{s_1, \dots, s_n\}$ and the associated connection 1-form $\omega = \omega_{ij}$, we have that $\Omega_{ij} = d\omega_{ij} - \omega_{ik} \wedge \omega_{kj}$, that is to say, $\Omega = d\omega - \omega \wedge \omega$. It follows that $d\Omega = 0$. Thus, Ω represents a cohomology class of M with values in the space of endomorphisms of E, and when changing local frames to $\tilde{s}_i = h_{ij} s_j$, we have the transformation rule $\tilde{\Omega}_{ij} = h_{ik} \Omega_{kl} h^{-1}_{lj}$, that is $\tilde{\Omega} = h \Omega h^{-1}$. This property implies that

$$\det\left(\mathbb{1} + \frac{i}{2\pi}\Omega\right) = 1 + c_1(M) + \cdots + c_n(M)$$

is a well-defined closed form on the manifold M. It represents the total Chern class $c(E)$ of the bundle E. The cohomology classes represented by the components of various degrees of $c(E)$ are independent of the particular connection ∇ chosen above.

Notice the explicit expressions

$$c_0 = 1\,, \quad c_1 = \frac{i}{2\pi} \operatorname{trace} \Omega\,, \quad c_2 = \frac{1}{8\pi^2} (\operatorname{trace} \Omega \wedge \Omega - \operatorname{trace} \Omega \wedge \operatorname{trace} \Omega)\,,$$

in terms of Ω.

Suppose now that M is a complex manifold. Let $E \to M$ be a complex vector bundle over M trivialized by a covering $\{U_\alpha\}$ of M by open sets. We say that this vector bundle is *holomorphic* iff the transition functions $f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{GL}(m, \mathbb{C})$ are all holomorphic mappings.

For a holomorphic bundle $E \to M$ over a complex manifold M, a Hermitian metric on the fiber determines a unique connection that preserves the Hermitian metric and whose connection 1-form is of type $(1, 0)$. Indeed, let us use a partition of unity subordinated to the cover $\{U_\alpha\}$ of M to construct a Hermitian metric $(\,\cdot\,, \,\cdot\,)$ on the fibers of E. If for a choice of a basis of sections we define $h_\alpha = (s_\alpha, s_\alpha)$, then the one form

$$\omega_\alpha = \partial h_\alpha h_\alpha^{-1}$$

is globally defined precisely because the transition functions $f_{\alpha\beta}$ are holomorphic. Indeed, by the Hermiticity of the metric, we have that $h_\alpha = f_{\alpha\beta} h_\beta \bar{f}_{\alpha\beta}$. and $df_{\alpha\beta} = \partial f_{\alpha\beta}$. Thus,

$$\omega_\alpha = df_{\alpha\beta} f^{-1}_{\alpha\beta} + f_{\alpha\beta} \omega_\beta f^{-1}_{\alpha\beta}$$

satisfies the transition rules of a connection one-form, that we call ∇_h. The Chern classes can now be computed using this canonical connection.

When the bundle is a holomorphic line bundle $L \to M$, the local Hermitian matrix h_α is of size 1×1. Therefore,

$$\Omega_\alpha = d\omega_\alpha - \omega_\alpha \wedge \omega_\alpha = d(\partial h_h h_\alpha^{-1}) = \overline{\partial}\partial \log h_\alpha = -\partial\overline{\partial} \log h_\alpha \, .$$

We thus obtain the first Chern class of the holomorphic line bundle,

$$c_1(L) = -\frac{i}{2\pi}\partial\overline{\partial}\log h \, ,$$

a $(1,1)$-form independent of the local frame s_α chosen to define it.

Given an almost complex manifold (M, J), the fibers of TM can be made into a complex vector space by defining $(a + ib)X = aX + bJX$. We thus obtain a complex vector bundle that is denoted by T_cM.

Definition 6. Let M be an almost complex manifold. The Chern classes of M are defined to be the Chern classes of the complex tangent bundle $T_cM \to M$.

Proposition 7. *Let (M, J, g) be a Kähler manifold, and let ρ be the Ricci form of g. Then*

$$\frac{1}{2\pi}\rho$$

represents $c_1(M)$, and this form is the curvature of the canonical line bundle $\kappa = \Lambda^n(T^*M)^{1,0}$.

Proof. By definition, the trace of the map $L \to iR(L, X)Y$ is $2\pi c_1(M)$ evaluated at (X, Y). But this is just $ir(X, Y) = \rho(X, Y)$. The first assertion follows.

The assertion about the curvature of κ follows from the remark about line bundles preceding the last definition above, and the fact that in complex coordinates, the Ricci form is given by

$$\rho = -i\partial\overline{\partial}\log\left(\det\left(g_{i\overline{j}}\right)\right) ,$$

as we saw in (7). □

The rigidity of the cohomology class represented by the Ricci form of any Kähler metric plays a significant rôle in the study of canonical metrics on a given manifold of Kähler type. For example, when seeking Kähler-Einstein metrics on this type of manifolds, the first Chern class $c_1(M, J)$ must be definite, or null, thus eliminating from consideration a large number of cases that do not satisfy this condition.

2.3. **Examples.** From the examples of almost complex manifolds given earlier, we now present examples of metrics adapted to the corresponding almost complex structures.

2.3.1. $\mathbb{CP}^n$. We have shown that the complex manifold $\mathbb{CP}^n$ may be covered by $n+1$ open sets $U_0, \ldots, U_n$. The holomorphic coordinates on U_l are defined in terms of the holomorphic coordinates $(\zeta^0, \zeta^1, \ldots, \zeta^n)$ on $\mathbb{C}^{n+1}$ by

$$z_l^j = \frac{\zeta^j}{\zeta^l}\,,\; j = 0, 1, \ldots, \widehat{l}, \ldots, n\,.$$

Then, we consider the locally defined $(1,1)$-form

$$\omega_l = \frac{i}{2}\partial\overline{\partial}\log\left(1 + z_l^1\bar{z}_l^1 + \cdots + z_l^n\bar{z}_l^n\right) = \frac{i}{2}\frac{\delta_{\alpha\beta}(1+\sum z_l^\gamma\bar{z}_l^\gamma) - z_l^\beta\bar{z}_l^\alpha}{(1+\sum z_l^\gamma\bar{z}_l^\gamma)^2}dz_l^\alpha\wedge d\bar{z}_l^\beta\,.$$

Since the transition function on the overlap of U_l and U_j is given by $1/z_l^j$, we have that

$$\begin{aligned}
\omega_j &= \frac{i}{2}\partial\overline{\partial}\log\left(1 + z_j^1\bar{z}_j^1 + \cdots + z_j^n\bar{z}_j^n\right)\\
&= \frac{i}{2}\partial\overline{\partial}\log\frac{(1 + z_j^1\bar{z}_j^1 + \cdots + z_j^n\bar{z}_j^n)}{z_l^j\bar{z}_l^j}\\
&= \frac{i}{2}\partial\overline{\partial}\log\left(1 + z_l^1\bar{z}_l^1 + \cdots + z_l^n\bar{z}_j^n\right)\\
&= \omega_l\,,
\end{aligned}$$

so the locally defined forms patch together to produce a form ω on $\mathbb{CP}^n$. It is rather clear that ω is closed, and fairly easy to see that it is positive. Thus, ω is the fundamental form of a Kähler metric g on $\mathbb{CP}^n$. This metric is commonly known as the Fubini-Study metric.

We may use the symmetries of g to verify that say, on U_0, we have that

$$d\mu = \frac{\omega^n}{n!} = \frac{i^n}{2^n}\frac{1}{(1+\sum z_0^\gamma\bar{z}_0^\gamma)^{n+1}}dz_0^1\wedge d\bar{z}_0^1\ldots dz_0^n\wedge d\bar{z}_0^n\,.$$

In then follows that the Ricci form of g is a multiple of ω,

$$\rho = 2(n+1)\omega\,,$$

and that the scalar curvature is

$$s = 4n(n+1)\,,$$

a positive constant.

The Fubini-Study metric is an example of an Einstein metric, one whose Ricci tensor is pointwise a multiple of the metric tensor itself. It is a consequence of the contracted differential Bianchi identity that if we have a Riemannian manifold (M, g) where $r_g = fg$ for some function f, if the real dimension of M is 3 or greater, then f must be constant.

Our calculation of ρ implies that

$$c_1(\mathbb{CP}^n) = \frac{n+1}{\pi}\omega\,.$$

This shows that $\mathbb{CP}^n$ is a complex manifold whose first Chern class is positive.

2.3.2. *Hypersurfaces of degree d in $\mathbb{CP}^3$.* Given a homogeneous polynomial $p(\zeta_0, \dots, \zeta_n)$ of degree d, we have seen that if

$$S_d\{[\zeta_0, \dots, \zeta_n] \in \mathbb{CP}^n \,:\, p(\zeta_0, \dots, \zeta_n) = 0\}$$

is smooth, then S_d is a well-defined complex submanifold of $\mathbb{CP}^n$ whose diffeomorphism class is independent of p. In that case, the Fubini-Study metric on $\mathbb{CP}^n$ induces a Kähler metric on S_d.

Let us then consider the polynomial $p_d(\zeta_0, \zeta_1, \zeta_2, \zeta_3) = \zeta_0^d + \zeta_1^d + \zeta_2^d + \zeta_3^d$, and the complex hypersurfaces

$$S_d = \{z = [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \,:\, p_d(z) = 0\}$$

in $\mathbb{CP}^3$. They are all Kähler manifolds.

The total Chern class of $\mathbb{CP}^3$ is given by $c(\mathbb{CP}^3) = (1+g)^4$ where g is the generator of $H^2(\mathbb{CP}^3; \mathbb{Z})$ such that $\langle g, [\mathbb{CP}^1]\rangle = 1$. If $i : S_d \hookrightarrow \mathbb{CP}^3$ is the inclusion map, let us set $x = i^* g$. Then we have that

$$c_1(S_d) = (4-d)x\,, \quad c_2(S_d) = (d^2 - 4d + 6)x^2\,.$$

Indeed, $T\mathbb{CP}^3 \mid_{S_d} = TS_d \oplus \nu S_d$, where $\nu(S_d)$ is the normal bundle of S_d. By Whitney's formula for the Chern class of the sum of vector bundles, we have that

$$c(T\mathbb{CP}^3 \mid_{S_d}) = (1+x)^4 = (1 + c_1(S_d) + c_2(S_d)) \cdot (1 + c_1(\nu S_d))\,,$$

which implies that

$$\begin{aligned} 1 + c_1(S_d) + c_2(S_d) &= (1+x)^4 \cdot (1 + c_1(\nu S_d))^{-1} \\ &= (1 + 4x + 6x^2) \cdot (1 - c_1(\nu S_d) + c_1^2(\nu S_d))\,. \end{aligned}$$

But we have that $c_1(\nu(S_d)) = dx$. The assertion made for $c_1(S_d)$ and $c_2(S_d)$ follows easily after using this fact in the expression above.

Now we may also compute the Euler characteristic χ and signature σ of these manifolds. For we have that $\langle x^2, [S_d]\rangle = d$ and thus $c_1^2[S_d] = (4-d)^2 d$. By Noether's formula for complex surfaces, we know that $c_1^2(S_d) + c_2(S_d) = 3(\sigma(S_d) + \chi(S_d))$, and therefore,

$$\chi(S_d) = (d^2 - 4d + 6)d\,, \quad \sigma(S_d) = \frac{d(4-d^2)}{2}\,.$$

2.3.3. *Calabi-Eckmann manifolds.* We have seen earlier that the product of spheres of odd dimension $\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$ is a complex manifold in a natural way, with complex structure J. Using this J and the standard metrics on the sphere factors, we can make of this an almost Hermitian manifold.

Let us recall that a submersion between manifolds E and M is a surjective map $\pi : E \to M$ of maximal rank. In such a case, for any $p \in M$, the submanifold $\pi^{-1}(p) \subset E$ is called the fiber of the submersion at p. When E and M are Riemannian, a submersion $\pi : E \to M$ is called Riemannian if $\pi_* : (TF)^{\perp}_q \to T_{\pi(q)}M$ is an isometry. Here $(TF)^{\perp}$ is the orthogonal complement of the tangent to the fiber TF in TE, usually called the horizontal space. The tangent to the fiber itself is called the vertical space.

The standard odd dimensional sphere $\mathbb{S}^{2n+1}$ admits a Riemannian submersion over $\mathbb{CP}^n$ endowed with the Fubini-Study metric. The fibers are totally geodesic submanifolds. We then use the product metric to define a Riemannian fibration $\pi : \mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1} \to \mathbb{CP}^n \times \mathbb{CP}^m$, whose fibers are totally geodesic tori. The metric $\langle \cdot \, , \cdot \rangle$ on $\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$ so obtained is obviously J-invariant. Indeed, the Riemannian fibration just defined corresponds to the holomorphic fibration defined by J.

If we have given a Riemannian submersion $\pi : E \to M$ and a function $\varphi : M \to \mathbb{R}$, we may dilate the metric on the vertical part of the fibration, and define $\langle \cdot \, , \cdot \rangle_{\varphi} = e^{2\varphi} \langle \cdot \, , \cdot \rangle_V + \langle \cdot \, , \cdot \rangle_H$, where the subindices V and H in the right side refer to the vertical and horizontal metrics, respectively. Since the horizontal part is left unchanged, with the new Riemannian structure on E, the map π is still a Riemannian submersion with the same horizontal-vertical decomposition. This general construction applies to the product of odd-dimensional spheres. Thus, given any function $\varphi : M \to \mathbb{R}$ on $M = \mathbb{CP}^n \times \mathbb{CP}^m$, we may introduce the metric $\langle \cdot \, , \cdot \rangle_{\varphi}$ on the product $E = \mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$ of the form

$$\langle \cdot \, , \cdot \rangle_{\varphi} = e^{2\varphi} \langle \cdot \, , \cdot \rangle_V + \langle \cdot \, , \cdot \rangle_H \, ,$$

where the subindices V and H in the right side refer to the vertical and horizontal metrics defined by $\langle \cdot \, , \cdot \rangle_0 = \langle \cdot \, , \cdot \rangle$, respectively. We obtain a family of Riemannian fibrations $\pi = \pi_{\varphi} : E \to M$ parametrized by φ. These metrics on E are clearly J-invariant, and since J is integrable, the triple $(E, J, \langle \cdot \, , \cdot \rangle_{\varphi})$ produces a family of Hermitian manifolds.

We discuss the details of the standard Hermitian submersion first, the one that corresponds to the product metric $g_1 \times g_2$ on the spherical factors of E. These metrics on the factors are induced by the metrics on $\mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$ and $\mathbb{R}^{2m+2} = \mathbb{C}^{m+1}$, respectively. In order to compute the various tensors of interest to us, we choose an orthonormal frame $\{v_j\}$ of the form $\{T_1, T_2, e_1, \ldots, e_n, Je_1, \ldots, Je_n, f_1, \ldots, f_m, Jf_1, \ldots, Jf_m\}$, where $\{T_1, T_2\}$ is the orthonormal frame of the fiber flat torus given by $\{J_1N_1, J_2N_2\}$, and $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ and $\{f_1, \ldots, f_m, Jf_1, \ldots, Jf_m\}$ are the horizontal lifts of orthonormal frames in the two complex projective space factors of the base. Observe that for $j > 2$, each v_j is horizontal.

The components of curvature tensor are given by the components of the curvature tensor in each factor. These, we know, are of the form $R_{skij} = \delta_{is}\delta_{jk} - \delta_{ik}\delta_{js}$. Let ω_1 and ω_2 be the Kähler forms of the Fubini-Study metrics on the complex projective space factors in the base M. We then see easily that

$$\rho^J = \pi_1^*\omega_1 + \pi_2^*\omega_2 \,,$$

where π_1 and π_2 are the projection maps to each factor., In fact, if we decompose the vector fields X and Y in their components tangential to the base and fiber as in §1.2.3, we see that the J-Ricci tensor is given by

$$r^J_{\mathbb{S}^{2n+1}\times\mathbb{S}^{2m+1}}(X, Y) = g_1(X_1, Y_1) + g_2(X_2, Y_2) \,,$$

and so, the J-scalar curvature is just

$$\sigma^J_{\mathbb{S}^{2n+1}\times\mathbb{S}^{2m+1}} = 2n + 2m \,,$$

a constant.

In the more general case, the expression of the J-Ricci tensor r^J_φ is a bit more complicated to describe. We do so below, exhibiting expressions for the Ricci tensor r_φ also. We invite the reader to draw a contrast between the two:

a) For vertical vectors $U = aT_1 + bT_2$ and $V = \tilde{a}T_1 + \tilde{b}T_2$, we have that

$$r^J_\varphi(U, V) = -e^{2\varphi}\|\nabla\varphi\|^2\langle U, V\rangle = -(a\tilde{a} + b\tilde{b})e^{2\varphi}\|\nabla\varphi\|^2 \,,$$

while

$$r_\varphi(U, V) = e^{2\varphi}\left(a\tilde{a}(2ne^{2\varphi} + \Delta\varphi - 2\|\nabla\varphi\|^2) + b\tilde{b}(2me^{2\varphi} + \Delta\varphi - 2\|\nabla\varphi\|^2)\right) \,.$$

b) For vectors $X = X_1 + X_2$ and $U = aT_1 + bT_2$, horizontal and vertical, respectively, we have that

$$r^J_\varphi(X, U) = e^{2\varphi}(\langle\nabla\varphi, aJX_1 + bJX_2\rangle + \langle\nabla\varphi, (2n+1)bX_1 - (2m+1)aX_2\rangle) \,,$$

while

$$r_\varphi(X,U) = 4e^{2\varphi}\langle \nabla\varphi, aJX_1 + bJX_2\rangle \,.$$

c) For horizontal vectors $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ we have that

$$r^J_\varphi(X,Y) = (1-e^{2\varphi})(r_{\mathbb{CP}^n}(X_1,Y_1)+r_{\mathbb{CP}^m}(X_2,Y_2))+e^{2\varphi}\sum_{j>2}\langle R(v_j,X)JY, Jv_j\rangle \,,$$

while

$$\begin{aligned} r_\varphi(X,Y) \; &= \; 2n\langle X_1,Y_1\rangle + 2m\langle X_2,Y_2\rangle + 2(1-e^{2\varphi})\langle X,Y\rangle - 2h_\varphi(X_1,Y_1) \\ & \quad -2h_\varphi(X_2,Y_2) - 2\langle\nabla\varphi,X_1\rangle\langle\nabla\varphi,Y_1\rangle - 2\langle\nabla\varphi,X_2\rangle\langle\nabla\varphi,Y_2\rangle \,. \end{aligned}$$

It is now easy to compute the J-scalar curvature:

$$s^J_\varphi = 2(n+m) - 2\|\nabla\varphi\|^2 + (1-e^{2\varphi})(2n(2n+1) + 2m(2m+1)) \,.$$

Notice the lack of symmetry of r^J_φ shown in (b), and also the contrasting difference between s^J_φ and the scalar curvature s_φ, given by

$$s_\varphi = 2n(2n+1) + 2m(2m+1) + 4\Delta\varphi - 6\|\nabla\varphi\|^2 + 2(n+m)(1-e^{2\varphi}) \,.$$

2.3.4. *The* 6*-sphere.* Let g be the metric in $\mathbb{S}^6$ induced by the canonical metric $\langle \;\;,\;\; \rangle$ in $\mathbb{R}^7$. We have seen that the "vector product" endomorphism

$$\begin{array}{rcl} J_x : T_x\mathbb{S}^6 & \longrightarrow & T_x\mathbb{S}^6 \\ Y & \mapsto & x\times Y \end{array} ,$$

defines an almost complex structure on $\mathbb{S}^6$. The metric g is J-invariant. Hence, $(\mathbb{S}^6, J, g)$ is an almost Hermitian manifold.

With respect to an orthonormal frame, the curvature tensor of any standard sphere is given by $R_{skij} = \delta_{is}\delta_{jk} - \delta_{ik}\delta_{js}$. It follows that

$$r_{\mathbb{S}^6} = 5g \,, \quad s_{\mathbb{S}^6} = 30 \,,$$

while

$$\rho^J_{\mathbb{S}^6} = \omega \,, \quad s^J = 6 \,,$$

where ω is the fundamental form of the pair (g,J). In fact, we have that

$$r^J = g \,.$$

Thus, $(\mathbb{S}^6, J, g)$ is a J-Einstein almost Hermitian manifold.

We propose to the reader, in the form of an exercise, the discussion of certain relationship between properties of the tensor r^J and integrability of the almost complex structure J. Answers to the first two parts of this exercise may be found in [4]. There is an alternative proof of part (c) that does not use (a) & (b), but it is less geometric than the one suggested here.

Exercise 3. Let (M, J, g) be a Hermitian manifold of complex dimension n, and let $\kappa = \Lambda^{n,0}_{\mathbb{C}} M = \Lambda^n (T^{1,0}M)^*$ be the canonical bundle of $(n, 0)$-forms. The Levi-Civita connection of g induces a connection D on κ, that as a line bundle, sits inside $\mathbb{C} \otimes \Lambda^n(M)$.

a) Prove that the curvature Ω_D is given by

$$\Omega_D = \sqrt{-1} R(\omega_g) + \Psi^* \wedge \Psi \, ,$$

where Ψ is the second fundamental form of the embedding of κ in the complexification of the bundle of n-forms on M.

b) Prove that $\Psi \in \Lambda^1(M) \otimes \mathrm{Hom}(\Lambda^{n,0}, (\Lambda^{n,0})^{\perp})$ is of type $(1, 0)$ and that

$$\Psi^* \wedge \Psi \leq 0 \, .$$

c) Use the results above and Exercise 2 to prove the following theorem: Let (M, J, g) be a J-Einstein complex manifold with positive J-Einstein coefficient and trivial second cohomology group. Then J is not an integrable almost complex structure.

Evidently, we may apply this result to conclude that the standard almost complex structure on $\mathbb{S}^6$ is not integrable. This way of deriving such a result inspires some thoughts between plausible integrable structures J on the sphere and certain properties of r^J for associated J-invariant metrics.

Remark 8. The result in the exercise can be reformulated by saying that that any compact J-Einstein complex manifold (M, J, g) of positive J-scalar curvature must be Kähler. The condition on the cohomology group assumed in part (c) of the exercise prevents this from being so.

3. Almost Hermitian metrics with constant J-scalar curvature

We now discuss the almost complex version of the Yamabe problem. This provides us with the first possible interpretation we can give to the term "canonical" in the problem of seeking canonical almost Hermitian metrics on a given almost complex manifold (M, J).

The solution to the almost complex Yamabe problem we present is associated with a conformal invariant that is universally bounded. We use this fact to define an invariant associated to (M, J). Its explicit computation is an open problem in all but one of the known examples of almost complex manifolds.

3.1. **The Yamabe problem for almost Hermitian manifolds.** We saw in §2 that on an almost Hermitian manifold (M, J, g) we may define the J-scalar curvature tensor s^J. This tensor ties naturally properties of g with properties of J.

Let g be any J-invariant metric. Any metric of the form $e^{2f}g$ is also a J-invariant metric. Thus, the entire conformal class $[g]$ of g consists of J-invariant metrics, and we may then ask if $[g]$ carries a canonical representative. Here we shall interpret the term canonical to mean that the J-scalar curvature of the metric in question is constant.

As such, the question just posed constitutes an almost complex version of the usual Yamabe problem, where one seeks instead a representative of $[g]$ whose scalar curvature is constant.

In the two dimensional case, the proposed problem corresponds to the fully solved uniformization problem for Riemann surfaces. Thus, we assume that the dimension of our manifold is strictly greater than 2.

The first step in a program to tackle our question is to find out how the J-scalar curvature transforms under conformal deformations of the metric. We have the following

Proposition 9. *Let (M, J, g) be a compact almost Hermitian manifold of dimension $n = 2m$. Suppose that $\tilde{g} = \varphi^{\frac{4}{n-2}} g = \varphi^{\frac{2}{m-1}} g$ is a conformal deformation of g, and let s^J and $\tilde{s}^J$ be the J-scalar curvatures of g and $\tilde{g}$, respectively. Then*

$$\frac{2}{m-1}\Delta_g \varphi + s^J \varphi = \tilde{s}^J \varphi^{\frac{m+1}{m-1}} . \tag{9}$$

Here Δ_g is the positive Laplacian of the background metric g.

Proof. Suppose firstly that $\tilde{g} = e^{2f}g$. Then the $(4,0)$ Riemannian curvature tensor (see [3], page 58; their convention for the curvature is the negative of ours) transforms as

$$\tilde{R} = e^{2f}(R + g \owedge (\nabla df - df \circ df + \frac{1}{2}|df|^2 g)) , \tag{10}$$

where $h \owedge k$ is the Kulkarni-Nomizu product of two symmetric 2-tensors.

Given an orthonormal frame $\{\tilde{e}_1, \ldots, \tilde{e}_{2m}\}$ of $\tilde{g}$, we want to calculate

$$\tilde{s}^J_{\tilde{g}} = \sum_{i,j} \tilde{R}(\tilde{e}_i, \tilde{e}_j, J\tilde{e}_j, J\tilde{e}_i) .$$

Let $\{e_1, \ldots, e_{2m}\}$ be an orthonormal frame for g. Then $\{e^{-f}e_1, \ldots, e^{-f}e_{2m}\}$ is an orthonormal frame for $\tilde{g}$. Using this frame and (10), we have that

$$\begin{aligned} \tilde{s}^J_{\tilde{g}} &= e^{-4f} \textstyle\sum_{i,j} \tilde{R}(e_i, e_j, Je_j, Je_i) \\ &= e^{-2f} \left(s^J_g + \textstyle\sum_{i,j} g \owedge (\nabla df - df \circ df + \frac{1}{2}|df|^2 g)(e_i, e_j, Je_j, Je_i) \right) . \end{aligned}$$

By the definition of the Kulkarni-Nomizu product of two symmetric 2-tensors and the fact that $g(e_i, Je_i) = 0$, given any symmetric tensor h we have that

$$\begin{aligned} \textstyle\sum_{i,j} g \owedge h(e_i, e_j, Je_j, Je_i) &= \textstyle\sum_{i,j} (g(e_i, Je_j)h(e_j, Je_i) + g(e_j, Je_i)h(e_i, Je_j)) \\ &= 2 \textstyle\sum_{i,j} g(e_i, Je_j)h(e_j, Je_i) \\ &= 2 \textstyle\sum_{i,j} \sum_{l,k} J^l{}_j J^k{}_i g(e_i, e_l)h(e_j, e_k) \\ &= -2 \textstyle\sum_j h(e_j, e_j) = -2\mathrm{trace}_g h \, . \end{aligned}$$

Applying this result to $h = \nabla df - df \circ df + \frac{1}{2}|df|^2 g$, we obtain that

$$\tilde{s}^J_{\tilde{g}} = e^{-2f}(s^J_g + 2\Delta_g f - 2(m-1)|df|^2) \, .$$

Setting $e^{2f} = \varphi^{\frac{2}{m-1}}$, we now get

$$\tilde{s}^J_{\tilde{g}} = \varphi^{-\frac{2}{m-1}} \left(s^J_g + \frac{2}{m-1} \frac{\Delta_g \varphi}{\varphi} \right) ,$$

and the desired result follows. □

Using this result, we attempt to answer our question employing techniques similar to those used to resolve the Yamabe problem. Indeed, let $N = 2n/(n-2) = 2m/(m-1)$. We parametrize metrics $\tilde{g}$ in $[g]$ by positive functions φ such that $\tilde{g} = \varphi^{\frac{2}{m-1}} g$, and consider the functional

$$\lambda^J_N(\varphi) = \frac{\displaystyle\int_M \tilde{s}^J_{\tilde{g}} d\mu_{\tilde{g}}}{\left(\displaystyle\int_M d\mu_{\tilde{g}} \right)^{2/N}} = \frac{\displaystyle\frac{2}{m-1} \int_M |d\varphi|^2 \, d\mu_g + \int_M s^J \varphi^2 d\mu_g}{\|\varphi\|^2_N} \, . \tag{11}$$

The idea now is understand this functional in further detail.

It is fairly easy to see that the quantity

$$\lambda^J(M, [g]) := \inf_{\varphi} \lambda^J_N(\varphi)$$

is a conformal invariant. One could hope to solve the almost complex Yamabe problem by finding a positive function φ that minimizes $\lambda^J_N(\varphi)$.

Indeed, since the Euler-Lagrange equation of $\lambda_N^J(\varphi)$ is given by

$$\frac{2}{m-1}\Delta_g\varphi + s^J\,\varphi = \frac{\lambda_N^J(\varphi)}{\|\varphi\|_{L^N}^N}\varphi^{N-1}\,, \tag{12}$$

a positive minimizer φ with L^N-norm equal to one would solve (9) with $\tilde{s}^J = \inf_\varphi \lambda_N^J(\varphi)$, and the metric $\tilde{g} = \varphi^{\frac{2}{m-1}}g$ would have J-scalar curvature equal to the constant $\inf_\varphi \lambda_N^J(\varphi)$. This is exactly what one can actually prove. We sketch the main points of the argument next, and refer the reader to [9] for all details.

Consider the following result of Aubin ([2], page 131):

Theorem 10. (Aubin) *Let $(M^{n=2m}, g)$ be a compact Riemannian manifold, and consider the equation*

$$2\frac{2m-1}{m-1}\Delta_g\varphi + h(x)\varphi = \lambda f(x)\varphi^{N-1}\,, \tag{13}$$

with h, f smooth functions, $f > 0$, $N = 2n/(n-2) = 2m/(m-1)$, and λ a real number. Let

$$I(\varphi) = \left[2\frac{2m-1}{m-1}\int |d\varphi|^2 d\mu_g + \int h\varphi^2 d\mu_g\right]\left(\int f\varphi^N d\mu_g\right)^{-2/N}\,,$$

and define $\nu = \inf_\varphi I(\varphi)$, the infimum taken over all non-zero φ's in $H^1(M)$. Then, $\nu \leq 2m(2m-1)\omega_{2m}^{1/m}[\sup f]^{-2/N}$, and if ν is strictly less than this upper bound, then equation (13) *has a positive smooth solution φ for $\lambda = \nu$. Here, ω_k is the volume of the k dimensional unit sphere in $\mathbb{R}^{k+1}$.*

We apply this result to the study of (12). We conclude the existence of the universal bound

$$\lambda^J(M,[g]) \leq 2m\omega_{2m}^{\frac{1}{m}} \tag{14}$$

for the conformal invariant $\lambda^J(M,[g])$. Moreover, if (M,J,g) is such that $\lambda^J(M,[g]) < 2m\omega_{2m}^{\frac{1}{m}}$, equation (12) admits a positive solution φ of L^N-norm 1, and the metric $\tilde{g} = \varphi^{2/(m-1)}g$ has J-scalar curvature equal to $\lambda^J(M,[g])$.

In this manner, we may turn our attention to the study of the conformal invariant $\lambda^J(M,[g])$ of any almost complex manifold (M,J,g), and attempt to describe those for which the universal upper bound is achieved. Potentially these are the only cases where one might face difficulties in solving the almost complex Yamabe problem by minimizing $\lambda_N^J(\varphi)$.

The analysis of $\lambda^J(M,[g])$ is accomplished in the following manner. First of all, consider the Weyl tensor W of an almost Hermitian metric g. In terms of fundamental form ω of (M,J,g), we have the very important relation

$$(2m-1)s^J - s = 2(2m-1)W(\omega^\#, \omega^\#) \tag{15}$$

between s^J and the scalar curvature s.

Given $p \in M$, notice that the sign of $W(\omega^\#, \omega^\#)(p)$ does not change under conformal deformations of g. Hence, we let $\mathcal{A}_H$ be the class of all almost Hermitian manifolds, and define the subclasses W^-, W^0 and W^+ by

$$\begin{array}{rcl} W^- & = & \{(M^{2m},J,g) \in \mathcal{A}_H : \; W(\omega^\#,\omega^\#)(p) < 0 \text{ for some } p \in M\} \\ W^0 & = & \{(M^{2m},J,g) \in \mathcal{A}_H : \; W(\omega^\#,\omega^\#) = 0\} \\ W^+ & = & \{(M^{2m},J,g) \in \mathcal{A}_H : \; W(\omega^\#,\omega^\#) \geq 0 \, , \; W(\omega^\#,\omega^\#) \neq 0\} \, , \end{array}$$

respectively. These three subclasses are pairwise disjoint, and $\mathcal{A}_H = W^- \cup W^0 \cup W^+$. Furthermore, if (M^{2m},J,g) is in one of these three subclasses, then its entire conformal class is in the same subclass.

Notice that W^0 contains more than conformally flat manifolds. It includes, for example, all Kähler Ricci-flat non-flat manifolds, and all almost Hermitian anti-self-dual 4-manifolds.

Then we have the following:

Theorem 11. *Let $(M^{2m},J,g) \in \mathcal{A}_H$. Then:*

a) *If $(M^{2m},J,g) \in W^-$, we have that $\lambda^J(M,[g]) < 2m\omega_{2m}^{1/m}$ and the almost complex Yamabe problem can be solved by a minimizer of the functional $\lambda_N^J(\varphi)$.*
b) *If $(M^{2m},J,g) \in W^0$, the almost complex Yamabe problem can be solved by a minimizer of $\lambda^J(\varphi)$. In fact, $\lambda^J(M,[g]) \leq 2m\omega_{2m}^{1/m}$, and the equality is achieved if and only if (M^{2m},g) is conformal to the 6 dimensional sphere $\mathbb{S}^6$ with its standard metric. In that case, J is an almost complex structure on $\mathbb{S}^6$ and we have that $s_g^J = 6$.*
c) *If $(M^{2m},J,g) \in W^+$, we have the strict inequality $\lambda^J(M,[g]) < 2m\omega_{2m}^{1/m}$, and the almost complex Yamabe problem can be solved by a minimizer of the functional $\lambda_N^J(\varphi)$.*

The statement of the theorem above parallels the different arguments used to prove the result for manifolds in W^-, W^0 and W^+, respectively. We give further details of the proof only for almost complex manifolds in W^+. This is the case where in order to achieve our goal, we used previously developed techniques in the most innovative fashion.

Let recall that the functions

$$u_\alpha(x) = \left(\frac{|x|^2+\alpha^2}{\alpha}\right)^{(2-n)/2} \tag{16}$$

realize the optimal Sobolev constant in $\mathbb{R}^n$. That is to say, we have that

$$a\|\nabla u_\alpha\|^2_{L^2(\mathbb{R}^n)} = \Lambda \|u_\alpha\|^2_{L^N(\mathbb{R}^n)}\,, \tag{17}$$

where $a = 4(n-1)/(n-2)$ and $\Lambda = n(n-1)\omega_n^{2/n}$. We also recall the following theorem of R. Graham (see [16]):

Theorem 12. *Let p be a point in M, $k \geq 0$, and let T be a symmetric $(k+2)$-tensor on T_pM. There exists a unique homogeneous polynomial f of degree $k+2$ in g-normal coordinates such that the metric $\tilde{g} = e^{2f}g$ satisfies*

$$\mathrm{Sym}(\tilde{\nabla}^k \tilde{r}_{ij}(p)) = T\,.$$

Let $(M^{2m}, J, g) \in W^+$. In order to prove that a minimizer of (11) exists, we just need to show that the conformal invariant $\lambda^J(M,[g])$ is strictly less than the universal bound $2m\omega_{2m}^{\frac{1}{m}}$.

Let us fix normal coordinates in a neighborhood of a point p to be specified later. We use (16), and consider test functions of the form $\varphi = \eta u_\alpha$, for η a radial cut-off function supported in the ball $B_{2\varepsilon}$, that is identically 1 in B_ε. By (11), we have that

$$\begin{aligned}
\lambda^J_N(\varphi)\,\|\varphi\|^2_N &= \frac{2}{m-1}\int_M |d\varphi|^2\,d\mu_g + \int_M s^J\varphi^2 d\mu_g \\
&= \frac{2}{m-1}\int_{B_{2\varepsilon}} (\eta^2|du_\alpha|^2 + 2\eta u_\alpha\langle d\eta, du_\alpha\rangle + u_\alpha^2|d\eta|^2)d\mu_g \\
&\quad + \int_{B_{2\varepsilon}} s^J\eta^2u_\alpha^2 d\mu_g\,.
\end{aligned}$$

Since in normal coordinates we have the expansion

$$\det g_{ij} = 1 - \frac{1}{3}r_{ij}x^ix^j - \frac{1}{6}r_{ij,k}x^ix^jx^k + O(|x|^4)\,,$$

by (17) we conclude that

$$\begin{aligned}\int_M \eta^2\, |du_\alpha|^2\, d\mu_g &= \int_{B_\varepsilon} |du_\alpha|^2\, d\mu_g + \int_{B_{2\varepsilon}-B_\varepsilon} \eta^2\, |du_\alpha|^2\, d\mu_g \\ &\leq m(m-1)\omega_{2m}^{\frac{1}{m}} \|\varphi\|^2_{L^N} - \int_{B_\varepsilon} |du_\alpha|^2\, \frac{r_{ij}x^i x^j}{3}\, dx + E(\varepsilon,\alpha)\,,\end{aligned}$$

where

$$E(\varepsilon,\alpha) = \int_{B_\varepsilon} |du_\alpha|^2\, O(|x|^3)\, dx + \int_{B_{2\varepsilon}-B_\varepsilon} \eta^2\, |du_\alpha|^2\, d\mu_g\,. \tag{18}$$

Thus,

$$\lambda_N^J(\varphi)\, \|\varphi\|^2_N \leq 2m\omega_{2m}^{\frac{1}{m}} \|\varphi\|^2_{L^N} + S(\varepsilon,\alpha) + \tilde{E}(\varepsilon,\alpha)\,, \tag{19}$$

where S is given by

$$S(\varepsilon,\alpha) = -\frac{2}{3(m-1)} \int_{B_\varepsilon} |du_\alpha|^2\, r_{ij}x^i x^j\, dx + \int_{B_\varepsilon} s^J u_\alpha^2 dx\,, \tag{20}$$

while $\tilde{E}$ is given by
(21)

$$\begin{aligned}\tilde{E} &= \frac{2}{m-1} \int_{B_{2\varepsilon}-B_\varepsilon} (2\eta u_\alpha \langle d\eta, du_\alpha\rangle + u_\alpha^2 |d\eta|^2) d\mu_g + \int_{B_{2\varepsilon}-B_\varepsilon} s^J \eta^2 u_\alpha^2 d\mu_g \\ &\qquad + \frac{2}{m-1} E(\varepsilon,\alpha)\,.\end{aligned}$$

We prove that choices of ε and α can be made such that $\tilde{E}$ is negligible in comparison with S, and furthermore, use Theorem 12 to make a choice of normal coordinates so that the contribution of S in (19) is negative. This will imply that the infimum of $\lambda_N^J(\varphi)$ over the unit sphere in $L^N(M)$ is strictly less than the universal bound $2m\omega_{2m}^{\frac{1}{m}}$, that in turn implies the solvability of the almost complex Yamabe problem.

In the analysis of $S(\varepsilon,\alpha)$, we carry out the integration in geodesic polar coordinates centered at the point p. For simplicity of notation, we set $n = 2m$ and $r = |x|$. Then

$$\begin{aligned} S(\varepsilon, \alpha) &= \omega_{n-1} \int_0^{\varepsilon} r^{n-1} \left(-\frac{r^2}{3m(m-1)} s\, |du_\alpha|^2 + s^J u_\alpha^2 \right) dr \\ &= \omega_{n-1} \int_0^{\varepsilon} r^{n-1} \left(\frac{r^2+\alpha^2}{\alpha} \right)^{-n} \left[\frac{r^4}{\alpha^2} \left(s^J - \frac{4(m-1)}{3m} s \right) + \right. \\ &\qquad \left. 2s^J r^2 + s^J \alpha^2 \right] dr \\ &= \omega_{n-1} \alpha^2 \int_0^{\frac{\varepsilon}{\alpha}} \sigma^{n-1} (1+\sigma^2)^{-n} \left[\sigma^4 \left(s^J - \frac{4(m-1)}{3m} s \right) + \right. \\ &\qquad \left. 2s^J \sigma^2 + s^J \right] d\sigma \, , \end{aligned}$$

which follows from (20) by simple manipulations plus the fact that $x^i x^j$ integrates over the unit sphere to $\delta^{ij}\omega_{n-1}/n$.

Let $\overline{W}$ be the symmetrization of the tensor $-\sum_i W(e_i, \cdot\,, J\cdot, Je_i)$. Its trace is equal to the function $W(\omega, \omega)$. We take as p any point of our manifold where this function is strictly positive, for instance, a maximum, and for convenience, we denote its value at the said point by $W(p)$. We then apply Theorem 12 to choose normal coordinates and a metric $\tilde{g} = e^{2f} g$ about p, such that

$$\tilde{r}_{ij}(p) = T \stackrel{def}{=} \ell \overline{W} \, ,$$

where ℓ is a positive parameter whose value shall be specified later on. Since f is a homogeneous polynomial of degree 2, we have that $T = \tilde{T}$ at p. From now on, we replace g by $\tilde{g}$, and continue with the argument started above.

By our choice of conformal normal coordinates, the value of the scalar curvature at p is given by $s(p) = \ell W(p)$ and, by (15),

$$s^J(p) = \frac{1}{2m-1} s(p) + 2W(p) = \frac{4m-2+\ell}{2m-1} W(p) \, ,$$

so we have that

$$\begin{aligned} \sigma^4 \left(s^J - \frac{4(m-1)}{3m} s \right) + 2s^J \sigma^2 + s^J &= W(p) \frac{4m-2+\ell}{2m-1} \left(1 + 2\sigma^2 + \right. \\ &\qquad \left. \left(1 - \tfrac{4(m-1)(2m-1)}{3m(4m-2+\ell)} \ell \right) \sigma^4 \right) . \end{aligned}$$

Given $\varepsilon > 0$, let us define

$$F^{\varepsilon}(\alpha, l) = \int_0^{\frac{\varepsilon}{\alpha}} \sigma^{n-1} (\sigma^2+1)^{-n} \left(1 + 2\sigma^2 + \left(1 - \frac{4(m-1)(2m-1)}{3m(4m-2+\ell)} \ell \right) \sigma^4 \right) d\sigma \, .$$

Then we have that

$$S(\varepsilon,\alpha) = \omega_{n-1}\alpha^2\frac{4m-2+\ell}{2m-1}F^{\varepsilon}(\alpha,\ell)\,,$$

and its asymptotic behaviour is manifestly dependent upon the behaviour of the improper integral

$$\int_0^{\infty}\sigma^{n-1}(\sigma^2{+}1)^{-n}\Big(1{+}2\sigma^2{+}\Big(1{-}\tfrac{4(m-1)(2m-1)}{3m}\Big)\sigma^4\Big)\,d\sigma = \lim_{\alpha\to 0^+}\lim_{\ell\to\infty}F^{\varepsilon}(\alpha,\ell)\,.$$

Lemma 13. *Let*

$$I_n(a) = \int_0^{a}\sigma^{n-1}(\sigma^2+1)^{-n}\left(1+2\sigma^2+\left(1-\tfrac{4(m-1)(2m-1)}{3m}\right)\sigma^4\right)d\sigma\,.$$

Then,

a) *For $n=2m>4$, $I_n(a)$ converges to a negative value as $a\to\infty$.*
b) *For $n=2m=4$, we have that*

$$\lim_{a\to\infty}\frac{I_4(a)}{\log a} = -1\,.$$

The proof of this Lemma is elementary. We use it to finish our argument.

Indeed, since $F^{\varepsilon}(\alpha,\ell)$ is a continuous function of (α,ℓ), we can choose α sufficiently small in comparison with ε and ℓ large enough so that, for some positive constant $C(m)$, we have that

$$S(\varepsilon,\alpha)\le\begin{cases}-C(m)W(p)\alpha^2 & \text{if } m>2\\ -C(m)W(p)\alpha^2\log(1/\alpha) & \text{if } m=2\end{cases}$$

The error term $\tilde{E}(\varepsilon,\alpha)$ in (21) may be estimated by the same arguments in [16] (see pages 50-51). In fact, we easily conclude that the first two integrals in the right side of (21) are of the order $O(\alpha^{2m-2})$, negligible in comparison with $S(\varepsilon,\alpha)$. This leaves us with the task of estimating the contribution to $\tilde{E}(\varepsilon,\alpha)$ given by $E(\varepsilon,\alpha)$. A rerun of the arguments above show that the integral corresponding to the $O(r^3)$ term in (18) is of the order $O(\alpha^3)$, while the integral over the domain $B_{2\varepsilon}-B_{\varepsilon}$ is of the order $O(\alpha^{2m-2})$ for values of α sufficiently small in comparison with ε.

Combining our estimates, we see that

$$\lambda_N^J(\varphi)\,\|\varphi\|_N^2\le\begin{cases}2m\omega_{2m}^{\frac{1}{m}}\|\varphi\|_{L^N}^2 - CW(p)\alpha^2+o(\alpha^2) & \text{if } m>2\\ 2m\omega_{2m}^{\frac{1}{m}}\|\varphi\|_{L^N}^2 - CW(p)\alpha^2\log(1/\alpha)+O(\alpha^2) & \text{if } m=2\end{cases}$$

and therefore, by selecting α sufficiently small, we conclude that

$$\lambda^J(M,[g]) = \inf \lambda^J_N(\varphi) < 2m\omega_{2m}^{\frac{1}{m}},$$

as desired.

Remark 14. For any (M,J,g) such that $\lambda^J(M,[g]) \leq 0$, the minimizer of $\lambda^J_N(\varphi)$ is unique.

In the positive case, the functional (11) may have other critical points, fact illustrated, for instance, by $\mathbb{CP}^n$ endowed with the Fubini-Study metric g. Indeed, since this is a Kähler metric of constant scalar curvature, it is a critical point of the functional $\lambda^J_N(\varphi)$. But this metric does not realize its infimum because its critical value is larger that the universal upper bound for the conformal invariant $\lambda^J(\mathbb{CP}^n,[g])$. The infimum is thus achieved by another critical point.

More explicitly, for the Fubini-Study metric we have that $s_g = 4n(n+1)$ and $\mu_g = \pi^n/n!$, so the value of λ^J_N for this metric is $4n(n+1)\pi/(n!)^{\frac{1}{n}}$. This number is strictly greater than $2n\omega_{2n}^{\frac{1}{n}} = 4n\pi 2^{\frac{1}{n}}/((2n-1)(2n-3)\cdots 3\cdot 1)^{\frac{1}{n}}$. Our Theorem 11 says that in the conformal class of g there exists another metric $\tilde{g}$ of volume one, whose J-scalar curvature $\tilde{s}^J_{\tilde{g}}$ is strictly less than $2n\omega_{2n}^{\frac{1}{n}} = 4n\pi 2^{\frac{1}{n}}/((2n-1)(2n-3)\cdots 3\cdot 1)^{\frac{1}{n}}$.

Exercise 4. (open problem) Describe the minimizer of the almost complex Yamabe functional for $\mathbb{CP}^n$ with its standard structure.

3.2. **The sigma constant of an almost complex manifold.** The existence of a universal upper bound for the conformal invariant $\lambda^J(M,[g])$ of any almost Hermitian manifold (M,J,g) leads naturally to an invariant of the pair (M,J). This invariant is analogous to the sigma constant that arises when studying the usual Yamabe problem. Accordingly, we call it the sigma constant of (M,J):

$$\sigma(M,J) = \sup_{[g]} \lambda^J(M,[g]) \leq 2m\omega_{2m}^{1/m}. \tag{22}$$

The analysis of this number makes the 6-dimensional sphere somewhat special. In fact, based on the work of R. Schoen [22] (that is the key ingredient in the proof of part (b) of Theorem 11), we can obtain the following

Theorem 15. *Let $(M,J,\tilde{g})$ be a six-dimensional almost complex manifold such that the constant $\sigma(M,J)$ is achieved by some J-invariant* *metric $\bar{g}$ and equals the universal upper bound $6\omega_6^{1/3}$. Then $(M,\bar{g})$ is conformal to $\mathbb{S}^6$ with its standard metric g, and J is any almost complex structure that leaves g invariant.*

Exercise 5. (open problem) Consider the Calabi-Eckmann manifolds $(\mathbb{S}^{2n+1}\times\mathbb{S}^{2m+1},J)$ with the metrics $g_\varphi=\langle\,\cdot\,,\,\cdot\,\rangle_\varphi$ introduced in §2.3.3. For constant φ, we obtain a family of conformal classes $[g_\varphi]$ parametrized by $\mathbb{R}$. Show that

$$F(\varphi)=\frac{\int_M \tilde{s}^J_{g_\varphi}d\mu_{g_\varphi}}{\left(\int_M d\mu_{g_\varphi}\right)^{2/N}}=(A+2(n+m)-Ae^{2\varphi})\left(\frac{4\pi^{n+m+2}e^{2\varphi}}{n!m!}\right)^{\frac{1}{n+m+1}}\,,$$

where $A=2n(2n+1)+2m(2m+1)$. This is non-positive for

$$\varphi\geq\frac{1}{2}\log\left(1+\frac{2(n+m)}{A}\right).$$

By the uniqueness of the minimizer in the negative case, conclude that

$$\sigma(\mathbb{S}^{2n+1}\times\mathbb{S}^{2m+1},J)\geq 0\,.$$

On the other hand, the maximum of $F(\varphi)$ is achieved at $A(n+m+1)e^{2\varphi}=A+2(n+m)$, but its value is strictly greater than the universal upper bound, given by

$$4(n+m+1)\pi\left(\frac{2}{(2(n+m+1)-1)(2(n+m+1)-3)\cdots3\cdot1}\right)^{\frac{1}{n+m+1}}.$$

What is $\sigma(\mathbb{S}^{2n+1}\times\mathbb{S}^{2m+1},J)$? Is it a positive number?

3.3. **The total J-scalar curvature.** The total J-scalar curvature of a J invariant metric g is defined by

$$s^J_{tot}=\int_M s^J d\mu_g\,.$$

The definition of the sigma constant of (M,J) follows a min-max path:

$$\sigma(M,J)=\sup_{[g]}\inf_{\tilde{g}\in[g]}\frac{\displaystyle\int_M \tilde{s}^J_{\tilde{g}}d\mu_{\tilde{g}}}{\left(\displaystyle\int_M d\mu_{\tilde{g}}\right)^{2/N}}\,.$$

Thus, if this constant is achieved by some J-invariant metric g, such a metric must be a critical point of the total J-scalar curvature functional over the space of J-invariant metrics of fixed volume. We attempt to understand what equation is satisfied by such a metric by computing the variation of s^J_{tot} along a path $g(t)$ of J-invariant metrics with $g(0)=g$. The result shows how intertwined are this critical metric and the almost complex structure.

Let $\{e_1, \ldots, e_{2n}\}$ be an orthonormal frame. In terms of the components of the Riemann curvature tensor and of J, the components of the J-Ricci tensor are given by

$$r^J_{ij} = -J^k_{\ j} R^s_{\ kli} J^l_{\ s} \tag{23}$$

from where it follows readily that

$$s^J = -g^{ij} J^k_{\ j} R^s_{\ kli} J^l_{\ s} = -J^{ki} R^s_{\ kli} J^l_{\ s} = -J^{ik} R_{islk} J^{sl}\,,$$

as we saw in (4). We use this expression to compute the desired variation.

If $h_{ij} = \dot{g}_{ij}$, we have that [18]

$$\dot{R}_{ijkl} = \nabla_k Z_{jil} - \nabla_l Z_{jik} + h_{ip} R^p_{\ jkl}\,,$$

where

$$Z_{ijk} = \frac{1}{2}(\nabla_i h_{jk} + \nabla_k h_{ij} - \nabla_j h_{ik})\,.$$

Consequently,

$$\dot{s}^J = -J^{it}J^{lm}(\nabla_m Z_{lit} - \nabla_t Z_{lim} + h_{ip} R^p_{\ lmt}) + h^{tp} J^i_{\ p} R_{ilmt} J^{lm} + h^{mp} J^l_{\ p} R_{ilmt} J^{it}\,.$$

By the symmetries of R, we have $J^i_{\ p} R_{ilmt} J^{lm} = -J^i_{\ p} R_{limt} J^{lm} = -r^J_{tp}$. Similarly, we see that $J^l_{\ p} R_{ilmt} J^{it} = J^l_{\ p} R_{iltm} J^{ti} = -r^J_{mp}$, and that $-J^{it}J^{lm} h_{ip} R^p_{\ lmt} = h^{ip} R_{plmt} J^t_{\ i} J^{lm} = -h^{ip} R_{mtlp} J^t_{\ i} J^{lm} = h^{ip} r^J_{ip} = (h, r^J)$, and so the balance of the three terms in the expression above that involve R is $-(h, r^J)$.

Since

$$\begin{aligned} \nabla_m Z_{lit} &= \frac{1}{2}\nabla_m(\nabla_l h_{it} + \nabla_t h_{li} - \nabla_i h_{lt})\,, \\ \nabla_t Z_{lim} &= \frac{1}{2}\nabla_t(\nabla_l h_{im} + \nabla_m h_{li} - \nabla_i h_{lm})\,, \end{aligned}$$

we obtain that

$$\begin{aligned} \nabla_m Z_{lit} - \nabla_t Z_{lim} &= \frac{1}{2}(\nabla_m \nabla_t - \nabla_t \nabla_m) h_{li} + \frac{1}{2}\nabla_m \nabla_l h_{it} + \frac{1}{2}\nabla_t \nabla_i h_{lm} \\ &\quad - \frac{1}{2}\nabla_m \nabla_i h_{lt} - \frac{1}{2}\nabla_t \nabla_l h_{im}\,. \end{aligned}$$

By the symmetries of J and h, we conclude that the contractions with $J^{it}J^{lm}$ of the second and third term in the right side of the expression above are zero. On the other hand, the last two have the same contraction with $J^{it}J^{lm}$. By the first Ricci identity, these observations and the ones above, we obtain that

$$\dot{s}^J = -\frac{1}{2}J^{it}J^{lm}(h_{ki}R^k{}_{lmt} + h_{lk}R^k{}_{imt}) + J^{it}J^{lm}\nabla_m\nabla_i h_{lt} - (h, r^J)\,.$$

By (23), we see that the contractions above involving the Riemann curvature tensor produce (h, r^J) in one case and $-(h, r^J)$ in the other, so we end up with

$$\dot{s}^J = J^{it}J^{lm}\nabla_m\nabla_i h_{lt} - (h, r^J)\,.$$

Now, the term $J^{it}J^{lm}\nabla_m\nabla_i h_{lt}$ reduces to a reasonably simple expression modulo divergences:

$$\begin{aligned} J^{it}J^{lm}\nabla_m\nabla_i h_{lt} &= \nabla_m\{J^{it}J^{lm}\nabla_i h_{lt}\} - \nabla_i\{(\nabla_m(J^{it}J^{lm}))h_{lt}\} + \\ &\qquad \{\nabla_i\nabla_m(J^{it}J^{lm})\}h_{lt}\,. \end{aligned}$$

But we have:

$$\begin{aligned} \nabla_i\nabla_m(J^{it}J^{lm}) &= (\nabla_i\nabla_m J^{it})J^{lm} + (\nabla_i J^{it})(\nabla_m J^{lm}) + (\nabla_m J^{it})(\nabla_i J^{lm}) \\ &\qquad + J^{it}(\nabla_i\nabla_m J^{lm})\,. \end{aligned}$$

Using the first Ricci identity once again, we obtain that

$$\nabla_i\nabla_m J^{it} = \nabla_m\nabla_i J^{it} - J^{is}R^t{}_{sim} - J^t{}_p R^p{}_{rim} g^{ir}\,,$$

and therefore,

$$(\nabla_i\nabla_m J^{it})J^{lm}h_{lt} = J^{lm}(\nabla_m\nabla_i J^{it})h_{lt} - (r^J, h) + (r \circ J, h)\,.$$

We now calculate the variation of s^J_{tot} by differentiating under the integral sign both, s^J and $d\mu_g$. We arrive at the expression

$$\begin{aligned} \dot{s}^J_{tot} &= \int_M \left(\frac{s^J}{2}g + r \circ J - 2r^J, h\right) d\mu_g + \int_M h_{lt}J^{lm}(\nabla_m\nabla_i J^{it})d\mu_g + \\ &\quad \int_M h_{lt}\Big((\nabla_i J^{it})(\nabla_m J^{lm}) + (\nabla_m J^{it})(\nabla_i J^{lm}) + J^{it}(\nabla_i\nabla_m J^{lm})\Big)d\mu_g\,. \end{aligned}$$

As a partial test, we see that if (M, J, g) is Kählerian, then the expression above reduces to the well-known formula for the variation of the total scalar curvature $\dot{s}_{tot} = \int \left(\frac{s}{2}g - r, h\right) d\mu_g$.

We can also compute this variation along a path $(g(t), J(t))$, where $J(t)$ itself is a path of almost complex structures with $J(0) = J$, and $g(t)$ is a path of metrics with $g(0) = g$, and $g(t)$ invariant under $J(t)$, for all t in a neighborhood of 0. If $K = \dot{J}(0)$, the extra contribution arising from the variation of J is given by $-2\int (K, \rho^J) d\mu_g$, where ρ^J is the J-Ricci form of (M, J, g).

Let us recall that the co-differential δ with respect to g is defined on k-multilinear forms by

$$(\delta A)(X_1, \ldots, X_{k-1}) = -\sum_{i=1}^{n} (\nabla_{e_i} A)(e_i, X_1, \ldots, X_{k-1}) \,, \tag{24}$$

where $\{e_1, \ldots, e_n\}$ denotes any local orthonormal frame of TM. We have the formula

$$\langle \delta A, B \rangle = \langle A, \nabla B \rangle \,,$$

for any section A of $\otimes^{k+1} T^*M$ and any section B of $\otimes^k T^*M$. Hence, if we interpret the result above invariantly, we arrive at the following

Theorem 16. *Suppose (M, J, g) is an almost Hermitian manifold with fundamental form ω. Let α be the symmetric 2-tensor*

$$\alpha(X, Y) = \nabla_{JX}(\delta\omega)Y + \nabla_{JY}(\delta\omega)X \,,$$

and β be the tensor

$$\beta(X, Y) = \mathrm{trace}\, L \to (\nabla_{(\nabla_L J)X} J)Y \,.$$

Then, the variation of s^J_{tot} at (g, J) in the direction of (h, K) is given by

$$\dot{s}^J_{tot} = \int_M \left(\frac{s^J}{2} g + r \circ J - 2r^J + \alpha - \beta - \delta\omega \otimes \delta\omega, h \right) d\mu_g - \int_M 2(K, \rho^J) d\mu_g \,.$$

Exercise 6. Consider the sphere $\mathbb{S}^6$ with the almost complex structure J derived from the octonion multiplication. Use the variational formula above to show that the standard metric on $\mathbb{S}^6$ is a critical point of s^J_{tot}, when defined over the space of J-invariant metrics on $\mathbb{S}^6$ of fixed volume.

4. Extremal metrics

Let (M, J) be a complex manifold of Kähler type, of complex dimension n. For convenience, let us heretofore identify Kähler metrics with their corresponding Kähler forms. We denote by $\mathfrak{M}$ the space of Kähler metrics on (M, J).

Given a positive class $\Omega \in H^{1,1}(M, \mathbb{R}) \subset H^2(M, \mathbb{R})$, we let $\mathfrak{M}_\Omega$ be the space of of Kähler metrics whose Kähler forms represent Ω, $\mathfrak{M}_\Omega = \{\omega \in \mathfrak{M} : [\omega] = \Omega\}$. In this chapter we discuss a program, proposed by Calabi, to find a canonical representative of $\mathfrak{M}_\Omega$. We also discuss some results concerned with the partial completion of this program.

4.1. **A bit of differential analysis of Kähler metrics.** Let us recall the space $\Lambda^{p,q}M$ of (p,q)-forms on a complex manifold (M,J). Then we know that $d = \partial + \overline{\partial}$, where ∂ and $\overline{\partial}$ act in the manner described in (1). Given a Hermitian metric g, we may consider the adjoints $\partial^* : C^\infty(\Lambda^{p+1,q}M) \to C^\infty(\Lambda^{p,q}M)$ and $\overline{\partial}^* : C^\infty(\Lambda^{p,q+1}M) \to C^\infty(\Lambda^{p,q}M)$. We thus obtain three second order operators,

$$\Delta = d\delta + \delta d\,, \quad \Delta_\partial = \partial\partial^* + \partial^*\partial\,, \quad \Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}\,,$$

the usual Laplacian and the ∂ and $\overline{\partial}$ Laplacians, respectively.

If the metric g is of Kähler type, these operators are related by $\Delta = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$, and we have the identities

$$\overline{\partial}^*\partial + \partial\overline{\partial}^* = 0\,, \quad \partial^*\overline{\partial} + \overline{\partial}\partial^* = 0\,.$$

Moreover, let ω be the Kähler form of g and $L : \Lambda^{p,q}M \to \Lambda^{p+1,q+1}M$ be the operator defined by $L(\alpha) = \alpha \wedge \omega$. Let us set $\Lambda = L^*$. Then we have the Kähler identities

$$\partial^* = i[\Lambda, \overline{\partial}]\,, \quad \overline{\partial}^* = -i[\Lambda, \partial]\,.$$

Hodge theorem can be stated by saying that there exists a unique elliptic pseudo-differential operator $G_d : \Lambda^p(M) \to \Lambda^p(M)$ such that if $H_d : \Lambda^p(M) \to \mathcal{H}^p_d(M)$ is the L^2-orthogonal projection onto the space of harmonic p-forms, then

$$\mathbb{1} = H_d + \Delta G_d\,.$$

It can be equivalently restated at the level of $\Lambda^{p,q}(M)$ for either Δ_∂ or $\Delta_{\overline{\partial}}$.

On a manifold (M,J) of Kähler type, the space of Kähler metrics $\mathfrak{M}_\Omega$ that represent a fixed cohomology class Ω is an affine space modeled on $C^\infty(M)$. Indeed, we have the following

Lemma 17. *Let (M,J,g) be a compact Kähler manifold. Assume that γ is a real form of type $(1,1)$ such that $\gamma = d\alpha$ for some real 1-form α. Then there exists a real-valued function f on M such that*

$$\gamma = i\partial\overline{\partial}f\,.$$

This result can be extended to exact real forms γ of type (p,p). The assertion in that case is that there exists a real form β of type $(p-1,p-1)$ such that $\gamma = i\partial\overline{\partial}\beta$.

Proof. Let us apply Hodge's decomposition theorem to the 1-form α. Then we have that

$$\alpha = \alpha_H + \Delta\beta\,,$$

for certain harmonic 1-form α_H and certain 1-form β. Since α is real, we may find a 1-form λ of type $(1,0)$ such that $\alpha = \lambda + \bar{\lambda}$. But λ itself can be written in terms of its Hodge's decomposition as $\lambda_H + \Delta\beta_\lambda$. Hence,

$$\alpha_H = \lambda_H + \bar{\lambda}_H \, , \quad \beta = \beta_\lambda + \bar{\beta}_\lambda \, .$$

The form $\gamma = d\alpha$ is of type $(1,1)$. Hence, $\bar{\partial}\Delta\bar{\beta}_\lambda = 0 = 2\bar{\partial}\bar{\partial}^*\bar{\partial}\bar{\beta}_\lambda$, and we obtain that

$$0 = \langle 2\bar{\partial}\bar{\partial}^*\bar{\partial}\bar{\beta}_\lambda, \bar{\partial}\beta_\lambda\rangle = 2\|\bar{\partial}^*\bar{\partial}\bar{\beta}_\lambda\|^2 \, .$$

Thus, $\bar{\partial}^*\bar{\partial}\bar{\beta}_\lambda = 0$, and

$$\bar{\lambda} = \bar{\lambda}_H + \Delta\bar{\beta}_\lambda = \bar{\lambda}_H + 2\bar{\partial}\bar{\partial}^*\bar{\beta}_\lambda \, .$$

We may obtain similarly that $\partial^*\partial\beta_\lambda = 0$. Therefore,

$$\gamma = d\alpha = d(\alpha_H + \Delta(\beta_\gamma + \bar{\beta}_\gamma)) = 2(\partial\bar{\partial}\bar{\partial}^*\bar{\beta}_\gamma + \bar{\partial}\partial\partial^*\beta_\gamma) = i\partial\overline{\partial}f \, ,$$

where f is the real-valued function defined by $if = 2(\overline{\partial}^*\bar{\beta}_\gamma - \partial^*\beta_\gamma)$. $\square$

Let us fix an element ω_0 of $\mathfrak{M}_\Omega$. Any other form ω in $\mathfrak{M}_\Omega$ is cohomologous to ω_0. Thus, $\omega - \omega_0$ is a real exact (1,1) form, and by Lemma 17, $\omega - \omega_0 = i\partial\overline{\partial}\varphi$, for some $\varphi \in C^\infty(M)$. In other words, at the expense of introducing an artificial origin ω_0, we have that

$$\mathfrak{M}_\Omega = \{\omega = \omega_0 + i\partial\overline{\partial}\varphi > 0\} \, , \tag{25}$$

showing that this space of metrics is parametrized by a suitable open neighborhood of the zero function in $C^\infty(M)$.

By (6) and Stokes' theorem, the volume of any metric in $\mathfrak{M}_\Omega$ is the constant $\Omega^n/n!$. Notice that any Kähler form ω can be represented locally as $\omega = i\partial\overline{\partial}f$, for some (locally defined) function f. Since the volume of M relative to ω is given by $[\omega]^n/n!$, there could not be a globally defined function f so that this representation would hold everywhere.

Proposition 18. *Let (M,J) be a manifold of Kähler type, and g be a Káhler metric on it, with Kähler form ω. Let $\omega_t = \omega + t\alpha$ be a path of Kähler forms defined on a suitable neighborhood of $t = 0$. Then we have the expansions*

$$\begin{array}{rcl} d\mu_t & = & (1 + t(\omega,\alpha))d\mu + O(t^2) \, , \\ \rho_t & = & \rho - it\partial\overline{\partial}(\omega,\alpha) + O(t^2) \, , \\ s_t & = & s + t(\Delta(\omega,\alpha) - 2(\rho,\alpha)) + O(t^2) \, , \end{array}$$

for the volume form, Ricci form, and scalar curvature of ω_t, respectively.

Proof. The volume form of any Kähler metric g is given by

$$d\mu_\omega = \frac{\omega^n}{n!}\,,$$

where ω is the Kähler form. Consider the curve of metrics g_t corresponding to the Kähler forms $\omega + t\alpha$. Then

$$d\mu_t = d\mu + t\alpha \wedge \frac{\omega^{n-1}}{(n-1)!} + O(t^2)\,.$$

If $*$ denotes the Hodge star operator, we have that

$$*\frac{\omega^{n-1}}{(n-1)!} = \omega\,.$$

Therefore,

$$d\mu_t = (1 + t(\omega,\alpha))d\mu + O(t^2)$$

as stated.

By (7) and the expansion above for the volume, we may easily obtain the expansion for the Ricci form:

$$\rho_t = \rho - it\partial\overline{\partial}(\omega,\alpha) + O(t^2)\,.$$

If we now take the trace of this expression, we obtain the expansion for the scalar curvature:

$$s_t = s + t(\Delta(\omega,\alpha) - 2(\rho,\alpha)) + O(t^2)\,.$$

□

Corollary 19. *Suppose that $\omega_t = \omega + it\partial\overline{\partial}\varphi$ is a path in $\mathfrak{M}_\Omega$. Then*

$$\begin{aligned} d\mu_t &= \left(1 - \tfrac{t}{2}\Delta\varphi\right) d\mu + O(t^2)\,, \\ \rho_t &= \rho + i\tfrac{t}{2}\partial\overline{\partial}\Delta\varphi + O(t^2)\,, \\ s_t &= s - t\left(\tfrac{1}{2}\Delta^2\varphi + 2(\rho, i\partial\overline{\partial}\varphi)\right) + O(t^2)\,. \end{aligned}$$

Remark 20. We may reinterpret more geometrically the operator $\frac{1}{2}\Delta^2\varphi + 2(\rho, i\partial\overline{\partial}\varphi)$ that appears in the expression above. Indeed, let $\alpha^{\#}$ be the vector field that corresponds to the 1-form α via the metric. Then we have the Bochner formula

$$\Delta\alpha = \delta\nabla\alpha + r(\alpha^{\#})\,.$$

Moreover, if $\nabla^+\alpha$ and $\nabla^-\alpha$ denote the J-invariant and J-anti-invariant parts of the covariant derivative $\nabla\alpha$, respectively, we have that

$$\nabla^-\alpha = -\frac{1}{2}\omega((\mathcal{L}_{\alpha^{\#}}J)\,\cdot\,,\,\cdot\,)$$

and that

$$\delta\nabla^{+}\alpha - \delta\nabla^{-}\alpha = r(\alpha^{\#})\,.$$

In particular, a vector field X is holomorphic (that is to say, X preserves J) iff $\nabla^{-}X^{\flat} = 0$. Here, $X^{\flat}$ is the 1-form that the metric makes correspond to the vector field X. Bochner formula can then be rewritten as

$$\delta\nabla^{-}\alpha = \frac{1}{2}\Delta\alpha + J\alpha^{\#}\lrcorner\rho\,,$$

and from this we obtain that

$$\delta\delta\nabla^{-}df = \frac{1}{2}\Delta^{2}f + 2(\rho, i\partial\overline{\partial}f) + (df, ds)$$

for any real valued function f.

In particular, if f annihilates this operator, then $0 = \langle\delta\delta\nabla^{-}df, f\rangle = \langle\delta\nabla^{-}df, df\rangle = \langle\nabla^{-}df, \nabla^{-}df\rangle$. Thus, f is a real valued solution of $\delta\delta\nabla^{-}df = 0$ iff df is a form dual to a holomorphic vector field, and $J\nabla f$ is Killing.

In the differential analysis we carry out in below, we could proceed using this geometric approach. However, we shall do so in a slightly different —but equivalent— manner. □

4.2. **Holomorphic vector fields.** The notion of a holomorphic vector field play a significant rôle in Calabi's program for the determination of canonical Kähler metrics. We thus pause to discuss this concept in some detail.

Let (M, J) be a complex manifold. A vector field X is said to be real *holomorphic* (or, simply, holomorphic) if it preserves J, that is to say, if

$$\mathcal{L}_X J = 0\,.$$

Notice that this condition is equivalent to saying that

$$[X, JY] = J[X, Y] \text{ for all } Y\,. \tag{26}$$

Since M is compact, the Lie algebra of the automorphism group of (M, J) coincides with the algebra of all real holomorphic vector fields.

Proposition 21. *A vector field X is real holomorphic iff its component of type $(1, 0)$ is a holomorphic vector field.*

Proof. Since the result is local, we choose complex coordinates $z^j = x^j + iy^j$ and assume that $X = \sum(a^j\partial_{x^j} + b^j\partial_{y^j})$ is real holomorphic. We must prove that

$$\frac{1}{2}(X - iJX) = \sum_j (a^j + ib^j)\partial_{z^j}$$

is holomorphic, that is to say, that $\overline{\partial}(a^j + ib^j) = 0$, or what is the same, that the functions a^j, b^j satisfy the Cauchy-Riemann equations.

Let us apply condition (26) to $Y = \partial_{x^k}$. A simple calculation shows that left and right side are equal if and only if

$$\frac{\partial a^j}{\partial x^k} = \frac{\partial b^j}{\partial y^k}\,, \quad \frac{\partial a^j}{\partial y^k} = -\frac{\partial b^j}{\partial x^k}\,,$$

which are precisely the Cauchy-Riemann equations of $a^j + ib^j$ as a function of (x^k, y^k). The result follows. □

Thus, when referring to a holomorphic vector field, we may think of it as a real vector field satisfying (26) for all real vector Y, or a holomorphic section of the holomorphic bundle $T^{(1,0)}M$.

Let $X^{1,0}$ the (1,0)-component of the vector field X. Notice that the $(1,0)$-component of JX is given by $iX^{1,0}$. If the section $X^{1,0}$ is holomorphic, so will be the section $JX^{1,0} = iX^{1,0}$. Indeed, under the identification of a real vector field X with its (1,0)-component, this assertion corresponds to the fact that if X is a real holomorphic vector field then so is JX, a simple consequence of the fact that the Nijenhuis tensor of J vanishes. Either way, the space $\mathfrak{h}$ of holomorphic vector fields has the structure of a complex Lie algebra. This algebra is isomorphic to the Lie algebra of the group $\mathrm{Aut}(M)$ of automorphisms of (M, J).

In the presence of a Kähler structure on (M, J), we may consider the set of holomorphic vector fields given by a holomorphy potential. This set turns out to be an ideal of $\mathfrak{h}$.

Indeed, given a function $\varphi : M \mapsto \mathbb{C}$, let us define the vector field $\partial^{\#}\varphi$ by the identity

$$g(\partial^{\#}\varphi, \,\cdot\,) = \overline{\partial}\varphi\,.$$

This operator assigns to a function φ the (1,0) component of its gradient, a vector field that, generally speaking, is not holomorphic. For that to be the case, we would need to impose the condition $\bar{\partial}\partial^{\#} f = 0$, condition equivalent to the fourth-order equation

$$(\bar{\partial}\partial^{\#})^{*}\bar{\partial}\partial^{\#} f = 0\,, \tag{27}$$

because $\langle f, (\bar{\partial}\partial^{\#})^{*}\bar{\partial}\partial^{\#} f\rangle_{L^2} = \|\bar{\partial}\partial^{\#} f\|^2_{L^2}$.

Functions satisfying (27) form a finite dimensional vector space. For if we compute the adjoint relative to the metric g, we see that

$$
(28)\qquad \begin{array}{rcl}
(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}\varphi & := & \nabla_j\nabla^{\bar k}\nabla_{\bar k}\nabla^j\varphi \\
& = & \nabla_j\nabla^{\bar k}\nabla^j\nabla_{\bar k}\varphi \\
& = & \nabla_j(\nabla^j\nabla^{\bar k}\nabla_{\bar k}\varphi - R^{\bar k j\bar\ell}{}_{\bar k}\nabla_{\bar\ell}\varphi) \\
& = & \nabla_j(\nabla^j\nabla^{\bar k}\nabla_{\bar k}\psi + r^{j\bar\ell}\nabla_{\bar\ell}\varphi) \\
& = & \frac{1}{4}\Delta^2\psi + r^{j\bar\ell}\nabla_j\nabla_{\bar\ell}\varphi + (\nabla_j r^{j\bar\ell})\nabla_{\bar\ell}\varphi \\
& = & \frac{1}{4}\Delta^2\varphi + (\rho, i\partial\overline{\partial}\varphi) + \frac{1}{2}(\nabla^{\bar\ell}s)\nabla_{\bar\ell}\varphi \\
& = & \frac{1}{4}(\Delta^2\varphi + 4(\rho, i\partial\overline{\partial}\varphi) + 2(\partial s)\lrcorner\partial^{\#}\varphi)\,,
\end{array}
$$

where we have used the contracted Bianchi identity $\nabla_j s = 2\nabla^{\bar k}r_{j\bar k}$. Hence, the operator $(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}$ is elliptic, and its kernel is a space of finite dimension.

Let us recall that the Albanese torus of (M, J) is defined as $\mathrm{Alb}(M) = H^0(M,\Omega^1)^*/H_1(M,\mathbb{Z})$, where $[\gamma] \in H_1(M,\mathbb{Z})$ is thought as the linear functional given by integration over a cycle representative γ. If p_0 is a given point of M, the Albanese map

$$a : M \mapsto \mathrm{Alb}(M)$$

is defined by

$$a(p)(\alpha) = \int_{p_0}^{p}\alpha\,.$$

This map induces isomorphisms

$$\frac{H_1(M,\mathbb{Z})}{\text{torsion}} \xrightarrow{a_*} H_1(\mathrm{Alb}(M),\mathbb{Z})$$

and

$$H^0(\mathrm{Alb}(M),\Omega^1) \xrightarrow{a^*} H^0(M,\Omega^1)\,,$$

respectively.

We have the following

Theorem 22. *Let (M, J, g) be a compact Kähler manifold and let Ξ be a holomorphic vector field on M. The following statements are equivalent:*

i) *Ξ has a zero somewhere on M.*
ii) *Ξ is tangent to the fibers of the Albanese map $M \mapsto \mathrm{Alb}(M)$.*
iii) *There exists a smooth function $f : M \to \mathbb{C}$ such that $\Xi = \partial^{\#}_g f$. (The function f depends on the Kähler metric g.)*

In particular, the set $\mathfrak{h}_0(M)$ *of holomorphic vector fields with zeroes is a linear subspace of* $\mathfrak{h}(M)$*, and the dimension of the space of holomorphic vector fields of the form* $\partial_g^{\#} f$ *is the same for all Kähler metrics* g *on* (M, J).

Proof. Let β be a holomorphic one form. Then $\beta(\Xi)$ is a holomorphic function on M. By compactness, this function is constant, and since Ξ has a zero, the constant must be zero. Now, any holomorphic one form β on M is the pull-back under the Albanese map of a holomorphic one form $\tilde{\beta}$ on $\mathrm{Alb}(M)$. Therefore, $\beta(\Xi) = 0 = \tilde{\beta}(a_* \Xi)$, showing that Ξ is in the kernel of the derivative of the Albanese map. This proves that (i) implies (ii).

We next prove that (ii) implies (iii). Suppose that Ξ is a holomorphic vector field on M which is everywhere tangent to the fibers of the Albanese map. Since every holomorphic 1-form β on M is the pull-back of a 1-form on the Albanese torus, it follows that $\beta(\Xi) \equiv 0$ for every global holomorphic 1-form β on M. Consider the 1-form $\alpha_\Xi = i \Xi \lrcorner \omega$ that corresponds to Ξ via the metric. This form is of type (0,1), and

$$-i d\alpha_\Xi = d(\Xi \lrcorner \omega) + \Xi \lrcorner d\omega = L_\Xi \omega \,,$$

is of type (1,1) so we must have $\overline{\partial} \alpha_\Xi = 0$.

Given any harmonic (0,1) form η, we have the L^2 inner product $\langle \alpha_\Xi, \eta \rangle = 0$. Indeed, there exists a holomorphic 1-form β such that $\eta = \bar{\beta}$, and

$$\langle \alpha_\Xi, \eta \rangle = \int (\alpha_\Xi, \bar{\eta}) d\mu_g = \int (\alpha_\Xi, \beta) d\mu_g = \int \beta(\Xi) d\mu_g = 0$$

because $\beta(\Xi)$ is the constant zero. Thus, in the Hodge decomposition of α_Ξ, the harmonic component is zero, and we have that

$$\alpha_\Xi = \overline{\partial} f + \overline{\partial}^* \zeta$$

On the other hand, $\|\overline{\partial}^* \zeta\| = \langle \zeta, \overline{\partial}\overline{\partial}^* \zeta \rangle = \langle \zeta, \overline{\partial}(\alpha_\Xi - \overline{\partial} f) \rangle = 0$ because $\overline{\partial} \alpha_\Xi = 0$. Hence, $\alpha_\Xi = \overline{\partial} f$, and $\Xi = \partial^{\#} f$.

We finally show that (iii) implies (i). Let us assume that $\Xi = \partial^{\#} f$ is a holomorphic vector field. Observe that $\Xi \bar{f} = \langle \Xi, \Xi \rangle \geq 0$. Using the compactness of M, define two real numbers $c, C \geq 0$ by $c = \min_M \Xi \bar{f} = \min_M \|\Xi\|^2$ and $C = \max_M |f|$. Since Ξ is holomorphic, its real and imaginary parts have commuting flows that are defined for all t. This implies the existence of a holomorphic action of $\mathbb{C}$ on M mapping d/dz to Ξ. Let $F : \mathbb{C} \mapsto M$ be an orbit of this action, and set $f_F = f \circ F$.

Then f_F is a smooth complex-valued function on $\mathbb{C}$ such that $d\bar{f_F}/dz$ and $|f_F|$ respectively take values in $[c,\infty)$ and $[0,C]$. If $D_r \subset \mathbb{C}$ is the closed disk of radius r centered at 0, by Stokes' theorem we obtain

$$\int_{\partial D_r} f_F\, dz = \int_{D_r} dg \wedge dz = \int_{D_r} \frac{df_F}{d\bar{z}} d\bar{z} \wedge dz = 2i \int_{D_r} \frac{df_F}{d\bar{z}} dx \wedge dy\,.$$

Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi} f_F(re^{i\theta})e^{i\theta}d\theta = \frac{1}{2\pi i r}\int_{\partial D_r} f_F\, dz = \frac{1}{\pi r}\int_{D_r}\frac{df_F}{d\bar{z}}dx\wedge dy \geq \frac{c\pi r^2}{\pi r} = cr\,.$$

Since $f_F(re^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k(r)e^{ik\theta}$, with $a_{-1}(r) = \frac{1}{2\pi}\int_0^{2\pi} f_F(re^{i\theta})e^{i\theta}d\theta$, the estimate above yields

$$\|f_F(re^{i\theta})\|_{L^2(S^1)} \geq cr\,.$$

In turn, this implies that

$$C \geq \max_{z\in\partial D_r} |f_F(z)| \geq cr\,,$$

for all $r > 0$. Dividing by r and taking the infimum, we conclude that $c = 0$. Thus, Ξ must have a zero. Thus, (iii) implies (i). □

Corollary 23. *Let (M, J) be a compact complex manifold of Kähler type. Then the set $\mathfrak{h}_0(M)$ consisting of holomorphic vector fields on M with zeroes is an ideal in the Lie algebra $\mathfrak{h}(M)$ of holomorphic vector fields on M, and the quotient algebra $\mathfrak{h}/\mathfrak{h}_0$ is Abelian.*

Proof. Consider the homomorphism

$$\begin{array}{ccc} \mathrm{Aut}(M,J) & \rightarrow & \mathrm{Aut}(\mathrm{Alb}(M)) \\ \Phi & \mapsto & \Phi_! \end{array}$$

defined so that $a \circ \Phi = \Phi_! \circ a$. This is a homomorphism of Lie groups. At the level of Lie algebras, it corresponds to a homomorphism from $\mathfrak{h}$ to the Abelian algebra of holomorphic vector fields on the Albanese torus. This homomorphism is just the push-forward of holomorphic vector fields via the Albanese map, and therefore its kernel is the set of the vector fields tangent to the fibers of the Albanese map. By Theorem 22, this is precisely $\mathfrak{h}_0$. Thus $\mathfrak{h}_0$ is an ideal, and the quotient Lie algebra $\mathfrak{h}/\mathfrak{h}_0$ is Abelian. □

Although the dimension of the solution space to (27) is independent of the Kähler metric g, care has to be exercised when accounting for the space of real-valued solutions to this equation. Let us recall that a Killing vector field X is one whose flow preserves the metric g, that is to say,

$L_X g = 0$. A Killing field must preserve the fundamental form ω because this form is the unique harmonic representative of its cohomology class. Hence, a Killing field must be holomorphic. The space $\mathfrak{z}$ of Killing vector fields forms a real Lie subalgebra of $\mathfrak{h}$.

We have the following:

Proposition 24. *Let (M, J, g) be a compact Kähler manifold. If f is a real valued solution of* (27)*, then* $\mathrm{Im}\partial^{\#} f$ *is a Killing vector field of g, and a Killing field arises in this way iff it has a zero.*

Proof. Suppose that $X - iJX = 2\partial^{\#} f$ is the holomorphic vector field that corresponds to a real valued solution of (27). That is to say, $X = \nabla f$ is real holomorphic. Since the Nijenhuis tensor vanishes, $J\nabla f$ is also holomorphic, so its flow preserves J. But $J\nabla f$ is a Hamiltonian vector field and, therefore, its flow must preserve the Kähler form ω. Since the metric $g(\,\cdot\,,\,\cdot\,)$ is given by $\omega(\,\cdot\,, J\,\cdot\,)$, the vector field $J\nabla f$ must preserve g. Thus, $J\nabla f$ is Killing.

Conversely, if X is a Killing field, it must preserve ω because ω is the unique harmonic form in its cohomology class. Therefore, the field preserves J also. By Proposition 21, this field must be the imaginary part of some holomorphic vector field $\Xi = JX + iX$. If X has a zero so does Ξ and, by Theorem 22, we may write $\Xi = \partial^{\#} f$ for some complex valued function $f = u + iv$. We have that $X = \mathrm{Im}\Xi = \nabla v - J\nabla u$, and

$$0 = L_X\omega = X \lrcorner d\omega + d(X \lrcorner \omega) = d((\nabla v - J\nabla u)\lrcorner \omega) = d(du - Jdv) = 2i\partial\overline{\partial} v\,.$$

Thus, we may take v to be identically zero and f to be real-valued. □

4.3. **Calabi functional.** The volume of M relative to any metric in the space $\mathfrak{M}_\Omega$ is equal to $\emptyset^n/n!$. As a mechanism to find canonical representative of this space of metrics, it is tempting to consider critical points of the total scalar curvature functional

$$\mathfrak{M}_\Omega \ni g \mapsto \int s_g d\mu_g\,.$$

That the mapping $g \mapsto \int s_g d\mu_g$ is not homogeneous of degree zero becomes irrelevant when seeking critical points in $\mathfrak{M}_\Omega$ because all metrics have the same volume. Nevertheless, this plan leads nowhere useful because the cohomology class represented by the Ricci form of any Kähler metric is always the same, regardless of what metric we consider, and that makes the functional above constant.

Indeed, ρ always represents $2\pi c_1$, and relation (8) implies that

$$s_{tot} = \int s_g d\mu_g = \frac{2}{(n-1)!}\int \rho\wedge\omega^{n-1} = \frac{4\pi}{(n-1)!}c_1\cup\Omega^{n-1}\,. \tag{29}$$

This expression only depends on J and Ω, and it is therefore constant on $\mathfrak{M}_\Omega$.

Instead, and with the thought in mind that $\mathfrak{M}_\Omega$ is an affine space modeled on $C^\infty(M)$, Calabi [6] proposed to study the functional

$$\begin{array}{ccc} \mathfrak{M}_\Omega & \stackrel{E}{\longrightarrow} & \mathbb{R} \\ \omega & \mapsto & \int_M s_\omega^2 d\mu_\omega \end{array}\,. \tag{30}$$

Its critical points have come to be known as *extremal Kähler metrics.* The hope is that the Euler-Lagrange equation that a critical point must satisfy should suffice to fix the functional parameter that defines it as an element of $\mathfrak{M}_\Omega$, selecting in this manner a canonical representative of that space.

4.4. **Euler-Lagrange equation for extremal metrics.** Let ω be a critical point of (30). Hence, for any infinitesimal variation of ω in $\mathfrak{M}_\Omega$, the corresponding variation of E must vanish.

Proposition 25. ([6]) *The first derivative of E at $\omega\in\mathfrak{M}_\Omega$ in the direction of $i\partial\overline{\partial}\varphi$ is given by*

$$\frac{d}{dt}E(\omega+it\partial\overline{\partial}\varphi)\mid_{t=0} = -4\int_M s(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}\varphi\, d\mu\,.$$

Proof. By the results in Corollary 19 and (28), we see that

$$\begin{aligned} \frac{d}{dt}E(\omega_t)\mid_{t=0} &= \int(2s\frac{ds}{dt}\mid_{t=0} -\frac{1}{2}s^2\Delta\varphi)d\mu \\ &= -\int s(\Delta^2\varphi+4(\rho,i\partial\overline{\partial}\varphi)+\frac{1}{2}s\Delta\varphi)d\mu \\ &= -\int s(\Delta^2\varphi+4(\rho,i\partial\overline{\partial}\varphi)+2\partial s\lrcorner\partial^{\#}\varphi)d\mu \\ &= -4\int s(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}\varphi\, d\mu\,, \end{aligned}$$

as stated. □

Corollary 26. *A Kähler metric $g\in\mathfrak{M}_\Omega$ is extremal iff the vector field $\partial_g^{\#}s_g$ is holomorphic, or equivalently, iff $J\nabla_g s_g$ is a Killing field.*

Remark 27. We could study the functionals

$$\begin{array}{ccc} \mathfrak{M}_\Omega & \stackrel{E_p}{\longrightarrow} & \mathbb{R} \\ \omega & \mapsto & \int_M s_\omega^p d\mu_\omega \end{array} , \tag{31}$$

for any $1 \leq p < \infty$. The arguments above show that

$$\frac{d}{dt} E_p(\omega_t) \mid_{t=0} = -2p \int s^{p-1} (\bar{\partial}\partial^{\#})^* \bar{\partial}\partial^{\#} \varphi \, d\mu \, .$$

Hence, the critical points of E_p are metrics whose scalar curvature s is such that $\partial^{\#} s^{p-1}$ is a holomorphic field. Evidently, this is always the case when $p = 1$, result consistent with the fact that (29) is a topological constant on $\mathfrak{M}_\Omega$. In the remaining cases, the resulting Euler-Lagrange equation shows that the $p = 2$ case is perhaps the most natural of all the functionals E_p we may consider. For only in that case, the holomorphic vector field associated with the critical metric is given by a linear expression in s.

Remark 28. Another very natural functional to consider is given by

$$\begin{array}{ccc} \mathfrak{M}_\Omega & \stackrel{F}{\longrightarrow} & \mathbb{R} \\ g & \mapsto & \int_M g(\nabla_g s_g, \nabla_g s_g) d\mu_g \end{array} , \tag{32}$$

the squared L^2-norm of the gradient of the scalar curvature. The Euler-Lagrange equation for critical points of F is

$$-4(\bar{\partial}\partial_g^{\#})^* \bar{\partial}\partial_g^{\#}(\Delta_g s_g) - 2(\partial s_g, \partial \Delta_g s_g)_g + \frac{1}{2}(\Delta_g s_g)^2 + 2(i\partial\overline{\partial} s_g, i\partial\overline{\partial} s_g)_g = 0 \, .$$

Notice that metrics of constant scalar curvature are critical points of (32). However, it is not clear if these are the only ones.

For reasons to be clarified later on, when seeking metrics in $\mathfrak{M}_\Omega$ of constant scalar curvature, F is perhaps a more natural Riemannian functional to study than E itself.

Exercise 7. (open problem) Study the plausible convergence of minimizing sequences of F. In particular, show that infimum of F on $\mathbb{CP}^2$ blown-up at one point cannot be achieved by a constant scalar curvature metric. What is the value of that infimum?

Calabi also computed the second derivative of E at a critical point. For convenience, let us call $L = L_g$ the operator $(\bar{\partial}\partial^{\#})^* \bar{\partial}\partial^{\#}$. We define the conjugate operator $\bar{L}$ by $\bar{L}\varphi = \overline{L\bar{\varphi}}$. Then we have

$$(\bar{L} - L)\varphi = \partial^{\#} s \lrcorner \partial\varphi - \partial^{\#}\varphi \lrcorner \partial s \,. \tag{33}$$

This identity implies that L and $\bar{L}$ coincide iff s is constant, and that for any Kähler metric, $Ls = \bar{L}s$. At an extremal metric, L and $\bar{L}$ commute, and we have that

$$(D^2 E)_g(\varphi, \psi) = 8 \int \bar{L}L\varphi\psi d\mu_g \,.$$

4.5. **The structure of the Lie algebra of holomorphic vector fields.** The automorphism group of a Kähler Einstein manifold (M, J, g) has a particular structure. That result was originally proved by Matsushima [20], and later on extended by Lichnérowicz [17] to the case of a metric g of constant scalar curvature. We discuss here a further extension, due to Calabi [7], to the case where g is extremal. We will see below that not all extremal metrics have constant scalar curvature.

Consider the map

$$\begin{array}{ccc} \operatorname{kernel} L & \stackrel{\partial^{\#}}{\rightarrow} & \mathfrak{h} \\ f & \rightarrow & \partial^{\#} f \end{array} .$$

By Theorem 22, its image $\mathfrak{h}_0 \cong \operatorname{kernel} L/\mathbb{C}$ is identified with the space of holomorphic vector fields that have zeroes.

The space $\mathfrak{h}_0$ is an ideal of the algebra $\mathfrak{h}$, and the quotient algebra $\mathfrak{h}/\mathfrak{h}_0$ is Abelian. Under the assumption that the manifold (M, J) carries an extremal metric g, we proceed to identify this Abelian algebra further, and study the detailed structure of $\mathfrak{h}_0$.

Let $\mathfrak{z}_0 \subset \mathfrak{z}$ be the image under $\partial^{\#}$ of the purely imaginary functions in $\operatorname{kernel} L$. This is just the space of Killing fields of the form $J\nabla\varphi$, for φ a real-valued solution of (27). By (33), we see that $\mathfrak{z}_0$ coincides with the purely imaginary elements of $\operatorname{kernel} L \cap \operatorname{kernel} \bar{L}/\mathbb{R}$. In fact,

Proposition 29. *The complexification $\mathfrak{z}_0 \oplus J\mathfrak{z}_0$ coincides with the commutator of $\partial^{\#} s$, that is to say,*

$$\mathfrak{z}_0 \oplus J\mathfrak{z}_0 = \{X \in \mathfrak{h}_0 : \, [\partial^{\#} s, X] = 0\} \,.$$

When the metric g is one of constant scalar curvature, this Proposition completely describes the algebra $\mathfrak{h}_0$ as the complexification of the subalgebra of Killing fields. However, if g is merely extremal, we have the following

Theorem 30. [7] *Let (M, J) be a compact complex manifold of Kähler type, with extremal Kähler metric g. Then the algebra $\mathfrak{h}$ of holomorphic vector fields admits the orthogonal decomposition*

$$\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{h}_0 \,, \tag{34}$$

where $\mathfrak{a}$ is the subalgebra of parallel holomorphic vector fields, and $\mathfrak{h}_0$ is the ideal of $\mathfrak{h}$ consisting of the image of $\partial_g^{\#}$: $\mathrm{kernel}[(\overline{\partial}\partial_g^{\#})^* \overline{\partial}\partial_g^{\#}] \rightarrow \mathfrak{h}$. In particular,

$$\mathfrak{z} = \mathfrak{a} \oplus \mathfrak{z}_0 \,.$$

The ideal $\mathfrak{h}_0$ admits the orthogonal decomposition

$$\mathfrak{h}_0 = \mathfrak{z}_0 \oplus J\mathfrak{z}_0 \oplus (\oplus_{\lambda > 0} \mathfrak{h}^{\lambda}) \,,$$

where $\mathfrak{h}^{\lambda} = \{X \in \mathfrak{h} : \; [\partial_g^{\#} s, X] = \lambda X\}$.

Proof. Let $\Xi \in \mathfrak{h}$ be a holomorphic field, and let α_{Ξ} be the (0,1)-form that corresponds to it via the metric g. Then, as we saw before, this form is $\overline{\partial}$-closed and, therefore, its Hodge decomposition is of the form

$$\alpha_{\Xi} = \alpha_{\Xi}^{h} + \overline{\partial} f \,,$$

for some harmonic form α_{Ξ}^{h} and some function f. Let Ξ_h be the (1,0) vector field that corresponds to the harmonic (0,1)-form α_{Ξ}^{h}. Since Ξ is holomorphic, we have

$$Lf = \nabla_j \nabla_k \nabla^k \nabla^j f = \nabla_j \nabla_k \nabla^k (\Xi^j - \Xi_h^j) = -\nabla_j \nabla_k \nabla^k \Xi_h^j \,.$$

By the Ricci identity,

$$\nabla_k \nabla^k \Xi_h^j = \nabla^k \nabla_k \Xi_h^j + R_k{}^{kj}{}_l \Xi_h^l, .$$

Since the form corresponding to Ξ_h is conjugate holomorphic, the first summand in the right of this expression is zero. On the other hand, using the contracted Bianchi identity, we see that $\nabla_j R_k{}^{kj}{}_l \Xi_h^l = \nabla^{\bar{l}} s (\alpha_{\Xi}^{h})_{\bar{l}}$. Hence,

$$Lf = -\nabla^{\bar{l}} s (\alpha_{\Xi}^{h})_{\bar{l}} = \overline{\partial^{\#} s}(\alpha_{\Xi}^{h}) \,,$$

an expression given by the duality pairing of the conjugate holomorphic 1-form α_{Ξ}^{h} with the conjugate holomorphic vector field $\overline{\partial^{\#} s}$. This must be a constant, and since $\langle Lf, 1\rangle = 0$, the constant is zero. Thus, $Lf = 0$ and $\partial^{\#} f$ is holomorphic.

Since Ξ and $\partial^{\#} f$ are holomorphic, we have that

$$\Xi = \Xi_h + \partial^{\#} f$$

with Ξ_h holomorphic. Thus, $\overline{\partial}\Xi^h = 0$. But we also have that $\partial \Xi^h = 0$ because α^h_{Ξ} is ∂-harmonic. Therefore, Ξ_h is a parallel holomorphic vector field. The decomposition of Ξ above is clearly unique. This completes the proof of (34).

Notice that given holomorphic fields $\Xi = \partial^{\#} f$ and $\tilde{\Xi} = \partial^{\#}\tilde{f}$ with potential functions f and $\tilde{f}$, respectively, we have that

$$[\Xi, \tilde{\Xi}] = \partial^{\#}(\partial^{\#} f(\tilde{f}) - \partial^{\#}\tilde{f}(f)) \, .$$

This shows that $[\Xi, \tilde{\Xi}]$ is also given by a potential function, argument that suffices to conclude that $\mathfrak{h}_0$ is itself a Lie algebra.

Parallel vector fields automatically satisfy Killing's equation. Thus, any parallel holomorphic field Ξ is of the form $\Xi = X + iY$ with X and Y infinitesimal generators of one parameter group of isometries of (M, J, g). Since the connection is torsion-free, we have that $[\mathfrak{a}, \mathfrak{a}] = 0$. Moreover, the Lie derivative L_{Ξ} in the direction of a Killing field Ξ is such that $L_{\Xi}(\partial^{\#} f) = \partial^{\#}(L_{\Xi} f)$. Then, $[\mathfrak{h}, \mathfrak{a}] \subset \mathfrak{h}_0$, and consequently, $[\mathfrak{h}, \mathfrak{h}_0] \subset \mathfrak{h}_0$. Thus, $\mathfrak{h}_0$ is an ideal of $\mathfrak{h}$, and by (34), the quotient $\mathfrak{h}/\mathfrak{h}_0$ is Abelian.

Let K be the kernel of $L_g = (\overline{\partial}\partial^{\#}_g)^*\overline{\partial}\partial^{\#}_g$. The image of this finite dimensional space under $\partial^{\#}$ is precisely $\mathfrak{h}_0$. Since the metric g is extremal, L_g commutes with $\bar{L}_g$. Therefore, $\bar{L}_g K \subset K$, and since $\bar{L}_g$ is a non-negative operator, its restriction to the invariant subspace K admits a decomposition into eigenspaces $E_\lambda = \{\varphi \in K : \; \bar{L}_g \varphi = \lambda\varphi\}$. By (33), we see that if $\varphi \in E_\lambda$ then

$$\lambda\varphi = \bar{L}_g\varphi = (\bar{L}_g - L_g)\varphi = \partial^{\#} s \lrcorner \partial\varphi - \partial^{\#}\varphi \lrcorner \partial s \, .$$

We apply the operator $\partial^{\#}$ to this identity. Since $\partial^{\#} s$ is a holomorphic field, we obtain that

$$\lambda \partial^{\#}\varphi = [\partial^{\#} s, \partial^{\#}\varphi] \, .$$

The desired result follows. $\square$

4.6. **The Futaki character.** Let $g \in \mathfrak{M}_{\Omega}$. By Lemma 17, its Ricci form ρ can be decomposed as

$$\rho = \rho_H + i\partial\overline{\partial}\psi \, ,$$

where ρ_H is harmonic and ψ is a real-valued function, the *Ricci potential* of the metric. This Ricci potential is explicitly given by $\psi = -2G(s) = -2G(s - s_0)$, where s and G are the scalar curvature and Green's operator of g, respectively.

On $\mathfrak{h}$, we define the function

$$X \mapsto \int X(\psi) d\mu_g \, .$$

Proposition 31. ([10, 7]) *The mapping above only depends upon Ω and not on the particular metric g in $\mathfrak{M}_\Omega$ used to define it.*

Proof. From the identity $\Delta \psi = s_0 - s$, we see that for $\omega_t = \omega + it\partial\overline{\partial}\varphi$, the variation of ψ satisfies the relation

$$2(i\partial\overline{\partial}\varphi, i\partial\overline{\partial}\psi) + \Delta\dot{\psi} = -\dot{s} = \frac{1}{2}\Delta^2\varphi + 2(\rho, i\partial\overline{\partial}\varphi) \, .$$

Hence,

$$\dot{\psi} - \frac{1}{v}\int \dot{\psi} d\mu = \frac{1}{2}\Delta\varphi + 2G(\rho_H, i\partial\overline{\partial}\varphi) \, ,$$

where v is the volume of M in the metric. Since ρ_H is harmonic, the last summand in the right side can be written as $-2G(\partial^*(\overline{\partial}^*(\varphi\rho_H)))$. For convenience, let us set $\eta = \overline{\partial}^*(\varphi\rho_H)$. Hence,

$$\frac{d}{dt}\int X(\psi_t) d\mu_{g_t} = \int X\left(\frac{1}{2}\Delta\varphi - 2G(\partial^*\eta)\right) - \frac{1}{2}\psi\Delta\varphi\Big) \, d\mu_g \, .$$

By the Ricci identity, we have that

$$\frac{1}{2}(\Delta\varphi)_\alpha = -\varphi_{,\beta\alpha}{}^{\beta} + \varphi_{,\beta}(\psi_{,\alpha}{}^{\beta} + (r_H)_\alpha{}^{\beta}) = -\varphi_{,\beta\alpha}{}^{\beta} + \varphi_{,\beta}\psi_{,\alpha}{}^{\beta} + \eta_\alpha \, ,$$

and so, after minor simplifications, we conclude that

$$\frac{d}{dt}\int X(\psi_t) d\mu_{g_t} = \int X^\alpha \left(\varphi_{,\beta}\psi_\alpha - \varphi_{,\beta\alpha}\right)^\beta d\mu_g + \int X^\alpha \left(\eta_\alpha - 2(G\partial^*\eta)_{,\alpha}\right) d\mu_g \, .$$

The first summand on the right above is zero because X is holomorphic. This is just a consequence of Stokes' theorem. The second summand is also zero since we have

$$\int X^\alpha \left(\eta_\alpha - 2(G\partial^*\eta)_{,\alpha}\right) d\mu_g = \int (\eta - \Delta G\eta, X^\flat) d\mu_g + (2\partial^*\partial G\eta, X^\flat) d\mu_g \, ,$$

and $\eta - \Delta G\eta = 0$ while $\partial X^\flat = 0$. □

We may then define the *Futaki invariant* in the following manner.

$$\begin{gathered} \mathfrak{F} : \mathfrak{h}(M) \times \mathcal{K} \longrightarrow \mathbb{C} \\ \mathfrak{F}(X, [\omega]) = \int_M X(\psi_\omega) d\mu = -\int_M X(Gs) d\mu \, , \end{gathered} \tag{35}$$

where the Kähler cone $\mathcal{K} = H^{1,1}(M, \mathbb{R})^+$ is the set of all cohomology classes represented by Kähler forms in (M, J). The previous Proposition shows that $\mathfrak{F}$ depends only on the Kähler class $[\omega]$ rather than the specific Kähler form ω chosen to represent it. In particular, we have that

$$\mathfrak{F}(X,[\omega]) = \mathfrak{F}(\Phi_* X, (\Phi^{-1})^*[\omega])$$

for any biholomorphism $\Phi : M \to M$, and so $\mathfrak{F}([X,Y],[\omega]) = 0$ for any pair of holomorphic vector fields X, Y in M. It is for this reason that $\mathfrak{F}(\,\cdot\,,[\omega])$ is often called the *Futaki character* of $\mathfrak{h}$.

Proposition 32. ([10, 7]) *Let (M,J,g) be a Kähler manifold with g an extremal Kähler metric. Then, g has constant scalar curvature iff $\mathfrak{F}(\,\cdot\,,[\omega]) = 0$.*

Proof. In one direction this result is obvious: a constant scalar curvature Kähler metric has trivial Ricci potential function.

Let us now consider a holomorphic vector field X of the form $X = \partial^{\#} f$. Then,

$$\mathfrak{F}(X,[\omega]) = -\int \partial^{\#} f (Gs) d\mu = -2 \int (\overline{\partial} f, \overline{\partial} Gs) d\mu = -2 \int f \overline{\partial}^* \overline{\partial} G s d\mu \, ,$$

and since $2\overline{\partial}^*\overline{\partial} = \Delta$, we have that

$$\mathfrak{F}(\partial^{\#} f, [\omega]) = -\int f(s - s_0) d\mu \, .$$

In particular, if g is an extremal metric, then $\partial^{\#} s$ is holomorphic and

$$\mathfrak{F}(\partial^{\#} s, [\omega]) = -\int (s - s_0)^2 d\mu \, .$$

Thus, if $\mathfrak{F}(\,\cdot\,,[\omega]) = 0$, then s must be a constant. □

Remark 33. If Ξ is a holomorphic vector field that is covariantly constant with respect to some Kähler form ω, then $\mathfrak{F}(\Xi,\Omega) = 0$ for any Ω in the Kähler cone. Indeed, $\Xi = X + iY$ for some Killing fields X, Y of (M,J,ω). Let $\mathbb{G}$ be the isometry group of this manifold. This is a compact group of biholomorphisms of (M,J). We can choose a representative $\tilde{\omega}$ of any given Ω that is invariant under $\mathbb{G}$. This means that X and Y are Killing fields of the associated metric $\tilde{g}$, and in particular, both the Green's operator and the scalar curvature of $\tilde{g}$ are invariant under $\mathbb{G}$. Therefore, $\mathfrak{F}(\Xi,\Omega) = -\int \Xi(Gs) d\mu = 0$.

Thus, we may use the decomposition (34) and the observation above to conclude that, when dealing with $\mathfrak{F}(\,\cdot\,,[\omega])$, it suffices to consider its restriction to vector fields in $\mathfrak{h}_0$.

4.7. **Holomorphy potentials.** By Calabi's Theorem 30, the isometry group of any extremal metric on (M, J) is a maximal compact subgroup of the identity component $\mathcal{A}$ of the biholomorphism group. Moreover, any two such groups are conjugate [13]. Throughout this section, we let G denote a maximal compact subgroup of $\mathcal{A}$, and let n be the complex dimension of (M, J).

We consider a cohomology class Ω in $H^{1,1}$ that can be represented by Kähler forms of Kähler metrics. Let g be a Kähler metric on M whose Kähler form represents Ω. Without loss of generality, we assume that g is G-invariant.

We denote by $L^2_{k,G}$ the Hilbert space of of G-invariant real-valued functions of class L^2_k, and consider G-invariant deformations of g preserving the Kähler class:

$$\tilde{\omega} = \omega + i\partial\overline{\partial}\varphi\,, \quad \varphi \in L^2_{k+4,G}\,,\ k > n. \tag{36}$$

Here, ω stands for the Kähler form of g, and the condition $k > n$ ensures that the scalar curvature of $\tilde{\omega}$ is a well-defined function in $L^2_{k,G}$. This last conclusion follows because the latter space is a Banach algebra under the assumed restriction on k. From now on, a function φ will be called *admissible* if $\tilde{\omega} = \omega + i\partial\overline{\partial}\varphi$ is positive.

Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{z} \subset \mathfrak{g}$ its center. We let $\mathfrak{z}_0 = \mathfrak{z} \cap \mathfrak{g}_0$, where $\mathfrak{g}_0 \subset \mathfrak{g}$ is the ideal of Killing fields which have zeroes. If $\tilde{g}$ is any G-invariant Kähler metric on (M, J), then each element of $\mathfrak{z}_0$ is of the form $J\,\nabla_{\tilde{g}} f$ for a real-valued function f solution of the equation (27) associated with the metric $\tilde{g}$. Moreover, $\mathfrak{z}_0$ thereby precisely corresponds to the set of real solutions f which are *invariant under* G, since

$$\partial^{\#} : \ker[(\bar{\partial}\partial^{\#}_{\tilde{g}})^* \bar{\partial}\partial_{\tilde{g}}] \to \mathfrak{h}_0$$

is a homomorphism of G-modules.

The restriction of $\ker(\overline{\partial}\partial^{\#}_{\tilde{g}})^* \overline{\partial}\partial_{\tilde{g}}$ to $L^2_{k+4,G}$ depends smoothly on the G-invariant metric $\tilde{g}$. Indeed, choose a basis $\{X_1, \ldots, X_m\}$ for $\mathfrak{z}_0$, and, for each $(1,1)$-form χ on (M, J), consider the set of functions

$$\begin{aligned} p_0(\chi) &= 1 \\ p_j(\chi) &= 2iG_g\overline{\partial}^*_g((JX_j + iX_j) \lrcorner \chi)\,, \quad j = 1, \ldots, m \end{aligned}$$

where G_g is the Green's operator of the metric g. If $\tilde{\omega}$ is the Kähler form of the G-invariant metric $\tilde{g}$, then $\partial^{\#}_{\tilde{g}} p_j(\tilde{\omega}) = JX_j + iX_j$, and the functions $\{p_j(\tilde{\omega})\}_{j=1}^m$ are real-valued and form a basis for $\ker(\bar{\partial}\partial^{\#}_{\tilde{g}})^* \bar{\partial}\partial_{\tilde{g}}$.

Moreover, for metrics $\tilde{\omega}$ as in (36), the map $\varphi \mapsto p_j(\omega + i\partial\overline{\partial}\varphi)$ is, for each j, bounded as a linear map from $L^2_{k+4,G}$ to $L^2_{k+3,G}$.

With respect to the background L^2 inner product, let

$$\{f^0_{\tilde{\omega}}, \ldots, f^m_{\tilde{\omega}}\} \tag{37}$$

be the orthonormal set extracted from $\{p_j(\tilde{\omega})\}$ by the Gram-Schmidt procedure. The set $\{f^j_{\tilde{\omega}}\}_{j=0}^m$ forms a basis for the vector space of real holomorphy potentials. We then let

$$\begin{array}{rcl} \pi_{\tilde{\omega}} : L^2_{k,G} & \rightarrow & L^2_{k,G} \\ u & \mapsto & \displaystyle\sum_{j=0}^m \langle f^j_{\tilde{\omega}}, u\rangle_{L^2} f^j_{\tilde{\omega}} \end{array} \tag{38}$$

denote the associated projector. By the regularity of the functions $\{p_1, \ldots, p_m\}$, this projection can be defined on $L^2_{k+j,G}$ for $j = 0, 1, 2, 3$, and for metrics as in (36), the map $\varphi \mapsto \pi_{\tilde{\omega}}$ is smooth from a suitable neighborhood of the origin in $L^2_{k+4,G}$ to the real Hilbert space $\mathrm{End}(L^2_{k+j,G}) \cong \bigotimes^2 L^2_{k+j,G}$.

This projection can be lifted to one at the level of $(1,1)$-forms. Indeed, let us denote by $\wedge^{1,1}_{k,G}$ the space of real forms of type $(1,1)$, invariant under G and of class L^2_k. Then, given any G-invariant metric $\tilde{g}$, there exists a unique continuous linear map

$$\Pi_{\tilde{\omega}} : \wedge^{1,1}_{k+2,G} \mapsto \wedge^{1,1}_{k+2,G}\,, \tag{39}$$

which intertwines the trace and the projection map $\pi_{\tilde{\omega}}$ in (38), and such that $\eta - \Pi_{\tilde{\omega}}\eta$ is cohomologous to zero for all $\eta \in \wedge^{1,1}_{k+2,G}$. For metrics $\tilde{\omega}$ as in (36), the map $\varphi \mapsto \Pi_{\tilde{\omega}}$ from $L^2_{k+4,G}$ to $\mathrm{End}(\wedge^{1,1}_{k+2,G})$ is smooth.

A metric $\tilde{\omega}$ is extremal if and only if $\tilde{s} = \pi_{\tilde{\omega}}\tilde{s}$, or equivalently, if and only if $\tilde{\rho} = \Pi_{\tilde{\omega}}\tilde{\rho}$. For arbitrary G-invariant metrics one may only say that $\rho = \Pi\rho + i\partial\overline{\partial}\psi$. The extremality condition implies that ψ is a constant.

Using the holomorphy potential of fields in $\mathfrak{h}_0(M)$, let us consider the bilinear form

$$\begin{array}{c} \mathfrak{B} : \mathfrak{h}_0(M) \times \mathfrak{h}_0(M) \times \mathcal{K} \longrightarrow \mathbb{C} \\ \mathfrak{B}(\Xi, \Theta, [\omega]) = \displaystyle\int_M f_\Xi f_\Theta d\mu\,, \end{array} \tag{40}$$

where in the right side, f_Ξ stands for the function such that $\partial^\# f_\Xi = \Xi$. This form has been studied by Mabuchi [19, 11], who has proven it only depends on the Kähler class $[\omega]$. This is analogous (and related) to the same property of the Futaki invariant. Evidently, if this form $\mathfrak{B}$ were non-degenerate and $\partial^{\#} s$ were a holomorphic vector field, (40) would identify this vector field with (minus) the Futaki character. Since for extremal metrics, the vector $\partial^{\#} s$ is the complexification of a Killing field, this conclusion can still be drawn if we have the non-degeneracy of $\mathfrak{B}$ when restricted to the smaller real Lie algebra of Killing fields with a zero. But this non-degeneracy is then obvious because $\mathfrak{B}(\Xi, \Theta, [\omega])$ is simply the L^2-inner product of the real valued potentials of the Killing fields Ξ and Θ [24], respectively. We thus have the following result.

Theorem 34. (Futaki) *Let g be any G-invariant metric representing a Kähler class Ω in (M, J), and consider the holomorphic vector field $X = \partial^{\#}(\pi_g s_g)$, where π_g is the projection* (38). *Then X only depends upon Ω and not on the particular metric g used to represent it, and is in the center of a reductive subalgebra of $\mathfrak{h}_0$.*

From now on, we shall refer to this vector field as X_Ω.

Proof. Restricted to $\mathfrak{z}_0$, the bilinear form $\mathfrak{B}$ is non-degenerate. This bilinear form identifies X with the functional

$$\Xi \mapsto -\int \Xi(G_g \pi_g s_g) d\mu_g \, ,$$

where G_g is the Green's operator of g. □

Theorem 35. *The Calabi functional* (30) *is bounded below by*

$$\int_M s_\omega^2 d\mu_\omega \geq E_\Omega := \int (\pi_\omega s_\omega)^2 d\mu_\omega \, . \tag{41}$$

The lower bound is achieved by a metric g if, and only if, g is extremal.

Remark 36. The field X_Ω may depend on the choice of a maximal compact subgroup G of the automorphism group of (M, J), but the value of $\mathfrak{F}(X_\Omega, \Omega)$ does not. This makes the lower bound above just a function of Ω that can be conveniently calculated in terms of quantities associated to G-invariant metrics.

4.8. **The extremal cone.** We now prove the following:

Theorem 37. *Let (M, J) be a compact complex manifold, and let $\mathcal{E} \subset \mathcal{K}$ be the subset of Kähler classes that can be represented by extremal Kähler metrics on M. Then $\mathcal{E}$ is open.*

This result implies the existence of many extremal metrics out of the knowledge of a single one. It is the only general result of this type that we know to date.

Theorem 37 will be a consequence of the inverse function theorem suitably applied to the non-linear extremal metric equation. As above, let G be a maximal compact Lie group of the the biholomorphism group $\mathcal{A}$ of (M,J), and $L^2_{k,G}$ be the real Hilbert space of G-invariant real-valued functions of class L^2_k. Every g-harmonic form is invariant under G, since the connected isometry group G obviously sends every harmonic form to a harmonic form in the same cohomology class.

Let $k>n$, and let $\mathcal{U}\subset \mathcal{H}^{1,1}(M)\times L^2_{k+4}(M)$ be the open neighborhood of $(0,0)$ consisting of pairs (α,φ) such that $\tilde{\omega}=\omega+\alpha+i\partial\overline{\partial}\varphi$ is the Kähler form of a C^2 Kähler metric. Here, $\mathcal{H}^{1,1}(M)$ is the space of real-valued g-harmonic (1,1)-forms on M.

We consider the scalar curvature map

$$\begin{array}{rccc} \mathcal{H}^{1,1}(M)\times L^2_{k+4}(M)\supset & \mathcal{U} & \stackrel{\mathcal{S}}{\longrightarrow} & L^2_k(M) \\ & (\alpha,\varphi) & \mapsto & s(\tilde{\omega})\,, \end{array} \tag{42}$$

where $s(\tilde{\omega})=s(\omega+\alpha+i\partial\overline{\partial}\varphi)$ is the scalar curvature of the metric with Kähler form $\omega+\alpha+i\partial\overline{\partial}\varphi$. This map is well-defined and C^1, with Fréchet derivative at the origin given by

$$D\mathcal{S}_{(0,0)}=\left[-2(\rho,\,\cdot\,)\qquad -\frac{1}{2}(\Delta^2+2r\cdot\nabla\nabla)\right]\,, \tag{43}$$

where $r\cdot$ denotes full contraction with the Ricci tensor of g.

We set $\mathcal{V}=\mathcal{U}\cap(\mathcal{H}^{1,1}(M)\times L^2_{k+4,G})$. Then, for $(\alpha,\varphi)\in\mathcal{V}$, the Kähler metric $\tilde{g}$ with Kähler form

$$\tilde{\omega}=\omega+\alpha+i\partial\overline{\partial}\varphi$$

is G-invariant, and hence its scalar curvature $\tilde{s}$ is G-invariant, as well. We consider the restricted map

$$\begin{array}{rccc} \mathcal{H}^{1,1}(M)\times L^2_{k+4,G}\supset & \mathcal{V} & \stackrel{\mathcal{S}_G}{\longrightarrow} & L^2_{k,G} \\ & (\alpha,\varphi) & \mapsto & s(\omega+\alpha+i\partial\overline{\partial}\varphi) \end{array}\,. \tag{44}$$

It is also a C^1 map, and its Fréchet derivative at $(0,0)$ is given by (43) restricted to $\mathcal{H}^{1,1}(M)\times L^2_{k+4,G}$.

For any integer ℓ, we let $I_\ell\subset L^2_{\ell,G}$ denote the orthogonal complement of the kernel of $(\bar{\partial}\partial^{\#}_g)^*\bar{\partial}\partial_g$, and set $\mathcal{W}=\mathcal{V}_0\cap(\mathcal{H}^{1,1}(M)\times I_{k+4})$, where

$\mathcal{V}_0$ is a neighborhood of $(0,0)$ in $\mathcal{H}^{1,1}(M) \times L^2_{k+4,G}$ such that

$$\ker(1-\pi_\omega)(1-\pi_{\tilde\omega}) = \ker(1-\pi_{\tilde\omega})\,,$$

whenever $\tilde\omega = \omega + \alpha + i\partial\overline{\partial}\varphi$ for some $(\alpha,\varphi) \in \mathcal{V}_0$. We the idea in mind that a Kähler form $\tilde\omega$ is extremal iff its scalar curvature is annihilated by $1-\pi_{\tilde\omega}$, we introduce the map

$$\mathcal{H}^{1,1}(M) \times I_{k+4} \supset \mathcal{W} \stackrel{\mathbf{S}}{\longrightarrow} \mathcal{H}^{1,1}(M) \times I_k$$

defined by

$$\mathbf{S}(\alpha,\varphi) = (\alpha, (1-\pi_\omega)(1-\pi_{\omega+\alpha+i\partial\overline{\partial}\varphi})\,\mathcal{S}_G(\alpha,\varphi)\,) \tag{45}$$

where $\mathcal{S}_G(\alpha,\varphi)$ is defined in (44) and $\pi_{\omega+\alpha+i\partial\overline{\partial}\varphi} = \pi_{\tilde\omega}$ is the projection defined in (38).

We have the following

Lemma 38. *Suppose the Kähler metric g is extremal. Then*

$$(1-\pi_\omega)\,(\frac{d}{dt}\pi_{\tilde\omega_t})\Big|_{t=0} s = (1-\pi_\omega)[2iG_g\overline{\partial}^*_g(\partial^\#_g s \lrcorner \alpha) + (\partial s \lrcorner \partial^\#_g \varphi)]\,,$$

where G_g is the Green's operator of g.

Proof. If s were constant, the left-hand-side of the expression in the statement above would vanish because $\pi_{\tilde\omega_t}1 \equiv 1$ for all t. But in this case the right-hand-side would also vanish, since we would then have $\partial^\#_g s = 0$.

Let us then assume that $\partial^\#_g s \not\equiv 0$. Since the extremal condition implies that $\mathrm{Im}\,\partial^\#_g s$ is a Killing vector field, we may then choose our basis $\{\xi_j\}$ for $\mathfrak{z}_0$ so that $J\xi_1 + i\xi_1 = \partial^\#_g s$. Now recall that any choice of basis gives rise to a family of t-dependent potentials $p_j(\tilde\omega_t) = 2iG_g\overline{\partial}^*_g((J\xi + i\xi)\lrcorner\tilde\omega_t)$, from which $f^0_{\tilde\omega_t}, \ldots, f^m_{\tilde\omega_t}$ are then obtained by the Gram-Schmidt procedure. With the choice of basis made as indicated, $f^0_{\tilde\omega_t} = (\mathrm{vol}(M))^{-1/2}$, while

$$f^1_{\tilde\omega_t} = \frac{p_1(\tilde\omega_t)}{\|p_1(\tilde\omega_t)\|_{L^2}}\,,$$

where $p_1(\tilde\omega_t) = 2iG_g\overline{\partial}^*_g(\partial^\#_g s \lrcorner \tilde\omega_t)$. Also notice that $p_1(\tilde\omega_0) = s - s_0$, where s_0 is the average value of s.

Now $\pi_{\tilde{w}_t} s = \sum_{j=0}^{m} \langle f^j_{\tilde{\omega}_t}, s\rangle_{L^2} f^j_{\tilde{\omega}_t}$, where $\{f^0_{\tilde{\omega}_t}, \ldots, f^m_{\tilde{\omega}_t}\}$ is the orthonormal set above. Consequently,

$$(1-\pi_\omega)\left(\frac{d}{dt}\pi_{\tilde{\omega}_t}\right)\mid_{t=0} s \;=\; \sum_j \langle f^j_\omega, s\rangle_{L^2}(1-\pi_\omega)\frac{d}{dt} f^j_{\tilde{\omega}_t}\mid_{t=0},$$

because each $f^j_{\tilde{\omega}_0} = f^j_\omega$ is in the kernel of $(1-\pi_\omega)$. Since s is perpendicular to each f^j_ω for $j > 1$, and since $\frac{d}{dt} f^0_{\tilde{\omega}_t} = 0$, the only surviving term in this expression corresponds to $j = 1$, and we have

$$\begin{aligned}(1-\pi_\omega)\left(\frac{d}{dt}\pi_{\tilde{\omega}_t}\right)\mid_{t=0} s &= \left\langle \frac{s-s_0}{\|s-s_0\|}, s\right\rangle (1-\pi_\omega)\frac{d}{dt} f^1_{\tilde{\omega}_t}\mid_{t=0} \\ &= \|s-s_0\|_{L^2}(1-\pi_\omega)\frac{d}{dt} f^1_{\tilde{\omega}_t}\mid_{t=0}.\end{aligned}$$

But

$$\frac{d}{dt} f^1_{\tilde{\omega}_t} \equiv \frac{1}{\|p_1(\tilde{\omega}_t)\|_{L^2}}\frac{d}{dt} p_1(\tilde{\omega}_t) \mod p_1(\tilde{\omega}_t),$$

and $p_1(\tilde{\omega}_0) = s - s_0$ is in the kernel of $(1-\pi_\omega)$. It therefore follows that

$$(1-\pi_\omega)\left(\frac{d}{dt}\pi_{\tilde{\omega}_t}\right)\mid_{t=0} s = (1-\pi_\omega)\frac{d}{dt} p_1(\tilde{\omega}_t)\mid_{t=0}.$$

We compute this last derivative using the expression for p_1 in terms of s. We see that

$$\begin{aligned}\frac{d}{dt} 2iG_g\overline{\partial}^*_g(\partial^\#_g s \lrcorner \tilde{\omega}_t)\mid_{t=0} &= 2iG_g\overline{\partial}^*_g(\partial^\#_g s\lrcorner\alpha) + 2iG_g\overline{\partial}^*_g(\partial^\#_g s\lrcorner i\partial\overline{\partial}\varphi) \\ &= 2iG_g\overline{\partial}^*_g(\partial^\#_g s\lrcorner\alpha) - 2iG_g\overline{\partial}^*_g\overline{\partial}(\partial^\#_g s\lrcorner i\partial\varphi) \\ &= 2iG_g\overline{\partial}^*_g(\partial^\#_g s\lrcorner\alpha) + (\partial^\#_g s\lrcorner\partial\varphi) + \text{constant},\end{aligned}$$

since $\partial^\#_g s$ is a holomorphic vector field and $2G_g\overline{\partial}^*_g\overline{\partial}$ is the identity on the orthogonal complement of the constants. But

$$\partial^\# s\lrcorner\partial\varphi = \overline{\partial\bar{s}\lrcorner\partial^\#\bar{\varphi}} = (\nabla s - iJ\,\nabla s)\lrcorner d\varphi$$

is real, since $J\nabla s$ is a Killing field and φ is real and G-invariant. Thus, our last expression is equal to $2iG_g\overline{\partial}^*_g(\partial^\#_g s\lrcorner\alpha) + (\partial s\lrcorner\partial^\#_g\varphi) + \text{constant}$, and the desired result follows because the constant is annihilated by $(1-\pi_\omega)$. □

Proposition 39. *For $k > n$, equation* (45) *defines a C^1 map whose Fréchet derivative at the origin is given by*
(46)

$$D\mathbf{S}_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1-\pi_\omega \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2(\rho, \,\cdot\,) - 2iG_g\overline{\partial}_g^*(\partial_g^{\#}\sigma \lrcorner \,\cdot\,) & -2(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#} \end{pmatrix} .$$

Proposition 40. *Let (M, J, g) be a compact extremal Kähler manifold. Then the map* $\mathbf{S}$ *defined in* (45) *becomes a diffeomorphism when restricted to a sufficiently small neighborhood of the origin.*

Proof. This result is a consequence of the inverse function theorem for Banach spaces, once we prove that $D\mathbf{S}_{(0,0)}$ has trivial kernel and cokernel.

Suppose that (α, φ) is in the kernel of $D\mathbf{S}_{(0,0)}$. By (46), we see that $\alpha = 0$ and that

$$(1 - \pi_\omega)(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}\varphi = 0 .$$

Thus, $(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}\varphi$ is a holomorphy potential, and consequently, it can be written as

$$(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}\varphi = \sum_j c_j f_\omega^j ,$$

in terms of the basis of holomorphy potentials constructed to define (38). Taking the inner product of this expression with f_ω^j, and dualizing the symmetric map $(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}$, we see that $c_j = 0$. Thus, $(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}\varphi = 0$. But $\varphi \in I_{k+4}$, space orthogonal to the kernel of $(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}$. So φ must be zero, and the kernel of $D\mathbf{S}_{(0,0)}$ consists of $(0, 0)$.

Suppose now that (β, ψ) is orthogonal to every element in the image of $D\mathbf{S}_{(0,0)}$. Then, it must be orthogonal to the image of $(0, \varphi)$ for any $\varphi \in I_{k+4}$, and therefore,

$$\langle (1 - \pi_\omega)(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}\varphi, \psi\rangle = \langle \varphi, (\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}(1 - \pi_\omega)\psi\rangle = 0$$

for all such φ. Hence, the component of $(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}(1-\pi_\omega)\psi$ perpendicular to the kernel of $(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}$ is zero, and thus, $(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}(1-\pi_\omega)\psi = \sum c_j f_\omega^j$. The same argument used above implies that $c_j = 0$, and so, $(1-\pi_\omega)\psi$ is the kernel of $(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}$. But the image of $1-\pi_\omega$ is orthogonal to this kernel. Hence, $\psi = 0$. Using this, we now conclude that β must be such that $\langle \beta, \alpha\rangle = 0$ for all harmonic (1,1)-form α, that is only possible if $\beta = 0$. This shows that the cokernel of $D\mathbf{S}_{(0,0)}$ is trivial. □

Proof of Theorem 37. Let $\mathcal{W}_0 \subset \mathcal{W}$ be a neighborhood of $(0, 0) \in \mathcal{H}^{1,1} \times I_{k+4}$ such that $\mathbf{S}\mid_{\mathcal{W}_0}$ is a diffeomorphism from $\mathcal{W}_0$ onto an open neighborhood of the origin in $\mathcal{H}^{1,1} \times I_k$. For any harmonic (1,1) form α in $\mathbf{S}(\mathcal{W}_0) \cap [\mathcal{H}^{1,1} \times \{0\}]$, we define $\varphi(\alpha)$ to be the projection onto I_{k+4} of $(\mathbf{S}\mid_{\mathcal{W}_0})^{-1}(\alpha)$. Then we have

$$(\alpha, 0) = \mathbf{S}(\alpha, \varphi(\alpha)) = (\alpha, (1-\pi_\omega)(1-\pi_{\tilde{\omega}})s(\alpha, \varphi(\alpha)) \, ,$$

where $\tilde{\omega} = \omega + \alpha + i\partial\overline{\partial}\varphi(\alpha)$. Since the kernel of $(1-\pi_\omega)(1-\pi_{\tilde{\omega}})$ equals to the kernel of $1-\pi_{\tilde{\omega}}$, it follows that $(1-\pi_{\tilde{\omega}})s(\alpha, \varphi(\alpha)) = 0$. Thus, the scalar curvature of $\tilde{\omega}$ is a holomorphy potential, and $\tilde{\omega}$ is an extremal metric representing the class $[\omega + \alpha]$. $\square$

5. A few examples of extremal metrics

In this chapter we present three type of examples of extremal Kähler metrics. In the first one, we show that on certain manifolds the extremal equation can be integrated fully on their universal covering. A suitable compactification then leads to examples of manifolds that carry extremal Kähler metrics with a lot of symmetries. These are the only known examples where the extremal equation can be so analyzed, essentially because we end up looking at an ordinary differential equation. In the second type of examples, we discuss the result of Yau and Aubin for the existence of Kähler-Einstein metrics on manifolds with negative first Chern class. This is done using the continuity method, and to date, it involves the only general technique known to prove existence. The last type of examples is presented as an application of our openness result. We study (an open subset of) the extremal cone in the blow-up of $\mathbb{CP}^2$ at three points.

Before embarking of this plan, and for the reader's convenience, we discuss the notion of blowing-up a point on a complex manifold.

We recall the tautological line bundle $\tau \to \mathbb{CP}^{n-1}$,

$$\tau = \{(l, p) \in \mathbb{CP}^{n-1} \times \mathbb{C}^n \, : \, p \in l\}$$

with projection map $(l, p) \mapsto l$. For example, when $n = 2$, τ consists of pairs of points $([u : v], (x, y))$ in $\mathbb{CP}^1 \times \mathbb{C}^2$ such that $xv = yu$. Since its transition functions are holomorphic, τ is a holomorphic line bundle.

The projection π onto the second factor $\mathbb{C}^n$ is an isomorphism away from $p = 0$. Indeed, $\pi^{-1}(p)$ is the set of all complex lines in $\mathbb{C}^n$ going through p and 0, which is unique if $p \neq 0$. For $p = 0$, $\pi^{-1}(0)$ is the set of all lines through the origin, $\pi^{-1}(0) = \mathbb{CP}^{n-1}$. This projection π is a biholomorphism between $\tau - \pi^{-1}\{0\}$ and $\mathbb{C}^n - \{0\}$.

Definition 41. The blow-up of $\mathbb{C}^n$ at 0 is the triple $(\tau, \mathbb{C}^n, \pi)$. If M is an arbitrary complex manifold and $P \in M$ is a point in it, consider a complex chart (U, φ) with U a neighborhood of P in M such that $\varphi(U) = V \subset \mathbb{C}^n$ and $\varphi(P) = 0$. The blow-up of M at P, M_P, is the complex manifold that is obtained by removing U from M and replacing it by the open set $\pi^{-1}(V) \subset \tau$. There is a projection map $\pi_P : M_P \mapsto M$ that defines a biholomorphism between $M_P - \{\pi_P^{-1}(P)\}$ and $M - \{P\}$. The hypersurface given by the inverse image of P, $E := \pi^{-1}(P) = \mathbb{CP}^n$, is called the exceptional divisor of the blow-up.

We would like to reinterpret this definition topologically. For that observe that the first Chern class $c_1(\tau)$ of the tautological bundle is -1. We have stated that as one of the axioms that characterize Chern classes. We may, however, use the algorithm to calculate these classes in terms of a connection, apply it to τ, and see that the result is consistent with this axiom. We show this in detail next.

Let us consider trivializing coordinates $\{(t^1, \ldots, t^n, z^1, \ldots, z^n)$ in τ over some affine open subset of $\mathbb{CP}^{n-1}$. We have that

$$t^i z^j - z^i t^j = 0\,, \text{ for all } i, j\,.$$

If we work nearby the zero section, and on the complement of $t_1 = 0$, we may use $z^1, t^2/t^1, \ldots, t^n/t^1$ as coordinates in the total space of the line bundle. We may define the section

$$e(t) = \left(1, \frac{t^2}{t^1}, \ldots, \frac{t^n}{t^1}\right)\,,$$

and a Hermitian metric on the bundle by declaring the norm of this section to be

$$h = (e, e) = 1 + \sum_{j \geq 2} w^j \bar{w}^j\,.$$

Here, for convenience, we have set $w^j = t^j/t^1$. The set $\{w^2, \ldots, w^n\}$ forms a coordinate system on the open subset of $\mathbb{CP}^{n-1}$ where $t^1 \neq 0$.

As we saw earlier, we can compute c_1 by

$$c_1 = -\frac{i}{2\pi} \partial \overline{\partial} \log h\,.$$

In order to prove the desired assertion, we just need to evaluate this (1,1)-form over a generator $[\mathbb{CP}^1]$ of the second homology group of $\mathbb{CP}^{n-1}$. But in performing this evaluation, we can take this homology representative to be given by the equations $w^j = 0$, $j = 3, \ldots, n$, because the complement of the affine space where we are working has zero measure. If we define the positive real number r by $r^2 = w^2\bar{w}^2$, we easily see that

$$c_1 \mid_{\mathbb{CP}^1} = -\frac{1}{\pi}\frac{r}{(1+r^2)^2} dr \wedge d\omega \, ,$$

where $d\omega$ is the volume form of the unit circle $\mathbb{S}^1$ inside $\mathbb{R}^2$. Hence,

$$\begin{array}{rcl} c_1([\mathbb{CP}^1]) & = & -\frac{2\pi}{\pi}\int_0^\infty \frac{r}{(1+r^2)^2} dr \\ & = & -1 \, . \end{array}$$

Proposition 42. *For any fixed point $Q \in \overline{\mathbb{CP}}^n$, the manifold $\overline{\mathbb{CP}}^n - \{Q\}$ is diffeomorphic to the total space of the tautological line bundle τ.*

Proof. Let us consider a hypersurface $H \cong \mathbb{CP}^{n-1}$ in $\overline{\mathbb{CP}}^n$ that does not contain the point Q. Given any point P, the complex line L_{PQ} intersects H at a single point, and therefore, we may define the map

$$\begin{array}{rcl} \pi : \overline{\mathbb{CP}}^n - \{Q\} & \mapsto & H \\ P & \rightarrow & L_{PQ} \cap H \end{array} .$$

that presents $\overline{\mathbb{CP}}^n - \{Q\}$ as a line bundle over $\mathbb{CP}^{n-1}$.

Any rational curve transversal to the fibers of this bundle and not contained in H must intersect H at a single point. Since this is an intersection of complex submanifolds, the intersection must be positive. Therefore,

$$\langle c_1(\overline{\mathbb{CP}}^n - \{Q\}), [\mathbb{CP}^1] \rangle = -1 \, ,$$

which shows that the first Chern class of this bundle and that of the tautological line bundle agree with one another. This fact characterizes these line bundles as diffeomorphic. □

Notice that the diffeomorphism above can be chosen to preserve the orientation of the fibers. Hence, if $\tilde{H}$ is the image of the zero section of τ, the induced diffeomorphism $\tilde{H} \mapsto H$ reverses the orientation because $\tilde{H}$ intersects the fibers positively, while H in $\overline{\mathbb{CP}}^n - \{P\}$ intersects the fibers negatively.

Furthermore, we see that a relatively compact neighborhood of the zero section of τ has closure whose boundary is topologically a sphere of dimension $2n-1$. By the very definition, we then conclude that as an oriented manifold, M_P is diffeomorphic to the connected sum $M \# \overline{\mathbb{CP}}^n$.

The exceptional divisor E of the blow-up of a complex surface M at one point is a curve of self-intersection -1. Complex surfaces without rational curves of self-intersection -1 are said to be minimal, and are modeled by $\mathbb{CP}^2$ and the Hirzebruch surfaces $\mathfrak{F}_n$, the projectivization of the bundle $H^n \oplus \mathbb{1}$, where H is a hyperplane bundle associated to a hyperplane section of $\mathbb{CP}^2$ (this bundle is the dual of the tautological line bundle). Hence, surfaces with -1 curves are obtained from these minimal models by blow-up operations.

The canonical bundles of M_P and M are related by $\kappa_{M_P} = \pi^*\kappa_M + (n-1)e$ where e is the line bundle defined by the exceptional divisor E. It then follows that for surfaces we have that

$$c_1(M_P) = \pi^* c_1(M) - e\,,$$

where e is the cohomology class Poincaré dual to E. If C is a curve that passes through P, the self-intersections of C and its lift $\hat{C}$ are related by

$$\hat{C}^2 = C^2 - 1\,.$$

The following result, to which we shall make references later on, classifies complex surfaces with $c_1 > 0$. It is due to N. Hitchin [12]

Proposition 43. *Let M be a complex surface. If $c_1(M) > 0$ then $M \cong \mathbb{S}^2 \times \mathbb{S}^2$ or M can be obtained from $\mathbb{CP}^2$ by blowing-up k points in general position, $k \leq 8$.*

Proof (*sketch*). Since $c_1 > 0$ is an integer class, M is an algebraic variety with all its plurigenera $\dim H^0(M, O(\kappa^m))$ equal to zero. By a theorem of Castelnuovo, M is rational (birationally equivalent to $\mathbb{CP}^2$). Then M must be obtained from $\mathbb{CP}^2$ via blow-ups and blow-downs.

A criterion of Nakai says that $c_1 > 0$ if, and only if, $c_1^2 > 0$ and $c_1 \cdot [C] > 0$ for any curve C.

For any complex surface S, if C is a curve of genus $g = g(C)$ with self-intersection $[C]^2$, we have Noether's relation

$$2g(C) - 2 = [C]^2 - c_1(S)[C]\,.$$

Consequently, if C is rational ($\cong \mathbb{CP}^1$), we have that

$$[C]^2 > -2\,,$$

whenever $c_1 > 0$. Since the Hirzebruch surface $\mathfrak{F}_n$ has a curve of self-intersection $-n$, $\mathfrak{F}_n$ cannot have $c_1 > 0$ for $n > 1$. Therefore, the minimal model for M with $c_1 > 0$ must be either $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$. Since the blow-up of an exceptional divisor gives us a curve of self-intersection -2, the expression above implies that M must be obtained by blow-ups of the minimal model at different points. Now the blow-up at a point of $\mathbb{CP}^1 \times \mathbb{CP}^1$ is equivalent to the blow-up at two points of

$\mathbb{CP}^2$. Then M must be obtained from $\mathbb{CP}^2$ by blowing-up k points, and since $c_1(M)^2 = 9-k$, we must have $k \leq 8$. □

5.1. **Extremal metrics on the blow-up of $\mathbb{CP}^n$ at one point.** Now we would like to construct examples of extremal metrics on $\mathbb{CP}^n \# \overline{\mathbb{CP}}^n$. These examples were originally discovered by Calabi [6], though the approach we use here to show this result is slightly different [23] from the one he originally used.

We initially search for extremal metrics on the non-compact manifold $\mathbb{C}^n$ with $\mathbb{U}(n)$ symmetry. We do so by seeking a Kähler potential function ϕ in $\mathbb{C}^n$ which only depends upon $u = z_1\bar{z}_1 + \cdots + z_n\bar{z}_n$. The $(1,1)$-form

$$\omega = i\partial\overline{\partial}\phi$$

will be the Kähler form of a Kähler metric provided it is positive. If we let the expression

$$\omega^{\wedge n} = i^n V dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 \wedge \ldots \wedge dz^n \wedge d\bar{z}^n$$

define the function V, then the Ricci form ρ is given by

$$\rho = -i\partial\overline{\partial} \log V \ .$$

We find the equation defining the scalar curvature s in terms of ϕ and $\psi = \log V$. By the $\mathbb{U}(n)$-invariance of the metric, we do not loose generality if we carry out the calculations at points with $z^2 = z^3 = \ldots = z^n = 0$, where $u = z^1\bar{z}^1$. Then, it is fairly easy to see that

$$s\dot{\phi}(\dot{\phi} + u\ddot{\phi}) = -2\dot{\phi}(\dot{\psi} + u\ddot{\psi}) - 2(n-1)\dot{\psi}(\dot{\phi} + u\ddot{\phi}) \ , \tag{47}$$

equation that is clearly equivalent to

$$\dot{s} = \frac{1}{(u\dot{\phi})^n}\frac{d}{du}\left(su^n\dot{\phi}^n + 2nu^n\dot{\phi}^{n-1}\dot{\psi}\right) \ .$$

Thus, the vector field $\partial^{\#}s$ is given by

$$\partial^{\#}s = \sum_j \frac{\dot{s}z^j}{\dot{\phi} + u\ddot{\phi}}\frac{\partial}{\partial z^j} = \sum_j \frac{z^j}{\dot{\phi} + u\ddot{\phi}}\frac{1}{(u\dot{\phi})^n}\frac{d}{du}\left(su^n\dot{\phi}^n + 2nu^n\dot{\phi}^{n-1}\dot{\psi}\right)\frac{\partial}{\partial z^j} \ ,$$

and the metric is extremal iff

$$\frac{d}{du}\frac{\dot{s}}{\dot{\phi} + u\ddot{\phi}} = 0 \, .$$

Hence, we must solve the non-linear differential equation

$$\frac{d}{du}s = c\frac{d}{du}(u\dot{\phi}) \ , \tag{48}$$

for some constant c.

If $c=0$ our metric, if it exists, will have constant scalar curvature. Otherwise, we will have an extremal metric of non-constant scalar curvature.

Integration of (48), together with the use of (47), results into the equation

$$
2\frac{d}{du}(u\dot{\psi}) + 2(n-1)\frac{\dot{\psi}}{\dot{\phi}}\frac{d}{du}(u\dot{\phi}) + c(u\dot{\phi})\frac{d}{du}(u\dot{\phi}) = -A\frac{d}{du}(u\dot{\phi}) \,, \tag{49}
$$

where A is a constant that arises in the integration process. This equation can be re-expressed as

$$
2\frac{d}{du}(\zeta^{n-1}\eta) + c\zeta^n\dot{\zeta} = -A\zeta^{n-1}\dot{\zeta} \,,
$$

where $\zeta = u\dot{\phi}$, $\eta = u\dot{\psi}$, respectively. Then,

$$
\eta + \frac{c}{2(n+1)}\zeta^2 = -\frac{A}{2n}\zeta + B\zeta^{1-n} \,, \tag{50}
$$

for B another constant of integration. If we now write V in terms of ζ and its derivative, we conclude that

$$
\eta = u\left(\frac{\ddot{\zeta}}{\dot{\zeta}} + \frac{(n-1)\dot{\zeta}}{\zeta} - \frac{n-1}{u}\right) \,,
$$

and using this identity into the previous expression, and writing the result as an equation for ζ, we obtain

$$
\frac{d}{du}\left(u\zeta^{n-1}\dot{\zeta}\right) - n\zeta^{n-1}\dot{\zeta} + \frac{c}{2(n+1)}\zeta^{n+1}\dot{\zeta} = -\frac{A}{2n}\zeta^n\dot{\zeta} + B\dot{\zeta} \,,
$$

which itself produces

$$
u\zeta^{n-1}\dot{\zeta} - \zeta^n + \frac{c}{2(n+1)(n+2)}\zeta^{n+2} = -\frac{A}{2n(n+1)}\zeta^{n+1} + B\zeta + C \,, \tag{51}
$$

for yet another constant of integration C.

In seeking metrics on the blow-up of $\mathbb{CP}^n$ at one point, we assume that the potential ϕ is a function of the form

$$
\phi(u) = a\log u + p(u) \,, \ a > 0 \,, \tag{52}
$$

where, in principle, we only required $p(u)$ to be C^2 (later on we shall see that equation (51) on such a function will imply that p is smooth). If we can then solve the extremal equation for $p(u)$, the resulting singular metric on $\mathbb{C}^n$ lifts to a smooth extremal metric on the blow-up of $\mathbb{C}^n$ at 0.

Notice that $V = n!\dot{\phi}^{n-1}(\dot{\phi} + u\ddot{\phi})$. Then, if

$$\dot{p}(0) = c_1\,, \quad \ddot{p}(0) = c_2\,,$$

the constants A, B and C are completely determined by (49), (50), and (51) in terms of these values. Indeed, we have

$$\left\{ \begin{aligned} A &= \frac{2(n-1)(n-2)}{a} - \frac{4c_2}{c_1^2} - ca\,, \\ B &= -\frac{a^{n-1}}{n}\left(2(n-1) + \frac{2c_2}{c_1^2}a + \frac{c}{2(n+1)}a^2\right), \\ C &= \frac{a^n}{n+1}\left(2(n-2) + \frac{2c_2}{c_1^2}a + \frac{c}{2(n+2)}a^2\right). \end{aligned} \right. \tag{53}$$

For ϕ as in (52), we have $\zeta = u\dot{\phi} = a + u\dot{p}$. If we write equation (51) in terms of p, after some simplifications, we obtain

$$\begin{aligned} (a+u\dot{p})^{n-1}u^2\ddot{p} = &-\frac{c}{2(n+1)(n+2)}((a+u\dot{p})^{n+2} - a^{n+2} - (n+2)a^{n+1}u\dot{p}) \\ &-\frac{A}{2n(n+1)}((a+u\dot{p})^{n+1} - a^{n+1} - (n+1)a^n u\dot{p}) \\ &+((a+u\dot{p})^n - a^n - na^{n-1}u\dot{p}) - u\dot{p}((a+u\dot{p})^{n-1} - a^{n-1})\,. \end{aligned}$$

Setting $w = \dot{p}$, this becomes

$$\begin{aligned} \dot{w} = &-\frac{c((a+uw)^{n+2} - a^{n+2} - (n+2)a^{n+1}uw)}{2(n+1)(n+2)u^2(a+uw)^{n-1}} \\ &-\frac{A((a+uw)^{n+1} - a^{n+1} - (n+1)a^n uw)}{2n(n+1)u^2(a+uw)^{n-1}} \\ &+\frac{a(a+uw)^{n-1} - a^n - (n-1)a^{n-1}uw}{u^2(a+uw)^{n-1}} \\ = &\ f(u,w)\,, \end{aligned} \tag{54}$$

where we let the expression define the function $f(u,w)$.

Since the function $f(u,w)$ is smooth, this equation (54) for $w = \dot{p}$ has a unique solution in a neighborhood of $u = 0$, provided one supplies an initial condition $w(0) = \dot{p}(0) = c_1$. By the theorem of existence and uniqueness to ordinary differential equations, there is one such smooth solution defined on a maximal domain $[0, \alpha)$. By integration we obtain p, and in principle, an extremal potential (52) defining an extremal metric in some subset of the blow-up of $\mathbb{C}^n$ at the origin.

We need to check that $\omega = i\partial\overline{\partial}\phi$ is the Kähler form of a Kähler metric. For that, we must require that the initial condition for $\dot{p}(u)$ be positive. For in that case, the solution to (54) remains positive for any u in $[0, \alpha)$. Indeed, if this were not the case, $w(u)$ would be zero at some positive value of u, which will force $\dot{w}(u)$ to be zero at that said value as well. By uniqueness of solutions of the differential equation (54), $w(u)$ would have to be identically zero.

We may find solutions that are defined on $[0, \infty)$. For if the constant c is strictly positive, the solution $w(u)$ cannot blow-up in finite time, that is to say, $w(u)$ cannot go to infinity for finite values of u. If $w \to +\infty$ at some nonzero u, the function $f(u, w)$ would be asymptotically equal to $-cuw^3/2(n+1)(n+2)$, which is negative. Therefore, when $c > 0$, the function w is decreasing when its value is large. Thus, $\alpha = \infty$ in these cases.

Hence, by merely assuming that $c_1 > 0$ and that $c > 0$, the solution to (54) exists for $u \in [0, \infty)$, and is strictly positive. Furthermore, $\dot{\phi} + u\ddot{\phi}$ must be positive. For is that were not the case, at some u, say u_0, we would have $\zeta_0 = u_0\dot{\phi}(u_0)$ a root of the polynomial defined by the terms in equation (51) which do not involve $\dot{\zeta}$. By uniqueness of solutions to ordinary differential equations, ζ would then have to be the constant ζ_0. That cannot be because ζ varies non-trivially nearby $u = 0$. Therefore, $\dot{\phi} + u\ddot{\phi} = \dot{p} + u\ddot{p} > 0$, and consequently, $V = n!\dot{\phi}^{n-1}(\dot{\phi} + u\ddot{\phi})$ is always positive. This shows that the form $\omega = i\partial\overline{\partial}\phi$ is non-degenerate and positive. So

$$p(u) = c_1 u + \int_0^u w(\tau)d\tau$$

is such that (52), translated to the blow-up $\overline{\mathbb{CP}^n}\#\mathbb{C}^n$ of $\mathbb{C}^n$ at the origin, is the potential function of an extremal Kähler metric of non-constant scalar curvature.

In order to obtain the desired result on $\mathbb{CP}^n\#\overline{\mathbb{CP}^n}$, we compactify suitably the open manifold above at infinity.

For that, notice that in addition to proving that the derivative of $u\dot{\phi}$ is positive, we have also shown that this function is bounded on $[0, \infty)$. Hence, $\lim_{u\to\infty} u\dot{\phi} = b > a$, and equation (51) implies that $\zeta = u(\dot{p} + u\ddot{p})$ converges to some limit as $u \to \infty$. This limit is necessarily zero, as we argue next.

For simplicity, let us set

$$P(\zeta) = -\frac{c}{2(n+1)(n+2)}\zeta^{n+2} - \frac{A}{2n(n+1)}\zeta^{n+1} + \zeta^n + B\zeta + C\,.$$

Then, (51) can be written as

$$u\zeta^{n-1}\dot{\zeta} = P(\zeta)\,.$$

It follows that $\zeta(0) = a$ and $\zeta(\infty) = b$ are both roots of $P(\zeta)$. Furthermore, they must be simple roots. The residue of $\zeta^{n-1}/P(\zeta)$ at $\zeta_0 = a$ or $\zeta_0 = b$ can be computed by the limit $\lim_{\zeta\to\zeta_0}\zeta^{n-1}/P'(\zeta)$. Given the value of the constants in (53), we see that the residue at $\zeta_0 = a$ is equal to 1, and we will see below that in order to extend the metric at $u = \infty$, the residue at $\zeta_0 = b$ must be -1.

Near $u = \infty$ the metric can be described explicitly. Let us introduce a new coordinate $r = \sqrt{\zeta}$, and choose a local orthonormal coframe $\sigma_1, \ldots, \sigma_{2n-1}$ for the sphere $\mathbb{S}^{2n-1}$ that agrees with $dx^2, dy^2, \ldots, dx^{2n-2}$, dy^{2n-2}, dy^1 at the point $(z^1, \ldots, z^n) = (1, 0, \ldots, 0)$. Here $z^k = x^k + iy^k$. Since

$$-i\omega = (\dot{\phi} + u\ddot{\phi})dz^1 \wedge d\bar{z}^1 + \dot{\phi}\sum_{j=2}^{n} dz^j \wedge d\bar{z}^j\,,$$

we obtain that

$$g = \frac{d\zeta}{du}\left(\frac{du^2}{4u} + u\sigma_{2n-1}^2\right) + r^2(\sigma_1^2 + \ldots + \sigma_{2n-2}^2)\,.$$

But

$$\frac{d\zeta}{du}\frac{du^2}{4u} = \frac{\zeta dr^2}{u\dot{\zeta}} = \frac{\zeta^n dr^2}{P(\zeta)} = \frac{r^{2n}}{P(r^2)}dr^2\,.$$

Similarly,

$$u\dot{\zeta}\sigma_{2n-1}^2 = \frac{P(r^2)}{r^{2(n-1)}}\sigma_{2n-1}^2\,.$$

Therefore,

$$g = \frac{r^{2n}}{P(r^2)}dr^2 + \frac{P(r^2)}{r^{2(n-1)}}\sigma_{2n-1}^2 + r^2(\sigma_1^2 + \ldots + \sigma_{2n-2}^2)\,.$$

The first two summands in the expression above should be viewed as the fiber metric of a bundle whose base space has metric given by the last summand. This base space is a $\mathbb{CP}^{n-1}$ with a Fubini-Study metric. Since $r^2 \to b$ as $u \to \infty$, its scalar curvature approaches $4n(n-1)/b$. This ad hoc argument can be fully justified if we show that the metric extends smoothly at ∞.

For that, we would like to find a new coordinate $v = \beta(r)$ such that the fiber metric above looks like

$$(1 + O(v^2))dv^2 + v^2(1 + O(v^2))\sigma^2_{2n-2}\,,$$

near $r^2 = b$. Let a_0 be the residue of $\zeta^{n-1}/P(\zeta)$ at $\zeta_0 = b$. Since

$$\frac{r^{2(n-1)}}{P(r^2)} = \frac{a_0}{r^2 - b} + Q(r^2 - b)\,,$$

with Q a regular function near b, we can choose

$$\beta^{'}(r) = -a_0 \frac{r}{\sqrt{b - r^2}}\,,$$

which implies that $v = \beta(r) = a_0\sqrt{b - r^2}$. Since we then want $P(r^2)/r^{2(n-1)}$ to be of the form $v^2(1 + O(v^2))$, we must have

$$a_0^2(b - r^2) = \frac{r^2 - b}{a_0}\,,$$

which then implies that $a_0 = -1$. Notice that the constants in (53) must then be related to c and b by

$$-b^{n-1} = -\frac{c}{2(n+1)}b^{n+1} - \frac{A}{2n}b^n + nb^{n-1} + B\,,$$

from which we obtain

$$\frac{2c_2}{c_1^2} = \frac{n}{b^n - a^n}\left[\frac{c}{2}\left(\frac{b^{n+1} + a^{n+1}}{n+1} - \frac{ab^n}{n}\right) + \frac{n-1}{n}\left(\frac{(n-2)b^n + 2a^n}{a}\right) - (n+1)b^{n-1}\right]\,.$$

This permits to express the constants in (53) in terms of the cohomological data a, b, and the dimension n. The free parameter c_2 is thus determined by the size of the exceptional curve and the size of the projective hyperplane at infinity.

With this restriction on the residue at $\zeta_0 = b$, the metric constructed extends smoothly across $u = \infty$, and defines an extremal metric on $\mathbb{CP}^n \# \overline{\mathbb{CP}^n}$.

We may easily compute the volume of the resulting manifold with this metric. This is just $2^n\mu(S^{2n-1})(b^n - a^n)/n$, where $\mu(S^{2n-1})$ is the volume of the $(2n-1)$-dimensional unit sphere.

5.2. **Kähler-Einstein metrics.** In attempting to give a compact manifold a canonical shape, we can seek on it a Riemannian metric g whose Ricci tensor r is a pointwise multiple of g itself: $r = fg$. Such a metric is said to be *Einstein*. Since the contracted second Bianchi identity implies that $ds = 2\delta r$, where s is the scalar curvature, we see that if $r = fg$ and the dimension of M is 3 or greater, the function f must be constant.

In the Kähler case, both the Ricci tensor and the metric are J-invariant, and the Einstein condition may be reformulated in terms of the Ricci and Kähler forms by requiring that

$$\rho = \frac{s}{2n}\omega \, . \tag{55}$$

Here n is the complex dimension and s is the (necessarily constant) scalar curvature. Since ρ represents a multiple of the first Chern class and ω is positive, for this equation to admit a solution it must be the case that $c_1 = c_1(M, J)$ be either positive, negative or zero.

Definition 44. The cohomology class $c_1 = c_1(M, J)$ is said to be positive (resp. negative) if there exists a (1,1)-form that represents it that is positive definite (resp. negative definite).

Notice that in seeking solutions of (55), the sign of the constant s is determined by the sign of the topological quantity s_{tot} given in (29). We thus define the space of metrics

$$\mathfrak{M}_{c_1} = \{\omega : \ \omega \in \mathfrak{M}, \ \frac{c_1 \cup [\omega]^{n-1}}{[\omega]^n}\omega \text{ represents } c_1\} \, .$$

Suppose then that ω is a Kähler metric in $\mathfrak{M}_{c_1}$. Is it possible to deform ω in $\mathfrak{M}_{c_1}$ to a metric $\tilde{\omega} = \omega_0 + i\partial\overline{\partial}\varphi$ that is Einstein? Calabi [8] conjectured this to be the case when $c_1 \leq 0$. He also conjectured it when $c_1 > 0$ and (M, J) has no non-trivial holomorphic vector fields. This restriction in the positive case needs to be added because of obstructions arising from the structure of $\mathfrak{h}$ and, for example, the blow-up of $\mathbb{CP}^2$ at one or two points are Kähler manifolds of positive first Chern class that could not possibly admit metrics of constant scalar curvature.

The one outlined above constitutes what is often called the *second* Calabi conjecture. On the other hand, the *first* conjecture says that given a Kähler metric ω, any real 2-form of type (1,1) that represents the cohomology class $2\pi c_1$ must be the Ricci form of one and only one Kähler metric cohomologous to ω.

Yau proved the first conjecture [26]. He used the technique that we describe here to present his proof of the second conjecture when $c_1 < 0$. This latter result was also proven independently by Aubin [1].

In the positive case, Calabi's second conjecture is still true for Kähler surfaces, as proven by Tian [25]. In fact, Tian showed that if there are no obstruction arising from the structure of $\mathfrak{h}$, any Kähler surface of positive first Chern class admits a Kähler Einstein metric. Tian himself discovered later on that the second conjecture is false.

Our presentation here is intended to illustrate a very useful technique in studying problems in global analysis.

5.2.1. *Monge-Ampère equation for extremal metrics.* We begin by finding a fully non-linear second order equation satisfied by the potential function $\varphi \in L^2_{k+4,G}$ of an extremal metric $\tilde{\omega} = \omega + i\partial\overline{\partial}\varphi$. For the time being, we do so in general, without making any assumptions on the relationship between $\Omega = [\omega]$ and c_1.

The cohomology class of the Ricci form of any Kähler metric is fixed. Using this fact, we can easily prove the following elementary result:

Lemma 45. *Let (M, J, ω) be any Kähler manifold with Kähler form ω and $v = [\omega]^n/n!$ be the volume of M of metrics representing the class $[\omega]$. Then the Ricci form ρ decomposes as*

$$\rho = \frac{s_{tot}}{2nv}\omega + \rho_0 + i\partial\overline{\partial} f\,, \tag{56}$$

where s_{tot} is given by (29), ρ_0 is a harmonic (1,1)-form of zero trace, and f is a smooth real valued function perpendicular to the constants.

If $\tilde{\omega} = \omega + i\partial\overline{\partial}\varphi$ is another metric representing the same cohomology class as that represented by ω, the corresponding decomposition of the Ricci form $\tilde{\rho}$ is given by

$$\tilde{\rho} = \frac{s_{tot}}{2nv}\tilde{\omega} + \tilde{\rho}_0 + i\partial\overline{\partial}\tilde{f}\,,$$

where $\tilde{\rho}_0 = \rho_0 + 2i\partial\overline{\partial}\tilde{G}(\mathrm{trace}_{\tilde{g}}\rho_0)$ and $\tilde{G}$ is the Green's operator of the metric $\tilde{\omega}$.

Proof. All the statements are straightforward, except perhaps for the fact that $\tilde{\rho}_0$ is $\tilde{\omega}$-harmonic and trace-free.

Observe that $\mathrm{trace}_{\tilde{g}}\rho_0$ has $\tilde{\omega}$-projection onto the constant identically equal to zero. This just follows from the fact that

$$\int \mathrm{trace}_{\tilde{g}}\rho_0\, d\tilde{\mu} = \int \mathrm{trace}_{g}\rho_0\, d\mu = 0\,.$$

And evidently $\tilde{\rho}_0$ is ∂ and $\bar{\partial}$ closed. We just need to prove that it is also $\bar{\partial}^*$ closed. But

$$\bar{\partial}^* \tilde{\rho}_0 = -i[\tilde{\Lambda}, \partial]\tilde{\rho} = i\partial\tilde{\Lambda}\tilde{\rho}_0 = i\partial(\mathrm{trace}_{\tilde{g}}\rho_0 - \mathrm{trace}_{\tilde{g}}\rho_0) = 0\,.$$

In obtaining the second equality above, we have used the fact that the harmonic component of $\mathrm{trace}_{\tilde{g}}\rho_0$ is zero. □

Since the cohomology class of the Ricci form of any Kähler metric is fixed, given two Kähler metrics ω and $\omega + i\partial\overline{\partial}\varphi$ representing the same cohomology class, we have that

$$\frac{\det\,(g_{i\bar{\jmath}} + \varphi_{i\bar{\jmath}})}{\det\,(g_{i\bar{\jmath}})} = e^F\,, \tag{57}$$

for some function F. By Lemma 45, we conclude that

$$\begin{aligned} \tilde{\rho} &= \rho - i\partial\overline{\partial}F \\ &= \frac{s_{tot}}{2nv}\omega + \rho_0 + i\partial\overline{\partial}f - i\partial\overline{\partial}F \\ &= \frac{s_{tot}}{2nv}\tilde{\omega} + \rho_0 + i\partial\overline{\partial}f - \frac{\sigma_{tot}}{2nv}i\partial\overline{\partial}\varphi - i\partial\overline{\partial}F\,. \end{aligned}$$

If we now take the trace of this expression, we obtain that

$$\tilde{s} = \frac{s_{tot}}{v} + 2\mathrm{trace}_{\tilde{g}}\rho_0 + \tilde{\Delta}\left(\frac{s_{tot}}{2nv}\varphi + F - f\right)\,.$$

Notice that by (29), the constant s_{tot} only depends on Ω and the complex structure.

If the metric $\tilde{g}$ is to be extremal, then $\tilde{s} = \pi_{\tilde{g}}\tilde{s}$, that is to say, $\tilde{s}$ must be a linear combination of the set of functions (37) spanning the space of real holomorphy potentials. Thus, the equation for the potential φ ensuring extremality of the metric can be written as

$$\tilde{s} = \pi_{\tilde{g}}\tilde{s} = \sum_{\jmath=0}^{m} c_\jmath f^\jmath_{\tilde{\omega}} = \frac{s_{tot}}{v} + 2\mathrm{trace}_{\tilde{g}}\rho_0 + \tilde{\Delta}\left(\frac{s_{tot}}{2nv}\varphi + F - f\right) \tag{58}$$

for some (real) constants $\{c_0, \ldots, c_m\}$. These constants are themselves functions of the potential φ.

We thus have the following

Proposition 46. *Let (M, J, Ω) be a compact polarized Kähler manifold of dimension n, and g be a metric representing the class Ω that is invariant under a maximal compact subgroup G of the biholomorphism group of (M, J). The metric*

$$\tilde{g}_{i\bar{\jmath}} = g_{i\bar{\jmath}} + \varphi_{i\bar{\jmath}}$$

is extremal if, and only if, φ is a G-invariant solution of the equation

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{-\frac{s_{tot}}{2nv}\varphi + f + G_{\tilde{g}}(\pi_{\tilde{g}}\tilde{s} - 2\mathrm{trace}_{\tilde{g}}\rho_0) + c}\,.$$

In this expression, $G_{\tilde{g}}$ is the Green's operator of the extremal metric, the function f and (1,1)*-form ρ_0 are those appearing in the decomposition* (56) *for the Ricci tensor of g, and c is a constant.*

5.2.2. *Kähler Einstein metrics: the continuity method.* Let us consider now a complex manifold (M, J) of Kähler type with $c_1 < 0$. In this case, the space of holomorphic vector fields is trivial, and the space of holomorphy potentials reduces to the constants. We consider the Monge-Ampère equation given by Proposition 46 with $\rho_0 = 0$ and absorb the constant c into φ itself. Thus, the equation for an extremal metric in the class of $-c_1$ is given by

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{-\frac{s_{tot}}{2nv}\varphi + f}\,.$$

Since $s_{tot}[\omega]/4\pi nv$ and $-c_1$ are multiples of each other, if such an extremal metric exists it must be Einstein.

Yau solved this equation by considering the one parameter family of equations

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{-\frac{s_{tot}}{2nv}\varphi + tf} \tag{59}$$

for values of t in the interval $[0, 1]$. He proved that the set

$$S = \{t \in [0,1] : \text{ (59) has an admissible solution } \varphi_t\} \tag{60}$$

is both, open and closed in the connected set $[0, 1]$. Since S is non-empty —$\varphi \equiv 0$ is the unique solution of (59) for $t = 0$— it must be the case that $S = [0, 1]$. The admissible function φ corresponding to $t = 1$ is the potential that deforms g to an Einstein metric.

The uniqueness of solutions to (59) was proven by Calabi [8] as an application of the maximum principle:

Proposition 47. [8] *Solutions to* (59) *are unique.*

Proof. Let φ^1 and φ^2 be two admissible solutions of the equation. Then we obtain that

$$\frac{\det(g_{i\bar{j}} + \varphi^1_{i\bar{j}} - \varphi^2_{i\bar{j}})}{\det(g_{i\bar{j}} + \varphi^2_{i\bar{j}})} = e^{-\frac{s_{tot}}{2nv}(\varphi^1 - \varphi^2)}\,.$$

At a point p where $\varphi^1 - \varphi^2$ achieves its maximum, $\varphi^1_{i\bar{j}} - \varphi^2_{i\bar{j}}$ is non-positive, and the quotient on the left side above must be less or equal than 1. This implies that

$$\max -\frac{s_{tot}}{2nv}(\varphi^1 - \varphi^2) = -\frac{s_{tot}}{2nv}(\varphi^1 - \varphi^2)(p) \leq 0\,,$$

that in turn implies that $\max\,(\varphi^1 - \varphi^2) \leq 0$ because s_{tot} is negative.

A similar argument shows that $\min\,(\varphi^1 - \varphi^2) \geq 0$. Thus, $\varphi^1 - \varphi^2 = 0$. □

5.2.3. *Openness of S.* We prove here that the set S in (60) is open. This result will be a consequence of the inverse function theorem.

Indeed, given a point $t_0 \in S$, let φ_{t_0} be the associated admissible solution. We consider the mapping

$$(t, \varphi) \mapsto (t, \log\det\,(g_{i\bar{j}} + \varphi_{i\bar{j}}) - \log\det\,(g_{i\bar{j}}) + \frac{s_{tot}}{2nv}\varphi)\,,$$

in a neighborhood of (t_0, φ_{t_0}). The function φ in the domain of this mapping can be taken to live in some appropriate Banach space of functions, for example, L^2_{k+2} with $k > n$.

We linearize the mapping at (t_0, φ_{t_0}). We prove the desired openness of S by showing that the linearization of the second component of the mapping at φ_{t_0} is an isomorphism, which suffices for this purpose. Suppose we vary φ_{t_0} infinitesimally in the direction of ψ. Since the Ricci potential of a Kähler metric is the logarithm of the determinant of the matrix of components of the metric, by Corollary 19 we see that derivative of the mapping in question at φ_{t_0} and in the direction of ψ is given by

$$\psi \mapsto -\frac{1}{2}\Delta_{t_0}\psi + \frac{s_{tot}}{2nv}\psi\,,$$

where Δ_{t_0} is the Laplacian of the metric $\omega + i\partial\overline{\partial}\varphi_{t_0}$. This operator is elliptic and symmetric. With its chosen domain and range, it is therefore Fredholm, and since $c_1 < 0$ and so $s_{tot} < 0$, its kernel is trivial. Hence, it is an isomorphism, as desired.

5.2.4. *Closedness of S.* In order to show that S is a closed set, Yau developed *a priori* t-independent estimates in $C^{k,\alpha}(M)$ for the solutions to (59). He then used these estimates to conclude that if $\{t_n\}$ is a sequence in S such that $t_n \nearrow t$, the sequence of solutions φ_{t_n} must be uniformly bounded in $C^{k,\alpha}(M)$. Since the inclusion $C^{k,\alpha_0}(M)$ into $C^{k,\alpha}(M)$ is compact for $\alpha_0 < \alpha$, this sequence has a subsequence, say φ_{t_n} itself, that converges to a function φ_t in $C^{k,\alpha_0}(M)$. The regularity of φ_{t_n} permits the passage to the limit within the non-linear equation. Hence, φ_t is a solution of (59) associated to t, and therefore, $t \in S$. Thus, S is closed.

The a priori estimates consists of estimates of type $C^{2,\alpha}$ for the solution to (59) that are independent of t and φ. With this estimates in hand, we can get a priori $C^{3,\alpha}$ estimates for φ in terms of the $C^{2,\alpha}$ bound. Indeed, locally $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \psi$ for some function ψ, and applying the derivative ∂_k to the logarithm of (59), we obtain that

$$-\frac{1}{2}\Delta_{\tilde{g}}(\partial_k(\psi+\varphi)) = -\frac{1}{2}\Delta_g(\partial_k \psi) - \frac{s_{tot}}{2nv}\partial_k \varphi + t\partial_k f\,,$$

where $\Delta_{\tilde{g}}$ is the Laplacian of the metric $g_{i\bar{j}}+\varphi_{i\bar{j}}$, and Δ_g the Laplacian of $g_{i\bar{j}}$. The operators in this equation are elliptic with coefficients in $C^{0,\alpha}$. By Schauder's interior estimates for this type of operators, we conclude that φ is in $C^{3,\alpha}$, with a norm that only depends upon $\|\varphi\|_{C^2,\alpha}$. Iteration of this argument will then show that φ is smooth.

We state the first estimate in the form of a proposition.

Proposition 48. *There exists a constant C such that any C^2 solution of* (59) *satisfies*

$$\sup_{x\in M} |\varphi(x)| \leq C\,.$$

This result is a very simple consequence of the maximum principle. The simplicity of this argument makes it very ironic that the estimate fails when $c_1 > 0$.

Proof. Let $p \in M$ where φ achieves its maximum. At this point, the matrix $\varphi_{i\bar{j}}$ is non-positive, and therefore, $\det(g_{i\bar{j}}+\varphi_{i\bar{j}})$ is less or equal than $\det(g_{i\bar{j}})$. By (59), it follows that

$$-\frac{s_{tot}}{2nv}\varphi(p) + tf(p) \leq 0\,.$$

We have seen that $c_1 < 0$ implies that $s_{tot} < 0$. Therefore,

$$\varphi(x) \leq \varphi(p) \leq -\frac{2nvt}{s_{tot}}(-f(p)) \leq -\frac{2nv}{s_{tot}}\sup(-f)\,.$$

Similarly, starting from a point where φ achieves it minimum, we obtain that

$$\varphi(x) \geq -\frac{2nv}{s_{tot}}\inf(-f)\,.$$

The desired C^0-estimate follows from these two estimates above. $\square$

Since $g_{i\bar{\jmath}} + \varphi_{i\bar{\jmath}}$ is a metric, its g-trace is positive. This fact leads to one of the bounds needed to obtain the C^2-estimate:

$$2n - \Delta_g \varphi \geq 0 \,.$$

We proceed now to show that $2n - \Delta_g \varphi$ is also bounded above, with a bound that is good enough for our purposes.

This is done by applying the maximum principle to the operator $\Delta_{\tilde{g}}$, the Laplacian of the metric $\omega + i\partial\overline{\partial}\varphi_t$, acting on a second order expression in φ. The action on this yet to be found expression is obtained by taking two traces of the curvature tensor, one with respect to the metric $\omega + i\partial\overline{\partial}\varphi$ and the other one with respect to the metric ω. The already proven C^0-estimate together with the equation (59) leads to the needed bound. Let us see the details.

In any complex coordinate system, the components of the curvature tensor of a Kähler metric are given by

$$R_{i\bar{\jmath}k\bar{l}} = -\partial_\imath \partial_{\bar{\jmath}} g_{k\bar{l}} + g^{p\overline{q}} \partial_\imath g_{k\overline{q}} \partial_{\bar{\jmath}} g_{p\bar{l}} \,.$$

Let us fix a point x in M, and let us choose *normal coordinates* for g such that $\tilde{g} = g_{i\bar{\jmath}} + \varphi_{i\bar{\jmath}}$ is diagonal at x. Then

$$\begin{aligned} \tilde{R}_{i\bar{\jmath}k\bar{l}} &= -\partial_\imath \partial_{\bar{\jmath}} g_{k\bar{l}} - \partial_\imath \partial_{\bar{\jmath}} \varphi_{k\bar{l}} + \tilde{g}^{p\overline{q}} \partial_\imath \tilde{g}_{k\overline{q}} \partial_{\bar{\jmath}} \tilde{g}_{p\bar{l}} \\ &= R_{i\bar{\jmath}k\bar{l}} - \partial_\imath \partial_{\bar{\jmath}} \varphi_{k\bar{l}} + \tilde{g}^{p\overline{q}} \partial_\imath \tilde{g}_{k\overline{q}} \partial_{\bar{\jmath}} \tilde{g}_{p\bar{l}} \,. \end{aligned}$$

Since $\rho_{i\bar{\jmath}} = g^{k\bar{l}} R_{i\bar{\jmath}k\bar{l}}$, the traces of the previous expression with respect to $\tilde{g}^{k\bar{l}}$ and $g^{i\bar{\jmath}}$, respectively, lead to

$$g^{i\bar{\jmath}} \tilde{\rho}_{i\bar{\jmath}} = \tilde{g}^{k\bar{l}} \rho_{k\bar{l}} + g^{i\bar{\jmath}} \tilde{g}^{k\bar{l}} \tilde{g}^{p\overline{q}} \partial_\imath \tilde{g}_{k\overline{q}} \partial_{\bar{\jmath}} \tilde{g}_{p\bar{l}} - g^{i\bar{\jmath}} \tilde{g}^{k\bar{l}} \partial_\imath \partial_{\bar{\jmath}} \varphi_{k\bar{l}} \,.$$

In the identity above, the fourth order term in φ can be re-written in terms of the Laplacians of the metrics:

$$\begin{aligned} g^{i\bar{\jmath}} \tilde{g}^{k\bar{l}} \partial_\imath \partial_{\bar{\jmath}} \varphi_{k\bar{l}} &= \tilde{g}^{k\bar{l}} \partial_k \partial_{\bar{l}} (g^{i\bar{\jmath}} \partial_\imath \partial_{\bar{\jmath}} \varphi) - \tilde{g}^{k\bar{l}} R^{i\bar{\jmath}}{}_{k\bar{l}} \varphi_{i\bar{\jmath}} \\ &= \frac{1}{4} \Delta_{\tilde{g}} \Delta_g \varphi - \tilde{g}^{k\bar{l}} R^{i\bar{\jmath}}{}_{k\bar{l}} \tilde{g}_{i\bar{\jmath}} + \tilde{g}^{k\bar{l}} \rho_{k\bar{l}} \,. \end{aligned}$$

This leads to the expression

$$\frac{1}{4} \Delta_{\tilde{g}} \Delta_g \varphi = -g^{i\bar{\jmath}} \tilde{\rho}_{i\bar{\jmath}} + g^{i\bar{\jmath}} \tilde{g}^{k\bar{l}} \tilde{g}^{p\overline{q}} \partial_\imath \tilde{g}_{k\overline{q}} \partial_{\bar{\jmath}} \tilde{g}_{p\bar{l}} + \tilde{g}^{k\bar{l}} R^{i\bar{\jmath}}{}_{k\bar{l}} \tilde{g}_{i\bar{\jmath}} \,. \tag{61}$$

At x we have that $1+\varphi_{\imath\bar{\imath}}>0$ for $\imath=1,\dots,n$. Since the harmonic component of $\tilde{\rho}$ is non-positive, we conclude that

$$\frac{g^{\imath\jmath}\tilde{\rho}_{i\jmath}}{n-\frac{1}{2}\Delta_g\varphi}=\frac{\delta^{ij}\tilde{\rho}_{i\jmath}}{1+\varphi_{1\bar{1}}+\cdots+\varphi_{n\bar{n}}}\leq\frac{(1-t)\sum_\imath f_{\imath\bar{\imath}}}{1+\varphi_{1\bar{1}}+\cdots+\varphi_{n\bar{n}}}\leq C\sum_\jmath\frac{1}{1+\varphi_{\jmath\bar{\jmath}}}\,,$$

for some constant C that only depends on g. On the other hand,

$$\tilde{g}^{k\bar{l}}R^{i\jmath}{}_{k\bar{l}}\tilde{g}_{i\jmath}=\sum_{\imath,\jmath}\frac{1+\varphi_{\imath\bar{\imath}}}{1+\varphi_{\jmath\bar{\jmath}}}R_{\imath\bar{\imath}\jmath\bar{\jmath}}\,,$$

and therefore,

$$\mid\tilde{g}^{k\bar{l}}R^{i\jmath}{}_{k\bar{l}}\tilde{g}_{i\jmath}\mid\leq C(n-\frac{1}{2}\Delta_g\varphi)\sum_\jmath\frac{1}{1+\varphi_{\jmath\bar{\jmath}}}$$

where the constant C only depends upon g.

By the Cauchy-Schwarz inequality, we can prove that

$$g^{i\jmath}\tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}\partial_\imath\tilde{g}_{k\bar{q}}\partial_{\bar{\jmath}}\tilde{g}_{p\bar{l}}\geq\frac{|\nabla_{\tilde{g}}(n-\frac{1}{2}\Delta_g\varphi)|^2}{n-\frac{1}{2}\Delta_g\varphi}\,.$$

Indeed, we have that

$$|\nabla_{\tilde{g}}(n-\frac{1}{2}\Delta_g\varphi)|^2=\sum_{i,j,k}\frac{\partial_k\varphi_{i\bar{\imath}}\partial_{\bar{k}}\varphi_{j\bar{\jmath}}}{1+\varphi_{k\bar{k}}}\leq\left(\sum_i\left(\sum_k\frac{1}{1+\varphi_{k\bar{k}}}|\partial_k\varphi_{i\bar{\imath}}|^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\,,$$

and right side of this expression can be estimated by

$$\left(\sum_i 1+\varphi_{i\bar{\imath}}\right)\left(\sum_{i,k}\frac{1}{1+\varphi_{i\bar{\imath}}}\frac{1}{1+\varphi_{k\bar{k}}}|\partial_k\varphi_{i\bar{\imath}}|^2\right)\,,$$

making now clear the assertion made.

We use the three inequalities proven above, and (61), in order to derive an inequality for $\Delta_{\tilde{g}}\log{(n-\frac{1}{2}\Delta_g\varphi)}$. We obtain

$$\begin{aligned}-\frac{1}{2}\Delta_{\tilde{g}}\log{(n-\frac{1}{2}\Delta_g\varphi)} &= \frac{-\frac{1}{2}\Delta_{\tilde{g}}(n-\frac{1}{2}\Delta_g\varphi)}{n-\frac{1}{2}\Delta_g\varphi}-\frac{|\nabla_{\tilde{g}}(n-\frac{1}{2}\Delta_g\varphi)|^2}{(n-\frac{1}{2}\Delta_g\varphi)^2}\\ &\geq \frac{1}{n-\frac{1}{2}\Delta_g\varphi}\left[-g^{i\jmath}\tilde{\rho}_{i\jmath}+\tilde{g}^{k\bar{l}}R^{i\jmath}{}_{k\bar{l}}\tilde{g}_{i\jmath}\right]\\ &\geq -C\sum_\jmath\frac{1}{1+\varphi_{\jmath\bar{\jmath}}}\,.\end{aligned}$$

Since

$$-\frac{1}{2}\Delta_{\tilde g}\varphi = \sum_{\jmath} \frac{\varphi_{\jmath\bar\jmath}}{1+\varphi_{\jmath\bar\jmath}} = n - \sum_{\jmath}\frac{1}{1+\varphi_{\jmath\bar\jmath}}\,,$$

we re-express the inequality above as

$$\frac{1}{2}\Delta_{\tilde g}(\log{(n-\frac{1}{2}\Delta_g\varphi)} - (C+1)\varphi) \;\le\; (C+1)n - \sum_{\jmath}\frac{1}{1+\varphi_{\jmath\bar\jmath}}\,. \tag{62}$$

Let A be the function

$$A = \log{(n-\frac{1}{2}\Delta_g\varphi)} - (C+1)\varphi\,.$$

At a point p where A achieves its maximum, we have that $\Delta_{\tilde g}A \ge 0$, and (62) implies the estimate

$$\sum_{\jmath}\frac{1}{1+\varphi_{\jmath\bar\jmath}} \le (C+1)n\,,$$

where the left side is evaluated at p. But for any set of positive real numbers $y_1,\dots,y_n$, we have that

$$\sum_{i=1}^{n} y_1\dots\hat y_i\dots y_m \le (\sum y_j)^{n-1}\,.$$

Therefore, by (59) and the estimate above, we see that

$$\left(n-\frac{1}{2}\Delta_g\varphi\right)e^{\frac{s_{tot}}{2nv}\varphi - tf} = \left(n-\frac{1}{2}\Delta_g\varphi\right)\Pi_j\frac{1}{1+\varphi_{\jmath\bar\jmath}} \le \left(\sum_{\jmath}\frac{1}{1+\varphi_{\jmath\bar\jmath}}\right)^{n-1} \le C\,,$$

where the functions are computed at p, the point where A achieves its maximum. Since we already have a C^0-bound for φ, we conclude that at this point p we have $n-\frac{1}{2}\Delta_g\varphi < C$. But log is an increasing function. So the maximum of A and that of $n-\frac{1}{2}\Delta_g\varphi$ can be estimated in terms of one another. We thus obtain an estimate from above for this latter expression. This, combined with the lower estimate obtained earlier, produces

$$0 < n - \frac{1}{2}\Delta_g\varphi < C\,,$$

for a constant C that only depends upon the metric g. The ellipticity of Δ_g now implies the C^2-estimates for φ.

Proposition 49. *There exists a constant C such that any C^2 solution of* (59) *satisfies the estimate*

$$\|\varphi\|_{C^2} \le C\,.$$

The result above does not finish the job. We need to improve the estimate to one in $C^{2,\alpha}$ for $\alpha > 0$. This have been done in various manners since Yau's paper appeared. In here, we sketch the original approach that consists of obtaining an a priori C^3 estimate which, in turn, leads to bounds on the C^1 norm of the coefficients of the metric $\tilde{g}$. The desired $C^{2,\alpha}$ estimate then follows by an application of Shauder's interior estimates for elliptic operators with reasonably behaved coefficients.

Notice that in order to obtain a C^3 estimate, one would need to control all derivatives of the form $\frac{\partial^3 \varphi}{\partial z^i \partial \bar{z}^j \partial z^k}$ and $\frac{\partial^3 \varphi}{\partial z^i \partial \bar{z}^j \partial \bar{z}^k}$. Since φ is real-valued, it will be enough to control the first type of derivatives.

Consider the tensorial expression

$$\psi = \tilde{g}^{i\bar{r}} \tilde{g}^{\bar{j}s} \tilde{g}^{k\bar{t}} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}} \, .$$

The key point is that

$$\Delta_{\tilde{g}}(\psi - C_1 \Delta_g \varphi) \leq C_2 - C_3 \psi \, ,$$

for constants that only depend upon f, the metric g, and their derivatives. We skip the argument that proves this inequality, and refer the reader to [21] for details.

Let p be a point where $\psi - C_1 \Delta_g \varphi$ achieves its maximum. At p, we will have the left-side of the expression above greater or equal than zero, and so $\psi(p) \leq C$. Thus,

$$(\psi - C_1 \Delta_g \varphi)(p) \leq C - C_1 \Delta_g \varphi(p) \, .$$

So, for an arbitrary point q, we have that

$$(\psi - C_1 \Delta_g \varphi)(q) \leq (\psi - C_1 \Delta_g \varphi)(p) \leq C - C_1 \Delta_g \varphi(p) \, ,$$

and

$$\psi(q) \leq C + C_1(\Delta_g \varphi(q) - \Delta_g \varphi(p)) \, .$$

We thus obtain that

$$\psi \leq C + C_1(\sup \Delta_g \varphi - \inf \Delta_g \varphi) \, ,$$

and the estimate for ψ follows by Proposition 49.

As explained earlier, this C^3-estimate suffices to finish the proof that the set S is closed.

Exercise 8. (open problem) Let (M, J) be a manifold of Kähler type with $c_1 < 0$. Suppose that g is a Kähler metric that is not in $\mathfrak{M}_{c_1}$. If g is not itself extremal, we may try to deform it to an extremal Kähler metric representing the same cohomology class by solving the Monge-Ampère equation

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{-\frac{s_{tot}}{2nv}\varphi + f - 2G_{\tilde{g}}(\mathrm{trace}_{\tilde{g}}\rho_0)} .$$

Here, ρ_0 and f are the components of the Ricci form ρ of g described by Lemma 45. Notice that we now have $\rho_0 \neq 0$.

Consider the 1-parameter family of equations

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{-\frac{s_{tot}}{2nv}\varphi + t(f - 2G_{\tilde{g}}(\mathrm{trace}_{\tilde{g}}\rho_0))} .$$

(1) Show that this equation has a unique solution nearby $t = 0$.
(2) Is it solvable for all $t \in [0, 1]$? Is it solvable for all $t \in [0, 1]$ under further topological assumptions besides the negativity of c_1?

Remark 50. Under the conditions stated in this exercise, the extremal metric (should it exist) must be one of constant scalar curvature. We conjecture that this is always the case. The application of the continuity method requires now the analysis of a zeroth-order pseudo-differential operator, at least when obtaining the *a priori* C^0-estimate.

5.3. **The extremal cone of** $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$**.** We now study in detail a neighborhood of c_1 in the Kähler cone of $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$. The neighborhood in question is obtained by applying Theorem 37, so each class in it is one that can be represented by an extremal metric. By computing the Futaki invariant of a given class, we find all those that admit representatives of constant scalar curvature, and perhaps more importantly, we find that their complement is rather large. This shows rather explicitly the existence of many new examples of extremal metrics of non-constant scalar curvature.

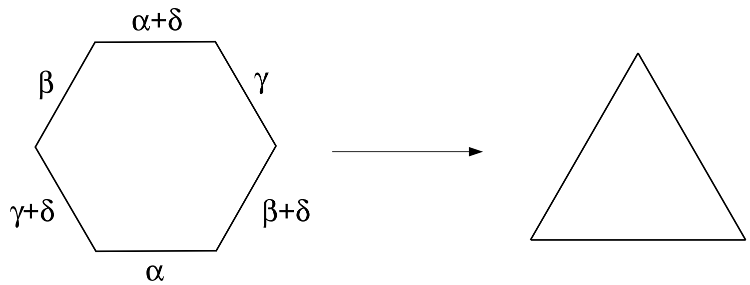
Suppose we were hoping to find non-constant scalar curvature extremal metrics on complex surfaces of positive Chern class. By Tian's proof [25] of the Calabi conjecture for surfaces with positive c_1, all of those for which the Lie algebra of the automorphism group is reductive admit Kähler-Einstein metrics representing c_1. Thus, all the manifolds $M_k = \mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$, $3 \leq k \leq 8$, carry Kähler-Einstein metrics, and we may apply Theorem 37 to obtain a neighborhood of c_1 consisting of extremal classes. If a class in this neighborhood does not have vanishing Futaki invariant, the extremal metric representing it is one of non-constant scalar curvature. Since any extremal metric on a manifold with no non-trivial holomorphic vector fields has constant scalar curvature, this plan will not work out for M_k, $4 \leq k \leq 8$, because in those cases, $\mathfrak{h}(M_k) = \{0\}$. This leaves us with M_3 as the only possibility to consider.

In our calculations, we shall make use of the following result [14], which we state without proof. Let us recall that an action of $\mathbb{C}^*$ on a compact manifold M is semi-free if the set of isotropy groups of the action is exactly $\{\{1\}, \mathbb{C}^*\}$.

Theorem 51. *Let (M, J) be any compact complex surface equipped with a semi-free holomorphic $\mathbb{C}^*$-action, and let $[\omega]$ be a Kähler class on M. By blowing up (at most twice) if necessary, arrange that the closure of the generic orbit is a smooth rational curve F of self-intersection 0. The induced action on the blow-up $\hat{M}$ then preserves a ruling $\hat{M} \mapsto \Sigma$ with generic fiber $F \cong \mathbb{CP}^1$. Let $s_0 = 8\pi c_1 \cdot [\omega]/[\omega]^2$ be the average scalar curvature of (M, J, ω), let $C_\pm \cong \Sigma$ denote the attractive (respectively, repulsive) fixed curve of the action on $\hat{M}$, and, for each isolated fixed point $p_j \in \hat{M}$, let E'_j (respectively, E_j) be the chain of rational curves obtained by following the action forward from p_j to C_+ (respectively, backward from p_j to C_-). Finally, let Ξ be the vector field on M corresponding to the standard generator of the Lie algebra of $\mathbb{C}^*$. Then the Futaki invariant $\mathfrak{F}(\Xi, [\omega])$ is given by*

$$\omega(F)\omega(C_- - C_+) + \frac{s_0(\omega(F))^3}{96\pi}\left[C_-^2 - C_+^2 + 6\frac{\omega(C_+ - C_-)}{\omega(F)} + \sum_j \left(\frac{\omega(E'_J - E_J)}{\omega(F)}\right)^3\right].$$

Let us consider homogeneous coordinates $[Z^0 : Z^1 : Z^2]$ in $\mathbb{CP}^2$. Without any loss of generality, we take the points $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$ to be those that will be blown-up to generate the manifold M_3. Since the blow-up of a complex surface at a point decreases by 1 the self-intersection of any curve that passes through the said point, we have that the three exceptional divisors together with the proper transform of the complex curves that join the blown-up points form a hexagon in M_3 consisting of curves of self-intersection -1. But the second Betti number of M_3 is 4, so these six curves must satisfy two relations among them: the three differences between opposite sides are homologous. Hence, for any Kähler class $[\omega]$ on M_3, the areas of these six curves are given by positive numbers α, β, γ, $\alpha + \delta$, $\beta + \delta$, and $\gamma + \delta$ as indicated in the following figure:



Let Ξ be the generator of the action

$$[Z_0 : Z_1 : Z_2] \mapsto [\zeta Z_0 : Z_1 : Z_2] \, .$$

For this action we have $C_{\pm}^2 = -1$, $\omega(C_-) = \alpha$, $\omega(C_+) = \alpha + \delta$, $\omega(F) = \beta + \gamma + \delta$, $\omega(E_1) = \gamma + \delta$, $\omega(E_1') = \beta$, $\omega(E_2) = \beta + \delta$, $\omega(E_2') = \gamma$.

The average scalar curvature is given by

$$s_0 = 8\pi \frac{c_1 \cdot [\omega]}{[\omega]^2} = 8\pi \frac{2(\alpha+\beta+\gamma)+3\delta}{\delta^2 + 2\delta(\alpha+\beta+\gamma) + 2(\alpha\beta+\beta\gamma+\alpha\gamma)} \, .$$

Thus, by Theorem 51, the Futaki character at Ξ, $\mathfrak{F}(\Xi, [\omega])$, is given by

$$-\delta(\beta+\gamma+\delta) + \frac{(2(\alpha+\beta+\gamma)+3\delta)[6\delta(\beta+\gamma+\delta)+(\beta-\gamma-\delta)^3+(\gamma-\beta-\delta)^3]}{12[\delta^2 + 2\delta(\alpha+\beta+\gamma) + 2(\alpha\beta+\beta\gamma+\alpha\gamma)]} \, .$$

We observe in passing that the knowledge of s_0 and $\mathfrak{F}(\Xi, [\omega])$ suffice to compute the energy $E_{[\omega]}$ of the class $[\omega]$. This turns out to be a rational function of the parameters $\alpha, \beta, \gamma, \delta$.

We may also write $\mathfrak{F}(\Xi, [\omega])$ as

$$\mathfrak{F}(\Xi, [\omega]) = \frac{2\delta}{[\omega]^2} \left[\sigma(\frac{\delta^2}{3} + \delta\alpha + \alpha^2) + p(2\delta + 4\alpha + \sigma) \right] \, ,$$

where $\sigma = (\beta - \alpha) + (\gamma - \alpha)$ y $p = (\beta - \alpha)(\gamma - \alpha)$, respectively. Then we have that the conditions

a) $\delta = 0$, or
b) $\alpha = \beta = \gamma$

suffice to ensure that $\mathfrak{F}(\,\cdot\,, [\omega]) = 0$. Indeed, next to Ξ, we only need one more field $\tilde{\Xi}$ to obtain a set of generators of the algebra of holomorphic vector fields of M_3, and this other generator can be chosen in such an way that its Futaki invariant is given by a formula as the one above where the rôles of α and β are interchanged. Of course, another possibility would be to interchange α and γ. But since one of the areas α, β or γ, is the smallest of them all, and since the signs of p and σ are apposite to each other, we may conclude that $\mathfrak{F}(\,\cdot\,,[\omega]) \equiv 0 \iff$ either (a) or (b) holds.

Furthermore, that (a) or (b) $\implies \mathfrak{F}(\,\cdot\,,[\omega]) \equiv 0$ can be seen as a consequence of the invariance of $\mathfrak{F}$ under the action of the biholomorphism group. For if we consider the finite group of biholomorphisms of M_3 given by

- $\mathbb{Z}_2$, generated by the Cremona transformation
$$\mathcal{C}([Z_0 : Z_1 : Z_2]) = [1/Z_0 : 1/Z_1 : 1/Z_2]\,,$$
and
- $\mathbb{Z}_3$, generated by the permutation
$$\mathcal{P}([Z_0 : Z_1 : Z_2]) = [Z_1 : Z_2 : Z_0]\,,$$

then a class $[\omega]$ satisfies (a) if, and only if, it is invariant under $\mathbb{Z}_2$, and satisfies (b) if, and only if, it is invariant under $\mathbb{Z}_3$. The transformations $\mathcal{C}$ y $\mathcal{P}$ act on $\mathfrak{h}(M_3)$ in such a way that 1 is an eigenvalue of neither. But if Φ is any biholomorphism and Υ is a holomorphic vector field such that $\Phi_*\Upsilon = \lambda\Upsilon$, then we have that
$$\mathfrak{F}(\Upsilon, (\Phi)^*[\omega]) = \mathfrak{F}(\Phi_*\Upsilon, [\omega]) = \lambda\mathfrak{F}(\Upsilon, [\omega])\,.$$
If $[\omega] = (\Phi)^*[\omega]$ and $\lambda \neq 1$, this shows that $\mathfrak{F}(\Upsilon, [\omega]) = 0$.

The result shows that $[\omega] = c_1$ is a Kähler class in M_3 for which *both* conditions (a) and (b) hold; then $\mathfrak{F}(\,\cdot\,, c_1) \equiv 0$. However, the half-line $\mathbb{R}^+ c_1$ is not isolated in the set of classes where $\mathfrak{F} = 0$, even though that for a generic class nearby c_1 we have that $\mathfrak{F} \neq 0$.

Exercise 9. (open problem) Of all complex surfaces with positive Chern class, $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$ is the only one where we do not know if there are extremal metrics or not.

a) Use Calabi's Theorem 30 to conclude that an extremal metric on $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$ must be invariant under the action of $\mathbb{S}^1 \times \mathbb{S}^1$.
b) Show that $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$ is diffeomorphic to the blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$ at one point.
c) Write the product metric on $\mathbb{CP}^1 \times \mathbb{CP}^1$ using geodesic polar coordinates (r, θ) and (s, ϕ) in each factor as
$$\omega_0 = \sin r\, dr \wedge d\theta + \sin s\, ds \wedge d\varphi\,,$$
and show that
$$\omega_t = \omega_0 + it\partial\overline{\partial} \log\left(3 + \cos r + \cos s - \cos r \cos s\right),$$

$t \in (0,2)$, defines a metric on the blow-up of this manifold at one point. This metric is clearly $\mathbb{S}^1 \times \mathbb{S}^1$-invariant. Show that it is not extremal.

d) Write down the extremal equation for an $\mathbb{S}^1 \times \mathbb{S}^1$-invariant metric representing the same cohomology class as that represented by ω_t. Can you solve the resulting equation?

6. Extremal metrics on almost Hermitian manifolds

We now sketch an extension of the extremal metric problem to almost Hermitian manifolds.

Let (M, J, g) be a compact almost Hermitian manifold. For any real valued function φ, we consider the symmetric tensor

$$g_{\varphi}(X,Y) = g(X,Y) + \frac{1}{4}(dJd\varphi(X,JY) + dJd\varphi(Y,JX)) \,, \tag{63}$$

If φ is sufficiently small, this tensor defines a J invariant metric, and calculated relative to it, $J^* = -J$. We denote by $\mathfrak{M} = \mathfrak{M}_{g,J}$ the set of all such metrics. We provide it with a topology that, in principle, is as strong as the C^4-topology of the φ-parameter.

Remark 52. This set-up is quite natural and worthy of consideration. A typical situation where it appears is the following: let (M, ω) be a symplectic manifold of dimension $2n$, and let J be an ω-tame almost complex structure, that is, J is an almost complex structure such that $\omega(X, JX) > 0$ for all non-zero X. Since such a J does not have to be a symplectic endomorphism of the tangent space, $\omega(X, JY)$ does not, in general, define a Riemannian metric. However, we can symmetrize it and consider the tensor

$$g_{\omega}(X,Y) = \frac{1}{2}((\omega(X,JY) + \omega(Y,JX)) \,.$$

It is a J invariant metric, and $J^* = -J$. With the structure defined by this metric and J, M is now an almost Hermitian manifold.

We can apply this same procedure to the symplectic structures defined by

$$\left\{ \tilde{\omega} = \omega + \frac{1}{2}dJd\varphi, \ \varphi \in C^{\infty}(M) \,, \text{ such that } \tilde{\omega}^n \neq 0 \right\} \,,$$

and obtain a metric $\tilde{g}$ for each symplectic structure $\tilde{\omega}$ of this type. The set of metrics that result are of the form (63), with $g = g_{\omega}$.

Remark 53. If J were integrable, we would have

$$i\partial\overline{\partial} = -\frac{1}{2}dJd\,.$$

This identity makes clear the relationship between the space of metrics we are introducing here, and those used earlier in the study of extremal metrics on polarized Kähler manifolds.

For any metric $\tilde{g} \in \mathfrak{M}_{g,J}$, we let $\tilde{s}^J$ and $d\mu_{\tilde{g}}$ be its J-scalar curvature and volume form, respectively. The functional

$$\begin{array}{ccc} \mathfrak{M}_{g,J} & \stackrel{C}{\longrightarrow} & \mathbb{R} \\ \tilde{g} & \mapsto & \displaystyle\int_M (\tilde{s}^J)^2 d\mu_{\tilde{g}}\,. \end{array} \tag{64}$$

is homogeneous of degree $n-2$, and, generally speaking, we do not expect to find critical points of it without suitably normalizing its domain. Since the volume of metrics in $\mathfrak{M}_{g,J}$ is not necessarily constant, we study the normalized functional obtained from C by dividing it by the $(1-\frac{2}{n})$-th power of the volume $\mu_{\tilde{g}}$:

$$\begin{array}{ccc} \mathfrak{M}_{g,J} & \stackrel{E}{\longrightarrow} & \mathbb{R} \\ \tilde{g} & \mapsto & \mu_{\tilde{g}}^{\frac{2-n}{n}} \displaystyle\int_M (\tilde{s}^J)^2 d\mu_{\tilde{g}}\,. \end{array} \tag{65}$$

6.1. **Variational formulae.** We give now a brief description of the variational formula for E. We hope that the metrics satisfying the resulting Euler-Lagrange equation will be intrinsically related to J, allowing the study of the latter through the understanding of the firsts. This is our main motivation in pursuing this line of ideas. However, we limit ourselves to the derivation of the formula, without showing any application.

It is often convenient to make use of the one form $\theta_g = J\delta_g\omega_g$, which is known as the *Lee form*. Let us introduce the operator d^c that on r-forms acts as $d^c = (-1)^r JdJ = -J^{-1}dJ$. Notice that on functions d^c is just the operator Jd. If Θ_g is the vector field that the metric g makes correspond to θ_g, we have that

$$(dd^c f, \omega)_g = \Delta_g f - \Theta_g f\,, \tag{66}$$

where Δ_g is the (positive) Laplace-Beltrami operator.

We now prove a formula for the variation of the volume element of metrics in $\mathfrak{M}_{g,J}$. It gives us the variation in t of the volume element of a one parameter family of metrics $g(t) \in \mathfrak{M}_{g,J}$ associated with a family of functions $\varphi(t)$.

Proposition 54. *Let $g(t)$ be a one-parameter family of metrics in $\mathfrak{M}_{g,J}$ associated with the one-parameter family of functions $\varphi(t)$. Then*

$$\begin{aligned} \tfrac{d}{dt} d\mu_{g(t)} &= \frac{1}{2}[\text{trace}: \ L \to -J\nabla_L(Jd\dot{\varphi}(t))] d\mu_{g(t)} \\ &= -\frac{1}{2}(dd^c\varphi(t), \omega_g(t))_{g(t)} \, d\mu_{g(t)} \\ &= -\frac{1}{2}\left[\left(\Delta_{g(t)} - \Theta_{g(t)}\right)\varphi\right] \, d\mu_{g(t)} \end{aligned}$$

where $d\mu_{g(t)}$, $\omega_{g(t)}$, ∇, and trace, *are the volume form, fundamental form, covariant derivative and trace operator of the metric $g(t)$, respectively, and $(\cdot\,,\cdot)$ is the pairing of 2-forms induced by $g(t)$.*

Proof. Let $d\mu(t)$ and Δ be the volume form and Laplacian of the metric $g(t)$, respectively. If $(x^1, \dots, x^{2n})$ is a coordinate system and $e_\imath$ denotes the local vector field $\partial_{x^\imath}$, then it is somewhat tedious but easy to see that

$$\frac{d}{dt} d\mu(t) = \frac{1}{2}\left[-\Delta\dot{\varphi}(t) + g^{ij}(t)(\nabla_{Je_\jmath} Je_\imath)\dot{\varphi}(t)\right] d\mu(t)\,,$$

from which the first formula follows easily calculating in normal coordinates.

In order to prove the second formula, let us consider an orthonormal basis of vector fields $\{e_i, Je_i\}_{i=1}^n$ (we suppress from the notation the dependence upon t). Since on functions we have $d^c = Jd$, we calculate the trace stated in the first formula, and obtain

$$\begin{aligned} \frac{d}{dt} d\mu_{g(t)} &= -\frac{1}{2}\sum_{i=1}^n (J\nabla_{e_i}(d^c\varphi))(e_i) - \frac{1}{2}\sum_{i=1}^n (J\nabla_{Je_i}(d^c\varphi))(Je_i) \\ &= \frac{1}{2}\sum_{i=1}^n (\nabla_{e_i}(d^c\varphi))(Je_i) - \frac{1}{2}\sum_{i=1}^n (\nabla_{Je_i}(d^c\varphi))(e_i) \\ &= \frac{1}{2}\sum_{i=1}^n (\nabla(d^c\varphi))(e_i, Je_i) - \frac{1}{2}\sum_{i=1}^n (\nabla(d^c\varphi))(Je_i, e_i) \\ &= \sum_{i=1}^n (dd^c\varphi)(e_i, Je_i) \\ &= -\frac{1}{2}(dd^c\varphi, \omega_g)\,. \end{aligned}$$

By (66), the third formula follows from the second. □

Corollary 55. *A metric g_0 in $\mathfrak{M}_{g,J}$ is a critical point of the volume functional if, and only if, $*J\delta_0\,\omega_0$ represents a cohomology class in the group $H^{2n-1}(M)$ (that is to say, iff $d\,(*J\delta_0\,\omega_0)=0$). Here, $*$ and δ_0 are the Hodge star operator and the dual of d (relative to g_0), respectively. In particular, any semi-Kähler metric in $\mathfrak{M}_{g,J}$ is a critical point of the volume functional.*

Proof. Let $g(t)$ be a deformation in $\mathfrak{M}_{g,J}$ of a metric g_0. By Proposition 54 and the Hermiticity of the metric, we have that

$$\begin{aligned}\frac{d}{dt}\mu_{g(t)} = -\frac{1}{2}\int_M (dd^c\varphi(t), \omega_{g_{(t)}})\, d\mu_{g(t)} &= -\frac{1}{2}\int_M (d^c\varphi(t), \delta_{g(t)}\omega_{g_{(t)}})\, d\mu_{g(t)} \\ &= \frac{1}{2}\int_M (d\varphi(t), J\delta_{g(t)}\omega_{g_{(t)}})\, d\mu_{g(t)}\,.\end{aligned}$$

Using the Hodge star operator and Stokes' theorem, we conclude that

$$\frac{d}{dt}\mu_{g(t)} = -\frac{1}{2}\int_M \varphi(t) d\,(*J\delta_{g(t)}\omega_{g_{(t)}})\,,$$

and the result follows.

Semi-Kähler metrics are defined as those whose fundamental form is in the kernel of the dual to d, that is to say, $\delta_0\omega_0$ itself is zero. Thus, semi-Kähler metrics are critical points of the volume as a functional on $\mathfrak{M}_{g,J}$. □

Remark 56. The six dimensional sphere $(\mathbb{S}^6, g, J)$ is an example of a semi-Kähler almost Hermitian manifold. Thus, the volume of $\mathbb{S}^6$, under deformations of g in $\mathfrak{M}_{g,J}$, is infinitesimally constant. □

Proposition 57. *Let (M, J, g) be an almost Hermitian manifold. Let α be the symmetric 2-tensor*

$$\alpha(X,Y) = \nabla_{JX}(\delta\omega)Y + \nabla_{JY}(\delta\omega)X\,,$$

and β be the tensor

$$\beta(X,Y) = \mathrm{trace}\, L \to (\nabla_{(\nabla_L J)X}J)Y\,.$$

Then, the variation of s^J at g in the direction of the J-invariant tensor h is given by

$$\frac{d}{dt}s^J(h) = -\delta(\delta h) - 2\delta((\delta(\omega^{\#}\otimes\omega^{\#}))\lrcorner_{1,2}h) + (r\circ J - 2r^J + \alpha - \beta - \delta\omega\otimes\delta\omega, h)\,.$$

Here $\lrcorner_{1,2}$ denotes the contraction in the first two indices.

Proof. Let $\{e_1, \ldots, e_{2n}\}$ be an orthonormal frame. Using (23), we have found (see §3.3) that

$$\dot{s}^J(h) = (h, r\circ J - 2r^J + \alpha - \beta - \delta\omega\otimes\delta\omega) + \nabla_m\{J^{it}J^{lm}\nabla_i h_{lt}\} - \nabla_i\{(\nabla_m(J^{it}J^{lm}))h_{lt}\}\,.$$

It remains to interpret invariantly the divergence terms above.

Since h is J-invariant, we have that

$$\nabla_m\{J^{it}J^{lm}\nabla_i h_{lt}\} = -\nabla_m\nabla_i(h_{mi}) - \nabla_m(h_{lt}\nabla_i(J^{it}J^{lm}))\,.$$

By the symmetry of h, the first summand on the right side of the expression above is $-\delta(\delta h)$. The second summand equals $-\nabla_i\{(\nabla_m(J^{it}J^{lm}))h_{lt}\}$. Thus,

$$\dot{s}^J(h) = (h, r\circ J - 2r^J + \alpha - \beta - \delta\omega\otimes\delta\omega) - \delta(\delta h) - 2\nabla_i\{(\nabla_m(J^{it}J^{lm}))h_{lt}\}\,.$$

Notice that $-(\nabla_m(J^{it}J^{lm}))h_{lt}$ is the i-th component of the tensor $(\delta(\omega^{\#}\otimes\omega^{\#}))\lrcorner_{1,2}h$. Here, $\lrcorner_{1,2}$ is the contraction in the first two indices. Therefore,

$$\dot{s}^J(h) = (h, r\circ J - 2r^J + \alpha - \beta - \delta\omega\otimes\delta\omega) - \delta(\delta h) - 2\delta((\delta(\omega^{\#}\otimes\omega^{\#}))\lrcorner_{1,2}h)\,,$$

as desired. □

Proposition 58. *Let (M, J, g) be an almost Hermitian manifold, and α and β the tensors*

$$\alpha(X, Y) = \nabla_{JX}(\delta\omega)Y + \nabla_{JY}(\delta\omega)X$$

and

$$\beta(X, Y) = \mathrm{trace}\, L \to (\nabla_{(\nabla_L J)X}J)Y\,,$$

respectively. Then, the variation of the functional E in (65) at g in the direction of φ is given by

$$E'(g)(\varphi) \;=\; \mu_g^{\frac{2-n}{n}}\left[\int 2s^J\left((r\circ J - 2r^J + \alpha - \beta - \delta\omega\otimes\delta\omega, h) - \delta(\delta h) \right.\right.$$
$$\left.\left. -2\delta((\delta(\omega^{\#}\otimes\omega^{\#}))\lrcorner_{1,2}h)\right) - \left(\frac{s^J}{4} + \frac{(2-n)\|s^J\|^2_{L^2}}{2n\mu_g}\right)\int (dd^c\varphi, \omega)\right] d\mu_g\,,$$

where h is the symmetric J-invariant two-tensor

$$h(X, Y) = \frac{1}{4}(dJd\varphi(X, JY) + dJd\varphi(Y, JX))\,.$$

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