# The Contact Process on Graphs

## DEDICATION

I dedicate this to my family and friends for their help, support and care.

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## PUBLICATIONS AND PRESENTATIONS

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ABSTRACT OF THE DISSERTATION

# THE CONTACT PROCESS ON GRAPHS

by

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We study the ergodic behavior of the contact process on infinite connected graphs of bounded degree. We show that the fundamental notion of complete convergence is not as well behaved as it was thought to be. In particular there are trees for which complete convergence holds in any number, finite or infinite, of separated intervals of values of the infection parameter and fails for the other values of this parameter. We then introduce a basic invariant probability measure related to the recurrence properties of the process, and an associated notion of convergence that we call "partial convergence". This notion is shown to be better behaved than complete convergence, and to hold in certain cases in which complete convergence fails. Relations between partial and complete convergence are presented, as well as tools to verify when these properties hold. For homogeneous graphs we show that whenever recurrence takes place (i.e., whenever local survival occurs) there are exactly two extremal invariant measures. We explore continuity properties of the survival probability and the recurrence probability. These order parameters are found to have a richer behavior than expected, with the possibility of the survival probability being discontinuous at or above the threshold for survival. A condition which guarantees the continuity of the survival probability above the survival point is introduced and exploited. The recurrence probability is shown to always be left-continuous above the recurrence point, and a necessary and sufficient condition for its right-continuity is introduced and exploited. It is shown that for homogeneous graphs the survival probability can only be discontinuous at the survival point, and the recurrence probability can only be discontinuous at the recurrence point. On the matter of complete convergence on homogeneous trees we present a new proof of Zhang's result that local survival implies complete convergence for the contact process on homogeneous trees.

# CHAPTER 1

# Introduction

## 1.1 Preliminaries.

In this work we consider the contact process on fairly arbitrary graphs. The only restrictions on the graphs being given by (G1) – (G3) in the next section. We present results addressing the characterization of the invariant probability measures and their domains of attraction. These are classical issues in the field of interacting particle systems, and are sometimes referred to as the study of the "ergodic behavior" of the process. Somewhat surprisingly, we find new results on these basic issues, in spite of the contact process having been extensively studied for more than twenty years. We believe that the reason for this is that most of the previous efforts had been concentrated on the cases in which the graph is homogeneous. In these cases most of the rich structure that will be presented is reduced to other known results. Still, some of the new results refer to homogeneous graphs (these results appear as Thm.2.2(i)), and include the fact that for such graphs under local survival there are exactly two extremal invariant measures.

Since its introduction by Harris in 1974 and until about 1990, the contact process had been mostly studied on the $d$-dimensional cubic lattice $\mathbb{Z}^d$. (In a harmless abuse of notation, we will also denote by $\mathbb{Z}^d$ the graph with this set as its vertex set and edges connecting each pair of vertices separated by Euclidean distance 1.) The ergodic behavior of the contact process on such graphs was completely characterized in the fundamental paper by Bezuidenhout and Grimmett (1990), who built on the extensive work of many others during the previous 15 years [see references in that paper]. A short time after that paper appeared, interest on the behavior of the contact process on other graphs, especially trees, was raised by Pemantle (1992). In that latter paper most of the analysis concerned the homogeneous trees of degree $b+1$, which we will denote by $\mathbb{T}_b$. It was shown that (when $b \geq 3$) the contact process on such trees has a subtler behavior than the one on $\mathbb{Z}^d$, $d \geq 1$, in that there are at least two different critical points and between them the system can survive in a global sense but not in a local sense. From that paper and the subsequent work by Madras and Schinazi (1992), Morrow, Schinazi and Zhang (1994), Durrett and Schinazi (1995), Wu (1995), Zhang (1996), Liggett (1996a), Stacey (1996), Liggett (1996b), Lalley and Selke (1998), Salzano and Schonmann (1998), Schonmann (1998) and Lalley (1999) a great deal of information became available about the ergodic behavior of these systems. For a review see Liggett (1999). In particular, the results have now been extended to all values of $b \geq 2$. Between the two critical points mentioned above, it is known that there are infinitely many extremal invariant probability measures, while above the second one of these points there are only two such measures and the complete convergence theorem, to be reviewed below, holds.

Our results in Chapter 2 and 3 show that on less regular graphs the ergodic behavior of the contact process can be richer than on $\mathbb{Z}^d$ or $\mathbb{T}_b$. An arbitrarily large number of critical points, and even infinitely many critical points, separating intervals where the ergodic behavior is qualitatively distinct, can occur. In particular, Conjecture 1 in Pemantle (1992) is disproved. A fundamental invariant probability measure is identified and studied. It is shown that this measure is always an extremal invariant measure, and while on $\mathbb{Z}^d$ and $\mathbb{T}_b$ it always coincides either with the lower or the upper invariant measure, this is not the case for other graphs. A notion, which we call "partial convergence", is introduced. While partial convergence is not as sharp a property as complete convergence is, it is shown to be nevertheless a better behaved notion than that latter one. Moreover in some situations we show that while complete convergence fails, partial convergence holds.

In Chapter 4 we study the continuity properties of the order parameters of the contact process on a graph (by "order parameters" we mean the probabilities of survival and of recurrence). We show that the survival probability may be discontinuous at or above the survival point. This contradicts Conjecture 3 of Pemantle (1992). We also present positive results, including a useful condition which guarantees the continuity of the survival probability above the survival point. The recurrence probability is shown to be always left-continuous above the recurrence point; and a necessary and sufficient condition for its right-continuity is also presented.

In Chapter 5 a new proof for Zhang's result that local survival implies complete convergence for the contact process on homogeneous trees is presented.

## 1.2 Notation and background.

Before we can proceed, we need to introduce a certain amount of notation; as we do it we will also review in greater detail some basic facts about the contact process. The graphs considered in this paper are supposed to be

**(G1)** infinite, since otherwise the issues discussed in this paper trivialize;

**(G2)** connected, since otherwise the features of interest can be studied on each connected component;

**(G3)** of bounded degree, i.e., each vertex belongs to at most $\kappa$ edges, for some $\kappa < \infty$. (This restriction is actually more than what we need, and could be replaced by the assumption that the process started from a finite set does not explode for all values of the infection parameter $\lambda$ introduced below.)

We denote by $\mathcal{G}$ the class of graphs which satisfy (G1), (G2) and (G3) above. For a graph $G \in \mathcal{G}$ we denote by $\mathcal{V}_G$ its set of vertices, also called sites in this work, to stick to the usual interacting-particle-system terminology. Pairs of sites which belong to a common edge of $G$ are said to be neighbors in $G$. One of the sites of $G$ is distinguished from the others and called its root, denoted simply by 0; in this work the choice of the root is usually arbitrary, in that the statements made will depend on the graph $G$ but not on the choice of its root. The notation $A \subset\subset \mathcal{V}_G$ means that $A$ is a finite subset of $\mathcal{V}_G$. We measure the distance between sites in $\mathcal{V}_G$ by the length of the minimal path along neighboring sites which joins them. The distance between two vertices $x, y \in \mathcal{V}_G$ is denoted by $d(x, y)$. The cardinality of a set $A \in \mathcal{V}_G$ is denoted by $|A|$. The ball of center $x \in \mathcal{V}_G$ and radius $N$ is denoted by $B(x, N)$. Clearly (G2) and (G3) imply that $\mathcal{V}_G$ is a countable set for all $G \in \mathcal{G}$.

A subgraph of a graph $G$ is another graph which has its set of vertices contained in the set of vertices of $G$ and its set of edges contained in the set of edges of $G$. An isomorphism between two graphs, $G_1$ and $G_2$, is a one-to-one mapping from $\mathcal{V}_{G_1}$ onto $\mathcal{V}_{G_2}$ which preserves the graph structure, i.e., such that the set of edges of $G_2$ can be obtained as the set of pairs of images of vertices of $G_1$ which form edges. An isomorphism between a graph $G$ and itself is called an automorphism of $G$. We say that a graph $G_1$ can be embedded as a subgraph of another graph $G_2$ in case there is an isomorphism between $G_1$ and a subgraph of $G_2$. A graph is said to be homogeneous if for each pair $x$ and $y$ of its vertices there is an automorphism of the graph which maps $x$ into $y$. The class of homogeneous graphs in $\mathcal{G}$ is denoted by $\mathcal{H}$. Typical examples of graphs in $\mathcal{H}$ are $\mathbb{Z}^d$ and $\mathbb{T}_b$, but, of course, there are others.

For instance if $G \in \mathcal{H}$ and we add edges to $G$, connecting all pairs of vertices which are at a given prescribed distance from each other, then the resulting graph is also in $\mathcal{H}$. Homogeneous graphs are also called transitive graphs. Everything that we say about homogeneous graphs in this work applies also (with essentially the same proofs) to the larger class of almost transitive graphs, defined as those graphs in $\mathcal{G}$ for which there is a finite set of vertices, $V_0$, with the property that each vertex of the graph can be mapped into one of the vertices of $V_0$ by an automorphism.

The contact process on the graph $G \in \mathcal{G}$ with infection parameter $\lambda > 0$ is a continuous time Markov process with state space $\{0,1\}^{\mathcal{V}_G}$. Elements of this state space are called configurations. When the configuration at a given site is 1 one says that there is a particle there or that the site is occupied or that the site is infected. Otherwise one says that the site is vacant or healthy. The contact process evolves according to the following local prescription.

**(i)** A particle at a site gives birth to new ones at each neighboring vacant site at rate $\lambda$.

**(ii)** Particles die at rate 1.

The assumption that $G$ has a bounded degree assures us that there is a well defined unique Markov process with these features, moreover it will satisfy the Feller property. For constructions of such processes and also for the proofs of the basic facts reviewed below, the reader can consult, e.g., Liggett (1985) or Durrett (1988).

We can think of an element $\eta$ of $\{0,1\}^{\mathcal{V}_G}$ either as a function from $\mathcal{V}_G$ to $\{0,1\}$, in which case the notation $\eta(x)$ will be used for the value of this function at $x \in \mathcal{V}_G$, or as the subset of $\mathcal{V}_G$ where this function takes the value 1. As usual, we take advantage of this flexibility in our notation, and no confusion should arise from this common practice.

The set $\{0,1\}$ is endowed with the discrete topology and $\{0,1\}^{\mathcal{V}_G}$ with the corresponding product topology and corresponding Borel $\sigma$-field. Probability distributions on the configuration space are determined then by their finite dimensional distributions, and the notion of weak convergence corresponds to the convergence of these finite dimensional distributions. We use the double arrow, $\Rightarrow$, to denote weak convergence. The probability measure which puts all mass on the configuration $\eta$ is denoted by $\delta_\eta$.

We denote by $(\xi_t^\mu \colon t \geq 0)$ or $(\xi_t^\mu)_{t\geq 0}$ the version of the contact process starting from a configuration which is randomly chosen according to the law $\mu$. When $\mu$ is concentrated on the configuration $\eta$ we write simply $(\xi_t^\eta \colon t \geq 0)$ or $(\xi_t^\eta)_{t\geq 0}$. Abusing notation one step further, we also write $(\xi_t^x \colon t \geq 0)$ or $(\xi_t^x)_{t\geq 0}$ for the contact process started from a single particle at $x \in \mathcal{V}_G$. Similar conventions on the notation are used systematically without further notice. When there is need to specify the graph $G$ or the value of $\lambda$ in the notation, this will be done in the following fashion: $(\xi_{G,\lambda;t}^\mu \colon t \geq 0)$ or $(\xi_{G,\lambda;t}^\mu)_{t\geq 0}$.

For fixed $t \geq 0$, the law of $\xi_t^\mu$ is denoted by $\mu_t = \mu S(t) = \mu S_{G,\lambda}(t)$. The set of invariant probability measures is denoted by $\mathcal{I} = \{\mu\colon \mu_t = \mu \text{ for all } t \geq 0\}$. This is a convex set, and the set of its extremal points is denoted by $\mathcal{I}_{\mathrm{e}}$. It is obvious that $\delta_\emptyset \in \mathcal{I}_{\mathrm{e}}$, regardless of the value of $\lambda$.

A basic property of the contact process is attractiveness. Endow $\{0,1\}^{\mathcal{V}_G}$ with the partial order given by writing $\eta \leq \zeta$ in case $\eta(x) \leq \zeta(x)$ for all $x \in \mathcal{V}_G$. Next endow the set of probability measures on $\{0,1\}^{\mathcal{V}_G}$ with the partial order given by writing $\mu_1 \leq \mu_2$ in case

$$\int f d\mu_1 \leq \int f d\mu_2,$$

for all continuous non-decreasing function $f\colon \{0,1\}^{\mathcal{V}_G} \to \mathbb{R}$. This is called the stochastic order. Attractiveness means that the stochastic order is preserved by the time evolution, i.e., if $\mu_1 \leq \mu_2$ then $\mu_1 S(t) \leq \mu_2 S(t)$ for all $t \geq 0$.

Attractiveness easily implies the following results. As $t \to \infty$, $\delta_{\mathcal{V}_G} S(t) \Rightarrow \overline{\nu}$. Here $\overline{\nu} \in \mathcal{I}_{\mathrm{e}}$ is called the upper invariant measure, while $\delta_\emptyset$ is called the lower invariant measure. Having $\delta_\emptyset = \overline{\nu}$ is equivalent to having $\mu S(t) \Rightarrow \delta_\emptyset$ for all laws $\mu$; the process is in this case said to be ergodic. If this happens, in particular, $\mathcal{I} = \{\delta_\emptyset\}$.

The contact process enjoys also a property which is stronger than attractiveness. This property is called additivity, and it states that the collection of processes $(\xi_t^A\colon t \geq 0)$, $A \subset \mathcal{V}_G$, can be constructed on a common probability space in such a way that the following relation holds.

$$\xi_t^{A \cup B} = \xi_t^A \cup \xi_t^B,$$

for all pairs of sets $A$ and $B$, and $t \geq 0$.

For this purpose, one can use the following graphical construction. We suppose that the contact process on the graph $G \in \mathcal{G}$, with infection parameter $\lambda > 0$ is constructed by means of Poisson death marks (at rate 1 for each site in $\mathcal{V}_G$) and Poisson arrows (at rate $\lambda$ for each oriented edge $(x, y)$ such that $\{x, y\} \in \mathcal{E}_G$). In order to use ergodicity we think of the graphical construction as being made on $G \times \mathbb{R}$, rather than just on $G \times \mathbb{R}_+$. We will use $\mathbb{P}_{G,\lambda} = \mathbb{P}$ to denote the probability measure corresponding to the graphical construction. Given two space-time points, $(x, s), (y, u) \in \mathcal{V}_G \times \mathbb{R}$, with $s < u$, we say that there is a path from $(x, s)$ to $(y, u)$ if there is a sequence of times $s = t_0 < t_1 < ... < t_n < t_{n+1} = u$ and spatial locations $x = x_0, x_1, ..., x_n = y$ so that for $i = 1, 2, ..., n$ there is an arrow from $x_{i-1}$ to $x_i$ at time $t_i$ and the vertical segments $\{x_i\} \times (t_i, t_{i+1})$ for $i = 0, 1, ..., n$ do not contain any death mark.

Two types of monotonicity are closely related to attractiveness and additivity. One is monotonicity in $\lambda$ and the other is monotonicity in the graph $G$. They can be combined in the following single statement. If $\lambda_1 \leq \lambda_2$, and $G_1$ is a subgraph of $G_2$, then for all $A \subset \mathcal{V}_{G_1}$ the processes $(\xi^A_{G_1,\lambda_1;t} : t \geq 0)$ and $(\xi^A_{G_2,\lambda_2;t} : t \geq 0)$ can be constructed on the same probability space in such a way that $\xi^A_{G_1,\lambda_1;t} \subset \xi^A_{G_2,\lambda_2;t}$ for all $t \geq 0$.

Another basic tool in the study of contact processes is their self-duality. This property can be expressed by

$$\mathbb{P}(\xi_t^A \cap B \neq \emptyset) = \mathbb{P}(\xi_t^B \cap A \neq \emptyset),$$

for all pairs of sets $A$ and $B$, and $t \geq 0$.

In order to introduce two basic critical points for the contact process on a graph, we define $\Omega_\infty^A = \{\xi_t^A \neq \emptyset, \text{ for all } t \geq 0\}$ as the event that the process $(\xi_t^A\colon\ t \geq 0)$ lives forever; and we set $\rho(A, \lambda) = \rho(A) = \mathbb{P}(\Omega_\infty^A)$. Another definition that we introduce here is $\Omega_r^A = \{\xi_t^A(0) = 1, \text{ for a unbounded set of values of } t\}$, as the event that there is recurrence; and we set $\beta(A, \lambda) = \beta(A) = \mathbb{P}(\Omega_r^A)$. When the argument $A$ is omitted in the functions $\beta$ and $\rho$, it should be understood that we are taking the set $A = \{0\}$. The positivity of $\rho(A, \lambda)$ for one finite set A clearly implies its positivity for all other such sets, and a similar remark is valid for $\beta(A, \lambda)$. When $\rho(\lambda)$ is positive one says that the contact process survives at $\lambda$, or, more precisely, that it survives globally at $\lambda$. Otherwise one says that the contact process dies out at $\lambda$. When $\beta(\lambda)$ is positive one says that the contact process is recurrent at $\lambda$, or that it survives locally at $\lambda$. Next we define the critical values

$$\lambda_s = \lambda_s(G) = \inf\{\lambda\colon \rho(\lambda) > 0\},$$

and

$$\lambda_r = \lambda_r(G) = \inf\{\lambda\colon \beta(\lambda) > 0\}.$$

Of course, the choice of the root for the graph $G$ is irrelevant in the definition of these critical points. Obviously we always have $\lambda_s \leq \lambda_r$. A standard comparison with a branching process shows that for all graphs in $\mathcal{G}$ we have $0 < \lambda_s$, and the remark that all these graphs have $\mathbb{Z}^+$ embedded into them, gives $\lambda_r \leq \lambda_r(\mathbb{Z}^+) < \infty$. (It is known that $\lambda_r(\mathbb{Z}^+) = \lambda_s(\mathbb{Z}^+) = \lambda_r(\mathbb{Z}) = \lambda_s(\mathbb{Z})$. For this see, e.g., Durrett and Griffeath (1983), or write down a proof based on the renormalization procedure of Bezuidenhout and Grimmett (1990).) We will refer to $\lambda_s(G)$ as the survival point of the graph $G$ and to $\lambda_r(G)$ as the recurrence point of this graph.

One should be careful with the distinction between finite and infinite sets above. Even when $\rho(\lambda) = 0$ we still trivially have $\rho(\eta, \lambda) = 1$ for all infinite sets $\eta$. Similarly, even when $\beta(\lambda) = 0$ we may have $\beta(\eta, \lambda) > 0$ for some infinite $\eta$. This happens, e.g., whenever $\rho(\lambda) > 0$ and $\eta = \mathcal{V}_G$, as can be easily checked using self-duality to see that $\mathbb{P}(\xi_t^{\mathcal{V}_G}(0) = 1) \geq \rho(\lambda)$ for all $t \geq 0$.

A fundamental notion is the following.

**Complete Convergence (cc).**

$$\text{For any } A \subset\subset \mathcal{V}_G,\ \xi_t^A \Rightarrow \big(1 - \rho(A)\big)\delta_\emptyset + \rho(A)\overline{\nu}, \quad \text{as } t \to \infty.$$

Or equivalently,

$$\text{for any } A, B \subset\subset \mathcal{V}_G,\ \mathbb{P}\big(\xi_t^A \cap B \neq \emptyset\big) \to \rho(A)\rho(B), \quad \text{as } t \to \infty.$$

For the equivalence between the two statements one should note that self-duality implies that

$$\overline{\nu}(\zeta \colon \zeta \cap A \neq \emptyset) = \mathbb{P}(\Omega_\infty^A) = \rho(A).$$

In the way that cc is being defined we are not requiring the system to survive, i.e., $\rho(\lambda)$ to be positive. With this definition of cc it holds trivially in case $\rho(\lambda) = 0$, or, in other words, when the system is ergodic. We introduce the notation s&cc (for "survival with complete convergence") to denote the statement that not only cc holds, but also $\rho(\lambda) > 0$.

We are not sure about the origin of the name "complete convergence", but it may be due to the fact that on $\mathbb{Z}^d$ if the statements in the definition of cc above hold in the way that they are presented, i.e., for finite initial configurations $A$, then the same is also true for all initial configurations. We will see in Thm.2.2(h) that this has to be replaced by a more general statement for general graphs in $\mathcal{G}$. Ironically, we will see then that it is still true that when cc holds, weak convergence always takes place for all initial configurations, justifying therefore the name "complete convergence".

The ergodic behavior of the contact process on $\mathbb{Z}^d$ can be summarized by saying that for every dimension $d$, $0 < \lambda_s = \lambda_r =: \lambda_c < \infty$; for $\lambda \leq \lambda_c$ the process is ergodic; while for $\lambda > \lambda_c$ there are exactly two extremal invariant measures, $\delta_\emptyset$ and $\overline{\nu}$, and cc holds.

The ergodic behavior of the contact process on $\mathbb{T}_b$ is richer. It is now known that $0 < \lambda_s < \lambda_r < \infty$ for each $b \geq 2$. For $\lambda \leq \lambda_s$ the process is ergodic; for $\lambda_s < \lambda \leq \lambda_r$ there are infinitely many measures in $\mathcal{I}_{\mathrm{e}}$, but $\beta(\lambda) = 0$, so that if the process is started from a set $A \subset\subset \mathcal{V}_G$ then $\xi_t^A \Rightarrow \delta_\emptyset$; finally for $\lambda > \lambda_r$ there are exactly two extremal invariant measures, $\delta_\emptyset$ and $\overline{\nu}$, and cc holds, moreover $\beta(\lambda) = \rho(\lambda)$.

The following notions related to the contact process on $\mathbb{T}_b$ will be useful. First we introduce some notation. We immerse the graph $\mathbb{Z}^+$ which has vertices $\{0, 1, 2, ...\}$ and edges connecting points which differ by 1 unit into $\mathbb{T}_b$, in an arbitrary fashion. This allows us to refer to the sites 0, 1, 2, ... of $\mathbb{T}_b$. The site 0 of $\mathbb{T}_b$ is called its root. An important subgraph of $\mathbb{T}_b$ is obtained from this tree by removing one of the neighbors of the root and defining it as the remaining connected component which contains the root. We will suppose that the removed vertex is not the vertex 1, so that the set of sites $\{0, 1, 2, ...\}$ is contained in the set of vertices of the new graph. This graph will be denoted by $T_b^+$.

Now set

$$u_{b,n}(\lambda) = u_n = \mathbb{P}\left(\xi^0_{\mathbb{T}_b;t}(n) = 1 \text{ for some } t \geq 0\right). \tag{1.1}$$

From the inequality $u_{n+m} \geq u_n u_m$ it follows that

$$\lim_{n\to\infty} (u_n)^{1/n} = \alpha = \alpha_b = \alpha_b(\lambda) = \sup\{(u_n)^{1/n} : \ n \geq 1\}. \tag{1.2}$$

(Our $\alpha$ was called $\rho$ in Liggett (1996b), but this conflicts with the standard use of $\rho$ for the survival probability; it was called $\beta$ in Lalley and Sellke (1998) and in Lalley (1999), but we used $\beta$ for the recurrence probability here, and therefore prefer to use a different notation here.)

From Lalley and Sellke (1998) it follows that when $\lambda \leq \lambda_r(\mathbb{T}_b)$ then one has $\alpha_b(\lambda) \leq 1/\sqrt{b}$. From Lalley (1999), we know that $\alpha_b(\cdot)$ is strictly increasing on

$(0, \lambda_r(\mathbb{T}_b)]$. Therefore,

$$\alpha_b(\lambda) < 1/\sqrt{b} \quad \text{for} \quad \lambda < \lambda_r(\mathbb{T}_b), \tag{1.3}$$

a result which will be very useful in this work.

# CHAPTER 2

## The second extremal invariant measure: general results.

### 2.1 The extremal invariant measure $\nu_r$.

We started the present investigation by asking ourselves some questions regarding cc and s&cc.

**(Q1)** Is it the case that for all graphs in $\mathcal{G}$ cc holds for all $\lambda > \lambda_r$?

**(Q2)** If s&cc holds for $G$ at $\lambda$ does it also hold for $G$ at every $\lambda' > \lambda$?

**(Q3)** If $G_0$ can be embedded as a subgraph of $G$ and s&cc holds for $G_0$ at $\lambda$ is it the case that s&cc also holds for $G$ at the same $\lambda$?

From what is known about the contact process on $\mathbb{Z}^d$ and on $\mathbb{T}_b$, it is clear that in these cases the answer to each one of these questions is "yes" (regarding (Q3), we mean here that we take both $G_0$ and $G$ as cubic lattices or as homogeneous trees). (Here the need to talk about s&cc, rather than cc, in (Q2) and (Q3) should be clear, since otherwise the answers are trivially "no", for a spurious reason.) Pemantle (1992) had conjectured that the answer to the first question would be "yes" for generic trees. Nevertheless the answer to the three questions above is in general "**no**", even if we restrict ourselves to trees.

The examples that we will present in order to answer the three questions above are actually surprisingly simple.

First we introduce some definitions and notation.

**Collage of graphs.** Suppose that $G_1, ..., G_n \in \mathcal{G}$ are disjoint graphs. We say that $G \in \mathcal{G}$ is a collage of $G_1, ..., G_n$ if the following conditions are satisfied:

**i)** The set of vertices of $G$ is $\mathcal{V}_G = (\cup_{i=1,...,n} \mathcal{V}_{G_i}) \cup V_0$, where $V_0$ is a finite set.

**ii)** The set of edges of $G$ is $(\cup_{i=1,...,n} \mathcal{E}_{G_i}) \cup E_0$, where $E_0$ is a finite set. We will use the notation $V_{\text{glue}}$ for the set of vertices which are endpoints of edges in $E_0$.

We will use the following notation: if $G_1$ and $G_2$ are two graphs which have disjoint sets of vertices, then $G_1 \vee G_2$ will denote the graph obtained by the collage of $G_1$ and $G_2$ connecting their roots, or more precisely, the graph in which the set of vertices is the union of the sets of vertices of $G_1$ and $G_2$ and the set of edges is the union of the set of edges of these two graphs plus an edge connecting their roots.

**Basic example.** The idea to construct an example with finitely many contact process transitions for the contact process is to choose $j > 2$ and $k$ sufficiently larger than $j$, so that we have

$$\lambda_s(\mathbb{T}_k) < \lambda_r(\mathbb{T}_k) < \lambda_s(\mathbb{T}_j) < \lambda_r(\mathbb{T}_j). \tag{2.1}$$

That these inequalities can all be satisfied by such a choice is an immediate consequence of what we have already reviewed about the contact process on these graphs and the fact that as $k \to \infty$, $\lambda_r(\mathbb{T}_k) \to 0$, as proven by Pemantle (1992). The desired tree is $\mathbb{T}_j \vee \mathbb{T}_k$. Under (2.1), the contact process on $\mathbb{T}_j \vee \mathbb{T}_k$ has the following features, which will be proven in the next chapter:

**(I)** In the interval $(\lambda_s(\mathbb{T}_j), \lambda_r(\mathbb{T}_j)]$ cc fails.

**(II)** In the intervals $(\lambda_r(\mathbb{T}_k), \lambda_s(\mathbb{T}_j)]$ and $(\lambda_r(\mathbb{T}_j), \infty)$ cc holds.

Together with the fact that s&cc holds for $\mathbb{T}_k$ in the interval to which (I) refers, these features of $\mathbb{T}_j \vee \mathbb{T}_k$ answer the three questions above in the negative.

(I) and (II) can be proven using Griffeath's equivalence, reviewed in Section 2.3. In Chapter 3 they will be proved, using the machinery developed in this chapter. We will nevertheless present now the intuitive reasons for (I) and also a heuristics which makes (II) at least plausible. For this purpose we first observe that intuitively cc means that if the system survives, then we see eventually $\overline{\nu}$. But in the situation of (I), the process can survive in $\mathbb{T}_j$ without ever reaching its root, since we are in the regime in which on this graph there is a positive probability of survival without recurrence. If this event happens we would have survival, but certainly not convergence to the non-trivial measure $\overline{\nu}$. Regarding (II), we start with the lower of the two intervals included there (the bounded one). On this interval the process dies out on $\mathbb{T}_j$, so that if there is going to be survival on $\mathbb{T}_j \vee \mathbb{T}_k$, then $\mathbb{T}_k$ must contain occupied sites at arbitrarily large times. But because the probability of survival without recurrence on a homogeneous tree is null above its recurrence point, the process will return to the root at arbitrarily large times. This means that for $\mathbb{T}_j \vee \mathbb{T}_k$ survival will also a.s. imply recurrence, i.e., $\rho(A) = \beta(A)$. This is not yet the same as saying that cc holds, but is a strong indication that if survival occurs then there should be convergence to some non-trivial invariant measure. Regarding the unbounded interval included in (II), it seems reasonable to expect cc to hold there since it holds then for both homogeneous trees, $\mathbb{T}_j$ and $\mathbb{T}_k$. Under survival there will be recurrence a.s., and so the occurrence of cc in this interval is at least as believable as the corresponding statement on the other interval discussed above. To wrap up this heuristic discussion it may be worth saying that what happens in the region covered by (I) is that the process, started from a finite set, can survive but hide in $\mathbb{T}_j$, where survival without recurrence is a possibility. In the regions covered by (II) there is no place to hide.

One way to rephrase the negative answer to questions (Q2) and (Q3) above is by saying that s&cc is not a monotone increasing property of either $\lambda$ or of the graph. For future reference we define the following notion.

**Monotone increasing Property.** A property of the contact process is said to be monotone increasing when both of the following hold.

**(a)** If the property holds for the contact process on a graph $G \in \mathcal{G}$ at some $\lambda$, then it also holds for the same graph for all $\lambda' > \lambda$.

**(b)** If the property holds for the contact process on some subgraph $G_0 \in \mathcal{G}$ of a graph $G \in \mathcal{G}$ at some value of $\lambda$, then it also holds for $G$ at the same $\lambda$.

We will also want to say that a property is $\mathcal{F}$-monotone increasing, for some family of graphs $\mathcal{F} \subset \mathcal{G}$, in case the statements in the definition above are true when $\mathcal{G}$ is replaced by $\mathcal{F}$ in each place where it appears in the definition.

It is natural to ask what the ergodic behavior of, e.g., the basic example is in the region covered by (I), where there is recurrence, but cc fails. The theory developed here will answer this question to a great extent.

Some of the main contributions here are the introduction of two objects. The first one is an extremal invariant probability measure for the contact process, which is distinct from $\delta_\emptyset$ and $\overline{\nu}$ at values of $\lambda$ where there is recurrence but $\beta(\lambda) < \rho(\lambda)$. This measure will be denoted by $\nu_r$ and is defined by setting for each finite $A$,

$$\nu_r(\zeta \colon \zeta \cap A \neq \emptyset) = \mathbb{P}(\Omega_r^A) = \beta(A).$$

The fact that $\nu_r$ is a probability measure is not immediately obvious; one needs an argument which shows that the probability of any cylinder set of configurations (i.e., any set of configurations in which the values at a finite set of sites are specified) is positive. The best argument that we found for this is as follows. Consider the additive coupling of the processes $(\xi_t^A \colon t \geq 0)$, with $A \subset \mathcal{V}_G$, and note that $\nu_r$ is the law of the random field indexed by $\mathcal{V}_G$ which takes the value 1 or 0 at $x \in \mathcal{V}_G$ according to whether $\Omega_r^x$ happens or not, respectively.

The invariance of $\nu_r$ derives from the following computation. For any $A \subset\subset \mathcal{V}_G$, by self-duality and the Markov property,

$$\begin{aligned}
\mathbb{P}(\xi_t^{\nu_r} \cap A \neq \emptyset) &= \sum_{B \subset\subset \mathcal{V}_G} \mathbb{P}(\xi_t^A = B)\nu_r(\zeta : \zeta \cap B \neq \emptyset) \\
&= \sum_{B \subset\subset \mathcal{V}_G} \mathbb{P}(\xi_t^A = B)\mathbb{P}(\Omega_r^B) \\
&= \mathbb{P}(\Omega_r^A) \\
&= \nu_r(\zeta : \ \zeta \cap A \neq \emptyset).
\end{aligned}$$

The basic properties of $\nu_r$ are collected in Thm.2.1 below. For future reference we introduce the following terminology, where r=s stands for "recurrence equals survival".

**Criterion r=s**

$$\nu_r = \overline{\nu}.$$

Or equivalently,

$$\text{for any } A \subset\subset \mathcal{V}_G, \quad \beta(A) = \rho(A).$$

Or still equivalently,

$$\text{for some non-empty } A \subset\subset \mathcal{V}_G, \quad \beta(A) = \rho(A).$$

The equivalence between the last two statements is a very simple and standard matter. In any case, we derive it next. If the third statement above holds, then, using the Markov property at time 1, for any $B \subset\subset \mathcal{V}_G$

$$0 = \rho(A) - \beta(A) = \mathbb{P}(\Omega_\infty^A \backslash \Omega_r^A) \geq \mathbb{P}(\Omega_\infty^B \backslash \Omega_r^B)\mathbb{P}(\xi_1^A = B).$$

This leads to

$$\rho(B) - \beta(B) = \mathbb{P}(\Omega_\infty^B \backslash \Omega_r^B) = 0.$$

The fact that in the definition above we restricted $A$ to be a finite set is of relevance. When $A$ is infinite clearly $\rho(A) = 1$, but even if r=s holds we may have $\beta(A) < 1$. One simple example with this feature is the tree $\mathbb{T}_k \vee \mathbb{Z}^+$, with $\lambda_r(\mathbb{T}_k) < \lambda_c(\mathbb{Z}^+) = \lambda_c(\mathbb{Z})$. To simplify an argument below, we suppose that $k$ is large enough for $\lambda_r(\mathbb{T}_k) < 1$. Take $\lambda$ between $\lambda_r(\mathbb{T}_k)$ and 1 and as initial configuration take $\mathbb{Z}^+$. Because $\lambda < 1$, a simple comparison with a biased random walk (obtained by not letting particles die unless they have at least one vacant neighboring site) shows that there is positive probability that $\mathbb{T}_k$ will never become infected, and while infection will always be present somewhere, it will disappear from every finite set eventually. Therefore $\beta(\mathbb{Z}^+) < 1 = \rho(\mathbb{Z}^+)$. That r=s holds, nevertheless, will be a consequence of Thm.2.4 and the fact that r=s holds for $\mathbb{T}_k$ above its recurrence point. An informal argument for the validity of r=s is contained in our informal discussion above of why (II) should hold. This example that we are discussing has the feature that in spite of cc holding, the process started from the infinite set $\mathbb{Z}^+$ does not converge to $(1-\rho(\mathbb{Z}^+))\delta_\emptyset + \rho(\mathbb{Z}^+)\overline{\nu}$, as one could naively expect [see Thm 2.2(h)].

**Theorem 2.1** *For each graph $G \in \mathcal{G}$ and each value of $\lambda > 0$, the following statements are true.*

*(a)* *(0–1 law) If* $\mathbb{P}(\Omega_r^0) > 0$ *(resp.* $= 0$*) then* $\lim_{N\to\infty} \mathbb{P}\Big(\Omega_r^{B(0,N)}\Big) = 1$ *(resp.* $= 0$*). In particular, if* $\mathbb{P}\Big(\Omega_r^0\Big) > 0$*, then* $\lim_{N\to\infty} \nu_r(\zeta\colon\ \zeta \cap B(0,N) = \emptyset) = 0$*, and* $\nu_r \perp \delta_\emptyset$*.*

*(b)* *For every* $\mu \in \mathcal{I}$ *such that* $\mu \perp \delta_\emptyset$ *the following order relation holds:*

$$\text{for every } A \subset\subset \mathcal{V}_G,\ \nu_r(\zeta\colon\ \zeta \cap A \neq \emptyset) \leq\ \mu(\zeta\colon\ \zeta \cap A \neq \emptyset).$$

*In particular this is the case for all* $\mu \in \mathcal{I}_e \backslash \{\delta_\emptyset\}$*.*

*(c)* $\nu_r \in \mathcal{I}_e$*.*

*(d)* *If the criterion r=s is satisfied, then* $\mathcal{I}_e = \{\delta_\emptyset, \overline{\nu}\}$*.*

To prove the theorem, define for every configuration $\eta$ and positive integer $N$ the stopping time $S_N^\eta = \inf\{t \geq 0\colon \xi_t^\eta \supset B(0,N)\}$ and let $\mathcal{F}_{S_N^\eta}$ be the associated $\sigma$-field. Define the event $\Omega^\eta(s,N) = \{S_N^\eta < s\} = \{\xi_t^\eta \supset B(0,N)$ for some $t < s\}$. To prove the 0-1 law in Thm2.1(a) the following lemma is used.

**Lemma 2.1** *For any configuration* $\eta$ *the following holds*

$$\lim_{N\to\infty} \lim_{s\to\infty} \mathbb{P}(\Omega_r^\eta \,\triangle\, \Omega^\eta(s,N)) = 0.$$

**Proof.** The event $\Omega^\eta(s,N)$ increases to $\Omega^\eta(N) = \{\xi_t^\eta \supset B(0,N),$ for some $t\}$ as $s \to \infty$. The event $\Omega^\eta(N)$ decreases to $\Omega^\eta(\infty) = \{\forall N,\ \xi_t^\eta \supset B(0,N),$ for some $t\}$ as $N \to \infty$. But it is easy to see that $\mathbb{P}\Big(\Omega^\eta(\infty) \,\triangle\, \Omega_r^\eta\Big) = 0$, so the result follows.

■

**Proof of Thm.2.1(a).** The two statements about $\nu_r$ follow from the 0-1 law since we have

$$\nu_r\Big(\zeta\colon\ \zeta \cap B(0,N) = \emptyset\Big) = 1 - \mathbb{P}\Big(\Omega_r^{B(0,N)}\Big)$$

and

$$\nu_r\Big(\emptyset\Big) = \lim_{N\to\infty} \nu_r\Big(\zeta\colon\ \zeta \cap B(0,N) = \emptyset\Big) = 0.$$

To prove the nontrivial part of the 0-1 law, set $\lim_{N\to\infty} \mathbb{P}\Big(\Omega_r^{B(0,N)}\Big) = \alpha$. Note that by attractiveness this is a monotone limit and for all finite $A \in \mathcal{V}_G$, $\mathbb{P}\Big(\Omega_r^A\Big) \leq \alpha$. By the previous lemma, the Markov Property and the last inequality,

$$\begin{aligned}
\mathbb{P}\Big(\Omega_r^0\Big) &= \lim_{N\to\infty} \lim_{s\to\infty} \mathbb{P}\Big(\Omega^0(s,N) \cap \Omega_r^0\Big) \\
&= \lim_{N\to\infty} \lim_{s\to\infty} \mathbb{P}\Big(\Omega^0(s,N)\Big) \mathbb{P}\Big(\Omega_r^0 \,\Big|\, \Omega^0(s,N)\Big) \\
&= \lim_{N\to\infty} \lim_{s\to\infty} \mathbb{P}\Big(\Omega^0(s,N)\Big) \sum_{A \subset\subset \mathcal{V}_G} \mathbb{P}\Big(\Omega_r^A\Big) \mathbb{P}\Big(\xi_s^0 = A \,\Big|\, \Omega^0(s,N)\Big) \\
&\leq \lim_{N\to\infty} \lim_{s\to\infty} \mathbb{P}\Big(\Omega^0(s,N)\Big)\alpha \\
&= \mathbb{P}\Big(\Omega_r^0\Big)\alpha.
\end{aligned}$$

If $\mathbb{P}(\Omega_r^0) > 0$, then the inequality above implies $\alpha \geq 1$. Hence $\alpha = 1$.

■

The following lemma will play a key role.

**Lemma 2.2** *For every law $\mu$, $A \subset \mathcal{V}_G$, $N \in \mathbb{N}$, $0 \leq \alpha \leq 1$, and $t \geq 0$,*

$$\mu_t(\zeta\colon \zeta \cap A \neq \emptyset) \geq \mathbb{P}(S_N^A < \alpha t) \inf_{u > (1-\alpha)t} \mu_u\Big(\zeta\colon\ \zeta \cap B(0,N) \neq \emptyset\Big).$$

**Proof.** By self-duality,

$$
\begin{aligned}
\mu_t(\zeta\colon \zeta\cap A\neq\emptyset) &= \int d\mu(\eta)\mathbb{P}(\xi_t^A\cap\eta\neq\emptyset)\\
&\geq \int d\mu(\eta)\mathbb{P}(\xi_t^A\cap\eta\neq\emptyset,\, S_N^A<\alpha t)\\
&= \int d\mu(\eta)\mathbb{E}\Big(\mathbb{E}\Big(\mathbb{1}_{\{\xi_t^A\cap\eta\neq\emptyset\}}\mathbb{1}_{\{S_N^A<\alpha t\}}\Big|\mathcal{F}_{S_N^A}\Big)\Big)\\
&= \mathbb{E}\Big(\mathbb{1}_{\{S_N^A<\alpha t\}}\int d\mu(\eta)\mathbb{E}\Big(\mathbb{1}_{\{\xi_t^A\cap\eta\neq\emptyset\}}\Big|\mathcal{F}_{S_N^A}\Big)\Big). \qquad (2.2)
\end{aligned}
$$

On $\{S_N^A<\alpha t\}$, the strong Markov property gives us

$$
\mathbb{E}\Big(\mathbb{1}_{\{\xi_t^A\cap\eta\neq\emptyset\}}\Big|\mathcal{F}_{S_N^A}\Big)=g\Big(\xi^A_{S_N^A},S_N^A,t,\eta\Big), \qquad (2.3)
$$

where $g(\zeta,s,t,\eta)=\mathbb{E}(\mathbb{1}_{\{\xi_{t-s}^\zeta\cap\eta\neq\emptyset\}})$. But on $\{S_N^A<\alpha t\}$ one has $\xi^A_{S_N^A}\supset B(0,N)$, so that by attractiveness

$$
\begin{aligned}
\int d\mu(\eta)g\Big(\xi^A_{S_N^A},S_N^A,t,\eta\Big) &\geq \int d\mu(\eta)\mathbb{P}\Big(\xi^{B(0,N)}_{t-S_N^A}\cap\eta\neq\emptyset\Big)\\
&\geq \inf_{u>(1-\alpha)t}\int d\mu(\eta)\mathbb{P}\Big(\xi_u^{B(0,N)}\cap\eta\neq\emptyset\Big). \qquad (2.4)
\end{aligned}
$$

From (2.2), (2.3), (2.4) and self-duality,

$$
\begin{aligned}
\mu_t(\zeta\colon\ \zeta\cap A\neq\emptyset) &\geq \mathbb{P}(S_N^A<\alpha t)\inf_{u>(1-\alpha)t}\int d\mu(\eta)\mathbb{P}\Big(\xi_u^{B(0,N)}\cap\eta\neq\emptyset\Big)\\
&= \mathbb{P}(S_N^A<\alpha t)\inf_{u>(1-\alpha)t}\mu_u\Big(\zeta\colon\ \zeta\cap B(0,N)\neq\emptyset\Big).
\end{aligned}
$$

∎

**Proof of Thm.2.1(b).** By Lemma 2.2 and the invariance of $\mu$ it follows that

$$
\mu(\zeta\colon\ \zeta\cap A\neq\emptyset)\geq\mathbb{P}(S_N^A<\alpha t)\mu\Big(\zeta\colon\ \zeta\cap B(0,N)\neq\emptyset\Big).
$$

Letting $t \to \infty$, then $N \to \infty$ and using Lemma 2.1 and the assumption that $\mu(\emptyset) = 0$, we obtain

$$\mu(\zeta\colon\ \zeta \cap A \neq \emptyset) \geq \mathbb{P}(\Omega_r^A) = \nu_r(\zeta\colon\ \zeta \cap A \neq \emptyset).$$

■

**Proof of Thm.2.1(c).** We already know that $\nu_r \in \mathcal{I}$. To see that $\nu_r \in \mathcal{I}_e$, by Thm.2.1(a), either $\nu_r = \delta_\emptyset$ or $\nu_r \perp \delta_\emptyset$. In the former case the proof is complete. In the latter case, suppose that $\nu_r = \alpha\mu_1 + (1-\alpha)\mu_2$, with $\mu_i \perp \delta_\emptyset$ and $\mu_i \in \mathcal{I}$, for some $\alpha \in (0,1)$. By Thm.2.1(b) for all finite $A \subset \mathcal{V}_G$

$$\mu_i(\zeta\colon \zeta \cap A \neq \emptyset) \geq \nu_r(\zeta\colon \zeta \cap A \neq \emptyset),\ i = 1, 2.$$

But this is impossible unless $\mu_1 = \mu_2 = \nu_r$.

■

**Proof of Thm.2.1(d).** This is an immediate consequence of Thm.2.1(b) and the fact that $\overline{\nu}$ is the largest element of $\mathcal{I}$ in the stochastic sense, and hence also in the sense of Thm.2.1(b).

■

In the order relation stated in Thm.2.1(b) above we say that $\nu_r$ is the second lowest extremal invariant measure for the contact process. The notion of partial order involved there is known not to be equivalent to the more commonly considered stochastic order, reviewed in the introduction. The order in Thm.2.1(b) is weaker than the stochastic order, and the following question was therefore raised.

**(Q4)** Can Thm.2.1(b) be strengthened by replacing the order which appears there with the stochastic order?

A partial result in this direction will be provided in Thm.2.2(e), but Enrique Andjel (private communication) answered this question affirmatively.

Regarding Thm.2.1(d), we will see in Thm.2.2(f) below that under cc, the condition r=s holds. On the other hand, this is not an interesting use of Thm.2.1(d), since in Thm.2.2(h) we will show that cc is actually stronger than the conclusion in Thm.2.1(d). One can ask:

**(Q5)** Are there examples in which r=s is satisfied, but cc is not?

The possible existence of such examples was one of the main motivations for singling Thm.2.1(d) out as an item of the Theorem above. The answer to this question is yes. The example will be a tree given as Example 3.3 in the next chapter. The main application that we have at the moment for Thm.2.1(d) is contained in Thm.2.2(i), where statements about homogeneous graphs are made.

A curious application of Thm.2.1(d) is the following. If $G_1$ and $G_2$ are graphs on which the contact process dies out at a certain value of $\lambda$, we tend to believe that also the contact process on $G_1 \vee G_2$ would die out at this value of $\lambda$. We are not able to prove it, but it is clear that survival without recurrence is impossible in this situation. Hence the criterion r=s is satisfied in this case and we can conclude that there are at most 2 extremal invariant probability measures. [See Thm.2.4 for details.]

## 2.2 Partial convergence.

The second main object introduced in this work is the following statement of weak convergence:

**Partial Convergence (pc).**

$$\text{For any } A \subset\subset \mathcal{V}_G, \quad \xi_t^A \Rightarrow \Big(1 - \beta(A)\Big)\delta_\emptyset + \beta(A)\nu_r, \quad \text{as } t \to \infty.$$

Or equivalently,

$$\text{for any } A, B \subset\subset \mathcal{V}_G, \quad \mathbb{P}\Big(\xi_t^A \cap B \neq \emptyset\Big) \to \beta(A)\beta(B), \quad \text{as } t \to \infty.$$

The name partial convergence stands for the fact that even when pc holds so that convergence takes place for the process started from any set $A \subset\subset \mathcal{V}_G$, the same can fail for initial sets which are infinite [see Thm.2.2(g)].

In analogy with s&cc, we define r&pc (for recurrence with partial convergence) as the property that pc holds and recurrence takes place. The following theorem gives the basic properties of pc and its relations with cc.

**Theorem 2.2** *For each graph $G \in \mathcal{G}$ and each value of $\lambda > 0$, the following statements are true.*

***(a)*** *For any $A, B \subset\subset \mathcal{V}_G$,*

$$\limsup_{t\to\infty} \mathbb{P}\Big(\xi_t^A \cap B \neq \emptyset\Big) \leq \beta(A)\beta(B). \tag{2.5}$$

*For any $\eta \subset \mathcal{V}_G$ and for any $B \subset\subset \mathcal{V}_G$,*

$$\limsup_{t\to\infty} \mathbb{P}\Big(\xi_t^\eta \cap B \neq \emptyset\Big) \leq \beta(\eta)\rho(B). \tag{2.6}$$

***(b)*** *r&pc is equivalent to*

$$\lim_{N\to\infty} \liminf_{t\to\infty} \mathbb{P}\Big(\xi_t^{B(0,N)} \cap B(0,N) \neq \emptyset\Big) = 1. \tag{2.7}$$

***(c)*** *The property r&pc is monotone increasing.*

***(d)*** *If pc holds, then for any law $\mu$ and any continuous non-negative non-decreasing function $f\colon \{0,1\}^{\mathcal{V}_G} \to \mathbb{R}$,*

$$\liminf_{t\to\infty} \int f(\zeta)d\mu_t(\zeta) \geq \int \beta(\eta)d\mu(\eta) \int f(\zeta)d\nu_r(\zeta).$$

*In particular, for any $\eta \subset \mathcal{V}_G$ and for any $B \subset\subset \mathcal{V}_G$,*

$$\liminf_{t\to\infty} \mathbb{P}\Big(\xi_t^\eta \cap B \neq \emptyset\Big) \geq \beta(\eta)\beta(B).$$

***(e)*** *If pc holds, then for every $\mu \in \mathcal{I}$ such that $\mu \perp \delta_\emptyset$ the following order relation holds: $\nu_r \leq \mu$, in the stochastic sense. In particular this is the case for all $\mu \in \mathcal{I}_e \backslash \{\delta_\emptyset\}$.*

***(f)*** *cc is equivalent to having simultaneously both pc and r=s. In particular, if s&cc holds, then r&pc also holds.*

***(g)*** *If r=s fails, then there is a configuration $\eta$ with infinitely many particles for which $\xi_t^\eta$ does not converge weakly as $t \to \infty$. By (f) above, this happens in particular if pc holds but cc fails.*

***(h)*** *If cc holds, then*

$$\text{for any } \eta \subset \mathcal{V}_G,\ \xi_t^\eta \Rightarrow \Big(1-\beta(\eta)\Big)\delta_\emptyset + \beta(\eta)\overline{\nu} \text{ as } t\to\infty.$$

*Or equivalently,*

$$\text{for any } \eta \subset \mathcal{V}_G \text{ and } B \subset\subset \mathcal{V}_G,\ \mathbb{P}\Big(\xi_t^\eta \cap B \neq \emptyset\Big) \to \beta(\eta)\rho(B) \text{ as } t\to\infty.$$

***(i)*** *If $G \in \mathcal{H}$, then whenever $\beta(\lambda) > 0$, the criterion r=s is satisfied. In particular we have the following for homogeneous graphs: (a) r&pc is equivalent to s&cc. (b) s&cc is a $\mathcal{H}$-monotone increasing property. (c) $\nu_r$ coincides with $\delta_\emptyset$ when $\beta(\lambda) = 0$ and with $\overline{\nu}$ when $\beta(\lambda) > 0$. (d) If $\beta(\lambda) > 0$, then $\mathcal{I}_e = \{\delta_\emptyset, \overline{\nu}\}$.*

**Lemma 2.3** *For any $A \subset\subset \mathcal{V}_G$ and any $\eta \subset \mathcal{V}_G$, it follows that*

$$\lim_{t\to\infty} \mathbb{P}\Big(\xi_t^\eta \cap A \neq \emptyset, (\Omega_r^\eta)^c\Big) = 0.$$

**Proof.** One has $\xi_t^\eta \cap A = \emptyset$ almost surely on $(\Omega_r^\eta)^c$ for $t$ large enough. So, on this set, $\mathbb{1}_{\{\xi_t^\eta \cap A \neq \emptyset\}} \to 0$ almost surely, as $t \to \infty$. The result follows then by the Dominated Convergence Theorem.

■

**Proof of Thm.2.2(a).** From Lemmas 2.1 and 2.3,

$$\limsup_{t\to\infty} \mathbb{P}\Big(\xi_t^A \cap B \neq \emptyset\Big) \leq$$
$$\limsup_{N\to\infty} \limsup_{u\to\infty} \limsup_{t\to\infty} \mathbb{P}\Big(\Omega^A(u,N), \xi_t^A \cap B \neq \emptyset\Big). \tag{2.8}$$

Using the Markov property and attractiveness, for $u < t$ and arbitrary $M$

$$\begin{aligned}\mathbb{P}\Big(\Omega^A(u,N), \xi_t^A \cap B \neq \emptyset\Big) &\leq \mathbb{P}\Big(\Omega^A(u,N)\Big)\Big\{\mathbb{P}\Big(\xi_{t-u}^{B(0,M)} \cap B \neq \emptyset\Big) + \\ &\qquad \mathbb{P}\Big(\xi_u^A \cap (B(0,M))^c \neq \emptyset |\ \Omega^A(u,N)\Big)\Big\}.\end{aligned}$$

Using self-duality for the first term inside the braces and Lemma 2.3,we obtain

$$\begin{aligned}\limsup_{t\to\infty} \mathbb{P}\Big(\Omega^A(u,N), \xi_t^A \cap B \neq \emptyset\Big) &\leq \mathbb{P}\Big(\Omega^A(u,N)\Big)\Big\{\beta(B) + \\ &\qquad \mathbb{P}\Big(\xi_u^A \cap (B(0,M))^c \neq \emptyset |\ \Omega^A(u,N)\Big)\Big\}.\end{aligned}$$

If in this inequality we let $M \to \infty$, then $u \to \infty$, then $N \to \infty$, the right-hand-side converges to $\beta(A)\beta(B)$, thanks to Lemma 2.1. Therefore the first statement in Thm.2.2(a) follows from (2.8).

To prove the second statement we use again Lemmas 2.1 and 2.3, the Markov property, attractiveness and self-duality to write

$$\begin{aligned}\limsup_{t\to\infty} \mathbb{P}\Big(\xi_t^\eta \cap B \neq \emptyset\Big) &\leq \limsup_{N\to\infty}\ \limsup_{u\to\infty}\ \limsup_{t\to\infty} \mathbb{P}\Big(\Omega^\eta(u,N), \xi_t^\eta \cap B \neq \emptyset\Big) \\ &\leq \limsup_{N\to\infty}\ \limsup_{u\to\infty}\ \limsup_{t\to\infty} \mathbb{P}\Big(\Omega^\eta(u,N)\Big)\mathbb{P}\Big(\xi_{t-u}^{\nu_G} \cap B \neq \emptyset\Big) \\ &= \beta(\eta)\rho(B).\end{aligned}$$

■

**Proof of Thm.2.2(b).** Suppose first that (2.7) holds. Then for large $N$, using (2.5), we have

$$\Big(\beta\Big(B(0,N)\Big)\Big)^2 \geq \limsup_{t\to\infty} \mathbb{P}(\xi_t^{B(0,N)} \cap B(0,N) \neq \emptyset) \geq \frac{1}{2},$$

which clearly implies recurrence.

To see that (2.7) implies pc, we will use Lemma 2.2 twice. In the first application of this lemma we replace $A$ with $B$, take $\mu = \delta_A$ and $\alpha \in (0,1)$. Using also self-duality, we have

$$\mathbb{P}\Big(\xi_t^A \cap B \neq \emptyset\Big) \geq \mathbb{P}\Big(S_N^B < \alpha t\Big) \inf_{u > (1-\alpha)t} \mathbb{P}\Big(\xi_u^{B(0,N)} \cap A \neq \emptyset\Big). \tag{2.9}$$

Using Lemma 2.2 again with $\mu = \delta_{B(0,N)}$,

$$\mathbb{P}\Big(\xi_u^{B(0,N)} \cap A \neq \emptyset\Big) \geq \mathbb{P}\Big(S_N^A < \alpha u\Big) \inf_{v > (1-\alpha)u} \mathbb{P}\Big(\xi_v^{B(0,N)} \cap B(0,N) \neq \emptyset\Big). \tag{2.10}$$

Combining (2.9) and (2.10) it follows that

$$\begin{aligned} \mathbb{P}\Big(\xi_t^A \cap B \neq \emptyset\Big) \;\; &\geq \;\; \mathbb{P}\Big(S_N^B < \alpha t\Big)\mathbb{P}\Big(S_N^A < \alpha(1-\alpha)t\Big) \times \\ &\inf_{v > (1-\alpha)^2 t} \mathbb{P}\Big(\xi_v^{B(0,N)} \cap B(0,N) \neq \emptyset\Big). \end{aligned} \tag{2.11}$$

pc follows now from (2.11), (2.7), Lemma 2.1 and (2.5).

To prove the other direction of Thm.2.2(b), note that pc implies

$$\lim_{t\to\infty} \mathbb{P}(\xi_t^{B(0,N)} \cap B(0,N) \neq \emptyset) = \Big(\beta\Big(B(0,N)\Big)\Big)^2.$$

Under recurrence $\beta\Big(B(0,N)\Big) \to 1$ as $N \to \infty$, by Thm.2.1(a), finishing the proof.

■

**Proof of Thm.2.2(c).** The condition (2.7), which in Thm.2.2(b) is presented as equivalent to r&pc, is manifestly monotone increasing.

■

**Proof of Thm.2.2(d).** Thanks to Fatou's lemma, there is no loss in generality in taking $\mu = \delta_\eta$, for some $\eta$. We can also suppose that the probability of recurrence is positive, since otherwise $\nu_r = \delta_\emptyset$ and there is nothing to prove. With these assumptions we have

$$\begin{aligned}
\liminf_{t\to\infty} \int f(\zeta)\, d\mu_t(\zeta) &\geq \liminf_{N\to\infty}\, \liminf_{u\to\infty}\, \liminf_{t\to\infty} \mathbb{E}\Big( f(\xi_t^\eta) \,\Big|\, S_N^\eta < u\Big) \mathbb{P}\Big(S_N^\eta < u\Big) \\
&\geq \liminf_{N\to\infty}\, \liminf_{u\to\infty} \Big[ \mathbb{P}\Big(S_N^\eta < u\Big) \liminf_{s\to\infty} \mathbb{E}\Big( f(\xi_s^{B(0,N)})\Big)\Big] \\
&\geq \liminf_{N\to\infty}\, \liminf_{u\to\infty} \mathbb{P}\Big(S_N^\eta < u\Big) \beta\Big(B(0,N)\Big) \int f(\zeta)\, d\nu_r(\zeta) \\
&= \beta(\eta) \int f(\zeta)\, d\nu_r(\zeta),
\end{aligned}$$

as desired. In the first inequality we used the fact that $f$ is non-negative. In the second inequality we used the strong Markov property at time $S_N^\eta$ and attractiveness. In the third one we used the hypothesis that pc holds and that $f$ is non-negative. In the final step we used Lemma 2.1 and Thm.2.1(a) with the assumption that the probability of recurrence is positive.

■

**Lemma 2.4** *If $\mu \in \mathcal{I}$ and $\mu \perp \delta_\emptyset$, then*

$$\int \beta(\eta)\, d\mu(\eta) = 1. \tag{2.12}$$

**Proof.** Taking $B = B(0,N)$, (2.6) can be rewritten as

$$1 - \beta(\eta)\rho\Big(B(0,N)\Big) \leq \liminf_{t\to\infty} \mathbb{P}\Big(\xi_t^\eta \cap B(0,N) = \emptyset\Big).$$

Integrating $\eta$ with respect to $\mu$, using Fatou's lemma and the invariance of $\mu$, we obtain

$$1 - \Big(\int \beta(\eta)\, d\mu(\eta)\Big)\rho\Big(B(0,N)\Big) \leq \mu(\zeta\colon\ \zeta \cap B(0,N) = \emptyset).$$

By letting $N \to \infty$ we obtain, from the hypothesis $\mu \perp \delta_{\emptyset}$,

$$1 - \Big(\int \beta(\eta)\, d\mu(\eta)\Big) \lim_{N\to\infty} \rho\Big(B(0,N)\Big) \leq 0.$$

But this can only happen if (2.12) holds.

■

**Proof of Thm.2.2(e).** From the previous lemma and Thm.2.2(d) we obtain, since $\mu \in \mathcal{I}$,

$$\int f(\zeta)\, d\mu(\zeta) \geq \int f(\zeta)\, d\nu_r(\zeta),$$

for all continuous non-negative and non-decreasing functions $f\colon \{0,1\}^{\mathcal{V}_G} \to \mathbb{R}$. The restriction to non-negative $f$ is irrelevant at this point, and we have proven the desired inequality between $\mu$ and $\nu_r$.

■

**Proof of Thm.2.2(f).** We will show that cc implies r=s; the rest is then immediate. Under cc, for all $A, B \subset\subset \mathcal{V}_G$,

$$\beta(A)\beta(B) \leq \rho(A)\rho(B) = \lim_{t\to\infty} \mathbb{P}(\xi_t^A \cap B \neq \emptyset) \leq \beta(A)\beta(B),$$

where the last inequality follows from (2.5). Taking $A = B$ we obtain $\beta(A) = \rho(A)$.

■

**Lemma 2.5** *If* $\left|\eta^c\right| < \infty$*, then* $\xi_t^\eta \Rightarrow \overline{\nu}$*, as* $t \to \infty$*.*

**Proof.** By self-duality, for any $A \subset\subset \mathcal{V}_G$

$$\begin{aligned}\mathbb{P}\Big(\xi_t^\eta \cap A \neq \emptyset\Big) &= \mathbb{P}\Big(\xi_t^A \cap \eta \neq \emptyset\Big) \geq \mathbb{P}\Big(\left|\xi_t^A\right| > \left|\eta^c\right|\Big) \\ &\geq \mathbb{P}\Big(\left|\xi_t^A\right| > \left|\eta^c\right|, \Omega_\infty^A\Big) \to \rho(A),\end{aligned}$$

as $t \to \infty$, since on $\Omega_\infty^A$, $\lim_{t\to\infty} \left|\xi_t^A\right| = \infty$ a.s..

But also

$$\mathbb{P}\Big(\xi_t^\eta \cap A \neq \emptyset\Big) \leq \mathbb{P}\Big(\xi_t^{\mathcal{V}_G} \cap A \neq \emptyset\Big) \to \rho(A), \text{ as } t \to \infty,$$

completing the argument.

■

**Proof of Thm.2.2(g).** From the hypothesis there is a set $C \subset\subset \mathcal{V}_G$ such that

$$\nu_r(\zeta\colon\ \zeta \cap C \neq \emptyset) = \beta(C) < \rho(C) = \overline{\nu}(\zeta\colon\ \zeta \cap C \neq \emptyset).$$

For $0 < M < N$ we use the notation $A(M,N) = B(0,N) \setminus B(0,M)$ for a annulus centered at 0. The configuration $\eta$ is taken as

$$\eta = \bigcup_{i=1}^{\infty} A(N_{2i-1}, N_{2i}),$$

where $0 = N_0 < N_1 < N_2 \ldots$ will be chosen properly. Also a sequence of times $t_j \nearrow \infty$ will be chosen, and the goal is to have for a small $\epsilon > 0$

$$\limsup_{i\to\infty} \mathbb{P}\Big(\xi_{t_{2i}}^\eta \cap C \neq \emptyset\Big) \leq \beta(C) + \epsilon, \tag{2.13}$$

and

$$\liminf_{i\to\infty} \mathbb{P}\Big(\xi^{\eta}_{t_{2i+1}} \cap C \neq \emptyset\Big) \geq \rho(C) - \epsilon. \tag{2.14}$$

Intuitively we want to let $N_j$ and $t_j$ increase very fast with $j$, so that: (i) at time $t_{2i}$ the set $C$ is not affected by what happens beyond $B(0, N_{2i+1})$ at time 0, so that this region could as well be totally vacant initially and (2.5) leads to (2.13). (ii) At time $t_{2i+1}$ the set $C$ is not affected by what happens beyond $B(0, N_{2i+2})$ at time 0, so that this region could as well be totally occupied initially, and (2.14) follows from the previous lemma.

To make this intuition precise define the following two sequences of configurations which approximate $\eta$ from opposite sides.

$$\eta^{(2k)} = \bigcup_{i=1}^{k} A(N_{2i-1}, N_{2i}) = \eta \cap B(0, N_{2k}), \quad k = 1, 2, \ldots$$

and

$$\eta^{(2k+1)} = \eta^{(2k)} \cup \Big(B(0, N_{2k+1})\Big)^{\mathrm{c}}, \quad k = 1, 2, \ldots$$

Choose $t_1$, $N_1$ and $N_2$ arbitrarily and proceed recursively as follows. Given $t_1, \ldots, t_{2i-1}$ and $N_1, \ldots, N_{2i}$ take $t_{2i}$ such that

$$\mathbb{P}\Big(\xi^{\eta^{(2i)}}_{t_{2i}} \cap C \neq \emptyset\Big) \leq \beta(C) + \epsilon/2,$$

which is possible by (2.5). Next take $N_{2i+1}$ such that

$$\mathbb{P}\Big(\xi^{\eta}_{t_{2i}} \cap C \neq \emptyset\Big) \leq \mathbb{P}\Big(\xi^{\eta^{(2i)}}_{t_{2i}} \cap C \neq \emptyset\Big) + \epsilon/2,$$

which is possible by the Feller property. Therefore (2.13) is assured.

Next take $t_{2i+1}$ such that

$$\mathbb{P}\Big(\xi_{t_{2i+1}}^{\eta^{(2i+1)}} \cap C \neq \emptyset\Big) \geq \rho(C) - \epsilon/2,$$

which is possible by the previous lemma. Finally take next $N_{2i+2}$ such that

$$\mathbb{P}\Big(\xi_{t_{2i+1}}^{\eta} \cap C \neq \emptyset\Big) \geq \mathbb{P}\Big(\xi_{t_{2i+1}}^{\eta^{(2i+1)}} \cap C \neq \emptyset\Big) - \epsilon/2.$$

Thus (2.14) is also assured.

■

**Proof of Thm.2.2(h).** Half of the claim is in (2.6). The other half is in the combination of Thm.2.2(f), which assures that pc and r=s hold, and Thm.2.2(d).

■

**Proof of Thm.2.2(i).** Suppose that $\beta(\lambda) > 0$. A proof that r=s holds in case $G$ is a homogeneous tree can be found in the proof of Thm.2.5 in Madras and Schinazi (1992). Their simple and intuitive argument applies as well to any homogeneous graph. It turns out that we can also present an alternative self-contained and equally simple proof, and for the reader's benefit we do it next.

Fix $N$ and let $F_N$ be the event that eventually a ball of radius $N$ centered somewhere will become fully occupied. Clearly $\mathbb{P}(\Omega_\infty^0 \cap (F_N)^c) = 0$. Using the strong Markov property, attractiveness and the hypothesis that $G \in \mathcal{H}$, we can write

$$\begin{aligned}\rho(\lambda) - \beta(\lambda) &= \mathbb{P}(\Omega_\infty^0 \cap (\Omega_r^0)^c) \leq \mathbb{P}(F_N \cap (\Omega_r^0)^c) \\ &\leq \sup_{x \in \mathcal{V}_G} \mathbb{P}((\Omega_r^{B(x,N)})^c) = \mathbb{P}\big((\Omega_r^{B(0,N)})^c\big).\end{aligned}$$

Letting $N \to \infty$, r=s follows now from Thm.2.1(a), since $\beta(\lambda) > 0$.

With this part done, the rest of the claim follows easily.

The statement (a) is immediate now, using Thm.2.2(f).

The statement (b) follows easily from (a) and Thm.2.2(c).

The statement (c) is immediate from the definitions of $\nu_r$ and of $\overline{\nu}$.

The statement (d) is immediate from Thm.2.1.(d).

■

The monotonicity of r&pc, as stated in Thm.2.2(c) above, is a very useful fact, since it provides a way for proving that pc holds, as we will see later on, when we prove some of the properties of the basic example and some other more general results.

Because of this monotonicity it is natural to define

$$\lambda_{\text{r\&pc}} = \inf\{\lambda : \text{ r\&pc holds}\}.$$

The following question can then be raised.

**(Q6)** Is it always the case that $\lambda_r = \lambda_{\text{r\&pc}}$ ?

The example 3.3 in the next chapter will show that this is not the case even for trees.

The parallel between the definitions of $\overline{\nu}$ and cc on one hand and of $\nu_r$ and pc on the other is evident and esthetically appealing (at least to the author and her advisor). When r=s holds, the parallel notions collapse into each other. The parallel is nevertheless broken by the fact that s&cc is not a monotone increasing property while r&pc is; we wonder if there is any intuitive reason behind this difference.

The facts about the contact process on homogeneous graphs, summarized in Thm.2.2(i), and the emphasis of the study of the contact process being put on the cases of the homogeneous graphs $\mathbb{Z}^d$ and more recently $\mathbb{T}_b$ may have been the reason why $\nu_r$ and pc have never been identified before as separate entities.

Because of the $\mathcal{H}$-monotonicity of s&cc, as stated in Thm.2.2(i), it is natural to define for the homogeneous graphs

$$\lambda_{s\&cc} = \inf\{\lambda : \text{ s\&cc holds}\},$$

and to ask the following question.

**(Q7)** Is it always the case that for homogeneous graphs $\lambda_r = \lambda_{s\&cc}$ ?

The answer is known to be positive in the basic cases of the cubic lattices, thanks to Bezuidenhout and Grimmett (1990), and of the homogeneous trees, thanks to Zhang (1996). Nevertheless the corresponding proofs are substantially different, and while each one of these proofs generalizes to some other homogeneous graphs, the complexity of these proofs and the use of the special structure of the graphs in them, make (Q7) seem as a very difficult question to settle rigorously.

Another related natural question is the following.

**(Q8)** Is it the case that for all homogeneous graphs if $\lambda_s < \lambda_r$ then for $\lambda_s < \lambda < \lambda_r$ there are infinitely many extremal invariant measures?

The answer is known to be positive in the case of the homogeneous trees, thanks to Durrett and Schinazi (1995). Liggett (1996b) has further results in this direction, which in particular indicate that the set of all extremal invariant measures may be difficult to characterize even for these relatively simple graphs. If the answers to (Q7) and (Q8) turn out to be positive, as we tend to expect, just for simplicity, then the qualitative ergodic behavior of the contact process on homogeneous graphs would basically always be the one found for homogeneous trees, but allowing also for the possibility that the survival and the recurrence points may coincide, with the intermediate phase being then absent, as is the case for the cubic lattices.

It is worth stressing that the fact that for homogeneous trees $\mathcal{I}_{\mathrm{e}} = \{\delta_{\emptyset}, \overline{\nu}\}$ above the recurrence point has its proof now greatly simplified. This fact is contained in Thm.2.2(i). The only other proof that we are aware of goes by proving first that cc holds, as done originally by Zhang (1996); and simplified in Chapter 5 of this thesis. This approach gives an important extra result, but is much more complicated.

The next results provide sufficient conditions for the r=s criterion to hold.

**Theorem 2.3** *Suppose that for a graph $G$ there exists $\delta > 0$ so that for every $x \in \mathcal{V}_G$,*

$$\mathbb{P}\Big(0 \in \xi_t^x \textit{ for some } t > 0\Big) \geq \delta > 0.$$

*Then r=s holds.*

This is a very intuitive result, since in the event of survival, the hypothesis of the theorem assures us that there is "a constant push towards the root". The proof is a direct rigorization of this intuition.

For use in the proof of this theorem, define for each configuration $\eta$ the stopping time $\tau^\eta = \inf\{t\colon \xi_t^\eta = \emptyset\}$.

**Proof of Thm.2.3.** Let $\eta$ be the initial configuration. For every $x \in \mathcal{V}_G$ take $T(x)$ such that

$$\mathbb{P}\Big(0 \in \xi_t^x \text{ for some } t \le T(x)\Big) \ge \delta/2. \tag{2.15}$$

Order the sites in $G$ and define a sequence of stopping times $(S_i)_{i=0,1,\dots}$, by first setting

$$S_0 = 0.$$

Then we proceed recursively in the following way. On $\{\tau^\eta > S_i + 1\}$ we define

$$S_{i+1} = S_i + 1 + T(x_i), \text{ where } x_i \text{ is the first site in } \xi^\eta_{S_i+1}.$$

On $\{\tau^\eta \le S_i + 1\}$ we define, quite arbitrarily, $S_{i+1} = S_i + 1$. Define the events

$$F_i = \Big\{0 \in \xi_t^\eta \text{ for some } t \in [S_i + 1, S_{i+1})\Big\}.$$

With this notation, it follows that

$$\begin{aligned}
\mathbb{P}\Big(\Omega_\infty^\eta, (\Omega_r^\eta)^c\Big) &\le \mathbb{P}\Big(\{\tau^\eta = \infty\}, \{F_i \text{ infinitely often }\}^c\Big) \\
&\le \sum_{j\ge 0} \mathbb{P}\Big(\bigcap_{i\ge j}\Big(\{\tau^\eta > S_i + 1\} \cap (F_i)^c\Big)\Big) \\
&= 0,
\end{aligned}$$

where in the last step use is being made of the Markov property and of (2.15).

■

**Theorem 2.4** *Suppose that $G$ is a collage of $G_1, ..., G_m$, then for each value of $\lambda > 0$ the condition r=s holds for $G$ if and only if it holds for each one of the graphs $G_i$, $i = 1, ..., m$.*

The intuitive reason behind this theorem is that if r=s fails for one $G_k$, then the contact process on $G$ can survive inside of $G_k$, without recurring, so that r=s should also fail for $G$. On the other hand, if r=s holds for all $G_i$, $i = 1, ..., m$, then we cannot have survival in one or more of the $G_i's$, without the infection recurring to each fixed site of these $G_i's$, so that r=s should also hold for $G$.

**Lemma 2.6** *The validity of r=s is equivalent to the statement that for all $A \subset\subset \mathcal{V}_G$, $\mathbb{P}\Big(\Omega_\infty^A,\ 0 \notin \xi_t^A$ for all $t > 0\Big) = 0$.*

**Proof.** Clearly r=s implies the other statement.

For the converse, suppose that r=s fails. Then $\mathbb{P}\Big(\Omega_\infty^0,\ (\Omega_r^0)^c\Big) > 0$ and therefore, for some large enough $T$, $\mathbb{P}\Big(\Omega_\infty^0,\ 0 \notin \xi_t^0 \text{ for all } t > T\Big) > 0$. Hence, for some $A \subset\subset \mathcal{V}_G$, $\mathbb{P}\Big(\Omega_\infty^0,\ 0 \notin \xi_t^0 \text{ for all } t > T,\ \xi_T^0 = A\Big) > 0$. Using the Markov property at time $T$, this inequality leads to $\mathbb{P}\Big(\Omega_\infty^A,\ 0 \notin \xi_t^A \text{ for all } t > 0\Big) > 0$.

■

**Proof of Thm.2.4.** We start proving that if the contact process on some $G_k$ does not satisfy r=s, then r=s and cc fail for the contact process on $G$. The previous lemma states that there is a $A \subset\subset \mathcal{V}_{G_k}$, such that for the contact process on $G_k$

starting from $A$, there is a positive probability of surviving without ever infecting the root. But then it is clear that the same property is true for the contact process on $G$ starting from the same set $A$. The proof is now completed by using the same lemma in the opposite direction.

Now we will show that if, for $i = 1, \ldots, m$, the contact process on each $G_i$ satisfy r=s, then the contact process on $G$ also satisfy r=s. For arbitrary $A \subset\subset \mathcal{V}_G$,

$$0 \le \rho(A) - \beta(A) \le \mathbb{P}\Big(\bigcup_{n=0}^{\infty} F_n\Big),$$

where $F_n$ is the event that $(\xi^A_{G;t} \colon t \ge 0)$ survives, but from time $n$ the sites $V_{\text{glue}} \bigcap_{i=1,\ldots,m} \mathcal{V}_{G_i}$ are never occupied. But on the event $F_n$ we must have after time $n$ survival starting from a finite set without ever hitting the finite set $V_{\text{glue}} \bigcap \mathcal{V}_{G_k}$ for the process restricted to some $G_k$. These are impossible events, since the contact processes on $G_i$, $i = 1, \ldots, m$, satisfy r=s.

■

The Thm.2.3 is used to prove the part of the next theorem that refers to recurrence, when $b \ge 2$; the part which refers to survival and the case $b = 1$ are already in the literature. This theorem refers to the tree $\mathbb{T}_b^+$, obtained from the tree $\mathbb{T}_b$ by removing one of the neighbors of the root and defining the new tree as the remaining connected component of $\mathbb{T}_b$ which contains its root.

**Theorem 2.5** *For each $b \ge 1$, $\mathbb{T}_b^+$ has the same survival point and the same recurrence point as $\mathbb{T}_b$. Moreover above the recurrence point cc is satisfied by the contact process on $\mathbb{T}_b^+$.*

**Proof of Thm.2.5.** As mentioned before the theorem was stated, the only parts that remain to be proven are the ones that concern $b \geq 2$ and the recurrence point and recurrence regime. By graph-monotonicity, it is clear that $\lambda_r(\mathbb{T}_b) \leq \lambda_r(\mathbb{T}_b^+)$, and we prove next the complementary inequality.

Think of $\mathbb{T}_b^+$ as a subgraph of $\mathbb{T}_b$, and take an arbitrary site $x \in \mathcal{V}_{\mathbb{T}_b^+}$. It is clear that when $\lambda > \lambda_r(\mathbb{T}_b)$

$$\mathbb{P}\Big(0 \in \xi^x_{\mathbb{T}_b^+;t} \text{ for some } t > 0\Big) \geq \beta_{\mathbb{T}_b}(\{x\}) = \beta_{\mathbb{T}_b}(\{0\}) > 0.$$

Therefore Thm.2.3 implies that r=s holds for $\mathbb{T}_b^+$, for such values of $\lambda$. The proof that $\lambda_r(\mathbb{T}_b^+) \leq \lambda_r(\mathbb{T}_b)$, is now completed by recalling that it is known that $\rho_{\mathbb{T}_b^+}(\{0\}, \lambda) > 0$, for $\lambda > \lambda_r(\mathbb{T}_b)$, since it is known that $\lambda_s(\mathbb{T}_b^+) = \lambda_s(\mathbb{T}_b) < \lambda_r(\mathbb{T}_b)$.

The argument above gave us already the validity of r=s when $\lambda > \lambda_r(\mathbb{T}_b^+)$, and therefore, by Thm.2.2(f) the proof of cc will be complete once we show that pc holds also in this regime. The argument for this is a good illustration of the usefulness of Thm.2.2(b). From Proposition 5 in Zhang (1996), we know that

$$\inf_{t \geq 0} \mathbb{P}(0 \in \xi^0_{\mathbb{T}_b^+;t}) > 0. \tag{2.16}$$

(In that paper, the set $U$ is a version of $\mathbb{T}_b^+$ and the $\liminf_{t\to\infty}$, which appears there, can clearly be replaced with the $\inf_{t\geq 0}$.) From this it is fairly easy to see that the condition (2.7), which in Thm.2.2(b) is presented as equivalent to r&pc, holds. First observe that each one of the $b^N$ sites in $\mathbb{T}_b^+$ which are exactly at distance $N$ from 0 can be thought of as the root of a subgraph of $\mathbb{T}_b^+$ which is isomorphic to $\mathbb{T}_b^+$.

These $b^N$ subgraphs have disjoint sets of vertices. We are interested in the process $(\xi^{B(0,N)}_{\mathbb{T}^+_b;t})$, and at time zero each one of the subgraphs that we just mentioned has therefore its root occupied. If we let the process in each one of these subgraphs evolve without being allowed to infect or be infected by other sites, then we are looking simply at a large number of independent copies of the process $(\xi^0_{\mathbb{T}^+_b;t})$. The validity of (2.7) follows now at once from (2.16).

■

Thm.2.3 will also be used in the proof of the following one. As before, $\lambda_c(\mathbb{Z})$ denotes the common value of $\lambda_s(\mathbb{Z})$ and $\lambda_r(\mathbb{Z})$.

**Theorem 2.6** *For every graph $G \in \mathcal{G}$, s&cc holds for $\lambda > \lambda_c(\mathbb{Z})$.*

This theorem may at first sight seem totally intuitive, but for the wrong reason. It is true that all graphs in $\mathcal{G}$ have $\mathbb{Z}^+$ embedded into them, and that for $\mathbb{Z}^+$ cc holds above its critical point, which coincides with $\lambda_c(\mathbb{Z})$. But as we know, the answer to question (Q3) is negative, and therefore we can not immediately conclude the statement in Thm.2.6 . In other words, Thm.2.6 states that the answer to (Q3) becomes "yes" if $G_0 = \mathbb{Z}^+$, and the nature of $\mathbb{Z}^+$ is crucial in this theorem.

**Proof of Thm.2.6.** Each graph $G \in \mathcal{G}$ has $\mathbb{Z}^+$ embedded into it as a subgraph. Since s&cc holds for $\mathbb{Z}^+$ above its recurrence point, which coincides with $\lambda_c(\mathbb{Z})$, we can conclude from Thm.2.2(f) and Thm.2.2(c) that r&pc holds for $G$. Thanks to Thm.2.2(f) we know now also that our task has been reduced to showing that r=s holds for $G$, this is what we do below.

Let $G_0$ be a graph which is isomorphic to $\mathbb{Z}^+$ and has no vertex in common with $G$. Set $G' = G \vee G_0$. Every site $x \in \mathcal{V}_G$ can be connected to the root of $G$ by a chain of neighboring sites in $G$. Therefore in $G'$, $x$ belongs to an infinite linear chain of neighboring sites in which only $x$ itself has 1 single neighbor, i.e., a subgraph $G_x$ isomorphic to $\mathbb{Z}^+$, with $x$ playing the role of its origin.

Because in $\mathbb{Z}^+$ survival without ever hitting a given fixed site is impossible (otherwise we would have survival without growth), we obtain for $\lambda > \lambda_s(\mathbb{Z}^+)$,

$$\begin{aligned}
\mathbb{P}\Big(0 \in \xi^x_{G;t} \text{ for some } t > 0\Big) &\geq \mathbb{P}\Big(0 \in \xi^x_{G';t} \text{ for some } t > 0\Big) \\
&\geq \mathbb{P}\Big(0 \in \xi^x_{G_x;t} \text{ for some } t > 0\Big) \\
&\geq \rho_{\mathbb{Z}^+}(\{0\}, \lambda) \\
&> 0.
\end{aligned}$$

Therefore Thm.2.3 assures us that r=s holds for $G$ provided $\lambda > \lambda_s(\mathbb{Z}^+) = \lambda_c(\mathbb{Z})$, as we wanted to show.

■

## 2.3 Notes.

In this section we will say something about how the work in this chapter came into existence and in so doing we will emphasize the connections with some earlier work.

In a first stage of this work we were trying to understand better the notion of complete convergence for the contact process on graphs. We wanted to decide whether Conjecture 1 in Pemantle (1992), which states that, on trees, cc should always hold above $\lambda_r$ was true, and we also wanted to prove what in our work is now called Thm.2.6. For this second task we checked that the technique introduced in the papers Schonmann (1987a) and Schonmann (1987b) could indeed be used to do it. This technique relies on a fundamental little result in Griffeath (1978), which states that cc is equivalent to the following. For all pairs of finite subsets of $\mathcal{V}_G$, $A$ and $B$, if $(\xi_t^A)$ and $(\overline{\xi}_t^B)$ are two independent versions of the contact process, starting from $A$ and $B$, respectively, then

$$\lim_{t\to\infty} \mathbb{P}(\Omega_\infty^A, \overline{\Omega}_\infty^B, \xi_t^A \cap \overline{\xi}_t^B = \emptyset) = 0. \tag{2.17}$$

To prove Thm.2.6 using this result, one can roughly say that on the intersection of the events $\Omega_\infty^A$ and $\overline{\Omega}_\infty^B$, each process will eventually contain lots of particles on some subgraph of $G$ which is isomorphic to $\mathbb{Z}^+$, with these two subgraphs being such that they have an intersection also isomorphic to $\mathbb{Z}^+$. At the random time when this happens one restricts each contact process to the corresponding subgraph and uses graph-monotonicity for comparisons with the true processes. We are above $\lambda_c(\mathbb{Z}^+) = \lambda_c(\mathbb{Z})$, so that the contact process on $\mathbb{Z}^+$ satisfies cc, and hence also (2.17). Using then the fact that above $\lambda_c(\mathbb{Z}^+) = \lambda_c(\mathbb{Z})$ the contact process on $\mathbb{Z}^+$ starting with many particles is likely to survive, one can finish the proof that (2.17) also holds for the contact process on $G$.

At this point our attention was called to the fact that the special structure of $\mathbb{Z}^+$ was crucial in such an argument, and that the answer to the corresponding (Q3), when $G_0$ is arbitrary could be negative. This led us to examples similar to what we are calling now our basic example and eventually to that one and to the understanding that the answers to (Q1), (Q2) and (Q3) are negative in general.

The question then arose as to what could be said about the ergodic behavior of the contact process for values of $\lambda$ above $\lambda_r$ for which cc fails. The introduction of $\nu_r$ and of the notion of pc came naturally at this point, by replacing $\Omega_\infty^A$ with $\Omega_r^A$ in cc, and trying to prove an analogue to Griffeath's result quoted above, concerning pc. This could be done, and we proved that pc is equivalent to the following. For all pairs of finite subsets of $\mathcal{V}_G$, $A$ and $B$, if $(\xi_t^A)$ and $(\overline{\xi}_t^B)$ are two independent versions of the contact process, starting from $A$ and $B$, respectively, then

$$\lim_{t\to\infty} \mathbb{P}(\Omega_r^A, \overline{\Omega}_r^B, \xi_t^A \cap \overline{\xi}_t^B = \emptyset) = 0. \tag{2.18}$$

Our first proof of the monotonicity of r&pc relied on the equivalence above and arguments of the nature of those presented above to prove Thm.2.6 based on Griffeath's equivalence. By successive simplifications of this proof, we eventually obtained Thm.2.2(b). We decided to omit the equivalence between pc and (2.18) from the main part of this work, because it seems to us now that any other application that this may have should also be obtainable from Thm.2.2(b), in an easier fashion.

In this connection we would like to stress that the machinery introduced in this chapter can be used to simplify proofs of cc. To our knowledge, cc has always been proved for the contact process on particular graphs either using Griffeath's equivalence above or some "shape theorem" for the region of $\mathcal{V}_G$ where the contact process started from a finite set agrees with the contact process started from the fully occupied lattice. This second method is more delicate, but it also provides one with a much more refined result than cc. Regarding the first method, one can now sometimes simplify such proofs by checking that pc holds, via the equivalent condition in Thm.2.2(b), and that r=s also holds (recall Thm.2.2(f)). For instance, the proof by Bezuidenhout and Grimmett (1990) of cc for the contact process on $\mathbb{Z}^d$ above the unique critical point becomes easier to understand: by Thm.2.2(i) and Thm.2.2(b) it is enough to verify that (2.7) holds. But this clearly follows from the dynamical renormalization scheme in Bezuidenhout and Grimmett (1990) (compare with the argument for cc in Section 5 of that paper). Also the proof by Zhang (1996) of cc for the contact process on homogeneous trees for $\lambda > \lambda_r$ is simplified. The use of Thm.2.2(b) in this case was presented in our proof of Thm.2.5, where we considered $\mathbb{T}_b^+$. The case of $\mathbb{T}_b$ is analogous (compare with Zhang's approach to verifying that Griffeath's equivalence holds).

# CHAPTER 3

## Examples of trees that answer some questions in Chapter 2

### 3.1 Collage of homogeneous trees.

**Example 3.1** *Trees with finitely many contact process transitions.*

The first example is the basic example of the previous chapter, $T_j \vee \mathbb{T}_k$ with $j \geq 2$ and $k$ sufficiently larger than $j$, so that we have

$$\lambda_s(\mathbb{T}_k) < \lambda_r(\mathbb{T}_k) < \lambda_s(\mathbb{T}_j) < \lambda_r(\mathbb{T}_j).$$

The contact process on $\mathbb{T}_j \vee \mathbb{T}_k$ has the following features:

**(I)** In the interval $\left(\lambda_s(\mathbb{T}_j), \lambda_r(\mathbb{T}_j)\right]$ cc fails.

**(II)** In the intervals $\left(\lambda_r(\mathbb{T}_k), \lambda_s(\mathbb{T}_j)\right]$ and $\left(\lambda_r(\mathbb{T}_j), \infty\right)$ cc holds.

**(III)** pc holds for all $\lambda > \lambda_r(\mathbb{T}_k)$, i.e., throughout the regions covered by the statements (I) and (II).

**(IV)** The critical parameters for survival and recurrence are $\lambda_s(\mathbb{T}_j \vee \mathbb{T}_k) = \lambda_s(\mathbb{T}_k)$ and $\lambda_r(\mathbb{T}_j \vee \mathbb{T}_k) = \lambda_r(\mathbb{T}_k)$.

It is clear that the basic example above can be generalized to produce a tree with a larger number of critical points, separating any number of alternating intervals on which cc holds or fails. To this end, it is enough to take a collage of trees $\mathbb{T}_{b_i}$, $i = 1, \ldots, n$, with an appropriate choice of $b_1, \ldots, b_n$, so that

$$\lambda_s(\mathbb{T}_{b_n}) < \lambda_r(\mathbb{T}_{b_n}) < \lambda_s(\mathbb{T}_{b_{n-1}}) < \lambda_r(\mathbb{T}_{b_{n-1}}) < \ldots$$
$$< \lambda_s(\mathbb{T}_{b_1}) < \lambda_r(\mathbb{T}_{b_1}). \quad (3.1)$$

**Proofs of (I),(II),(III) and (IV).** To prove (III), by Thm.2.2(c), it is enough to see that pc holds for the subgraph $\mathbb{T}_k$ of our graph $\mathbb{T}_k \vee \mathbb{T}_j$. But this is true since for homogeneous trees above the recurrence point we have cc and therefore, by Thm.2.2(f), also pc. To obtain (I) and (II), first note that from Thm.2.2(f) applied to the homogeneous trees $T_j$ and $\mathbb{T}_k$ and from Thm.2.4 we learn that in (I) we do not have r=s on $\mathbb{T}_k \vee \mathbb{T}_j$ while in (II) we have r=s on $\mathbb{T}_k \vee \mathbb{T}_j$. Using again Thm.2.2(f), now for $\mathbb{T}_k \vee \mathbb{T}_j$, and (III) we obtain (I) and (II). The proof of (IV) is a particular case of the theorem below. Note that $\mathbb{T}_b$ is a collage of two copies of $\mathbb{T}_b^+$, so that a collage of copies of $\mathbb{T}_{b_1}, \mathbb{T}_{b_2}, \ldots, \mathbb{T}_{b_m}$ fits the hypothesis of the theorem.

**Theorem 3.1** *Suppose that $G$ is a collage of copies of $\mathbb{T}_{b_1}^+, \mathbb{T}_{b_2}^+, \ldots, \mathbb{T}_{b_n}^+$, and set $B = \max\{b_i : i = 1, \ldots, n\}$. Then $\lambda_s(G) = \lambda_s(\mathbb{T}_B)$ and $\lambda_r(G) = \lambda_r(\mathbb{T}_B)$. If $B \geq 2$, then also $\rho_G(\lambda_s(G)) = 0$.*

(Corollary 4.4.6 and Corollary 4.5.2 contain further information about such graphs.) The following lemma will be used in the proof of Thm.3.1.

**Lemma 3.1** *Suppose that $G$ is a collage of $n$ copies of $\mathbb{T}_b^+$, and that $\lambda < \lambda_r(\mathbb{T}_b)$. Then $\beta_G(\lambda) = 0$.*

**Proof.** To prove this lemma we introduce a notion of "cutting time intervals", and then use ergodicity to show that such "cutting time intervals" will a.s. be present.

We will use the notation in the definition of "collage" of graphs, so that, in particular, $G_i$, $i = 1, ..., n$ are disjoint subgraphs of $G$ and each one is isomorphic to $T_b^+$. Let $0_i$, $i = 1, ..., n$, be the roots of these graphs. With no loss in generality, we will suppose that $n \geq 1$ (to assure that $V_{\text{glue}} \neq \emptyset$), that $V_{\text{glue}} \cap \mathcal{V}_{G_i} = 0_i$, $i = 1, ..., n$ (this can always be obtained by enlarging $V_0$ and $n$), and that $0 \in V_{\text{glue}}$.

For each vertex $x$ which belongs to $G_i$ for some $i$, we define for $j \in \{0, 1, ...\}$,

$$\begin{aligned} E_x^{\uparrow j} &= \{\xi_t^{j+\frac{1}{2},x}(0_i) = 1 \text{ for some } t \geq j + 1/2\}, \\ E_x^{\downarrow j} &= \{\hat{\xi}_t^{j+1/2,x}(0_i) = 1 \text{ for some } t \leq j + \frac{1}{2}\}, \\ E_x^j &= E_x^{\uparrow j} \cap E_x^{\downarrow j}. \end{aligned}$$

For $j \in \{0, 1, ...\}$, the time interval $[j, j + 1]$ is called a cutting time interval in case during this time interval, in the graphical construction, there is a death mark at each site in $V_0$, no arrows have end-points at these sites and the event $\cap_{i=1}^n \cap_{x \in G_i} \left(E_x^j\right)^c$ happens.

It is clear that if $[j, j + 1]$ is a cutting time interval, then $\xi_t^0(0) = 0$ for all $t \geq j + 1$. Therefore, if there is any cutting time interval then $(\Omega_r^0)^c$ happens. By ergodicity of the Poisson processes in the graphical construction, the proof that

$\beta_G(\lambda) = 0$ is reduced to showing that

$$\mathbb{P}([0,1] \text{ is a cutting time interval}) > 0. \tag{3.2}$$

This can be done as follows. Note that if $x \in G_i$,

$$\mathbb{P}(E_x^0) = \mathbb{P}(E_x^{\uparrow 0})\mathbb{P}(E_x^{\downarrow 0}) = (u(d(x,0_i)))^2 \leq \Big(\alpha_b(\lambda)\Big)^{2d(x,0_i)}.$$

Since $\alpha_b(\lambda) < b^{-1/2}$, by (1.3), we have now $\sum_{i=1}^{n} \sum_{x \in G_i} \mathbb{P}(E_x^0) < \infty$. But the events $(E_x^0)^c$, as well as the finitely many events involving the sites in $V_0$ which must happen for $[0,1]$ to be a cutting time interval are all decreasing events. Therefore (3.2) follows from Lemma 3.1(a) in Madras, Schinazi and Schonmann (1994).

■

**Proof of Thm.3.1.** Obviously $\lambda_s(G) \leq \lambda_s(\mathbb{T}_B)$ and $\lambda_r(G) \leq \lambda_r(\mathbb{T}_B)$.

Since $G$ is a subgraph of a collage of $n$ copies of $\mathbb{T}_B^+$, Lemma 3.1 implies that

$$\beta_G(\lambda) = 0 \ \text{ for } \ \lambda < \lambda_r(\mathbb{T}_B). \tag{3.3}$$

The proof that $\lambda_r(G) = \lambda_r(\mathbb{T}_B)$ is therefore complete.

If $\lambda \leq \lambda_s(\mathbb{T}_B)$, then the contact process on each $\mathbb{T}_{b_i}^+$, $i = 1, ..., n$ dies out, and hence satisfies r=s. Thm.2.4 then gives that r=s also holds for the contact process on $G$. But since $\lambda_s(\mathbb{T}_B) \leq \lambda_r(\mathbb{T}_B)$ (*resp.* $\lambda_s(\mathbb{T}_B) < \lambda_r(\mathbb{T}_B)$ in case $B \geq 2$), (3.3) implies now that for $\lambda < \lambda_s(\mathbb{T}_B)$ (*resp.* $\lambda \leq \lambda_s(\mathbb{T}_B)$ in case $B \geq 2$) we have $\rho_G(\lambda) = \beta_G(\lambda) = 0$. This completes the proof.

■

The condition that $B \geq 2$, needed for the final conclusion in Thm.3.1 may be just technical. The case which has been excluded is that in which $G$ is a collage of copies of $\mathbb{T}_1^+ = \mathbb{Z}^+$. From the part of Thm.3.1 which can still be applied in this case, we know that $\lambda_s(G) = \lambda_r(G) = \lambda_c(\mathbb{Z})$. The open question is then whether $\rho_G(\lambda_s(G)) = 0$, or not. Surprisingly enough, even the following apparently much simpler question seems to also be open. Consider the graph $G'$ obtained by starting with $\mathbb{Z}$ and adding to it a unique extra site and a unique extra edge connecting this extra site to the origin of $\mathbb{Z}$. Clearly $\lambda_s(G') = \lambda_r(G') = \lambda_c(\mathbb{Z})$ (since $\mathbb{Z}$ is a subgraph of $G'$, which is a subgraph of a collage of three copies of $\mathbb{Z}^+$). But is it the case that $\rho_{G'}(\lambda_s(G')) = 0$, or not? This question is akin to Open Problem 1 in Madras, Schinazi and Schonmann (1994).

## 3.2 Infinitely many transitions.

**Example 3.2** *Trees with infinitely many contact process transitions.*

We will give an example of a tree in $\mathcal{G}$ on which, as $\lambda$ increases, cc alternates between holding and failing infinitely many times. First we introduce some "building blocks".

For each positive integer $l$, define $\mathbb{T}_{2,l}$ as the tree given by the rule that for $n = 0, 1, 2, \ldots$ the vertices at distance $nl$ from the origin have degree three and all the other vertices have degree two. Notice that the tree $\mathbb{T}_{2,1}$ is the same as the homogeneous tree $\mathbb{T}_2$. To define $\mathbb{T}_{2,l}^+$, remove one edge connected to the origin of $\mathbb{T}_{2,l}$ and call $\mathbb{T}_{2,l}^+$ the connected component containing the origin.

For any positive integer $l$, the tree $\mathbb{T}_{2,l}$ has:

$$0 < \lambda_s(\mathbb{T}_{2,l}) < \lambda_r(\mathbb{T}_{2,l}) < \lambda_c(\mathbb{Z}). \tag{3.4}$$

The second inequality comes from Stacey (1996). The third one comes from Aizenman and Grimmett (1991) once we observe that $\mathbb{T}_{2,l}$ has a subgraph, $\mathbb{Z}^{(l)}$, constructed as follows. Label each vertex of $\mathbb{Z}$ according to its distance to the origin and for each integer $k$ add a new vertex and a new edge connecting it to the vertex $kl$ in $\mathbb{Z}$. The same arguments used in the proof of Thm.2.5 show that:

$$\lambda_s(\mathbb{T}_{2,l}^+) = \lambda_s(\mathbb{T}_{2,l}) \quad \text{and} \quad \lambda_r(\mathbb{T}_{2,l}^+) = \lambda_r(\mathbb{T}_{2,l}).$$

Also arguments similar to those used for the homogeneous trees [see Morrow, Schinazi and Zhang (1994) and Zhang (1996) or Chapter 5] give

$$\rho_{\mathbb{T}_{2,l}^+}\Big(\lambda_s(\mathbb{T}_{2,l})\Big) = 0 \quad \text{and} \quad \beta_{\mathbb{T}_{2,l}^+}\Big(\lambda_r(\mathbb{T}_{2,l})\Big) = 0.$$

To construct a tree with infinitely many transitions, we will choose a suitable sequence $\{l_i\}_{i=1,2,\ldots}$. The tree $T\Big(l_1, l_2, \ldots\Big)$ is then constructed as follows. Start with $\mathbb{Z}^+$, label each of its sites according to its distance to the origin. For each positive $k$ add an edge connecting the site $\sum_{i=1}^{k} l_i$ to the root of a copy of $\mathbb{T}_{2,l_k}^+$.

Define $\mathcal{G}_l$ to be the set of all trees with vertices of degree either two or three, with no infinite chains of vertices of degree 2 and where the distance between any two vertices of degree three is at least $l$. Let $\tilde{\lambda}(l) = \inf\Big\{\lambda_s(G)\colon G \in \mathcal{G}_l\Big\}$. Clearly $\tilde{\lambda}(l) \le \lambda_s(\mathbb{T}_{2,l})$ for any $l$, since $\mathbb{T}^+_{2,l} \in \mathcal{G}_l$.

The main technical task here will be the proof of the following:

**Lemma 3.2** $\lim_{l\to\infty} \tilde{\lambda}(l) = \lambda_c(\mathbb{Z})$.

This lemma, and (3.4) assures us that we can take $l_1 < l_2 < l_3 < \dots$ such that the following is satisfied:

$$\begin{aligned}\lambda_s(\mathbb{T}_{2,l_1}) < \lambda_r(T_{2,l_1}) \quad &< \quad \tilde{\lambda}(l_2) \le \lambda_s(\mathbb{T}_{2,l_2}) \\ &< \lambda_r(\mathbb{T}_{2,l_2}) < \tilde{\lambda}(l_3) \le \cdots < \lambda_c(\mathbb{Z}). \qquad (3.5)\end{aligned}$$

Of course, the choice of $\{l_i\}_{i=1,2,\dots}$ can be made so that $l_{i+1}$ is a multiple of $l_i$, for $i = 1, 2, \dots$

**Theorem 3.2** *Suppose that $l_1, l_2, \dots$ are such that (3.5) is satisfied. Then for the contact process on $T\Big(l_1, l_2, \dots\Big)$ we have:*

***(a)*** *cc does not hold in $\Big(\lambda_s(\mathbb{T}_{2,l_j}),\ \lambda_r(\mathbb{T}_{2,l_j})\Big]$, $j = 1, 2, \dots$*

***(b)*** *s&cc holds in $\Big(\lambda_r(\mathbb{T}_{2,l_j}),\ \tilde{\lambda}(l_{j+1})\Big)$, $j = 1, 2, \dots$ Moreover, if the sequence $\{l_i\}_{i=1,2,\dots}$ is such that $l_{i+1}$ is a multiple of $l_i$, for $i = 1, 2, \dots$, then s&cc holds in $\Big(\lambda_r(\mathbb{T}_{2,l_j}),\ \lambda_s(\mathbb{T}_{2,j+1})\Big]$.*

**Remark:** We conjecture, but have not been able to prove that $\tilde{\lambda}(l) = \lambda_s(\mathbb{T}_{2,l})$ for every positive integer $l$.

**Proof of Thm.3.2.** In order to prove (a), let $\lambda \in \Big(\lambda_s(\mathbb{T}_{2,l_j}),\ \lambda_r(\mathbb{T}_{2,l_j})\Big]$, for some $j$. In this interval we know that the contact process on $\mathbb{T}^+_{2,l_j}$ does not satisfy r=s. The tree $T(l_1, l_2, \dots)$ is a collage of $\mathbb{T}^+_{2,l_j}$ and another tree. Since the first one does not satisfies r=s, by Thm.2.4, the same happens to $T(l_1, l_2, \dots)$. It follows from Thm.2.2(f) that in this case $T(l_1, l_2, \dots)$ does not satisfy cc.

For (b), fix $\lambda \in \Big(\lambda_r(\mathbb{T}_{2,l_j}),\ \tilde{\lambda}(l_{j+1})\Big)$, for some $j$. Notice that $T(l_1, l_2, \dots)$ is a collage of $\mathbb{T}^+_{2,l_1}, \dots, \mathbb{T}^+_{2,l_j}$ and $T(l_{j+1}, l_{j+2}, \dots)$. For such values of $\lambda$ we have that the process on $T(l_{j+1}, l_{j+2}, \dots)$ dies out since $T(l_{j+1}, l_{j+2}, \dots) \in \mathcal{G}_{l_{j+1}}$ and $\lambda < \tilde{\lambda}(l_{j+1})$. The contact process on each $\mathbb{T}^+_{2,l_i}$, $i = 1, \dots, j$ satisfies s&cc since $\lambda > \lambda_r(\mathbb{T}_{2,l_i})$ [same proof as in Zhang(1996) or here in Chapter 5]. The process on a collage of a finite number of trees where the process dies out in some of them and in the others satisfies s&cc, also satisfies s&cc. For this, combine Thms.2.2(c), 2.2(f) and Thm.2.4. Thus, the contact process on $T(l_1, l_2, \dots)$ satisfies s&cc.

In the case $l_{i+1}$ is a multiple of $l_i$, for $i = 1, 2, \dots$ then $T(l_{j+1}, l_{j+2}, \dots)$ is a subgraph of $\mathbb{T}_{2,l_{j+1}}$, and hence the contact process on $T(l_{j+1}, l_{j+2}, \dots)$ dies out when $\lambda \leq \lambda_s(\mathbb{T}_{2,l_{j+1}})$. So, the proof of (b) also gives the final claim in the theorem.

■

**Proof of Lemma 3.2.** Fix $\lambda < \lambda_c(\mathbb{Z})$ and let $G \in \mathcal{G}_l$. We want to show the extinction of the contact process on $G$ when $l$ is large.

We will compare the contact process $\left(\xi^0_t\right)_{t\geq 0}$ on $G$ to a process $\left(\tilde{\xi}^0_t\right)_{t\geq 0}$ where the states of $\left(\tilde{\xi}^0_t\right)_{t\geq 0}$ are finite collections of particles on $G$. Each particle has a type and there is no more than one particle of each type in each site.

Let $\mathcal{V}^{(3)}$ be the set of vertices of $G$ of degree three. Particles of each type have a site in $\mathcal{V}^{(3)}$ they call their home.

Particles of different type evolve as independent contact processes unless a particle tries to infect a site $y \in \mathcal{V}^{(3)}$ different from its home. In this case, a particle with a type still not used is created at $y$, having $y$ as its home. Particles of the same type have the same home. At time 0 the process $\left(\tilde{\xi}_t^x\right)_{t\geq 0}$ has just one particle at $x$, of type 0 and it has $x$ as its home.

Note that

$$\xi_t^0 \leq \tilde{\xi}_t^0, \text{ in the sense that: } \forall x \in \mathcal{V}_G, \ \forall t \geq 0,$$

$$\xi_t^0(x) = 1 \ \Rightarrow \ \tilde{\xi}_t^0 \text{ has at least one particle at } x.$$

To each type of particle we associate a generation number. Particles of type 0 have generation number 0. When a particle of a type which is of generation $n$ creates a new type of particle, this new type of particle is said to be of generation $n+1$. Let $N_k$ be the number of types of particles of generation $k$ ever created.

Given $x \in \mathcal{V}^{(3)}$, let $\mu_x$ be the expected number of types of particles of generation 1 created in the process $\left(\tilde{\xi}_t^x\right)_{t\geq 0}$.

Set

$$\mu(G) = \sup_{x \in \mathcal{V}^{(3)}} \mu_x.$$

Since it is clear that the multi-valued process $\left(\tilde{\xi}_t^x\right)_{t\geq 0}$ cannot survive forever restricted to a finite set of vertices, in the event of survival, infinitely many different types of particles must be created. But the next result will show that this is not the case when $\lambda < \lambda_c(\mathbb{Z})$.

Clearly

$$E\Big(N_k \Big| N_{k-1}\Big) \leq \mu N_{k-1}.$$

So, $E(N_k) \leq \mu^k$. Using the Lemma 3.3 below, we can take $l$ large enough so that $\mu < 1$. Therefore the number of types of particles ever created will be a.s. finite.

■

In the proof of the lemmas below, we will use several well-known exponential estimates for the sub-critical contact process [see Liggett (1985)]. For later reference we recall them now. Suppose $\lambda < \lambda_c(\mathbb{Z})$, then for some positive finite constants $c_1$, and $c_2$,

**(a)** $P\Big(\xi^0_{\mathbb{Z};t} \neq \emptyset\Big) \leq c_1 \exp\{-c_2 t\}$.

**(b)** $P\Big(\big(\xi^0_{\mathbb{Z};t}\big)_{t\geq 0} \text{ reaches the site } k\Big) \leq c_1 \exp\{-c_2 k\}$.

**(c)** $P\Big(\inf \xi^{\mathbb{Z}^+}_{\mathbb{Z};t} \leq \exp\{c_2 t\} \text{ for some } t \geq t_0\Big) \leq c_1 \exp\{-c_2 t_0\}$.

**Lemma 3.3** *Suppose* $\lambda < \lambda_c(\mathbb{Z})$. *Then* $\lim_{l\to\infty} \sup_{G\in\mathcal{G}_l} \mu(G) = 0$.

**Proof.** Given $G \in \mathcal{G}_l$ and $x \in \mathcal{V}^{(3)}$, we define the star centered at $x$, $S_x$, as the maximal connected subgraph of $G$ which has $x$ as the only vertex of $\mathcal{V}^{(3)}$. The tips of $S_x$ are the vertices of this subgraph which have degree one in $S_x$. The set of tips of $S_x$ will be denoted by $\mathcal{T}_x$.

Note that

$$\mu_x = \lambda \sum_{y \in \mathcal{T}_x} \mathbb{E}\Big( \int_0^\infty \mathbb{1}_{\left\{\xi^x_{S_x;t}(y)=1\right\}} dt \Big). \tag{3.6}$$

Also, for $y \in \mathcal{T}_x$, using self-duality and (b),

$$\begin{aligned} \mathbb{E}\int_0^l \mathbb{1}_{\left\{\xi^x_{S_x;t}(y)=1\right\}} dt &\leq \mathbb{E}\int_0^l \mathbb{1}_{\left\{\left(\xi^y_{S_x;s}\right)_{s\geq 0} \text{ reaches } x\right\}} dt \\ &\leq (c_1 l) \exp\{-c_2 l\}. \end{aligned} \tag{3.7}$$

Let $S$ be the tree in which one vertex, the root, has degree three and all other vertices have degree two.

If we think of $S_x$ as a subgraph of $S$, Lemma 3.4 below gives:

$$\begin{aligned} \mathbb{E}\int_l^\infty \mathbb{1}_{\left\{\xi^x_{S_x;t}(y)=1\right\}} dt &\leq \mathbb{E}\int_l^\infty \mathbb{1}_{\left\{\xi^S_{S;t}(y)=1\right\}} dt \\ &\leq \int_l^\infty c_3 \exp\{-c_4 t/\log t\} dt. \end{aligned} \tag{3.8}$$

The lemma follows from (3.6), (3.7) and (3.8).

■

**Lemma 3.4** *Suppose $\lambda < \lambda_c(\mathbb{Z})$. Then there are positive finite constants $c_3$ and $c_4$ such that for every vertex $y$ of $S$ and $t \geq 0$,*

$$P\Big(\xi^S_{S;t}(y) = 1\Big) \leq c_3 \exp\{-c_4 t/\log t\}.$$

**Proof.** Let $H_r$ be the set of the configurations in which all vertices of $S$ within distance $r$ of 0 are vacant.

Set

$$T = \sup\{t \colon \xi^S_{S;t} \notin H_t\}.$$

We will first argue that:

$$P\Big(T > t\Big) \leq c_3 \exp\{-c_4 t/\log t\}. \tag{3.9}$$

Let $A_n$ be the event that if we start the contact process on $S$ at time $(n-1)C\log t$ from the configuration with all vertices occupied, then the process will be in the set $H_{2t}$ at time $nC\log t$. The events $A_1, A_2, \ldots$ are independent and for large enough $C$, we can use (c) to obtain

$$P(A_n) \geq \frac{1}{1+3\lambda} \Bigg[ P\Big\{(\xi^{\mathbb{Z}^+}_{\mathbb{Z},t})_{t\geq 0} \text{ is contained in } \mathbb{Z}^+ \text{ and at time } C\log t \text{ the sites } \{0,1,\ldots,\lfloor 2t\rfloor\} \text{ are empty}\Big\}\Bigg]^3 \geq \epsilon,$$

where $\epsilon > 0$ does not depend on $t$ or $n$. The term $1/(1+3\lambda)$ is the probability that the particle that was at the center of the star $S$ at the initial time $(n-1)C\log t$ dies before infecting any neighboring vertex. The other part comes from the fact that discarding the origin of the star, we can compare the process with three independent processes in $\mathbb{Z}^+$, starting with $\mathbb{Z}^+$ full of particles and then further compare each one of these with a process in $\mathbb{Z}$.

Using the strong Markov property at the moment $T'_{2t}$ when the process $\left(\xi^S_{S;s}\right)_{s\geq 0}$ hits $H_{2t}$ and (b) and (c), we obtain:

$$
\begin{aligned}
P\Big(T \geq t\Big) &\leq P\Big(T'_{2t} \geq t\Big) + P\Big(T'_{2t} < T,\ T'_{2t} < t\Big) \\
&\leq P\Big(A^c_1 \cap A^c_2 \cap \cdots \cap A^c_{\lfloor t/C\log t\rfloor}\Big) + c_5 \exp\{-c_6 t\} \\
&\leq (1-\epsilon)^{\{t/C\log t\}-1} + c_5 exp\{-c_6 t\} \\
&\leq c_3 \exp\{-c_4 t/\log t\}.
\end{aligned}
$$

Note now that if $d(y,0) < t$, then our thesis follows immediately from (3.9). Otherwise, using self-duality and thinking of $\mathbb{Z}^+$ as a subgraph of $S$,

$$
\begin{aligned}
P\Big(\xi^S_{S,t}(y) = 1\Big) &\leq P\Big(\left(\xi^y_{\mathbb{Z}^+;s}\right)_{s\geq 0} \text{ reaches } 0\Big) + P\Big(\xi^y_{\mathbb{Z}^+;t} \neq \emptyset\Big) \\
&\leq c_5 \exp\left(-c_6 d(y,0)\right) + c_7 \exp\left(-c_8 t\right) \\
&\leq c_9 \exp\left(-c_{10} t\right),
\end{aligned}
$$

where we used (b) and (a).

■

## 3.3 Spherically symmetric trees with peculiar properties.

**Example 3.3** *Tree where the process satisfies r=s but not cc; and recurrence without pc.*

With this example we will answer (Q5) and (Q6). First we recall why these questions are relevant. From Thm.2.2(f), we know that cc is equivalent to having simultaneously pc and r=s. The Example 3.1 provides us with a case in which, for some values of $\lambda$, we have pc, while cc fails. It is natural to ask (Q5), i.e., if there are examples in which r=s is satisfied, but cc is not.

From Thm.2.2(c), we know that r&pc is a monotone increasing property. It is natural then to define

$$\lambda_{r\&pc} = \inf\{\lambda : \text{ r\&pc holds}\}.$$

It is equally natural to ask (Q6): if it is always the case that $\lambda_r = \lambda_{r\&pc}$. As we proved in the previous sections, this equality of critical points holds for the examples 3.1 and 3.2.

**Theorem 3.3** *There is a spherically symmetric tree for which there is a non-degenerate interval of values of $\lambda$ on which*

***(a)*** $\liminf_{t\to\infty} \mathbb{P}(\xi_t^0(0) = 1) = 0$.

***(b)*** *The contact process survives and r=s holds. In other words, $\rho(\lambda) = \beta(\lambda) > 0$.*

***(c)*** $\limsup_{t\to\infty} \mathbb{P}(\xi_t^0(0) = 1) > 0$.

Note that under the conditions of this theorem, pc fails, since under pc the limit in (a) would be $\left(\beta(\lambda)\right)^2$, which is positive by (b). Therefore the answer to (Q6) is negative. Since pc fails, also cc fails. Thus, the answer to (Q5) is positive.

Note that from (a) and (c) we know that the law of $\xi_t^0$ does not converge as $t \to \infty$. Nevertheless the set of invariant distributions for this contact process is very simple for the values of $\lambda$ being considered. Since we know that r=s holds, Thm.2.1(d) tells us that there are exactly two extremal invariant distributions: $\mathcal{I}_e = \{\delta_\emptyset, \overline{\nu}\}$.

**Proof of Thm.3.3.** To construct the required spherically symmetric tree, G, we take two numbers $b < B$, so that $\lambda_r(\mathbb{T}_B) < \lambda_s(\mathbb{T}_b)$. We observe that this is possible, because $\lambda_r(\mathbb{T}_B) \to 0$ as $B \to \infty$, by the estimates in Pemantle (1992). Fix a sequence $0 = N_0 < N_1 < N_2 < \dots$, to be specified later. Our spherically symmetric tree is defined by the branching numbers $\{b_i\}_{i=0,1,\dots}$, given by:

$$b_i = \begin{cases} B, & \text{if } N_{2j} \le i < N_{2j+1} \text{ for some } j \in \{0,1,...\}, \\ b, & \text{otherwise.} \end{cases}$$

The interval of values of the parameter $\lambda$ in the statement of the proposition is $(\lambda_s(\mathbb{T}_b), \lambda_r(\mathbb{T}_b))$.

Two sequences of spherically symmetric trees, which may be thought of as approximating our $G$ from opposite sides will be introduced next.

For $j = 0, 1...$, we define $G^{(2j)}$ as the spherically symmetric tree which has the same branching numbers $b_i$ as $G$ has, for $i < N_{2j}$, and for $i \ge N_{2j}$ has $b_i = b$.

For $j = 0, 1...$, we define $G^{(2j+1)}$ as the spherically symmetric tree which has the same branching numbers $b_i$ as $G$ has, for $i < N_{2j+1}$, and for $i \ge N_{2j+1}$ has $b_i = B$.

For $\lambda \in (\lambda_s(\mathbb{T}_b), \lambda_r(\mathbb{T}_b))$ the contact process on each graph $G^{(2j+1)}$, $j = 0, 1...$ satisfies cc. To see this first recall that cc is equivalent to having pc and r=s

(Thm.2.2(f)). Recall also that since $\lambda > \lambda_r(\mathbb{T}_B)$, cc holds on $\mathbb{T}_B^+$ (Thm.2.5), so both r&pc and r=s hold on $\mathbb{T}_B^+$. Observe now that the graphs $G^{(2j+1)}$ are collages of copies of $\mathbb{T}_B^+$. Therefore r=s holds for the contact process on each such graph by Thm.2.4. On the other hand, since r&pc is a monotone increasing property (Thm.2.2(c)), it must also be satisfied by the contact process on the graph $G^{(2j+1)}$, which has $\mathbb{T}_B^+$ as a subgraph.

We will explain below how the sequence $0 = N_0 < N_1 < N_2 < \dots$, and a sequence of times, $t_1 < t_2 < \dots$ can be chosen so that

$$\lim_{j\to\infty} \mathbb{P}(\xi^0_{t_{2j}}(0) = 1) = 0, \tag{3.10}$$

and

$$\begin{aligned} \mathbb{P}(\xi^x_t(0) = 1 \text{ for some } t > 0) &\geq \inf_j \inf_{x:\, d(x,0)<N_{2j}} \mathbb{P}(\xi^x_{t_{2j+1}}(0) = 1) \\ &\geq \frac{1}{2}\left(\rho_{\mathbb{T}_b^+}\right)^2 > 0, \end{aligned} \tag{3.11}$$

in particular

$$\liminf_{j\to\infty} \mathbb{P}(\xi^0_{t_{2j+1}}(0) = 1) > 0. \tag{3.12}$$

Before we explain how to choose the sequences $0 = N_0 < N_1 < N_2 < \dots$, and $t_1 < t_2 < \dots$, we explain why (3.10), (3.11) and (3.12) solve our problem. Clearly (3.10) implies (a), and (3.12) implies (c). The statement in (b) that the contact process survives is trivial, since $\mathbb{T}_b^+$ is a subgraph of $G$. The statement that r=s holds is a consequence of (3.11) and Thm.2.3, since for any site $x$,

$$\mathbb{P}(\xi^x_t(0) = 1 \text{ for some } t > 0) \geq \inf_j \inf_{x:\, d(x,0)<N_{2j}} \mathbb{P}(\xi^x_{t_{2j+1}}(0) = 1).$$

Now we return to the choice of the sequences $0 = N_0 < N_1 < N_2 < \dots$, and $t_1 < t_2 < \dots$. Choose $t_1$ and $N_1$ arbitrarily and proceed recursively as follows. Given $t_1, \dots, t_{2j-1}$, and $N_1, \dots, N_{2j-1}$ take $t_{2j}$ such that

$$\mathbb{P}\Big(\xi^0_{G^{(2j)};t_{2j}}(0) = 1\Big) \leq \frac{1}{j},$$

which is possible by Lemma 3.1, since $G^{(2j)}$ is a collage of copies of $\mathbb{T}_b^+$. Next take $N_{2j}$ such that

$$\mathbb{P}\Big(\xi^0_{G;t_{2j}}(0) = 1\Big) \leq \mathbb{P}\Big(\xi^0_{G^{(2j)};t_{2j}}(0) = 1\Big) + \frac{1}{j},$$

which is clearly possible, since $t_{2j}$ is held fixed and $G$ and $G^{(2j)}$ are identical up to generation $N_{2j}$. Therefore (3.10) is assured.

Next take $t_{2j+1}$ such that

$$\inf_{x:\, d(x,0)<N_{2j}} \mathbb{P}\Big(\xi^x_{G^{(2j+1)};t_{2j+1}}(0) = 1\Big) \geq \frac{3}{4}\Big(\rho_{\mathbb{T}_b^+}\Big)^2,$$

which is possible because the contact process on $G^{(2j+1)}$ satisfies cc and the probability of survival on this graph, starting from any single occupied site is bounded below by $\rho_{\mathbb{T}_b^+}$.

Finally take next $N_{2j+1}$ such that

$$\inf_{x:\, d(x,0)<N_{2j}} \mathbb{P}\Big(\xi^x_{G;t_{2j+1}}(0) = 1\Big) \geq \mathbb{P}\Big(\xi^x_{G^{(2j+1)};t_{2j+1}}(0) = 1\Big) - \frac{1}{4}\Big(\rho_{\mathbb{T}_b^+}\Big)^2,$$

which is clearly possible, since $t_{2j+1}$ is held fixed and $G$ and $G^{(2j+1)}$ are identical up to generation $N_{2j+1}$. Thus (3.11) is also assured.

■

# CHAPTER 4

## Continuity properties of $\rho(\cdot)$ and $\beta(\cdot)$

### 4.1 Preliminaries.

In statistical mechanics there are two distinct ways to search for "transition points". One of these relies on finding values of the parameters of the model at which the set of equilibrium distributions changes qualitatively. The other one is based on finding values of these parameters at which quantities of relevance (called "order parameters") have some sort of non-smooth behavior, e.g., a discontinuity, or a discontinuity of a derivative of some order, or just a lack of analyticity. When studying an interacting particle system, like the contact process, the first of these ideas can be expanded to encompass any sort of qualitative modification in the ergodic behavior (even when the set of invariant distributions itself does not present any qualitative modification). For instance a value of the parameter $\lambda$ immediately below which, say, cc fails, and immediately above which cc holds would be considered a "transition point". Regarding "order parameters" for the contact process, it is very natural to consider the behavior of the functions $\rho(\cdot)$ and $\beta(\cdot)$, and to try to locate any value of $\lambda$ at which they are not "smooth". Of course, $\lambda_s$ is a transition point in the sense that $\rho(\cdot)$ changes there from being 0 to being positive and it is also a transition point in the sense that it separates the region in which there is a unique invariant distribution from the one where this is no longer the case. Similarly $\lambda_r$ is a transition point in the sense that $\beta(\cdot)$ changes there from being 0 to being positive. For the contact process on a homogeneous tree (with $b \geq 2$) this point also separates a region where there are infinitely many extremal invariant distributions from one where there are exactly two of these.

An interesting direction, in which we will nevertheless not go in this work, would be to relate in a precise way, if possible, the two notions of phase transition discussed above. It should be clear, though, that the presence of discontinuities and other types of non-analytic behavior of the functions $\rho(\cdot)$ and $\beta(\cdot)$ is a matter of intrinsic interest. Here we will only address issues of continuity of these functions.

In the most studied cases, the complete answer is known. First, on $\mathbb{Z}^d$, the contact process has a continuous $\rho(\lambda) = \beta(\lambda)$ for all $\lambda > 0$ (this was one of the major open problems about the contact process, until it was solved by Bezuidenhout and Grimmett (1990)). Second, on $\mathbb{T}_b$, with $b \geq 2$, the contact process has a continuous $\rho(\lambda)$ for all $\lambda > 0$ [see Morrow, Schinazi and Zhang (1994)], while $\beta(\lambda) = 0$ on $(0, \lambda_r(\mathbb{T}_b)]$ and $\beta(\lambda) = \rho(\lambda)$ on $(\lambda_r(\mathbb{T}_b), \infty)$ (recall that r=s holds there), so that $\beta(\cdot)$ is also continuous on this open interval, but it is discontinuous at $\lambda_r(\mathbb{T}_b)$.

The theorems which we state and prove in the next two Sections, 4.2 and 4.3, address the continuity properties of $\rho(\cdot)$ and $\beta(\cdot)$, respectively, on general graphs in $\mathcal{G}$. In Section 4.4 we present a sufficient condition for continuity of $\rho(\cdot)$ above $\lambda_s$ and apply it. In Section 4.5 we present a necessary and sufficient condition for right-continuity of $\beta(\cdot)$ and apply it.

The continuity properties of $\rho(\cdot)$ and $\beta(\cdot)$ on $\mathbb{Z}^d$ and on $\mathbb{T}_b$ will be seen to be special, as compared to what can happen on general graphs, but will also be seen to be typical of what happens on homogeneous graphs [see Corollary 4.4.1].

We introduce next some technical tools. We will sometimes need to couple versions of the contact process with values ranging in an interval of the type $(0, \lambda]$. This will be done in a standard fashion, by enlarging the probability space of Poisson processes on which $\mathbb{P}_{G,\lambda}$ is defined (the graphical construction), introducing independent random variables uniformly distributed on $(0, 1)$ associated to each one of the arrows in the graphical construction with infection rate $\lambda$. The corresponding probability measure will be denoted by $\bar{\mathbb{P}}_{G,\lambda}$, and the contact process with infection parameter $\lambda' < \lambda$ can be obtained by only keeping the arrows associated to random variables which take value at most $\lambda'/\lambda$. We will say in this situation that we are keeping the arrows only up to level $\lambda'$.

It is a standard matter to use the coupling above to prove the continuity in $\lambda$ of various probabilities which concern the contact process started from some finite set and depend only on what happens up to a fixed time. For instance, this is the case of $\mathbb{P}_{G,\lambda}(\xi_t^0 \neq \emptyset)$, seen as a function of $\lambda$, with $t$ fixed. We will use the term "finite-time-continuity", when referring to results of this nature.

Suppose that $G'$ is a subgraph of $G$, $\lambda' \leq \lambda$, $x, y \in \mathcal{V}_{G'}$ and $s < u$. We will use the notation $\{A \xrightarrow{G',\lambda'} B\}$ for the event that there is a path from $A$ to $B$ in the graphical construction when we keep the arrows only up to level $\lambda'$, and moreover this path only passes through sites and edges of $G'$. In this notation $A$ and $B$ can be subsets or elements of $\mathcal{V}_G \times \mathbb{R}$. The event that there is such a path, starting from $A$ and reaching space-time locations at arbitrarily large times will be denoted by $\{A \xrightarrow{G',\lambda'} \infty\}$. In this notation, if $G'$ is omitted, it is understood that $G' = G$, and if $\lambda'$ is omitted, it is understood that $\lambda' = \lambda$. If $G'$ is the largest subgraph of $G$ which has $\mathcal{V}_{G'} = C \subset \mathcal{V}_G$, then in the notation above we can replace $G'$ with $C$.

A minor issue which nevertheless requires our attention is whether the choice of the root of $G$ plays any role in the continuity properties which we will be discussing. The following simple proposition settles the question, as expected, in the negative, and also relates continuity properties of $\rho(\cdot)$ to those of the distribution $\overline{\nu}$ and continuity properties of $\beta(\cdot)$ to those of the distribution $\nu_r$.

**Proposition 4.1.1** *Suppose $G \in \mathcal{G}$ and $\lambda > 0$. Then*

***(a)*** *Either $\rho(A, \cdot)$ is left-continuous at $\lambda$ for all non-empty $A \subset\subset \mathcal{V}_G$ or for no such $A$.*

***(b)*** *Either $\rho(A, \cdot)$ is right-continuous at $\lambda$ for all non-empty $A \subset\subset \mathcal{V}_G$ or for no such $A$.*

***(c)*** *Either $\beta(A, \cdot)$ is left-continuous at $\lambda$ for all non-empty $A \subset\subset \mathcal{V}_G$ or for no such $A$.*

***(d)*** *Either $\beta(A,\cdot)$ is right-continuous at $\lambda$ for all non-empty $A \subset\subset \mathcal{V}_G$ or for no such $A$.*

**Proof of Proposition 4.1.1.** We will only prove (a), the other claims having analogous proofs.

Set

$$\Delta(A,\lambda) = \rho(A,\lambda) - \lim_{\lambda' \nearrow \lambda} \rho(A,\lambda').$$

By monotonicity in $\lambda$, it is clear that $\Delta(A,\lambda) \geq 0$, so that left-continuity of $\rho(A,\cdot)$ at $\lambda$ is equivalent to the statement that $\Delta(A,\lambda) \leq 0$.

A standard application of the Markov property, of finite-time-continuity and of the dominated convergence theorem gives, for $A \subset\subset \mathcal{V}_G$,

$$\Delta(A,\lambda) = \sum_{B \subset\subset \mathcal{V}_G} \mathbb{P}_{G,\lambda}(\xi_1^A = B)\, \Delta(B,\lambda). \tag{4.1}$$

That (4.1) suffices for our purpose should be clear, since if $\Delta(B,\lambda) > 0$ for some $B \subset\subset \mathcal{V}_G$, this equation gives us that $\Delta(A,\lambda) > 0$ for all non-empty $A \subset\subset \mathcal{V}_G$ (the fact that $G$ is connected is being used to assure us that all the probabilities which appear in (4.1) are strictly positive when $A \neq \emptyset$).

■

## 4.2 General continuity properties of $\rho(\cdot)$.

**Theorem 4.2.1** *For every $G \in \mathcal{G}$ the function $\rho(\cdot)$ is right-continuous. On the other hand, there are trees in $\mathcal{G}$ for which $\rho(\cdot)$ is discontinuous at $\lambda_s$, and there are trees in $\mathcal{G}$ for which $\rho(\cdot)$ is discontinuous at some $\lambda > \lambda_s$.*

**Proof of Thm.4.2.1.** The right-continuity of $\rho(\cdot)$ is a well-known fact and easy to prove. Set $\rho^{(n)}(\lambda) = \mathbb{P}_{G,\lambda}(\xi_n^0 \neq \emptyset)$. For each $n$ this is a continuous function of $\lambda$, by finite-time-continuity. But $\rho(\lambda) = \inf_n \rho^{(n)}(\lambda)$, so that $\rho(\cdot)$ is upper-semi-continuous. Since $\rho(\cdot)$ is a non-decreasing function also, it is right-continuous.

We describe next a tree for which we will show that there is positive probability of survival at the survival point, so that $\rho(\cdot)$ is discontinuous there. This tree is obtained from a tree $\mathbb{T}_b^+$, with some $b \geq 2$, by removing certain sites from it (and the edges which have at least one endpoint at a removed site). To explain how we remove sites from $\mathbb{T}_b^+$ to obtain the new tree, it is convenient to label the sites of $\mathbb{T}_b^+$ in a certain standard way. The origin will be labeled $(0)$. The sites in generation 1 will be labeled $(0,1)$, $(0,2)$,...,$(0,b)$. ... The sites in generation $n$ which are descendents of the site of generation $n-1$ labeled $(0, g_1, g_2, ..., g_{n-1})$ will be labeled $(0, g_1, g_2, ..., g_{n-1}, 1)$, $(0, g_1, g_2, ..., g_{n-1}, 2)$, ...,$(0, g_1, g_2, ..., g_{n-1}, b)$. We think of the sites in each generation of the tree $\mathbb{T}_b^+$ as being ordered lexicographically, with basis on the labels (which then run from $(0, 0, 0, ..., 0)$, to $(0, b, b, \dots, b)$).

Suppose that we are given two sequences of strictly positive integer numbers: $\mathbf{l} = (l_i)_{i=1,2,...}$ and $\mathbf{k} = (k_i)_{i=1,2,...}$ which satisfy $k_i \leq b^{l_i}$. We will denote by $[\mathbb{T}_b^+; \mathbf{l}, \mathbf{k}]$ the tree which is obtained from $\mathbb{T}_b^+$ by deleting sites and edges from this tree in the following recursive fashion. In the first step select the first $k_1$ sites in generation $l_1$ and call them (1)-head sites; now delete all the descendents of the sites in generation $l_1$ which are not (1)-head sites. In the $n$-th step, $n = 2, 3, ...$, take the tree which was obtained in the $(n-1)$-th step and for each $(n-1)$-head site select the first $k_n$ of its descendents which are in generation $l_1 + \cdots + l_n$ and call them $(n)$-head sites; now delete all the descendents of the sites in generation $l_1 + \cdots + l_n$ which are not $(n)$-head sites. The graph $[\mathbb{T}_b^+; \mathbf{l}, \mathbf{k}]$ is the remaining graph after the procedure just described is applied indefinitely.

We will show that for a certain choice of the sequences $\mathbf{l}$ and $\mathbf{k}$ we have

$$\rho_{[\mathbb{T}_b^+;\mathbf{l},\mathbf{k}]}(\lambda_s([\mathbb{T}_b^+; \mathbf{l}, \mathbf{k}])) > 0, \tag{4.2}$$

and

$$\beta_{[\mathbb{T}_b^+;\mathbf{l},\mathbf{k}]}(\lambda_s([\mathbb{T}_b^+; \mathbf{l}, \mathbf{k}])) = 0. \tag{4.3}$$

The relevance of this second claim will become clear later, when we use this graph as a building block to obtain another one for which $\rho(\cdot)$ is discontinuous at some $\lambda > \lambda_s$.

Let $\lambda^*$ be an arbitrary point in the interval $(\lambda_s(\mathbb{T}_b^+), \lambda_r(\mathbb{T}_b^+)]$. We will make the choices of the sequences $\mathbf{l} = (l_i)_{i=1,2,...}$ and $\mathbf{k} = (k_i)_{i=1,2,...}$ in such a way that

$$\lambda_s([\mathbb{T}_b^+; \mathbf{l}, \mathbf{k}]) = \lambda^*. \tag{4.4}$$

For this purpose we introduce first a new quantity related to $u_{b,n}(\lambda)$. Define

$\tilde{u}_{b,n}(\lambda)$ and $s(n)$ by

$$\tilde{u}_{b,n}(\lambda) = \max_{s>0} \mathbb{P}_{\mathbb{T}_b^+,\lambda}((0,0) \overset{B(0,n)}{\longrightarrow} (n,s)) = \mathbb{P}_{\mathbb{T}_b^+,\lambda}((0,0) \overset{B(0,n)}{\longrightarrow} (n,s(n))).$$

Clearly such a maximizing $s(n)$ exists; if it is not unique, we can define $s(n)$ as the minimal one. Obviously $\tilde{u}_{b,n}(\lambda) \leq u_{b,n}(\lambda)$, but nevertheless

$$\lim_{n\to\infty} (\tilde{u}_{b,n}(\lambda))^{1/n} = \alpha_b(\lambda). \tag{4.5}$$

This result is a marginal strengthening of Lemma 2.1 and can be proven in essentially the same way as that lemma. We know from Lalley and Sellke (1998) and Lalley (1999) that for any $\lambda_s(\mathbb{T}_b) < \lambda' < \lambda^*$ we have

$$1/b < \alpha_b(\lambda') < \alpha_b(\lambda^*) \leq 1/\sqrt{b}. \tag{4.6}$$

(The first inequality follows from the strict monotonicity of $\alpha_b(\cdot)$ in $(0, \lambda_r(\mathbb{T}_b)]$ and the fact that if $\alpha_b(\lambda) < 1/b$, then the expected number of sites of $\mathbb{T}_b$ ever to be infected in the process $\{\xi^0_{\mathbb{T}_b,\lambda;t} \colon t \geq 0\}$ is finite, so that $\rho_{\mathbb{T}_b}(\lambda) = 0$.)

The choice of the sequence $\mathbf{l} = (l_i)_{i=1,2,...}$ will be made later, but we anticipate that it will satisfy

$$\lim_{i\to\infty} l_i = \infty. \tag{4.7}$$

Supposing $\mathbf{l}$ given, we specify $\mathbf{k}$ via

$$k_i = \lfloor c/\tilde{u}_{b,l_i}(\lambda^*) \rfloor,$$

where $c > 1$ is an arbitrary constant. Note that (4.5) and (4.6) guarantee that $k_i \leq b^{l_i}$ (as required, for the construction of $[\mathbb{T}_b^+; \mathbf{l}, \mathbf{k}]$ to make sense), provided only that $l_i \geq L_1$, $i = 1, 2, ...$, where $L_1 < \infty$ is some appropriate constant.

It is not hard to show that the choice above yields

$$\rho_{[\mathbb{T}_b^+;\mathbf{l},\mathbf{k}]}(\lambda') = 0 \quad \text{for all } \lambda' < \lambda^*. \tag{4.8}$$

To this end, observe that in $[\mathbb{T}_b^+; \mathbf{l}, \mathbf{k}]$ there are exactly $k_1 k_2 \cdots k_n$ $(n)$-head sites and they are at distance $l_1 + \cdots + l_n$ from the root. Let $E_n$ be the event that the process $\{\xi_t^0 \colon t \geq 0\}$ ever infects one of these sites. Then, since $[\mathbb{T}_b^+; \mathbf{l}, \mathbf{k}]$ is a subgraph of $\mathbb{T}_b$,

$$\begin{aligned}
\rho_{[\mathbb{T}_b^+;\mathbf{l},\mathbf{k}]}(\lambda') &= \lim_{n\to\infty} \mathbb{P}_{[\mathbb{T}_b^+;\mathbf{l},\mathbf{k}],\lambda'}(E_n) \leq \limsup_{n\to\infty} k_1 k_2 \cdots k_n \, u_{b,l_1+\cdots+l_n}(\lambda') \\
&\leq \limsup_{n\to\infty} k_1 k_2 \cdots k_n \, (\alpha_b(\lambda'))^{l_1+\cdots+l_n} \leq \limsup_{n\to\infty} \prod_{i=1}^{n} c \frac{(\alpha_b(\lambda'))^{l_i}}{\tilde{u}_{b,l_i}(\lambda^*)},
\end{aligned}$$

where (1.2) was used in the second inequality. Thanks to (4.5), 4.6), and (4.7), the $n$-th term in this product vanishes as $n \to \infty$, and therefore (4.8) follows.

In order to complete the proof of the claim (4.2), it remains now to show that we can make the choice of $\mathbf{l}$, satisfying (4.7), in such a way that

$$\rho_{[\mathbb{T}_b^+;\mathbf{l},\mathbf{k}]}(\lambda^*) > 0. \tag{4.9}$$

For this purpose, we consider now the following modification of the contact process on $[\mathbb{T}_b^+; \mathbf{l}, \mathbf{k}]$ started from a single particle at the origin. Until time $s(l_1)$ we run the usual contact process on this graph. At time $s(l_1)$ we remove all particles except for those which are at (1)-head sites; from this time on we keep the set $B(0, l_1 - 1)$ free of particles, and until time $s(l_1) + s(l_2)$ we let the system evolve in the remaining sites with the usual contact process rules. Recursively, at time

$s(l_1) + \cdots + s(l_n)$ we remove all particles except for those which are at $(n)$-head sites; from this time on we keep the set $B(0, l_1 + \cdots + l_n - 1)$ free of particles, and until time $s(l_1) + \cdots + s(l_{n+1})$ we let the system evolve in the remaining sites with the usual contact process rules.

Let $Z_n$ be the number of particles in the process described above at time $s(l_1) + \cdots + s(l_n)$. Obviously

$$\rho_{[\mathbb{T}_b^+;\mathbf{l},\mathbf{k}]}(\lambda^*) \geq \mathbb{P}(Z_n > 0 \text{ for all } n \geq 1). \tag{4.10}$$

The process $\left(Z_n\right)_{n=1,...}$ is a time-dependent branching process. Each one of the particles counted in $Z_{n-1}$ gives rise, independently and with the same distribution, to a random number of offspring, which are particles counted in $Z_n$. The average number of offspring of each particle of $Z_{n-1}$ is $\mu_n \geq k_n \tilde{u}_{b,l_n}(\lambda^*)$. Note that if we take $c' \in (1, c)$, then there is $L_2 \in [L_1, \infty)$ so that if $l_n \geq L_2$, we have $\mu_n \geq c'$. If $\left(Z_n\right)_{n=1,...}$ were a branching process, this would be enough to conclude that the right-hand-side of (4.10) is positive. In order to use branching process theory to handle the process $\left(Z_n\right)_{n=1,...}$, we will take the sequence $\mathbf{l}$ increasing by steps, with long stretches in which it is constant. More precisely, we will take

$$1 \leq n_1 < n_2 < n_3 < ...$$

and

$$\begin{aligned} l_1 = l_2 = \ldots &= l_{n_1} < l_{n_1+1} = l_{n_1+2} = \cdots = l_{n_1+n_2} \\ &< l_{n_1+n_2+1} = l_{n_1+n_2+2} = \cdots = l_{n_1+n_2+n_3} < ... \end{aligned}$$

Define $\bar{l}_j = l_{n_1+\cdots+n_j}$ and $\bar{k}_j = k_{n_1+\cdots+n_j} = \lfloor c/u_{b,\bar{l}_j}(\lambda^*)\rfloor$. Let also $\mathbf{l}^{(j)} = (\bar{l}_j, \bar{l}_j, ...)$ and $\mathbf{k}^{(j)} = (\bar{k}_j, \bar{k}_j, ...)$ be constant sequences. Let $\left(Z_n^{(j)}\right)_{n=1,...}$ be defined as the process $\left(Z_n\right)_{n=1,...}$, but for the tree $[\mathbb{T}_b^+; \mathbf{l}^{(j)}, \mathbf{k}^{(j)}]$ instead of the tree $[\mathbb{T}_b^+; \mathbf{l}, \mathbf{k}]$. For each $j$, the process $\left(Z_n^{(j)}\right)_{n=1,...}$ is a branching process with offspring distribution having mean $\mu_j = \bar{k}_j \tilde{u}_{b,\bar{l}_j}(\lambda^*) \geq c' > 1$, provided that $l_j \geq L_2$. Since the offspring distribution has a finite support (namely $\{0, 1, ..., \bar{k}_j\}$), and in particular a finite second moment, it follows from standard branching-process theory [see, e.g., example 4.3 in Sect. 4.4, p. 254 of Durrett (1996)] that for some random variable $X_j$ with mean $\mathbb{E}(X_j) = 1$,

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{Z_n^{(j)}}{(\mu_j)^n} = X_j\right) = 1.$$

In particular, there is $\epsilon_j > 0$, and $N_j < \infty$ such that

$$\mathbb{P}\left(Z_n^{(j)} \geq \frac{(\mu_j)^n}{2}\right) \geq \epsilon_j, \tag{4.11}$$

for all $n \geq N_j$.

Back to the process $\left(Z_n\right)_{n=1,...}$, it is easy to see that from (4.11) we obtain

$$\begin{aligned}
\mathbb{P}(Z_n > 0 \text{ for all } n \geq 1) &\geq \mathbb{P}\left(Z_{n_1} \geq \frac{(\mu_1)^{n_1}}{2}\right) \times \\
&\prod_{j=1}^{\infty} \mathbb{P}\left(Z_{n_1+\cdots+n_{j+1}} \geq \frac{(\mu_{j+1})^{n_{j+1}}}{2} \;\middle|\; Z_{n_1} \geq \frac{(\mu_1)^{n_1}}{2}, \ldots, Z_{n_1+\cdots+n_j} \geq \frac{(\mu_j)^{n_j}}{2}\right) \\
&\geq \mathbb{P}\left(Z_{n_1}^{(1)} \geq \frac{(\mu_1)^{n_1}}{2}\right) \prod_{j=1}^{\infty} \left\{1 - \left(1 - \mathbb{P}\left(Z_{n_{j+1}}^{(j+1)} \geq \frac{(\mu_{j+1})^{n_{j+1}}}{2}\right)\right)^{\frac{(\mu_j)^{n_j}}{2}}\right\} \\
&\geq \epsilon_1 \prod_{j=1}^{\infty} \left\{1 - (1-\epsilon_{j+1})^{\frac{(c')^{n_j}}{2}}\right\},
\end{aligned}$$

provided that $n_j \geq N_j$, for $j \geq 1$. This infinite product can be assured to be positive if we make our choices, for instance, in the following fashion. We will take $\bar{l}_j = L_2 + j$, this gives us values for $\epsilon_j$ and $N_j$, $j \geq 1$. And the infinite product above is clearly positive if $n_j$ (besides satisfying $n_j \geq N_j$ for each $j$) is chosen to grow fast enough. This establishes the choice of $\mathbf{l}$ and shows that, thanks to the comparison (4.10), (4.9) holds.

From (4.9) and (4.8) we have (4.4) and (4.2). This completes the proof that there is a graph on which the contact process survives at the survival point.

The claim (4.3) is an immediate consequence of the fact that $[\mathbb{T}_b^+; \mathbf{l}, \mathbf{k}]$ is a subgraph of $\mathbb{T}_b^+$, and $\lambda^* \leq \lambda_r(\mathbb{T}_b^+)$. Suppose now that $b \leq B$ and set $G = [\mathbb{T}_b^+; \mathbf{l}, \mathbf{k}] \vee \mathbb{T}_B$. Clearly $\lambda_s(G) \leq \lambda_s(\mathbb{T}_B) < \lambda_s([\mathbb{T}_b^+; \mathbf{l}, \mathbf{k}])$ and from Proposition 4.2.1 below, (4.2) and (4.3) we obtain that $\rho_G(\cdot)$ is discontinuous at $\lambda_s([\mathbb{T}_b^+; \mathbf{l}, \mathbf{k}])$. This completes the proof of Thm.4.2.1.

■

**Proposition 4.2.1** *Suppose that $G = G_1 \vee G_2$ and the contact process on $G_1$ does not satisfy r=s when the infection parameter takes the value $\lambda_s(G_1)$. Then $\rho_G(\cdot)$ is discontinuous at $\lambda_s(G_1)$.*

**Proof of Prop.4.2.1.** For convenience, we will take for the origin of $G$ the origin of $G_1$.

Consider the coupling $\bar{\mathbb{P}}_{G,\lambda_s(G_1)}$ of versions of the contact process with values ranging in the interval $(0, \lambda_s(G_1)]$, as reviewed in Section 4.1. Let $E$ be the event that when we keep all the arrows in this construction the contact process started from a single particle at the origin survives, but no site of $G_2$ is ever infected. The hypothesis on $G_1$ assures us that $\bar{\mathbb{P}}_{G,\lambda_s(G_1)}(E) > 0$.

For $\lambda < \lambda_s(G_1)$ we have

$$\begin{aligned}\rho_G(\lambda_s(G_1)) - \rho_G(\lambda) &= \bar{\mathbb{P}}_{G,\lambda_s(G_1)}\left((0,0) \xrightarrow{\lambda_s(G_1)} \infty \ , \ \left\{(0,0) \xrightarrow{\lambda} \infty\right\}^c\right) \\ &\geq \bar{\mathbb{P}}_{G,\lambda_s(G_1)}(E) - \bar{\mathbb{P}}_{G,\lambda_s(G_1)}\left((0,0) \xrightarrow{G_1,\lambda} \infty\right) \\ &= \bar{\mathbb{P}}_{G,\lambda_s(G_1)}(E) - \rho_{G_1}(\lambda) = \bar{\mathbb{P}}_{G,\lambda_s(G_1)}(E).\end{aligned}$$

Hence

$$\lim_{\lambda \nearrow \lambda_s(G_1)} \rho_G(\lambda) \leq \rho_G(\lambda_s(G_1)) - \bar{\mathbb{P}}_{G,\lambda_s(G_1)}(E) < \rho_G(\lambda_s(G_1)).$$

■

We will now present another tree, which has the property that $\rho(\cdot)$ is discontinuous at $\lambda_s$. This tree is somewhat simpler than $[\mathbb{T}_b^+; \mathbf{l}, \mathbf{k}]$, but satisfies r=s, at $\lambda_s$, so that we cannot use it, as we did with $[\mathbb{T}_b^+; \mathbf{l}, \mathbf{k}]$, to construct another tree for which $\rho(\cdot)$ is discontinuous at some $\lambda > \lambda_s$. Moreover, the complete proof that $\rho(\lambda_s) > 0$ for our new tree is rather long and involved, and will only be sketched here. The example is adapted from Example B, Proposition 1.3 of Madras, Schinazi and Schonmann (1994), and we refer the reader to that paper for the details of the proof. Later in this work we will refer to the example which we introduce below as the "desert-oasis example".

Suppose that we are given two sequences of strictly positive integer numbers: $\mathbf{d} = (d_i)_{i=1,2,\ldots}$ and $\mathbf{o} = (o_i)_{i=1,2,\ldots}$. We will denote by $[\mathbf{d}, \mathbf{o}]$ the tree obtained from $\mathbb{Z}^+$ by adding sites and edges to this tree in the following fashion. First partition the sites of $\mathbb{Z}^+$ into two sets: desert-sites and oasis-sites. The first $d_1$ sites of $\mathbb{Z}^+$ are declared to be desert-sites, then the next $o_1$ sites of $\mathbb{Z}^+$ are declared to be oasis-sites, then the next $d_2$ sites of $\mathbb{Z}^+$ are declared to be desert-sites, then the next $o_2$ sites of $\mathbb{Z}^+$ are declared to be oasis-sites, and so on. A new site is added to the tree in association with each oasis-site, and connected to this oasis-site by means of a new edge. This completes the construction. The added sites are leaves of the tree, and will be called palm-sites.

For our purpose we will choose $d_i = i^3$ and $o_i = i^2$. The tree $[\mathbf{d}, \mathbf{o}]$ turns out to have then the following features:

$$\lambda_s([\mathbf{d}, \mathbf{o}]) = \lambda_r([\mathbf{d}, \mathbf{o}]) = \lambda_s(\mathbb{Z}), \tag{4.12}$$

and

$$\rho_{[\mathbf{d},\mathbf{o}]}(\lambda_s([\mathbf{d}, \mathbf{o}])) > 0. \tag{4.13}$$

Moreover r=s holds for the contact process on $[\mathbf{d}, \mathbf{o}]$ for all values of $\lambda$.

We will explain below why the claims above hold. But before we can do it we need to introduce another graph, which may be seen as a doubly infinite oasis. This graph will be denoted by $O$, and is obtained from $\mathbb{Z}$ in the same fashion in which $[\mathbf{d}, \mathbf{o}]$ is obtained from $\mathbb{Z}^+$, but with all sites of $\mathbb{Z}$ being declared to be oasis-sites. More precisely, to each site of $\mathbb{Z}$ we associate a new site, and connect it to that site of $\mathbb{Z}$. The resulting graph is $O$.

From Aizenman and Grimmett (1991) [see the end of Section 2 there], we know that

$$\lambda_s(O) < \lambda_s(\mathbb{Z}).$$

Moreover, $O$ can be studied by means of the dynamic rescaling approach found on Bezuidenhout and Grimmett (1990), which gives, for instance $\lambda_s(O) = \lambda_r(O)$, and the validity of cc for the contact process on $O$ above $\lambda_s(O)$.

The idea of the proof of (4.12) and (4.13) is as follows. Think of $[\mathbf{d}, \mathbf{o}]$ as a sequence of desert stretches and oasis stretches. If $\lambda < \lambda_s(\mathbb{Z})$, then the time needed for infection to cross a desert stretch of length $i^3$ grows with $i$ as $\exp(C_1 i^3)$ (this follows from self-duality and exponential estimates on the survival time of the sub critical one-dimensional contact process – see for instance Chapter 6 of Liggett (1985)), while clearly infection can typically only persist in an oasis stretch of length $i^2$ for at most a time of order $\exp(C_2 i^2)$. Therefore the infection should eventually disappear, i.e.,

$$\rho_{[\mathbf{d},\mathbf{o}]}(\lambda) = 0 \quad \text{for } \lambda < \lambda_s(\mathbb{Z}). \tag{4.14}$$

On the other hand, when $\lambda = \lambda_s(\mathbb{Z})$, then because $\lambda_s(O) < \lambda_s(\mathbb{Z})$, infection should persist in an oasis stretch of length $i^2$ for at least a time of order $\exp(Ci^2)$ [a rigorous version of this claim can be proved in a standard fashion using the dynamic rescaling scheme of Bezuidenhout and Grimmett (1990)]. But the time needed to cross a desert stretch of length $i^3$ grows then only as a power of this length, i.e., as $i^{C_3}$ for some $C_3$ [see (3.4) and the proof of Lemma 3.2 in Madras, Schinazi and Schonmann (1994), which are based on estimates in Durrett, Schonmann and

Tanaka (1989)]. This allows the system to survive with positive probability, i.e.,

$$\rho_{[\mathbf{d},\mathbf{o}]}(\lambda_s(\mathbb{Z})) > 0, \tag{4.15}$$

with infection crossing from oasis to oasis, through desert, to the right.

The claims (4.12) and (4.13) are equivalent to (4.14) and (4.15). Next we sketch the proof that r=s holds for the contact process on $[\mathbf{d},\mathbf{o}]$ at $\lambda_s([\mathbf{d},\mathbf{o}])$ (that it holds above $\lambda_s([\mathbf{d},\mathbf{o}])$ is clear, for instance from the fact that cc holds then, by Thm.2.6).

The same argument from Madras, Schinazi and Schonmann (1994) which gives (4.15) also implies that there is $\delta > 0$ such that for all oasis-site $x$,

$$\mathbb{P}_{[\mathbf{d},\mathbf{o}],\lambda_s([\mathbf{d},\mathbf{o}])}\Big(0 \in \xi_t^x \text{ for some } t > 0\Big) \geq \delta. \tag{4.16}$$

Intuitively, in the same way that the infection can cross from oasis to oasis, through desert, to the right, it can also move in a similar fashion in the opposite direction. Now, if (4.16) were true for all site $x$ in the tree, then Thm.2.3 would give the validity of r=s at $\lambda_s([\mathbf{d},\mathbf{o}])$. Nevertheless, in spite of our only having (4.16) for oasis-sites $x$, this is enough to obtain the desired conclusion, because the set of all oasis-sites has the property that if the process survives, then sites in this set will be visited at arbitrarily large times a.s.. It is easy to adapt the proof of Thm.2.3, to obtain the conclusion that r=s holds, from the knowledge that (4.16) is satisfied for all site $x$ in a set with the property just described.

## 4.3 General continuity properties of $\beta(\cdot)$.

We start with the following result.

**Theorem 4.3.1** *For every $G \in \mathcal{G}$ the function $\beta(\cdot)$ is left-continuous on $(\lambda_r, \infty)$. On the other hand, there are trees in $\mathcal{G}$ for which $\beta(\cdot)$ is not right-continuous at $\lambda_r$, there are trees in $\mathcal{G}$ for which $\beta(\cdot)$ is not left-continuous at $\lambda_r$, and there are trees in $\mathcal{G}$ for which $\beta(\cdot)$ is discontinuous at some $\lambda > \lambda_r$.*

**Proof of Thm.4.3.1.** To prove the left-continuity of $\beta(\cdot)$ on $(\lambda_r, \infty)$, recall the definition:

$$\text{For } A \subset \mathcal{V}_G, \text{ and } s, R > 0, \text{ set } \Omega^\eta(t,R) = \{\xi_s^\eta \supset B(0,R) \text{ for some } s < t\}.$$

By Lemma 2.1,

$$\lim_{R\to\infty} \lim_{t\to\infty} \mathbb{P}(\Omega^A(t,R)) = \mathbb{P}(\Omega_r^A). \tag{4.17}$$

We will also use the fact that by Thm.2.1(a) we have that if $\beta(\lambda) > 0$, then

$$\lim_{R\to\infty} \mathbb{P}\Big(\Omega_r^{B(0,R)}\Big) = 1. \tag{4.18}$$

Suppose $\lambda > \lambda_r$. If $\tilde{\lambda} < \lambda' < \lambda$, then for any $R$ and $t$, the Strong Markov property, attractiveness and monotonicity in $\lambda$ imply that

$$\begin{aligned} \beta_G(\lambda') &= \mathbb{P}_{G,\lambda'}(\Omega_r^0) \geq \mathbb{P}_{G,\lambda'}(\Omega^0(t,R))\mathbb{P}_{G,\lambda'}(\Omega_r^{B(0,R)}) \\ &\geq \mathbb{P}_{G,\lambda'}(\Omega^0(t,R))\mathbb{P}_{G,\tilde{\lambda}}(\Omega_r^{B(0,R)}). \end{aligned}$$

By finite-time-continuity,

$$\lim_{\lambda' \nearrow \lambda} \mathbb{P}_{G,\lambda'}(\Omega^0(t,R)) = \mathbb{P}_{G,\lambda}(\Omega^0(t,R)).$$

Therefore

$$\liminf_{\lambda' \nearrow \lambda} \beta_G(\lambda') \geq \mathbb{P}_{G,\lambda}(\Omega^0(t,R))\mathbb{P}_{G,\tilde{\lambda}}(\Omega_\infty^{B(0,R)}).$$

Now let first $t \to \infty$ and then $R \to \infty$ and use (4.17) and (4.18) to obtain

$$\liminf_{\lambda' \nearrow \lambda} \beta_G(\lambda') \geq \beta_G(\lambda).$$

Since $\beta_G(\lambda') \leq \beta_G(\lambda)$, for all $\lambda' < \lambda$, the proof of left-continuity of $\beta_G(\cdot)$ at $\lambda$ is complete.

The homogeneous trees $\mathbb{T}_b$, with $b \geq 2$, provide well-known examples for which $\beta(\cdot)$ is not right-continuous at $\lambda_r$.

The desert-oasis example of Section 4.2 provides an example in which $\beta(\cdot)$ is not left-continuous at $\lambda_r$. Indeed, we know that for this example $\rho(\cdot)$ is discontinuous from the left at $\lambda_s = \lambda_r$ and that r=s holds for every $\lambda$.

Our final task is to provide an example for which $\beta(\cdot)$ is not right-continuous at some $\lambda > \lambda_r$. The example 3.1 works for this purpose. So, set $G = \mathbb{T}_j \vee \mathbb{T}_k$ with $j \geq 2$ and $k$ sufficiently larger than $j$, so that we have

$$\lambda_s(\mathbb{T}_k) < \lambda_r(\mathbb{T}_k) < \lambda_s(\mathbb{T}_j) < \lambda_r(\mathbb{T}_j). \tag{4.19}$$

From Thm.3.1 we know that $\lambda_r(G) = \lambda_r(\mathbb{T}_k)$ (actually, we only need to know that $\lambda_r(G) \leq \lambda_r(\mathbb{T}_k)$, which is trivially true). We will show that for some $x \in \mathcal{V}_{\mathbb{T}_j} \subset \mathcal{V}_G$, the function $\beta(\{x\},\cdot)$ is not right-continuous at $\lambda_r(\mathbb{T}_j)$. Thanks to Proposition 4.1.1, this is all we have to show.

Under recurrence, each site of $G$ will eventually be infected. Therefore, if $n$ denotes the distance between $x$ and the root of $\mathbb{T}_j$, we have from (1.2),

$$\begin{aligned} \beta_G(\{x\},\lambda_r(\mathbb{T}_j)) &\leq \mathbb{P}_{G,\lambda_r(\mathbb{T}_j)}(0\in\xi_t^x \text{ for some } t>0) \\ &= \mathbb{P}_{\mathbb{T}_j,\lambda_r(\mathbb{T}_j)}(0\in\xi_t^x \text{ for some } t>0) \\ &= u_{j,n}(\lambda) \\ &\leq (\alpha_j(\lambda_r(\mathbb{T}_j))^n. \end{aligned} \tag{4.20}$$

Since from Lalley and Sellke (1998) we know that $\alpha_j(\lambda_r(\mathbb{T}_j)) \leq 1/\sqrt{j}$, the upper bound in (4.20) vanishes as $n\to\infty$.

Suppose now that $\lambda > \lambda_r(\mathbb{T}_j)$. If the origin of $\mathbb{T}_j$ is ever infected, and before recovering it infects the origin of $\mathbb{T}_k$, then the conditional probability of recurrence is at least $\beta_{\mathbb{T}_k}(\lambda)$. Therefore

$$\begin{aligned} \beta_G(\{x\},\lambda) &\geq \mathbb{P}_{\mathbb{T}_j,\lambda}(0\in\xi_t^x \text{ for some } t>0)\frac{\lambda}{\lambda+1}\beta_{\mathbb{T}_k}(\lambda) \\ &\geq \beta_{\mathbb{T}_j}(\lambda)\frac{\lambda}{\lambda+1}\beta_{\mathbb{T}_k}(\lambda) \\ &= \rho_{\mathbb{T}_j}(\lambda)\frac{\lambda}{\lambda+1}\rho_{\mathbb{T}_k}(\lambda). \end{aligned}$$

Hence

$$\liminf_{\lambda\searrow\lambda_r(\mathbb{T}_j)} \beta_G(\{x\},\lambda) \geq \rho_{\mathbb{T}_j}(\lambda_r(\mathbb{T}_j))\frac{\lambda_r(\mathbb{T}_j)}{\lambda_r(\mathbb{T}_j)+1}\rho_{\mathbb{T}_k}(\lambda_r(\mathbb{T}_j)) > 0. \tag{4.21}$$

If $n$ is large enough, the claimed discontinuity of $\beta_G(\{x\},\cdot)$ at $\lambda_r(\mathbb{T}_j^+)$ follows from the comparison between (4.20) and (4.21).

■

## 4.4 A sufficient condition for continuity of $\rho(\cdot)$.

We will say that the contact process on $G$ *survives uniformly* at $\lambda$ on a set $A \subset \mathcal{V}_G$ if

$$\lim_{R\to\infty} \inf_{x\in A} \mathbb{P}_{G,\lambda}(\Omega_\infty^{B(x,R)}) = 1.$$

In case $A = \mathcal{V}_G$ and the condition above is satisfied, then we just say that the contact process on $G$ survives uniformly at $\lambda$.

If $\rho(\lambda) > 0$, then the contact process on $G$ survives uniformly at $\lambda$ on every finite set $A \subset \mathcal{V}_G$. To see this, observe that there is no loss in considering $A = \{x\}$ to be a singleton, and note that

$$\lim_{R\to\infty} \mathbb{P}_{G,\lambda}(\Omega_\infty^{B(x,R)}) = \lim_{R\to\infty} \overline{\nu}\{\eta\colon \eta \cap B(x,R) \neq \emptyset\} = 1, \tag{4.22}$$

where we used the fact that $\overline{\nu}(\emptyset) = 0$.

If $\rho(\lambda) > 0$, a set $A \subset \mathcal{V}_G$ will be called a *recurrence set* for the contact process on $G$ at $\lambda$ if

$$\mathbb{P}_{G,\lambda}(\Omega_\infty^0\ ,\ \{t\colon \xi_t^0 \cap A \neq \emptyset\} \text{ is bounded}) = 0.$$

Note that if $\rho_G(\lambda) > 0$, then having r=s for the contact process on $G$ at $\lambda$ is equivalent to saying that $\{0\}$ is a recurrence set for the contact process on $G$ at $\lambda$.

**Theorem 4.4.1** *If there exists $A \subset \mathcal{V}_G$ which is a recurrence set for the contact process on $G$ at $\lambda$, and on which the contact process on $G$ survives uniformly at some $\tilde{\lambda} < \lambda$, then $\rho_G(\cdot)$ is continuous at $\lambda$.*

**Proof of Thm.4.4.1.** Thanks to Thm.4.2.1, only left-continuity has to be shown.

Set

$$E_{R,t} = \{\xi_s^0 \supset B(x,R) \text{ for some } x \in A \text{ and } 0 \leq s \leq t\}.$$

If $\tilde{\lambda} < \lambda' < \lambda$, then for any $R$ and $t$, the Strong Markov property, attractiveness and monotonicity in $\lambda$ imply that

$$\begin{aligned} \rho_G(\lambda') = \mathbb{P}_{G,\lambda'}(\Omega_\infty^0) &\geq \mathbb{P}_{G,\lambda'}(E_{R,t}) \inf_{x\in A} \mathbb{P}_{G,\lambda'}(\Omega_\infty^{B(x,R)}) \\ &\geq \mathbb{P}_{G,\lambda'}(E_{R,t}) \inf_{x\in A} \mathbb{P}_{G,\tilde{\lambda}}(\Omega_\infty^{B(x,R)}). \end{aligned}$$

By finite-time-continuity,

$$\lim_{\lambda' \nearrow \lambda} \mathbb{P}_{G,\lambda'}(E_{R,t}) = \mathbb{P}_{G,\lambda}(E_{R,t}).$$

Therefore

$$\liminf_{\lambda' \nearrow \lambda} \rho_G(\lambda') \geq \mathbb{P}_{G,\lambda}(E_{R,t}) \inf_{x\in A} \mathbb{P}_{G,\tilde{\lambda}}(\Omega_\infty^{B(x,R)}). \tag{4.23}$$

Since $A$ is a recurrence set for the contact process on $G$ at $\lambda$, for any $R$ we have

$$\liminf_{t\to\infty} \mathbb{P}_{G,\lambda}(E_{R,t}) \geq \rho_G(\lambda).$$

So, in (4.23), let first $t \to \infty$ and then $R \to \infty$ to obtain

$$\liminf_{\lambda' \nearrow \lambda} \rho_G(\lambda') \geq \rho_G(\lambda),$$

where the hypothesis that the contact process on $G$ survives uniformly at $\tilde{\lambda}$ on $A$ was used.

Since $\rho_G(\lambda') \leq \rho_G(\lambda)$, for all $\lambda' < \lambda$, the proof is complete.

■

It is interesting to compare the similarities between this proof and that of the left-continuity of $\beta(\cdot)$ above $\lambda_r$ (Thm.4.3.1).

In the rest of this section we will apply Thm.4.4.1 to various examples and collect several of its consequences.

Some of the simplest applications of Thm.4.4.1 concern cases in which we can take $A = \mathcal{V}_G$. Regardless of what $G$ is, if $\rho_G(\lambda) > 0$, the set $\mathcal{V}_G$ is, tautologically, a recurrence set. Therefore, to apply the theorem at some point $\lambda$ we only have to check whether the contact process on $G$ survives uniformly at some $\tilde{\lambda} < \lambda$. The following is a case in which this is immediate.

**Corollary 4.4.1** *If $G \in \mathcal{H}$, then $\rho_G(\cdot)$ can only be discontinuous at $\lambda_s(G)$, and $\beta_G(\cdot)$ can only be discontinuous at $\lambda_r(G)$.*

**Proof.** If $\lambda > \lambda_s$ the homogeneity of $G$ reduces the statement of uniform survival of the contact process on $G$ at some $\tilde{\lambda} \in (\lambda_s, \lambda)$, to the statement that

$$\lim_{R\to\infty} \mathbb{P}_{G,\tilde{\lambda}}(\Omega_\infty^{B(0,R)}) = 1,$$

which is a particular case of (4.22). So the claim concerning the function $\rho_G(\cdot)$ is a consequence of Thm.4.4.1.

Regarding the function $\beta_G(\cdot)$, just note that below $\lambda_r(G)$ this function is identically 0, and above this point it coincides with the function $\rho_G(\cdot)$, by Thm.2.2(i). But since $\lambda_s(G) \le \lambda_r(G)$, we have just shown that $\rho_G(\cdot)$ is continuous above $\lambda_r(G)$.

■

In the two basic examples of graphs in $\mathcal{H}$, $\mathbb{Z}^d$ and $\mathbb{T}_b$, it is also known that $\rho(\cdot)$ is continuous at $\lambda_s$. It is natural to ask whether this is the case for all homogeneous graphs. Unfortunately we are unable to answer this question.

It may be worthwhile to point out that the proof given above that for the contact process on $\mathbb{T}_b$, $\rho(\cdot)$ is continuous above $\lambda_s$ is different from the proof contained in the papers by Pemantle (1992) and Morrow, Schinazi and Zhang (1994).

Not only for homogeneous graphs can we apply Thm.4.4.1 with $A = \mathcal{V}_G$. Another example is that of $\mathbb{T}_b^+$. For these graphs it is indeed easy to see that the contact process survives uniformly at any $\lambda > \lambda_s(\mathbb{T}_b^+)$. For this purpose first observe that each site $x \in \mathbb{T}_b^+$ can be seen as the root of a subgraph $G_x$ which is isomorphic to $\mathbb{T}_b^+$ itself. Note that this isomorphism maps $B(x,R) \cap G_x$ onto $B(0,R)$ and therefore

$$\lim_{R\to\infty} \inf_{x\in\mathcal{V}_{\mathbb{T}_b^+}} \mathbb{P}_{\mathbb{T}_b^+,\lambda}(\Omega_\infty^{B(x,R)}) \ge \lim_{R\to\infty} \mathbb{P}_{\mathbb{T}_b^+,\lambda}(\Omega_\infty^{B(0,R)}) = 1,$$

by (4.22).

Another consequence of Thm.4.4.1 follows.

**Corollary 4.4.2** *If the criterion r=s holds for the contact process on $G$ at $\lambda > \lambda_s(G)$, then $\rho_G(\cdot)$ is continuous at $\lambda$.*

**Proof.** Take $A = \{0\}$ and note that from the remarks made when the notions of uniform survival and of recurrent set were introduced, we have the hypothesis of Thm.4.4.1 satisfied.

∎

The hypothesis that $\lambda > \lambda_s(G)$ in Corollary 4.4.2 is crucial. Note that the desert oasis example of Section 4.2 satisfies r=s at $\lambda_s$, but $\rho(\cdot)$ is discontinuous there. Since for that example r=s also holds for all $\lambda > \lambda_s = \lambda_r =: \lambda_c$, we conclude that for it $\rho(\cdot) = \beta(\cdot)$ is discontinuous only at $\lambda_c$.

From Thm.2.6 we know that for any graph $G \in \mathcal{G}$, cc (and hence r=s) holds when $\lambda$ is above the unique critical point, $\lambda_c(\mathbb{Z})$, for the contact process on $\mathbb{Z}$. Therefore we also have the following.

**Corollary 4.4.3** *For any graph $G \in \mathcal{G}$, $\rho_G(\cdot) = \beta_G(\cdot)$ is continuous on $[\lambda_c(\mathbb{Z}), \infty)$.*

The desert-oasis example of Section 4.2 shows that Corollary 4.4.3 is optimal in the sense that $\lambda_c(\mathbb{Z})$ cannot be replaced in that proposition with any smaller number.

To state the next corollary to Thm.4.4.1, we need a new definition. Suppose that $\Lambda$ is a subset of $(0, \infty)$. We say that the contact process on a graph $G \in \mathcal{G}$ has the *uniform survivability property on* $\Lambda$ in case there exists $A \subset \mathcal{V}_G$ which is a recurrence set for the contact process on $G$ at all $\lambda \in \Lambda$, and on which the contact process on $G$ survives uniformly at all $\lambda \in \Lambda$. In case $\Lambda = (\lambda_s(G), \infty)$, we say that the contact process on $G$ has the *supercritical uniform survivability property.* In case $\Lambda = \{\lambda\colon \rho_G(\lambda) > 0\}$, we simply say that the contact process on $G$ has the *uniform survivability property.*

Examples of graphs with the uniform survivability property are the homogeneous graphs, and the graphs $\mathbb{T}_b^+$. The example 3.1 is an example of a graph which does not have this property (but is a collage of graphs with the property).

**Corollary 4.4.4** *If $G \in \mathcal{G}$ is a collage of graphs $G_1, G_2, \ldots, G_n$ which have the supercritical uniform survivability property, then the function $\rho_G(\cdot)$ is continuous except possibly at the points $\lambda_s(G_1), \lambda_s(G_2), \ldots, \lambda_s(G_n)$, and $\lambda_s(G)$. If in addition the contact process on each one of the graphs $G_i$, $i = 1, 2, \ldots, n$ satisfies r=s at its survival point (for instance, if $\rho_{G_i}(\lambda_s(G_i)) = 0$), then $\rho_G(\cdot)$ is continuous, except possibly at the point $\lambda_s(G)$.*

**Proof.** We only have to consider the case $\lambda > \lambda_s(G)$. Suppose also that we have $\lambda \notin \{\lambda_s(G_1), \lambda_s(G_2), \ldots, \lambda_s(G_n)\}$ and set

$$I_{\mathrm{S}} = \{i\colon \lambda_s(G_i) < \lambda\} \quad \text{and} \quad I_{\mathrm{D}} = \{1, 2, ..., n\} \backslash I_{\mathrm{S}}.$$

We denote by $A_i$ the subset of $\mathcal{V}_{G_i}$ which is a recurrence set for the contact process on $G_i$ at all $\lambda > \lambda_s(G_i)$, and on which the contact process on $G_i$ survives uniformly at all $\lambda > \lambda_s(G_i)$.

We will first shown that the set $V_{\mathrm{glue}} \cup (\cup_{i \in I_{\mathrm{S}}} A_i)$ is a recurrence set for the contact process on $G$ at $\lambda$. Indeed, on the event $\Omega_\infty^0$ the set $V_{\mathrm{glue}} \cup (\cup_{i \in I_{\mathrm{S}}} A_i)$ must be infected at arbitrarily large times, $\mathbb{P}_{G,\lambda}$-a.s.. This is so because the complementary part of $\Omega^0_\infty$ can be covered by a countable union of events of the following type: For one of the $i \in \{1, ..., n\}$, the contact process on the graph $G_i$ starting at the integer time $k$ from a finite set $B \in \mathcal{V}_{G_i}$ survives, but never reaches any site in the set $V_{\text{glue}} \cup (\cup_{i \in I_{\text{S}}} A_i)$. But such events, indexed by $i$, $k$ and $B$ have probability 0. To see this, there are two cases to consider: If $i \in I_{\text{D}}$, then the probability of this event is bounded above by $\rho_{G_i}(B, \lambda) = 0$. If $i \in I_{\text{S}}$, then the probability of the referred event is bounded above by the probability that the contact process on $G_i$, started from $B$ survives but never reaches the set $A_i$; this probability is zero by the definition of the supercritical uniform survivability property.

It is clear now from the finiteness of $V_{\text{glue}}$ and $I_{\text{S}}$, (4.22), the definition of $I_{\text{S}}$ and the hypothesis of supercritical uniform survivability on the graphs $G_1, G_2, \ldots, G_n$ that the contact process on $G$ survives uniformly at $\tilde{\lambda} < \lambda$ on the set $V_{\text{glue}} \cup (\cup_{i \in I_{\text{S}}} A_i)$, provided that $\tilde{\lambda}$ is chosen close enough to $\lambda$.

The continuity of $\rho_G(\cdot)$ at $\lambda$ follows from combining the conclusions of the two paragraphs above, therefore verifying the hypothesis of Theorem 4.4.1. This completes the proof of the first claim in the corollary.

If the contact process on each one of the graphs $G_1, G_2, \ldots, G_n$ satisfies r=s at its survival point, then for any $\lambda > \lambda_s(G)$ the argument above with a minor modification still applies to give the continuity of $\rho_G(\cdot)$ at $\lambda$. This minor modification is that when $i \in I_{\text{D}} \cap \{i \colon \lambda_s(G_i) = \lambda\}$ the probability of the event indexed by $i$, $k$ and $B$ used above should now be bounded above by the probability that the contact process on $G_i$ started from $B$ survives without ever infecting the sites in $V_{\text{glue}} \cap \mathcal{V}_{G_i}$; such probabilities are 0, since r=s is supposed to hold.

■

The corollary above should be contrasted with Proposition 4.2.1. Combining the two results we can state the following.

**Corollary 4.4.5** *Suppose that $G = G_1 \vee G_2$, with $\lambda_s(G_2) < \lambda_s(G_1)$, and suppose also that $G_2$ satisfies the supercritical uniform survivability property. Then $\rho_G(\cdot)$ is continuous at $\lambda_s(G_1)$ if and only if the contact process on $G_1$ satisfies r=s when the infection parameter takes the value $\lambda_s(G_1)$.*

The following is a somewhat surprising application of the corollary above. Suppose that $G_2 \in \mathcal{G}$ has the supercritical uniform survivability property and $\lambda_s(G_2) < \lambda_s(\mathbb{Z})$ (for instance $G_2 = \mathbb{Z}^2$ or $G_2 = \mathbb{T}_2$), and let $G_1$ be the desert oasis example of Section 4.2. In this case $G_1$ satisfies r=s when the infection parameter takes the value $\lambda_s(G_1) = \lambda_s(\mathbb{Z})$, and therefore $\rho_{G_1 \vee G_2}(\cdot)$ is continuous at this point. This may seem surprising, because one could expect the discontinuity in $\rho_{G_1}(\cdot)$ at $\lambda_s(G_1)$ to reflect in a discontinuity of $\rho_{G_1 \vee G_2}(\cdot)$ at the same point, but this turns out not to be the case.

Another interesting particular case in which Corollary 4.4.4 applies is the following.

**Corollary 4.4.6** *If $G \in \mathcal{G}$ is a collage of copies of $\mathbb{T}^+_{b_1}, \mathbb{T}^+_{b_2}, \ldots, \mathbb{T}^+_{b_n}$, then $\rho_G(\cdot)$ is continuous, except possibly at $\lambda_s(G)$. If in addition* $\max\{b_i\colon\ i = 1, \ldots, n\} \geq 2$, *then $\rho_G(\cdot)$ is continuous.*

**Proof.** From Corollary 4.4.4 we obtain the first statement, and combining this conclusion with Thm.3.1, we obtain the second statement.

∎

Note that the example 3.1 is covered by the corollary above, so that in spite of its various transitions from one type of ergodic behavior to another one, it has a continuous $\rho(\cdot)$. It is natural at this point to ask what the continuity properties of $\beta(\cdot)$ are for this example. This problem will be treated in the next section.

## 4.5 A necessary and sufficient condition for right-continuity of $\beta(\cdot)$.

For $G \in \mathcal{G}$, $A, S \subset\subset \mathcal{V}_G$ and $\lambda > 0$, we define

$$\gamma_G^S(A, \lambda) = \mathbb{P}_{G,\lambda}(\xi_t^A \cap S \neq \emptyset \text{ for some } t \geq 0) = \mathbb{P}_{G,\lambda}(A \times \{0\} \longrightarrow S \times [0, \infty)).$$

Since $\gamma_G^S(A, \lambda) = \sup_{T \geq 0} \mathbb{P}_{G,\lambda}(\xi_t^A \cap S \neq \emptyset \text{ for some } t \in [0, T])$, we conclude that the function $\gamma_G^S(A, \cdot)$ is left-continuous everywhere (by the same sort of argument used to prove the right-continuity of $\rho_G(\cdot)$). Regarding its right-continuity, we have the result stated as the next theorem. (Note that this theorem has no analogue for left-continuity, since there are graphs in $\mathcal{G}$ for which $\beta_G(\cdot)$ is not left-continuous at $\lambda_r(G)$.)

**Theorem 4.5.1** *Suppose that $G \in \mathcal{G}$ and $S \subset\subset \mathcal{V}_G$, $S \neq \emptyset$. Then $\beta_G(\cdot)$ is right-continuous at $\lambda$ if and only if for each $A \subset\subset \mathcal{V}_G$ the function $\gamma_G^S(A, \cdot)$ is right-continuous at $\lambda$.*

**Proof of Thm.4.5.1.** We first prove the "if" part. For arbitrary $\lambda', T, N > 0$, the Markov property gives

$$\begin{aligned} 1 - \beta_G(\lambda') &\geq \mathbb{P}_{G,\lambda'}(\xi_t^A \cap S = \emptyset \text{ for all } t \geq T) \\ &= \sum_{A \subset\subset \mathcal{V}_G} \mathbb{P}_{G,\lambda'}(\xi_T^0 = A)(1 - \gamma_G^S(A, \lambda')) \\ &\geq \sum_{\substack{A \subset\subset \mathcal{V}_G \\ A \subset B(0,N)}} \mathbb{P}_{G,\lambda'}(\xi_T^0 = A)(1 - \gamma_G^S(A, \lambda')). \end{aligned}$$

Therefore

$$1 - \limsup_{\lambda' \searrow \lambda} \beta_G(\lambda') \geq \sum_{\substack{A \subset\subset \mathcal{V}_G \\ A \subset B(0,N)}} \mathbb{P}_{G,\lambda}(\xi_T^0 = A)(1 - \gamma_G^S(A, \lambda)).$$

Since $N$ is arbitrary, this implies

$$\begin{aligned} 1 - \limsup_{\lambda' \searrow \lambda} \beta_G(\lambda') &\geq \sum_{A \subset\subset \mathcal{V}_G} \mathbb{P}_{G,\lambda}(\xi_T^0 = A)(1 - \gamma_G^S(A, \lambda)) \\ &= \mathbb{P}_{G,\lambda}(\xi_t^A \cap S = \emptyset \text{ for all } t \geq T). \end{aligned}$$

Letting $T \to \infty$ yields now

$$\limsup_{\lambda' \searrow \lambda} \beta_G(\lambda') \leq 1 - (1 - \beta_G(\lambda)) = \beta_G(\lambda).$$

Since $\beta_G(\cdot)$ is a non-decreasing function this shows that it is right-continuous at $\lambda$, finishing the proof of the "if" part.

We turn now to the proof of the "only if" part. We will use the coupling $\bar{\mathbb{P}}_{G,\tilde{\lambda}}$, where $\tilde{\lambda} > \lambda$ is arbitrary. The notation $\Omega^A_{r,\lambda}$ will denote the event that when the arrows are kept only up to level $\lambda$, the contact process started from $A$ is recurrent, i.e., has each site infected at arbitrarily large times. For $\lambda < \lambda' < \tilde{\lambda}$,

$$\begin{aligned}
0 &\le \gamma^S_G(A,\lambda') - \gamma^S_G(A,\lambda) \\
&= \mathbb{P}_{G,\tilde{\lambda}}\left(\left\{A \times \{0\} \xrightarrow{\lambda'} S \times [0,\infty)\right\} \cap \left\{A \times \{0\} \xrightarrow{\lambda} S \times [0,\infty)\right\}^c\right) \\
&\le \mathbb{P}_{G,\tilde{\lambda}}\left(\Omega^A_{r,\lambda} \cap \left\{A \times \{0\} \xrightarrow{\lambda'} S \times [0,\infty)\right\} \cap \left\{A \times \{0\} \xrightarrow{\lambda} S \times [0,\infty)\right\}^c\right) \\
&+ \mathbb{P}_{G,\tilde{\lambda}}\left(\left(\Omega^A_{r,\tilde{\lambda}}\right)^c \cap \left\{A \times \{0\} \xrightarrow{\lambda'} S \times [0,\infty)\right\} \cap \left\{A \times \{0\} \xrightarrow{\lambda} S \times [0,\infty)\right\}^c\right) \\
&+ \mathbb{P}_{G,\tilde{\lambda}}\left(\left(\Omega^A_{r,\lambda}\right)^c \cap \Omega^A_{r,\tilde{\lambda}}\right).
\end{aligned}$$

The first term in the right-hand-side is clearly null. We introduce an arbitrary $T > 0$ and break down the second term into two parts, to write

$$\begin{aligned}
0 &\le \gamma^S_G(A,\lambda') - \gamma^S_G(A,\lambda) \\
&\le \mathbb{P}_{G,\tilde{\lambda}}\left(\left\{A \times \{0\} \xrightarrow{\lambda'} S \times [0,T)\right\} \cap \left\{A \times \{0\} \xrightarrow{\lambda} S \times [0,\infty)\right\}^c\right) \\
&\quad + \mathbb{P}_{G,\tilde{\lambda}}\left(\left(\Omega^A_{r,\tilde{\lambda}}\right)^c \cap \left\{A \times \{0\} \xrightarrow{\lambda'} S \times [T,\infty)\right\}\right) \\
&\quad + \mathbb{P}_{G,\tilde{\lambda}}\left(\left(\Omega^A_{r,\lambda}\right)^c \cap \Omega^A_{r,\tilde{\lambda}}\right) \\
&\le \mathbb{P}_{G,\tilde{\lambda}}\left(\left\{A \times \{0\} \xrightarrow{\lambda'} S \times [0,T)\right\} \cap \left\{A \times \{0\} \xrightarrow{\lambda} S \times [0,T)\right\}^c\right) \\
&\quad + \mathbb{P}_{G,\tilde{\lambda}}\left(\left(\Omega^A_{r,\tilde{\lambda}}\right)^c \cap \left\{A \times \{0\} \xrightarrow{\tilde{\lambda}} S \times [T,\infty)\right\}\right) \\
&\quad + \mathbb{P}_{G,\tilde{\lambda}}\left(\left(\Omega^A_{r,\lambda}\right)^c \cap \Omega^A_{r,\tilde{\lambda}}\right).
\end{aligned}$$

As we let $\lambda' \searrow \lambda$, the first term in the right-hand-side vanishes, by finite-time-continuity. As we then let $T \to \infty$, the second term in the right-hand-side vanishes, since the corresponding event decreases to the empty set. Finally, as we then let $\tilde{\lambda} \searrow \lambda$, the third term in the right-hand-side vanishes, since it equals $\beta_G(A, \tilde{\lambda}) - \beta_G(A, \lambda)$, which vanishes by hypothesis. The conclusion, as desired, is that

$$\lim_{\lambda' \searrow \lambda} \gamma_G^S(A, \lambda') = \gamma_G^S(A, \lambda).$$

■

The theorem above may at first sight seem to be difficult to apply, in order to prove the right-continuity of $\beta_G(\cdot)$ for any specific $G$. The corollaries below show otherwise. The point is that since the theorem provides an equivalence, it can be used to extend the known right-continuity of $\beta_G(\cdot)$ for certain graphs, in certain intervals of values of $\lambda$, to other graphs built using those ones.

**Corollary 4.5.1** *Suppose that $G$ is a collage of $G_1, ..., G_n$, then $\beta_G(\cdot)$ is right-continuous at $\lambda$ if and only if $\beta_{G_i}(\cdot)$ is right-continuous at $\lambda$ for each one of the graphs $G_i$, $i = 1, ..., n$.*

**Proof.** It is easy to see that for any $A \subset\subset \mathcal{V}_G$,

$$\gamma_G^{V_{\text{glue}}}(A, \cdot) \;=\; 1 - \prod_{i=1}^{n} \left(1 - \gamma_{G_i}^{V_{\text{glue}} \cap \mathcal{V}_{G_i}}(A \cap \mathcal{V}_{G_i}, \cdot)\right). \tag{4.24}$$

If for all $i \in \{1, ..., n\}$ and $A_i \subset\subset \mathcal{V}_{G_i}$ the function $\gamma_{G_i}^{V_{\text{glue}} \cap \mathcal{V}_{G_i}}(A_i, \cdot)$ is right-continuous at $\lambda$, then from (4.24) the same is true for all functions $\gamma_G^{V_{\text{glue}}}(A, \cdot)$, $A \subset\subset \mathcal{V}_G$.

On the other hand, if for some $i \in \{1, ..., n\}$ and for some $A_i \subset \mathcal{V}_{G_i}$ the function $\gamma_{G_i}^{V_{\text{glue}} \cap \mathcal{V}_{G_i}}(A_i, \cdot)$ is not right-continuous at $\lambda$, then take $A = A_i \subset\subset \mathcal{V}_G$, and note that (4.24) gives then

$$\gamma_G^{V_{\text{glue}}}(A, \cdot) \;=\; \gamma_{G_i}^{V_{\text{glue}} \cap \mathcal{V}_{G_i}}(A_i, \cdot).$$

In particular $\gamma_G^{V_{\text{glue}}}(A, \cdot)$ is not right-continuous at $\lambda$.

Our proof is complete, by referring to the equivalence provided by Theorem 4.5.1.

■

As with Thm.4.5.1, there is no analogue of Corollary 4.5.1 for left-continuity, as the following example shows. Suppose that $G = \mathbb{Z}^2 \vee [\mathbf{d}, \mathbf{o}]$, where $[\mathbf{d}, \mathbf{o}]$ is the desert-oasis example of Section 4.2. We know that $\beta_{\mathbb{Z}^2}(\cdot)$ is continuous everywhere and that $\beta_{[\mathbf{d},\mathbf{o}]}(\cdot)$ is continuous except at the point $\lambda_s([\mathbf{d}, \mathbf{o}]) = \lambda_r([\mathbf{d}, \mathbf{o}]) = \lambda_s(\mathbb{Z}) > \lambda_s(\mathbb{Z}^2)$, where it is right-continuous but not left-continuous. From Corollary 4.4.5 and the discussion which followed it we know that $\rho_G(\cdot)$ is continuous at $\lambda_s(\mathbb{Z})$. We know also that the contact processes on $\mathbb{Z}^2$ and on $[\mathbf{d}, \mathbf{o}]$ satisfy r=s for all $\lambda$. Therefore, by Thm.2.4, so does the contact process on $G$. Hence $\beta_G(\cdot)$ is continuous at $\lambda_s(\mathbb{Z})$.

An interesting application of Thm.4.5.1 follows.

**Corollary 4.5.2** *If $G \in \mathcal{G}$ is a collage of copies of $\mathbb{T}_{b_1}^+, \mathbb{T}_{b_2}^+, \ldots, \mathbb{T}_{b_n}^+$, with $b_i \geq 2$, then $\beta_G(\cdot)$ is continuous, except at the points $\lambda_r(G_i)$, $i = 1, ..., n$, where it is not right-continuous (it is left-continuous at these points, except possibly at the smallest* *of them, which coincides with $\lambda_r(G)$).*

**Proof.** For the contact process on $\mathbb{T}_b^+$, $b \geq 2$, we know by Thm.2.6 that cc, and hence r=s, holds above $\lambda_r(\mathbb{T}_b^+)$. From Corollary 4.4.2 we learn that $\beta_{\mathbb{T}_b^+}(\cdot) = \rho_{\mathbb{T}_b^+}(\cdot)$ is continuous above this point. But since $\lambda_s(\mathbb{T}_b^+) < \lambda_r(\mathbb{T}_b^+)$,

$$\lim_{\lambda \searrow \lambda_r(\mathbb{T}_b^+)} \beta_{\mathbb{T}_b^+}(\lambda) = \rho_{\mathbb{T}_b^+}(\lambda_r(\mathbb{T}_b^+)) > 0 = \beta_{\mathbb{T}_b^+}(\lambda_r(\mathbb{T}_b^+)),$$

so that $\beta_{\mathbb{T}_b^+}(\cdot)$ is not right-continuous at $\lambda_r(\mathbb{T}_b^+)$.

From Corollary 4.5.1 we obtain now the claim in the corollary, except for the part in parenthesis. That part has already been proven in Thm.3.1 and Thm.4.3.1.

■

From Thm.3.1, Corollary 4.4.6 and Corollary 4.5.2, a great deal of information has been learned about the contact process on graphs which are collages of copies of severed homogeneous trees. When we apply these results for instance to the Example 3.1, $\mathbb{T}_j \vee \mathbb{T}_k$, with (2.1) satisfied, we learn that its survival point is $\lambda_s(\mathbb{T}_j)$, its recurrence point is $\lambda_r(\mathbb{T}_j)$, its survival probability is a continuous function of $\lambda$ and its recurrence probability is discontinuous precisely at the points $\lambda_r(\mathbb{T}_j)$ and $\lambda_r(\mathbb{T}_k)$.

Note that the arguments presented in this section provided a second proof of the claim in Thm.4.3.1, that there are trees in $\mathcal{G}$ for which $\beta(\cdot)$ is discontinuous at some $\lambda > \lambda_r$.

# CHAPTER 5

## A new proof that local survival implies complete convergence for the contact process on homogeneous trees.

In this chapter we will consider the contact process with infection parameter $\lambda > 0$, on the homogeneous tree of degree $\mathbb{T}_b$. The case in which $b = 1$ corresponds to the linear chain $\mathbb{Z}$, and will not be considered here, so that we assume that $b \geq 2$. As usual in this case we write $\lambda_s = \lambda_1$ and $\lambda_r = \lambda_2$. Note that it is known that at $\lambda_1$ the system dies out, and that at $\lambda_2$ it survives globally but not locally. The second of these two facts was first proven by Zhang (1996), but a simpler argument appeared in Lalley and Sellke (1998). It is interesting to point out that Zhang's proof relied on the same machinery developed to proof the complete convergence theorem above $\lambda_2$, the result whose proof we simplify in the current chapter.

In this chapter we will focus on the following complete convergence theorem, originally proven by Zhang (1996).

**Theorem 5.1 (Zhang)** *For the contact process on $\mathbb{T}_b$ if $\lambda > \lambda_2$ then for any $\eta \in \{0,1\}^{\mathcal{V}_{\mathbb{T}_b}}$*

$$\xi_t^\eta \Rightarrow \Big(1 - \rho(\eta)\Big)\delta_\emptyset + \rho(\eta)\overline{\nu}, \tag{5.1}$$

*as $t \to \infty$.*

The technically most difficult part of Zhang's proof of complete convergence for the contact process on $\mathbb{T}_b$ is his proof of Proposition 5 in his paper, which states that when $\lambda > \lambda_2$,

$$\inf_{t \geq 0} \mathbb{P}\left(\xi^0_{\mathbb{T}_b^+;t}(0) = 1\right) > 0. \tag{5.2}$$

After proving this, Zhang completes his proof of Thm.5.1 by using a classical result in Griffeath (1978) which establishes a necessary and sufficient condition for complete convergence to hold based on the behavior of two independent copies of the contact process (see Section 2.3). In Chapter 2 we provided a minor simplification of Zhang's proof of Thm.5.1 in that we showed how the machinery which we developed there can replace the result of Griffeath (1978) in proving complete convergence on $\mathbb{T}_b$, once (5.2) is available.

In this chapter we give a self-contained, relatively short and simple proof of the following result. We recall that the definition of $u_n$ and $\alpha$ are, respectively, (1.1) and (1.2).

**Proposition 5.1** *If $\alpha(\lambda) > 1/\sqrt{b}$ then (5.2) holds.*

When $\lambda > \lambda_2$ we have $u_n \geq \beta(\{0\}) > 0$ for each $n$. Therefore $\alpha(\lambda) = 1$ and from Proposition 5.1 the estimate (5.2) follows. In this way Zhang's proof of Thm.5.1 is greatly simplified. It is worth stressing that Zhang's renormalization procedure [see p. 1418 and followings in his paper] is not needed in our proof.

From Proposition 5.1, one also obtains the following immediate consequence, of a different nature.

**Theorem 5.2 (Lalley and Sellke)** *If* $\lambda < \lambda_2$ *then* $\alpha(\lambda) \leq 1/\sqrt{b}$.

We shall use the definition of an increasing event that was given by Bezuidenhout and Grimmett (1991). Briefly, an event $E$ is said to be increasing if the following holds: for any realization of the graphical construction that is in $E$, every other realization obtained from it by the addition of arrows or the suppression of $\delta$ marks is also in $E$. The Harris-FKG inequality says that if $E$ and $F$ are both increasing events, then

$$P(E \cap F) \geq P(E)P(F).$$

The following object will be important in our proof of Proposition 5.1.

$$\begin{aligned} Y_{n,s} &= \mathbb{P}\left(\text{There is a path from } (0,0) to (n,s) \text{ inside } \mathbb{T}_b^+ \times \mathbb{R}_+\right) \\ &= \mathbb{P}\left(\xi^0_{\mathbb{T}_b^+;s}(n) = 1\right). \end{aligned}$$

**Lemma 5.1** *There is a sequence* $(s(n))_{n\geq 1}$ *such that* $\lim_{n\to\infty}\left(Y_{n,s(n)}\right)^{1/n} = \alpha$.

**Proof.** Clearly for any sequence $s(n)$

$$\limsup_{n\to\infty}\left(Y_{n,s(n)}\right)^{1/n} \leq \limsup_{n\to\infty}\left(u_n\right)^{1/n} = \alpha.$$

This is not a very useful inequality. We are really after the complementary inequality, which we address next. Define

$$\begin{aligned} V_{m,k} &= \mathbb{P}\left(\text{There is a path from } (0,0) \text{ to } \{m\} \times \mathbb{R}_+ \text{ inside } B(0,k) \times [0,k]\right) \\ &= \mathbb{P}\left(\xi^0_{B(0,k);t}(m) = 1 \text{ for some } t \leq k\right). \end{aligned}$$

Clearly

$$\lim_{k\to\infty} V_{m,k} = u_m. \tag{5.3}$$

Next define

$$\begin{aligned} W_{n,k} &= \mathbb{P}\Big(\text{There is a path from } (0,0) \text{ to } \{n\}\times\mathbb{R}_+ \text{ inside } \mathbb{T}_b^+ \times [0,kn]\Big) \\ &= \mathbb{P}\left(\xi^0_{\mathbb{T}_b^+;t}(n) = 1 \text{ for some } t \le kn\right). \end{aligned}$$

We will argue next that

$$W_{n,k} \ge C_{m,k}\left(V_{m,k}\right)^{\lceil n/m \rceil}, \tag{5.4}$$

where $C_{m,k}$ is a positive quantity which does not depend on $n$. For this purpose set $I = \min\{i \in \{1,2,...\}:\ im > k\} = \min\{i \in \{1,2,...\}:\ B(im,k) \subset \mathbb{T}_b^+\}$. Consider now the sequence of sites $x_1 = Im, x_2 = (I+1)m, x_3 = (I+2)m, ..., x_J = \lceil n/m \rceil m$. By pasting together a sequence of paths and using the translation invariance of the graphical construction and the strong Markov property of the underlying Poisson processes, one readily obtains 5.4). In this argument, the first path is somewhat different from the others and it can go from $(0,0)$ to $(x_1,1)$ without exiting $\mathbb{T}_b^+ \times \mathbb{R}_+$. The second path should go from $(x_1,1)$ to $(x_2,T_2)$, for some random time $T_2$ which satisfies $T_2 - 1 \le k$, and it should not exit $B(x_1,k)$. The third path should go from $(x_2,T_2)$ to $(x_3,T_3)$, for some random time $T_3$ which satisfies $T_3 - T_2 \le k$, and it should not exit $B(x_2,k)$. By now the way to choose the other paths should be clear. The reason for (5.4) should also be clear, with the factor $C_{m,k} > 0$ being the probability of the existence of the first path in this construction.

Define

$$\overline{W}_{n,k} =$$

$$\max_{j=0,\dots,nk-1} \mathbb{P}\left(\text{There is a path from } (0,0) \text{ to } \{n\} \times [j,j+1] \text{ inside } \mathbb{T}_b^+ \times \mathbb{R}_+\right)$$

$$= \max_{j=0,\dots,nk-1} \mathbb{P}\left(\xi^0_{\mathbb{T}_b^+;t}(n) = 1 \text{ for some } t \in [j,j+1]\right).$$

The event in this definition and the event that there is no death mark in $\{n\} \times [j,j+1]$ are both increasing, therefore, for a proper choice of $s(n)$,

$$Y_{n,s(n)} \geq e^{-1}\overline{W}_{n,k} \geq e^{-1}\frac{1}{kn}W_{n,k}. \tag{5.5}$$

Using now (5.5), (5.4), (5.3) and the definition of $\alpha$,

$$\begin{aligned}
\liminf_{n\to\infty}\left(Y_{n,s(n)}\right)^{1/n} &= \liminf_{m\to\infty}\liminf_{k\to\infty}\liminf_{n\to\infty}\left(Y_{n,s(n)}\right)^{1/n} \\
&\geq \liminf_{m\to\infty}\liminf_{k\to\infty}\left(V_{m,k}\right)^{1/m} \\
&= \liminf_{m\to\infty}\left(u_m\right)^{1/m} \\
&= \alpha,
\end{aligned}$$

finishing the proof.

■

**Proof of Prop.5.1.** Suppose that $\alpha(\lambda) > 1/\sqrt{b}$. Thanks to Lemma 5.1, we can take $n$ and $s$ such that

$$\left(Y_{n,s}\right)^{1/n} = a > 1/\sqrt{b}. \tag{5.6}$$

We will show that for a proper choice of a positive integer $l$,

$$\inf_{i=0,1,2,\dots} \mathbb{P}\left(\xi^0_{\mathbb{T}_b^+;2ils}(0) = 1\right) > 0. \tag{5.7}$$

This clearly suffices for our purposes, since then

$$\inf_{t\geq 0}\mathbb{P}\left(\xi^0_{\mathbb{T}_b^+;t}(0)=1\right)\geq e^{-2ls}\inf_{i=0,1,2,\dots}\mathbb{P}\left(\xi^0_{\mathbb{T}_b^+;2ils}(0)=1\right)>0,$$

by the same reasoning behind the first inequality in (5.5).

In order to prove (5.7), we consider now the following modification of the contact process on $\mathbb{T}_b^+$. Until time $s$ we run the usual contact process on this graph started from a single particle at the origin. At time $s$ we remove all particles except for those which are in $B(0,n)\backslash B(0,n-1)$; from this time on we keep the set $B(0,n-1)$ free of particles, but until time $2s$ we let the system evolve in the remaining vertices with the usual contact process rules. At time $2s$ we remove all particles except for those which are in $B(0,2n)\backslash B(0,2n-1)$; from this time on we keep the set $B(0,2n-1)$ free of particles, but until time $3s$ we let the system evolve in the remaining vertices with the usual contact process rules. The modification should now be clear. For each $j$, at time $js$ we remove all particles except for those which are in $B(0,jn)\backslash B(0,jn-1)$; from this time on we keep the set $B(0,jn-1)$ free of particles, but until time $(j+1)s$ we let the system evolve in the remaining vertices with the usual contact process rules.

Let $Z_j$ be the number of particles in this modified process at time $js$ (all of them are in $B(0,jn)\backslash B(0,jn-1)$). It is clear that $(Z_j)_{j=0,1,2,\dots}$ is a branching process with mean offspring number $b^nY_{n,s}=(ba)^n$. Since the offspring distribution has a finite support (namely $\{0,1,...,b^n\}$), and in particular a finite second moment, it follows from standard branching-process theory [see, e.g., example 4.3 in Sect. 4.4,

p. 254 of Durrett (1996)] that for some random variable $X$ with mean $\mathbb{E}(X) = 1$

$$\frac{Z_j}{(ba)^{nj}} \to X \qquad \text{a.s.} \qquad \text{as } j \to \infty.$$

In particular, there is $\epsilon > 0$ such that

$$\mathbb{P}\left(Z_l \geq \frac{(ba)^{nl}}{2}\right) \geq \epsilon, \tag{5.8}$$

for all large enough $l$.

Choose now $l$ large enough for (5.8) to hold, and also so that

$$\left(1 - \frac{\epsilon}{2}a^{nl}\right)^{\frac{(ba)^{nl}}{2}} \leq \frac{1}{2}. \tag{5.9}$$

This last requirement can be fulfilled because

$$\begin{aligned}
\lim_{l\to\infty} \left(1 - \frac{\epsilon}{2}a^{nl}\right)^{\frac{(ba)^{nl}}{2}} &\leq \lim_{l\to\infty} \exp\left(-\frac{\epsilon}{2}a^{nl} \cdot \frac{(ba)^{nl}}{2}\right) \\
&= \lim_{l\to\infty} \exp\left(-\frac{\epsilon}{4}(ba^2)^{nl}\right) \\
&= 0,
\end{aligned}$$

since $ba^2 > 1$ by (5.6).

Define now

$$r_i = \mathbb{P}\left(\xi^0_{\mathbb{T}_b^+;2ils}(0) = 1\right).$$

We will show inductively in $i$ that

$$r_i \geq \frac{\epsilon}{2}a^{nl}, \tag{5.10}$$

verifying therefore the validity of (5.7).

For $i = 0$ inequality (5.10) is clearly true, and we will show now that if it is true for $i$ it is true for $i+1$. From the Markov property and attractiveness of the contact process we have

$$\begin{aligned} r_{i+1} &\geq \mathbb{P}\left(\xi^0_{\mathbb{T}_b^+;(2i+1)ls}(x) = 1 \text{ for some } x \in B(0,nl)\backslash B(0,nl-1)\right) Y_{ln,ls} \\ &\geq \mathbb{P}\left(\xi^0_{\mathbb{T}_b^+;(2i+1)ls}(x) = 1 \text{ for some } x \in B(0,nl)\backslash B(0,nl-1)\right) (Y_{n,s})^l. \end{aligned}$$

It is clear that the contact process on $\mathbb{T}_b^+$ dominates the modified process introduced above, immediately before the definition of $Z_j$, in the usual sense that it has a particle at any space-time location in which this other one has a particle. The same is true if we consider a somewhat different modification of the contact process on $\mathbb{T}_b^+$, in which until time $ls$ we make the same modifications as before, but from this time on we keep only the sites in $B(0,nl-1)$ free of particles, and we let the system evolve in the remaining vertices with the usual contact process rules. Using these observations, the last display above yields

$$r_{i+1} \geq \mathbb{P}\left(Z_l \geq \frac{(ba)^{nl}}{2}\right)\left(1-(1-r_i)^{\frac{(ba)^{nl}}{2}}\right)(Y_{n,s})^l \geq \frac{\epsilon}{2}a^{nl},$$

where in the second inequality we are using (5.8), the induction hypothesis (5.10), (5.9) and (5.6). This completes the proof of Proposition 5.1.

■

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