The Index Formula for Dirac Operators: an Introduction

Contents

Preface

This set of lecture notes aims to be a leisurely introduction to the impressive collection of mathematical ideas introduced by M. Atiyah and I. Singer in the early sixties, collectively known as 'Spin Geometry', and centering around their celebrated formula for the index of elliptic operators. The text was designed to serve as an introduction to the many excellent monographs available (among these we cite [BGV], [D], [G], [LM], [R], [S], [T]), and the potential reader should be warned that as such it only scratches the surface of this fascinating (and very effective!) theory.

Our style of presentation tends to be sloppy at many points, so as to conform to the introductory character of the notes. In other words, the adjective 'leisurely' above is taken seriously here: as a rule, informal discussions are preferred to technically rounded expositions. We hope that the assiduous reader will be able to fill the inevitable gaps, possibly after consulting more authoritative sources.

Throughout the text emphasis has been given on those aspects of the theory which have immediate applications to Riemannian Geometry. This option dictated our choice of restricting the presentation to twisted Dirac operators over spin manifolds, even though we include in a final chapter a necessarily brief discussion on the index formula for general elliptic operators. This approach reinforces the prominence of Dirac type operators in Atiyah-Singer's theory since besides admitting a direct definition in terms of reasonably familiar geometric objects, their understanding provides a significant, if not essential, step in the formulation and proof of the index formula in the general setting.

The classical Atiyah-Singer index formula for Dirac operators expresses a certain topological invariant of a closed spin manifold, the so-called $\hat{A}$-genus, as the Fredholm index of the associated Dirac operator acting on the space of spinors, thus providing an effective, fruitful link between geometric and analytical aspects of the underlying manifold. Here we follow the modern trend and approach this fundamental formula by using heat equation methods. After the continued efforts of many researchers[1] it is possible by now to present an illuminating proof of this result resting ultimately upon some fairly elementary aspects of the structure of Clifford algebras. In this setting the heat flow associated to the Dirac Laplacian happens to be the interpolating mechanism that allows us to express a manifestly global invariant (the index) as an integral over the manifold of a universal differential form (a characteristic class) locally computable from geometric data (a Riemannian metric). In this regard we have chosen here to follow the method due to E. Getzler (as explained in [BGV] and [R]) as this seems to be the most suitable in a first contact with index theory. We stress that a complete immersion into the subtleties of this approach would certainly make the presentation unnecessarily arid at some points, so we content ourselves with a somewhat detailed sketch of the main ideas involved in the argument.

These notes are organized as follows. In chapters 1-3 we review the basic differential-geometric material, namely, smooth manifolds, Lie groups, principal and vector bundles and the corresponding connections. In Chapter 4, we let the classical Hodge-de Rham theory, which expresses certain topological invariants of manifolds (the Betti numbers) in terms of geometric data, to pose as a warm-up for the general theory. The pertinent representation theory of Clifford algebras and spin groups is described in Chapter 5 and then used in Chapter 6 to construct the spinor bundle and the Dirac operator, under the assumption that the underlying manifold is spin. A more general construction involving twisted Dirac operators appears in Chapter 8, together with the formulation of the index formula; this uses the material on characteristic classes developed in Chapter 7. In Chapter 9, the central theme is the heat flow associated to the Dirac Laplacian. We indicate, following Getzler, how it can be used to implement the 'fantastic cancellations' leading to the proof of the Atiyah-Singer index formula for Dirac oeprators, first established in the early sixties by means of topological tools. We also include in this chapter a heat equation proof of the main theorem in Hodge theory. In Chapter 10 we indicate how Hirzebruch's signature and Chern-Gauss-Bonnet formulas can be deduced by suitably twisting the spinor bundle. We also present here a famous result due to M. Gromov and H. B. Lawson providing a geometric obstruction to the existence of metrics of positive scalar curvature on certain Riemannian manifolds and a brief discussion of the material on four dimensional spin^c structures needed for the formulation of the celebrated Seiberg-Witten equations. Finally, in Chapter 11, we briefly indicate how the

[1]We have chosen not to include detailed information on the historical development of the subject; in this regard, the interested reader should consult the monographs mentioned above and the references therein.

index formula for general elliptic operators follows from the Dirac case via K-theory.

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Chapter 1

Manifolds and Lie groups

This chapter contains an informal review of the foundational material we shall need later. Our presentation is intended basically to fix notation, so that most definitions and proofs are merely sketched and we refer to the many excellent available sources (most notably[1] [GP], [KN], [S] and [W]) for the details.

1.1 Smooth manifolds and de Rham theory

A *manifold* $\mathbf{M}$ is a Hausdorff topological space with countable basis which has been sewn together from pieces of a Euclidean space of a given dimension n. In case the maps providing the sewing (coordinate charts) are diffeomorphisms between open subsets of $\mathbb{R}^n$, we say that $\mathbf{M}$ is a *smooth manifold* of *dimension* n. More generally, if the sewing maps are diffeomorphism between open subsets of closed half-spaces in $\mathbb{R}^n$ then the *boundary* $\partial\mathbf{M}$ of $\mathbf{M}$ is well defined. This is a smooth (boundaryless) manifold of dimension $n-1$.

The gist of the definition is the possibility of transferring the basic calculus constructions from $\mathbb{R}^n$ to this new arena. For example, using the local identifications to $\mathbb{R}^n$ provided by the definition, we can associate to each point $x \in \mathbf{M}$ the *tangent space* $T_x\mathbf{M}$ of tangent vectors to $\mathbf{M}$ at x and $T\mathbf{M}$, the *tangent bundle* of $\mathbf{M}$, is conceived by assembling together all of these tangent spaces:

$$T\mathbf{M} = \bigcup_{x \in \mathbf{M}} T_x\mathbf{M}.$$

[1] Taken together, these are in fact general references for the chapters 1 to 4.

It turns out that $T\mathbf{M}$ is itself a (smooth) manifold of dimension $2n$ and in fact it has a preferred way to be sewn together: $T\mathbf{M}$ is a vector bundle in the sense of Definition 3.1 below with the obvious projection map $\lambda : T\mathbf{M} \to \mathbf{M}$. Now, smooth maps $f : \mathbf{M} \to \mathbf{N}$ between manifolds can be defined and any such map lifts to a map $f_* : T\mathbf{M} \to T\mathbf{N}$, the *derivative map* of f. By definition, f_* preserves the linear structure on the 'fibers' of λ and, for $x \in \mathbf{M}$, $f_*(x) : T_x\mathbf{M} \to T_{f(x)}\mathbf{N}$ is the *derivative* of f at x, which is the best linear approximation to f around x. If $\alpha : (-\varepsilon, \varepsilon) \to \mathbf{M}$ is a curve satisfying $\alpha(0) = p$ and $\alpha'(0) = v$ for some $v \in T_x\mathbf{M}$ then

$$f_*(v) = \frac{d}{dt} f(\alpha(t))|_{t=0},$$

so f_* is a convenient way of recording all the directional derivatives of f. We recall that if a map $f : \mathbf{M} \to \mathbf{N}$ as above is invertible and the inverse map is smooth as well we say that f is a *diffeomorphism*, case in which one should consider $\mathbf{M}$ and $\mathbf{N}$ as equivalent objects.

A map $X : \mathbf{M} \to T\mathbf{M}$ such that $\lambda \circ X = \mathrm{Id}_{\mathbf{M}}$ is called a *vector field* over $\mathbf{M}$ and the space of vector fields over $\mathbf{M}$ is denoted by $\mathcal{X}(\mathbf{M})$. Locally, i.e. in terms of a coordinate chart $x = (x_1, \ldots, x_n)$, a vector field is given by

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i},$$

where the a_i's are smooth functions and $\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$ is the standard pointwise basis for tangent vectors. As the notation suggests, we may think of vector fields as derivations over the space of smooth functions: $X(f) = f_*(X)$. We also denote this by $\mathcal{L}_X f$ and call it the *Lie derivative* of f with respect to X. This point of view makes it natural the definition of the *Lie bracket* of $X, Y \in \mathcal{X}(\mathbf{M})$ by the formula

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

Direct computations imply that the Lie bracket is skew-symmetric ($[X, Y] = -[Y, X]$) and satisfies the *Jacobi identity*

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

Vector fields can be integrated once a point $x \in \mathbf{M}$ is given. There exists then a unique map $\varphi : (a_x, b_x) \to \mathbf{M}$ defined on a maximal open interval containing 0 and satisfying $\varphi(0) = x$ and $\partial\varphi/\partial t = X \circ \varphi$ on (a_x, b_x). We say that (the image of) φ is the *orbit* of X through x. In case we can take $(a_x, b_x) = \mathbb{R}$ for any $x \in \mathbf{M}$, X is said to be *complete*.

We can also consider the space $T_x^*\mathbf{M}$ of all linear maps $T_x\mathbf{M} \to \mathbb{R}$ and define the *cotangent bundle*

$$T^*\mathbf{M} = \bigcup_{x\in\mathbf{M}} T_x^*\mathbf{M}.$$

A rule that to each $x \in \mathbf{M}$ assigns an element in $T_x^*\mathbf{M}$, and does it in a smooth way, is termed a *differential* 1*-form*. Locally, a differential 1-form η is given by

$$\eta = \sum_{i=1}^{n} a_i dx_i,$$

where the a_i's are smooth functions on $\mathbf{M}$ and $\{dx_1, \ldots, dx_n\}$ is the standard pointwise basis for 1-forms dual to $\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$. More generally, for each $0 \leq p \leq n$, we can take advantage of the exterior product operation and consider objects locally defined by expressions like

$$\eta = \sum_I a_I dx_I,$$

where the sum ranges over the set of multi-indices $I = \{1 \leq i_1 < \ldots < i_p \leq n\}$, $1 \leq p \leq n$, and $dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ (here, the wedge means exterior product). These define global objects (in case they transform properly under change of coordinates) which are called *differential p-forms*. The space of differential p-forms on $\mathbf{M}$ is represented by $\mathcal{A}^p(\mathbf{M})$, a vector space under pointwise operations. We also consider $\mathcal{A}(\mathbf{M}) = \oplus_{p=0}^n \mathcal{A}^p(\mathbf{M})$, which is a graded skew-commutative algebra in the sense that

$$\eta \wedge \eta' = (-1)^{pq}\eta' \wedge \eta, \ \ \eta \in \mathcal{A}^p(\mathbf{M}), \ \eta' \in \mathcal{A}^q(\mathbf{M}). \tag{1.1.1}$$

The important notion of orientability can be expressed in terms of differential forms. More precisely, $\mathbf{M}$ is *orientable* if and only if there exists a nowhere vanishing element $\eta \in \mathcal{A}^n(\mathbf{M})$, $n = \dim \mathbf{M}$ (recall[2] that $\dim \Lambda^n\mathbb{R}^n = 1$). An *orientation* for $\mathbf{M}$ is just a choice of such an element.

There is a fundamental linear first order differential operator $d : \mathcal{A}^p(\mathbf{M}) \to \mathcal{A}^{p+1}(\mathbf{M})$, the *exterior differential*, defined locally by

$$d\eta = \sum_I da_I \wedge dx_I. \tag{1.1.2}$$

It is not hard to verify that the composite operator vanishes:

$$d^2 = 0, \tag{1.1.3}$$

[2]If V is a finite dimensional vector space, we denote by $\Lambda^p V$ the space of multilinear alternating maps $\underbrace{V \times \ldots \times V}_{p \text{ times}} \to \mathbb{R}$.

and, as we shall see, this will have a paramount importance in what follows. Moreover, d satisfies the following Leibniz type rule

$$d(\eta \wedge \eta') = d\eta \wedge \eta' + (-1)^p \eta \wedge d\eta', \;\; \eta \in \mathcal{A}^p(\mathbf{M}), \;\; \eta' \in \mathcal{A}^q(\mathbf{M}). \tag{1.1.4}$$

It turns out that n-forms can be integrated over $\mathbf{M}$ (admitting orientability) generating the extended real number

$$\int_{\mathbf{M}} \eta.$$

Naturally, we assume that this (improper) integral converges, and we shall meet this by letting $\mathbf{M}$ to be compact, for example. More generally, if $\eta \in \mathcal{A}^p(\mathbf{M})$ and $f : \mathbf{M}' \to \mathbf{M}$ is a map, we can form $f^*\eta \in \mathcal{A}^p(\mathbf{M}')$, the *pullback* of η under f, and integrate this over $\mathbf{M}'$ in case $\dim \mathbf{M}' = p$. In particular, if $\eta \in \mathcal{A}^{n-1}(\mathbf{M})$ we can make sense of $\int_{\partial \mathbf{M}} \eta$ and the celebrated *Stokes theorem* says that

$$\int_{\partial \mathbf{M}} \eta = \int_{\mathbf{M}} d\eta \tag{1.1.5}$$

in case $\mathbf{M}$ is compact. In particular, if $\mathbf{M}$ is *closed*, i.e. compact and without boundary, we have

$$\int_{\mathbf{M}} d\eta = 0. \tag{1.1.6}$$

Notice that for $f : \mathbf{M}' \to \mathbf{M}$ as above we have the compatibility relations:

$$f^*d = df^*, \;\; f^*(\eta \wedge \eta') = f^*\eta \wedge f^*\eta', \;\; \eta, \eta' \in \mathcal{A}(\mathbf{M}). \tag{1.1.7}$$

At this point we recall that part of the business in Algebraic Topology (see for example [W]) is to associate to a general smooth n-dimensional manifold $\mathbf{M}$ a series of modules, represented by $H_p^{\mathrm{sing}}(\mathbf{M}, \Lambda)$, $p = 0, 1, ..., n$. These are the *singular homology groups* of $\mathbf{M}$ with coefficients in the abelian group Λ.[3] Roughly speaking, the definition is as follows. We consider formal finite linear combinations, with real coefficients, of continuous maps $j : \Delta_p \to \mathbf{M}$, where $\Delta_p \subset \mathbb{R}^{p+1}$ is the standard p-simplex, in order to form the space of singular chains $C_p(\mathbf{M})$. These are connected by boundary homomorphisms $\partial : C_p(\mathbf{M}) \to C_{p-1}(\mathbf{M})$ satisfying a property similar to (1.1.3):

$$\partial^2 = 0,$$

[3] In this section we shall restrict ourselves to the case $\Lambda = \mathbb{R}$ and write simply $H_p^{\mathrm{sing}}(\mathbf{M}) = H_p^{\mathrm{sing}}(\mathbf{M}, \mathbb{R})$.

and one can then define

$$H_p^{\text{sing}}(\mathbf{M}) = \frac{\ker\left(\partial : C_p(\mathbf{M}) \to C_{p-1}(\mathbf{M})\right)}{\operatorname{im}\left(\partial : C_{p+1}(\mathbf{M}) \to C_p(\mathbf{M})\right)},$$

by taking the quotient of the space of *p-cycles* by the space of *p-boundaries*. Continuous maps between topological spaces induce homomorphisms at the singular homology level so that each $H_p(\mathbf{M})$, or more precisely, its dimension $b_p(\mathbf{M})$, the p^{th} *Betti number* of $\mathbf{M}$, is a topological invariant of $\mathbf{M}$.[4]

Now, given that $\mathbf{M}$ is a smooth manifold, we can also consider the *piecewise smooth real singular homology groups* by repeating the above construct with piecewise smooth p-chains $j : \Delta_p \to \mathbf{M}$, but these gives us nothing new since a standard approximation argument implies that the resulting homology theory is naturally isomorphic to the previous one. For us, this has the advantage of being able to represent the singular (topological) homology of $\mathbf{M}$ by means of piecewise smooth cycles, a point of view we shall adopt from now on.

We now observe that (1.1.3) suggests a similar construction at the level of differential forms. More precisely, the sequence of arrows

$$0 \to \mathcal{A}^0(\mathbf{M}) \xrightarrow{d} \mathcal{A}^1(\mathbf{M}) \xrightarrow{d} \dots \xrightarrow{d} \mathcal{A}^n(\mathbf{M}) \to 0, \qquad (1.1.8)$$

the so-called de *de Rham complex*, leads us to consider the *de Rham cohomology groups*

$$H^p_{\text{dR}}(\mathbf{M}) = \frac{\ker\left(d : \mathcal{A}^p(\mathbf{M}) \to \mathcal{A}^{p+1}(\mathbf{M})\right)}{\operatorname{im}\left(d : \mathcal{A}^{p-1}(\mathbf{M}) \to \mathcal{A}^p(\mathbf{M})\right)},$$

by quotienting the space of *closed p-forms* by the subspace of *exact p-forms*. Notice that, by (1.1.7), any $f : \mathbf{M}' \to \mathbf{M}$ induces a homomorfism $f^* : H^p_{\text{dR}}(\mathbf{M}) \to H^p_{\text{dR}}(\mathbf{M}')$ which behaves in the expected way under compositions, so that the de Rham groups are differential-geometric invariants.

Now, the striking (and fundamental) result is *de Rham theorem* to the effect that

$$H^p_{\text{dR}}(\mathbf{M}) \cong H^p_{\text{sing}}(\mathbf{M}),$$

i.e. the de Rham groups are *isomorphic* to the singular cohomology groups of $\mathbf{M}$ (of course, 'cohomology' here means that we have dualized the singular homology groups). In fact, this isomorphism is natural in the sense that it is induced by integration of closed p-forms over p-cycles (or more precisely

[4]From now on we assume that $\mathbf{M}$ is compact as this implies that each $b_p(\mathbf{M})$ is finite. We remark moreover that the Betti numbers are in fact homotopy type invariants of manifolds.

over piecewise smooth approximations of such p-chains). One should emphasize that, as remarked above, in principle the de Rham groups are tied to the differentiable structure of $\mathbf{M}$ and the fact that they recover topological invariants of $\mathbf{M}$ is certainly a cornerstone result in Differential Topology.

This is perhaps the right moment to insert a few comments on Poincaré duality and the ring structure on the total de Rham cohomology

$$H^*_{\mathrm{dR}}(\mathbf{M}) = \oplus_{p=0}^n H^p_{\mathrm{dR}}(\mathbf{M}).$$

In effect, given closed forms $\eta \in \mathcal{A}^p(\mathbf{M})$ and $\eta' \in \mathcal{A}^q(\mathbf{M})$, $\eta \wedge \eta'$ is closed (by (1.1.4)) and, as a simple computation shows, its de Rham cohomology class $[\eta \wedge \eta']$ does depend only on the corresponding cohomology classes $[\eta]$ and $[\eta']$ and not on the particular closed forms representing them. This defines a product

$$H^p_{\mathrm{dR}}(\mathbf{M}) \times H^q_{\mathrm{dR}}(\mathbf{M}) \to H^{p+q}_{\mathrm{dR}}(\mathbf{M}),$$

which extends to a ring structure

$$H^*_{\mathrm{dR}}(\mathbf{M}) \times H^*_{\mathrm{dR}}(\mathbf{M}) \to H^*_{\mathrm{dR}}(\mathbf{M}).$$

Notice that, by (1.1.7), for $f : \mathbf{M}' \to \mathbf{M}'$ the induced homomorphism f^* is a ring homomorphism indeed.

In this respect, a fundamental result that we shall prove in Chapter 9 by entirely analytical methods is *Poincaré duality*, which says that the bilinear pairing

$$H^p_{\mathrm{dR}}(\mathbf{M}) \times H^{n-p}_{\mathrm{dR}}(\mathbf{M}) \to H^n_{\mathrm{dR}}(\mathbf{M}) \cong \mathbb{R}$$

given by

$$(\eta, \eta') \mapsto \int_{\mathbf{M}} \eta \wedge \eta',$$

is non-degenerate (here we assume $\mathbf{M}$ oriented). In particular,

$$b_p(\mathbf{M}) = b_{n-p}(\mathbf{M}). \tag{1.1.9}$$

As a consequence, let the *Euler characteristic* of $\mathbf{M}$ be given by

$$\chi(\mathbf{M}) = \sum_{i=0}^{n} (-1)^i b_i(\mathbf{M}). \tag{1.1.10}$$

In case n is odd one has $\chi(\mathbf{M}) = 0$ by (1.1.9). In case n is even, the celebrated Gauss-Chern-Bonnet formula (see Theorem 10.1.1) expresses $\chi(\mathbf{M})$ as the integral over $\mathbf{M}$ of a canonical geometrically defined n-form. This is an example of an index theorem in Differential Geometry predating by many years the Atiyah-Singer index formula.

We finish this brief survey on the differential topology of manifolds by showing how degree theory follows from Poincaré duality. First, if $\Delta \subset \mathbf{M}$ is a p-cycle, we can associate to any closed p-form ω over $\mathbf{M}$ the real number $\int_\Delta \omega$. By (1.1.5) this descends to a linear map $T_\Delta : H^p_{\mathrm{dR}}(\mathbf{M}, \mathbb{R}) \to \mathbb{R}$ and by Poincaré duality there exists a closed $(n-p)$-form η_Δ, well determined up to cohomology, such that T_Δ is given by integration against η_Δ:

$$T_\Delta(\omega) = \int_{\mathbf{M}} \omega \wedge \eta_\Delta. \tag{1.1.11}$$

We then say that η_Δ (or more precisely its cohomology class) is the *Poincaré dual* of Δ. A special case occurs when it is given a map $f : \mathbf{M} \to \mathbf{N}$ between closed oriented manifolds of dimension n and we choose a generic point $y \in \mathbf{M}$ so that $f^{-1}(y) \subset \mathbf{M}$ is a finite collection of points defining a 0-cycle in $\mathbf{M}$. Its Poincaré dual is an n-form, say η_y, so that if $h : \mathbf{M} \to \mathbb{R}$ we have

$$\int_{\mathbf{M}} h\eta_y \overset{(1.1.11)}{=} \int_{f^{-1}(y)} h = \sum_{x \in f^{-1}(y)} \pm h(x),$$

where the signs come from the fact that in the definition of the 0-cycle above we ascribe to any $x \in f^{-1}(y)$ a sign depending on whether f preserves or not the orientation at the given point. In particular, $h \equiv 1$ defines the *degree* of f by

$$\deg f = \sum_{x \in f^{-1}(y)} \pm 1.$$

It is possible to show that this integer does not depend on the declared choices and is a homotopy invariant of f. Moreover, the following *transformation of variables formula* holds:

$$\int_{\mathbf{M}} f^*\beta = \deg f \int_{\mathbf{N}} \beta, \tag{1.1.12}$$

for any $\beta \in \mathcal{A}^n(\mathbf{N})$.

1.2 Lie groups

Lie groups made their way in Mathematics as a convenient device for the description of continuous symmetries of spaces. Building upon the foundational work of Lie, a compreensive theory has emerged with many ramifications and applications. The purpose of this section is to review the most elementary aspects of Lie group theory in a way that fits to our purposes.

Definition 1.2.1 *A Lie group is a smooth manifold* $\mathbf{G}$ *endowed with a map* $\mu : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ *giving a group structure to* $\mathbf{G}$*. We also assume that the corresponding inversion map* $\nu : \mathbf{G} \to \mathbf{G}$*,* $\nu(g) = g^{-1}$*, is a diffeomorphim.*

Very often we abuse notation and write $\mu(g, g') = gg'$ for $g, g' \in \mathbf{G}$. Also, we denote by e the identity element in $\mathbf{G}$.

In general, given a field $\mathbb{K}$, we denote by $\mathrm{M}_n(\mathbb{K})$ the space of $n \times n$ matrices with entries in $\mathbb{K}$. It follows that $\mathrm{GL}_n(\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R}); \det A \neq 0\}$, the *real general linear group*, is a Lie group of dimension n^2 with $\mu(A, B) = AB$, the usual product of matrices. Clearly, the real line $\mathbb{R}$ and the unit circle $S^1 = \{z \in \mathbb{C}; |z| = 1\}$, with the obvious operations, are Lie groups as well. And we still can form products of these, generating, for example, the *torus* T^n, the direct product of n circles.

Another way of constructing Lie groups is to consider certain subgroups of known Lie groups.

Definition 1.2.2 *A Lie subgroup* $\mathbf{H}$ *of a Lie group* $\mathbf{G}$ *is a subgroup of* $\mathbf{G}$ *which is also a submanifold of* $\mathbf{G}$*. If* $\mathbf{H} = \mathbb{R}$*, we say that the inclusion* $i : \mathbb{R} \to \mathbf{G}$ *defines a one-parameter subgroup of* $\mathbf{G}$*.*

For instance, if $\mathbf{G} \subset \mathrm{GL}_n(\mathbb{R})$ we say that $\mathbf{G}$ is a *linear group*. Even though there exist Lie groups which are not linear groups, most of the Lie groups appearing in geometry can be realized (by their very definition indeed!) as linear groups. We just mention the examples $\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}); \det A = 1\}$, the *special linear group*, $\mathrm{O}_n = \{A \in \mathrm{GL}_n(\mathbb{R}); A^t A = \mathrm{Id}\}$, the *orthogonal group*, and $\mathrm{SO}_n = \{A \in \mathrm{O}_n; \det A = 1\}$, the *rotation group*. As we shall see in Chapter 4, the last example plays a key role in (oriented) Riemannian Geometry.

The following example is very instructive. Consider the torus $T^2 = S^1 \times S^1$ as a plane square whose sides of length 1 have been identified in the usual way. For any $m, n \in \mathbb{R}$, consider the map $H : \mathbb{R} \to T^2$, $H(t) = (mt, nt)$, where we use mod 1 arithmetic. Clearly, each such H is a one-parameter subgroup. Now, if $m/n \in \mathbb{Q}$, it is easy to show the existence of t_0 such that $H(t_0) = H(0)$ so that (the image of) H defines a periodic subgroup of T, in fact a Lie subgroup isomorphic to S^1 (see Definition 1.2.3 below). On the other hand, if $m/n \notin \mathbb{Q}$, one checks that H never closes up and in fact the closure of this subgroup is the whole torus. In particular, the subgroup is not closed. This shows that Lie subgroups can display a very wild behavior when considered as submanifolds.

It is a basic fact however that a closed (in the topological sense) subgroup $\mathbf{H} \subset \mathbf{G}$ is automatically a Lie subgroup of $\mathbf{G}$ (and an embedded submanifold actually). Notice that this assures that the examples $\mathrm{SL}_n(\mathbb{R})$, O_n and SO_n above are Lie groups indeed. Another gallery of examples shows up if we remark that the *complex general linear group* $\mathrm{GL}_n(\mathbb{C}) = \{A \in \mathrm{M}_n(\mathbb{C}); \det A \neq 0\}$ is naturally contained in $\mathrm{GL}_{2n}(\mathbb{R})$ and this allows us to consider the *unitary group* $\mathrm{U}_n = \{A \in \mathrm{GL}_n(\mathbb{C}); \overline{A}^t A = 1\}$ ($\overline{A}^t$ denotes the conjugate transpose of A) and the *special unitary group* $\mathrm{SU}_n = \{A \in \mathrm{U}_n; \det A = 1\}$, which are basic in complex geometry.

An interesting question at this point is: what are the one-parameter subgroups of a given Lie group $\mathbf{G}$? To approach this, we recall the following

Definition 1.2.3 *A Lie group homomorphism is a map $\psi : \mathbf{G} \to \mathbf{G}'$ satisfying $\psi(gg') = \psi(g)\psi(g')$, $g, g' \in \mathbf{G}$. If ψ is a diffeomorphism then we say that it is a Lie group isomorphism.*

Now, the question above becomes the problem of describing the Lie group homomorphisms $\psi : \mathbb{R} \to \mathbf{G}$. This is achieved as follows. Define the maps $R_g : \mathbf{G} \to \mathbf{G}$ and $L_g : \mathbf{G} \to \mathbf{G}$, the *right* and *left translations* by $g \in \mathbf{G}$, respectively, by $L_g(g') = gg'$ and $R_g(g') = g'g$, $g' \in \mathbf{G}$. As can be readily checked, the translations are diffeomorphisms.

Definition 1.2.4 *A vector field $X \in \mathcal{X}(\mathbf{G})$ is said to be left (right) invariant if $L_{g_*}X = X$ ($R_{g_*}X = X$) for any $g \in \mathbf{G}$.*

We remark that the notation $L_{g_*}X = X$, for instance, is a shorthand for $L_{g_*}X_{g'} = X_{gg'}$.

A left invariant vector field is fully determined by its value at a given point $g \in \mathbf{G}$. In particular, if we take $g = e$, the space of left invariant vector fields, denoted by $\mathfrak{g}$, gets identified to $T_e\mathbf{G}$. Since the Lie bracket of two left invariant vector fields is still left invariant (because $L_{g_*}[X, Y] = [L_{g_*}X, L_{g_*}Y] = [X, Y]$), it follows that $[\,,\,]$ induces a Lie algebra structure on $\mathfrak{g}$. We then say that $\mathfrak{g}$ is the *Lie algebra*[5] of $\mathbf{G}$. Formally, a *Lie algebra* is a finite dimensional (real or complex) vector space $\mathfrak{g}$ endowed with a bilinear product $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the conditions:

1. $[X, Y] = -[Y, X]$, $\;X, Y \in \mathfrak{g}$ (skew-symmetry);
2. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$, $\;X, Y, Z \in \mathfrak{g}$ (Jacobi identity).

We see without difficulty that the Lie bracket of elements in the Lie algebra of a linear Lie group is given by the usual matrix bracket: $[A, B] =$

[5]In general, we shall adopt the usual convention of denoting the Lie algebra of a Lie group by the small German version of the string of letters representing the given Lie group.

$AB - BA$. A good exercise at this point is to determine the Lie algebra of all Lie groups listed so far. As a by-product, the dimensions of the corresponding groups can be explicitly computed. For example, SO_n has dimension $n(n-1)/2$ and, after choosing a positive orthonormal basis $\{\mathrm{e}_1, \ldots, \mathrm{e}_n\}$ for $\mathbb{R}^n$, an explicit basis for $\mathfrak{so}_n$ is given by the skew-symmetric transformations $\mathrm{e}_i \wedge \mathrm{e}_j : \mathbb{R}^n \to \mathbb{R}^n$, $i < j$, where for $v, w \in \mathbb{R}^n$,

$$v \wedge w(x) = \langle v, x\rangle w - \langle w, x\rangle v, \quad x \in \mathbb{R}^n. \tag{1.2.13}$$

Since the standard inner product $\langle\, ,\, \rangle$ on $\mathbb{R}^n$ establishes a correspondence between skew-symmetric transformations and skew-symmetric bilinear forms on $\mathbb{R}^n$, this furnishes the canonical isomorphism

$$\mathfrak{so}_n \cong \Lambda^2 \mathbb{R}^n. \tag{1.2.14}$$

It is obvious that if $\mathbf{H} \subset \mathbf{G}$, the Lie algebra $\mathfrak{h}$ of $\mathbf{H}$ is a *Lie subalgebra* of $\mathfrak{g}$ in the sense that the bracket of elements in $\mathfrak{h}$ remains in $\mathfrak{h}$. In particular, the Lie algebra of a one-parameter subgroup of $\mathbf{G}$ is a line in $\mathfrak{g}$ passing through 0.

Here is the answer to the question posed above on one-parameter subgroups.

Proposition 1.2.1 *There is a one-to-one correspondence between one-parameter subgroups of* $\mathbf{G}$ *and orbits of elements of* $\mathfrak{g}$ *(considered as left invariants vector fields) passing through* e.

Proof. Given a one-parameter subgroup $\psi : \mathbb{R} \to \mathbf{G}$, consider $X = \psi'(0) \in \mathfrak{g}$. Then ψ is the orbit through e of X, considered as a left invariant vector field.⌋

Given that left invariant vector fields are always complete, this proposition has as a very important consequence the construction of the *exponential map* $\exp : \mathfrak{g} \to \mathbf{G}$ by $\exp(X) = \psi_X(1)$, where ψ_X is the unique one-parameter subgroup of $\mathbf{G}$ such that $\psi_X'(0) = X$. In the case of linear Lie groups, this is given by the usual series

$$\exp X = \sum_{i=0}^{\infty} \frac{X^i}{i!}. \tag{1.2.15}$$

In general, we see that any one-parameter subgroup ψ is of the form $\psi(t) = \exp tX$, for some $X \in \mathfrak{g}$.

A straightforward computation gives

Proposition 1.2.2 $\exp_*(0) = \mathrm{Id} : \mathfrak{g} \to \mathfrak{g}$. *In particular, there exists a neighborhood* U *of* $0 \in \mathfrak{g}$ *such that* $\exp$ *restricted to* U *is a diffeomorphism onto its image* $V \subset \mathbf{G}$.

The map $\exp|_U$ above is a natural coordinate chart around the identity element of a Lie group, and we see how useful it is when trying to elucidate the relationship between Lie groups and respective Lie algebras. For instance, writing elements $g, g' \in U$ as $g = \exp tX$ and $g' = \exp tY$, one checks (after restricting our attention from now on to linear groups, which allows us to use (1.2.15), and discarding terms of third order or higher in t) that

$$gg' = \exp tX \exp tY = \exp\left(t(X+Y) + \frac{1}{2}t^2[X,Y]\right), \tag{1.2.16}$$

and one sees that the bracket precisely measures the deviation of exp being a homomorphism or, what amounts to be the same, the obstruction to $\mathbf{G}$ being abelian. Moreover, if $\mathbf{G}$ happens to be abelian, then $\mathfrak{g}$ is also abelian in the sense that the bracket is identically zero. As an application of (1.2.16), we have

Proposition 1.2.3 *If $\phi : \mathbf{G} \to \mathbf{G}'$ is a Lie group homomorphism then $\varphi = \phi_*(e) : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra homomorphism in the sense that*

$$\varphi\left([X,Y]_{\mathfrak{g}}\right) = [\varphi(X), \varphi(Y)]_{\mathfrak{g}'}, \quad X, Y \in \mathfrak{g}.$$

Moreover, $\phi(\exp_{\mathbf{G}} X) = \exp_{\mathbf{G}'} \varphi(X)$, $X \in \mathfrak{g}$.

Proof. Obviously, $\phi(\exp_{\mathbf{G}} tX)$ is a one-parameter subgroup of $\mathbf{G}'$ for any $X \in \mathfrak{g}$. So we can write $\phi(\exp_{\mathbf{G}} X) = \exp_{\mathbf{G}'} Y$ for some $Y \in \mathfrak{g}'$ and the last assertion follows since clearly $Y = \varphi(X)$. As for the first one, notice that

$$\phi(\exp_{\mathbf{G}} X \exp_{\mathbf{G}} Y) = \phi(\exp_{\mathbf{G}} X)\phi(\exp_{\mathbf{G}} Y).$$

Now expand using (1.2.16) and compare second order terms.$\rfloor$

If we apply this to $\mathrm{Ad}_g : \mathbf{G} \to \mathbf{G}$, the *adjoint action* by g, given by $\mathrm{Ad}_g(g') = gg'g^{-1}$, we find that

$$\exp(\mathrm{Ad}_g X) = g \exp(X) g^{-1}, \tag{1.2.17}$$

since the *adjoint representation* $\mathrm{Ad}_g = \mathrm{Ad}_{g_*}(e) : \mathfrak{g} \to \mathfrak{g}$, at least for linear groups, is also given by conjugation by g. Starting from (1.2.17), it is not hard to show that

$$\frac{d}{dt}\mathrm{Ad}_{\exp tX} Y|_{t=0} = [X,Y], \tag{1.2.18}$$

providing yet another interpretation for the bracket. By its turn, this has an interesting application we shall explore later. Assume we have a bilinear map $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ which is *invariant* in the sense that $B(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = B(X,Y)$

for any $g \in \mathbf{G}$ and $X, Y \in \mathfrak{g}$. Taking derivatives with respect to g and using (1.2.18) we find that

$$B([Z,X],Y) + B(X,[Z,Y]) = 0, \tag{1.2.19}$$

for any other $Z \in \mathfrak{g}$. We shall need a variant of this in our discussion of Chern-Weil theory in Chapter 7.

We now say a few words about the *Maurer-Cartan form* ϑ of a Lie group $\mathbf{G}$. This is a $\mathfrak{g}$-valued 1-form on $\mathbf{G}$ defined by $\vartheta(X) = {L_{g^{-1}}}_*(X)$ if $X \in T_g\mathbf{G}$. Clearly, ϑ is uniquely determined by the conditions: *i*) ϑ is left invariant in the sense that ${L_g}^*\vartheta = \vartheta$; *ii*) ϑ is the identity on $\mathfrak{g} = T_e\mathbf{G}$. As can be easily checked, if $g : \mathbf{G} \to \mathrm{GL}_n(\mathbb{R})$ denotes the defining embedding of a linear group we have

$$\vartheta = g^{-1}dg. \tag{1.2.20}$$

For us, the importance of ϑ is due to the fact that it determines the non-homogeneous term in the transformation rule for a connection on a principal or vector bundle (see the proof of Proposition 3.2.2).

Left invariant forms also play a central role when discussing integration on Lie groups. Let $\{X_1, \ldots, X_n\}$, $n = \dim \mathbf{G}$, be a basis for $\mathfrak{g}$ and $\{\varpi_1, \ldots, \varpi_n\} \subset \mathfrak{g}^*$ be the corresponding dual basis. Clearly, each ϖ_i extends uniquely as a *left invariant* 1-form, i.e. $L_g^*\varpi_i = \varpi_i$, $g \in \mathbf{G}$. We can then form $\varpi = \varpi_1 \wedge \ldots \wedge \varpi_n$ and this is a left invariant n-form in the sense that $\wedge^n L_g^*\varpi = \varpi$.[6] Clearly, ϖ is everywhere nonzero and uniquely determined up to a constant. Moreover, there exists a Lie group homomorphism $\mu : \mathbf{G} \to \mathbb{R}^+$ such that $\wedge^n R_g^*\varpi = \mu(g)\varpi$. If $\mathbf{G}$ is compact we necessarily have $\mu \equiv 1$ and integration of ϖ over Borelean subsets $U \subset \mathbf{G}$ defines a *bi-invariant* measure dg on $\mathbf{G}$, which means that $dg(U) = dg(L_{g'}(U)) = dg(R_{g'}(U))$ for $g' \in \mathbf{G}$ and any such U. This measure is uniquely determined up to a positive constant and, when normalized to satisfy $dg(\mathbf{G}) = 1$, it is called the *Haar measure* of $\mathbf{G}$.

1.3 Group actions on manifolds

We now turn to actions of Lie groups on manifolds.

Definition 1.3.1 *An action of a Lie group* $\mathbf{G}$ *(or a* $\mathbf{G}$*-action) on a manifold* $\mathbf{M}$ *is a smooth map* $\Psi : \mathbf{G} \times \mathbf{M} \to \mathbf{M}$ *satisfying:*

[6] If $A : V \to V$ is a linear map we define $\wedge^p A : \Lambda^p V \to \Lambda^p V$ on simple elements by $\wedge^p A(v_1 \wedge \ldots \wedge v_p) = Av_1 \wedge \ldots \wedge Av_p$.

1. $\Psi(e,x) = x, \ \ x \in \mathbf{M}$;
2. $\Psi(g, \Psi(g', x)) = \Psi(gg', x), \ \ g, g' \in \mathbf{G}, \ x \in \mathbf{M}$.

We also say that **M** *is a* **G**-*space.*

As usual, we write gx as a shorthand for $\Psi(g,x)$ if no confusion arises. The conditions above can thus be rewritten as

$$ex = e \ \text{ and } \ g(g'x) = (gg')x.$$

The definition can be rephrased as saying that the action Ψ induces a homomorphism $\overline{\Psi} : \mathbf{G} \to \text{Diff}(\mathbf{M})$ by $\overline{\Psi}(g)(x) = gx$. Hence, the concept of action formalizes the idea of transformations groups.

Technically speaking, the above defines a *left* **G**-action on **M** and similarly we can consider *right* actions. Clearly, the definitions are interchangeable since $\tilde{\Psi}(x,g) = g^{-1}x$ defines a right action and which case we take is just a matter of taste.

A special case deserves some attention and in fact will play a prominent role in these notes. If $\mathbf{M} = V$, a (real or complex) vector space, and the **G**-action on V is by *linear* transformations (so that $\overline{\Psi} : \mathbf{G} \to \text{Aut}(V)$) we say that Ψ is a *representation* (or **G**-*representation* if emphasis is needed). For obvious reasons we also reserve the terminology **G**-*module* for V. We say that the representations $\Psi_i : \mathbf{G} \to V_i$, $i = 0, 1$, are *equivalent* if there exists an isomorphism $A : V_0 \to V_1$ such that the diagram

$$\begin{array}{ccc} V_0 & \xrightarrow{\Psi_0(g)} & V_0 \\ A \downarrow & & \downarrow A \\ V_1 & \xrightarrow{\Psi_1(g)} & V_1 \end{array}$$

commutes for any $g \in \mathbf{G}$. A basic question of course is the classification of **G**-representations up to equivalence but we do not have much to say about this here.

A **G**-action on a manifold gives rise to an *infinitesimal representation* of $\mathfrak{g}$ on $\mathcal{X}(\mathbf{M})$ as follows. To each $X \in \mathfrak{g}$ we associate $\hat{X} \in \mathcal{X}(\mathbf{M})$ by the rule

$$\hat{X}_x = \frac{d}{dt} \exp(tX)x|_{t=0}, \ \ x \in \mathbf{M}. \tag{1.3.21}$$

As can be easily checked, this is a linear homomorphism.

Among the many possibilities for a **G**-action, one would like to single out for later reference the following one. A **G**-action on **M** is said to be *free* if for any $x \in \mathbf{M}$, $\mathbf{G}_x = \{g \in \mathbf{G}; gx = x\}$, the *isotropy group* at x, reduces to the identity element. In this case, the rule $X \in \mathfrak{g} \mapsto \hat{X}_x \in T_x\mathbf{M}$, obtained by composing the infinitesimal representation with evaluation at some $x \in \mathbf{M}$, is linear injective. In effect, if $\hat{X}_x = 0$ for some $X \neq 0$ in $\mathfrak{g}$, we would have $(\exp tX)x = x$, for any $t \in \mathbb{R}$, a contradiction to Proposition 1.2.2.

Chapter 2

Principal bundles

Even though not essential for a presentation of the fundamentals of Spin Geometry, principal bundles constitute a natural framework for the discussion of a number of concepts we shall introduce later (spin bundles, characteristic classes, etc.) In this chapter, we review the basic theory of principal bundles with an emphasis toward the geometric aspects.

2.1 The concept of a principal bundle

The local model for a principal bundle is the product $U \times \mathbf{G}$, where $U \subset \mathbb{R}^n$ is open and $\mathbf{G}$ is a Lie group. One should be aware however that a hidden symmetry is present here, namely, the usual action of $\mathbf{G}$ on itself by right translations is trivially extended to $U \times \mathbf{G}$ so that the corresponding orbit space[1] $U \times \mathbf{G}/\mathbf{G}$ is naturally diffeomorphic to U. A principal bundle is then just a bunch of such local models coherently related by suitable transition maps.

Definition 2.1.1 *A principal bundle is given by a manifold P (the total space) upon which a Lie group $\mathbf{G}$ acts freely from the right (represent the action by $R : P \times \mathbf{G} \to P$, say) such that the orbit projection $\pi : P \to P/\mathbf{G}$ defines a manifold $\mathbf{M} = P/\mathbf{G}$ (the base space) and is locally trivial in the sense that for each $x \in \mathbf{M}$ there exists a neighborhood $x \in U \subset \mathbf{M}$ and a map $\phi : \pi^{-1}(U) \to U \times \mathbf{G}$ (a trivializing chart) with the property that $\phi(p) = (\pi(p), \varphi(p))$, $p \in \pi^{-1}(U)$, for*

[1] If $\mathbf{M}$ is a $\mathbf{G}$-space and $x \in \mathbf{M}$, the *orbit* through x is the set (actually a submanifold) $\mathbf{G}x = \{gx; g \in \mathbf{G}\}$. Usually we denote by $\mathbf{M}/\mathbf{G}$ the space of orbits of a $\mathbf{G}$-action on $\mathbf{M}$.

some $\varphi : \pi^{-1}(U) \to \mathbf{G}$ satisfying $\varphi(pg) = \varphi(p)g$, $g \in \mathbf{G}$.[2] *When emphasis on $\mathbf{G}$ is required, we say that P is a principal $\mathbf{G}$-bundle and that $\mathbf{G}$ is the structural group of P.*

A *bundle map* between principal $\mathbf{G}$-bundles over $\mathbf{M}$ is a map $f : P \to P'$ satisfying $f(pg) = f(p)g$, $p \in P$, $g \in \mathbf{G}$. Notice that f induces a map $\overline{f} : \mathbf{M} \to \mathbf{M}$. If $\overline{f} = \mathrm{Id}_{\mathbf{M}}$we say that f is a *bundle isomorphism*. In this case we also say that f performs an *equivalence* between P and P'. If a principal bundle P is equivalent to the usual product bundle $\mathbf{M} \times \mathbf{G}$, we say that P is *trivial.* We remark that, for a general principal bundle P, a local trivializing chart $\phi : \pi^{-1}(U) \to U \times \mathbf{G}$ induces a preferred local section $\sigma : U \to \pi^{-1}(U)$, namely $\sigma(x) = \phi^{-1}(x, e)$, $x \in U$. Conversely, given a local section σ as before, we construct a local trivilization by the rule $\phi(\sigma(x)g) = (x, g)$. In this way, local sections are in one-to-one correspondence with local trivializations. In particular, a principal bundle is trivial if and only if it admits a *global* section. Naturally, a crucial question here is to tell whether a given principal bundle is trivial or not. More generally, one would like to classify principal $\mathbf{G}$-bundles up to equivalence (see Section 6.1 for more on this point).

We shall find it convenient sometimes to restrict principal $\mathbf{G}$-bundles to submanifolds $\mathbf{M}' \subset \mathbf{M}$. This forms a new principal $\mathbf{G}$-bundle $P|_{\mathbf{M}'}$ whose total space is $\pi^{-1}(\mathbf{M}')$. Another useful construction is to start with a principal $\mathbf{G}$-bundle over $\mathbf{M}$ and to consider a Lie subgroup $\mathbf{G}' \subset \mathbf{G}$. We say that a principal $\mathbf{G}'$-bundle P' over $\mathbf{M}$ is a *reduction* of P if there is an embedding $P' \overset{i}{\hookrightarrow} P$ satisfying $i(pg) = i(p)g$, $g \in \mathbf{G}'$, and inducing an identification $P'/\mathbf{G}' \cong \mathbf{M}$.

Very often it is instructive, and sometimes even mandatory, to define a principal bundle by means of its transition functions. Consider a principal $\mathbf{G}$-bundle $\pi : P \to \mathbf{M}$ and assume that $\mathbf{M}$ admits a covering by open sets

$$\mathbf{M} = \bigcup_{\alpha \in \Lambda} U_\alpha$$

such that $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbf{G}$ is a trivializing chart. Recall that associated to each ϕ_α we have the corresponding preferred local section σ_α given by $\sigma_\alpha(x) = \phi_\alpha^{-1}(x, e)$, $x \in U_\alpha$. Now if $U_\alpha \cap U_\beta \neq \emptyset$, we define the corresponding *transition function* $\xi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbf{G}$ by the rule

$$\sigma_\alpha(x)\xi_{\alpha\beta}(x) = \sigma_\beta(x), \;\; x \in U_\alpha \cap U_\beta. \tag{2.1.1}$$

If $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, these clearly satisfy the *cocycle condition*

$$\xi_{\alpha\beta}(x)\xi_{\beta\gamma}(x) = \xi_{\alpha\gamma}(x), \;\; x \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{2.1.2}$$

[2]Here we abuse notation so that $R(p, g) = pg$. Notice also that we reserve the same notation both for the right action of $\mathbf{G}$ on P and the action of $\mathbf{G}$ on itself by right translations.

Conversely, it is not hard to show that the transition functions, together with the additional conditions (2.1.2), completely determine P up to equivalence.

The notion of reduction introduced above can be rephrased in terms of transition functions. In effect, a reduction of P (with structural group $\mathbf{G}$) to P' (with structural group $\mathbf{G}'$) just means that the transition functions of P can be chosen to take values in $\mathbf{G}'$. In particular, we see that reduction is a strictly topological issue having to do with the way $\mathbf{G}'$ sits inside $\mathbf{G}$.

2.2 Connections on principal bundles

We now consider the geometric aspects of principal bundles. It follows immediately from the definition of a principal $\mathbf{G}$-bundle that each fiber $\pi^{-1}(x)$, $x \in \mathbf{M}$, being an orbit of the free action of $\mathbf{G}$ on P, is diffeomorphic to $\mathbf{G}$ so that P is foliated by copies of $\mathbf{G}$, the leaves of the *vertical foliation*. If $p \in P$ we denote by $\mathbf{G}^{(p)}$ the fiber containing p.

We now discuss an important class of vector fields over the total space of a principal bundle. To each $X \in \mathfrak{g}$, we can associate a vector field $\hat{X}$ over P by the recipe described in (1.3.21). Clearly, $\hat{X}$ is tangent to the vertical foliation everywhere and it is called a *canonical vertical vector field*.[3] Moreover, we have

Proposition 2.2.1 *For each $p \in P$, the map*

$$X \in \mathfrak{g} \mapsto \hat{X}_p \in T_p\mathbf{G}^{(p)}$$

is a linear isomorphism.

Proof. This follows from the fact that the $\mathbf{G}$-action on P is free (see the discussion at the end of Section 1.3).⌋

Thus, passing from $\mathbf{G}^{(p)}$ to its tangent space $\mathcal{V}_p$, we obtain a canonical integrable (in the sense of Frobenius) distribution $\mathcal{V}$ on P. Now, the extra piece of structure needed to implement geometry on P is given by

Definition 2.2.1 *A connection on P is a $\mathbf{G}$-invariant choice of a complementary distribution to $\mathcal{V}$. More precisely, a connection is defined by prescribing a smooth family $p \in P \mapsto \mathcal{K}_p \subset T_pP$ of subspaces such that:*

1. $T_pP = \mathcal{V}_p \oplus \mathcal{K}_p$ *(direct sum);*
2. $R_{g_*}(\mathcal{K}_p) = \mathcal{K}_{pg}$, $g \in \mathbf{G}$, $p \in P$.

[3]In general, if a vector field Y on P is tangent everywhere to the vertical foliation, we say simply that Y is *vertical*.

We remark that connections *always* exist. Since we are not going to use this here, we refer to [KN] for a proof.

The above definition is not very satisfactory from a computational viewpoint and a reformulation is in order. Thus, given a connection $\mathcal{K}$ as above, we define a $\mathfrak{g}$-valued 1-form on P by

$$\begin{cases} \omega(X) &= X, \quad X \in \mathcal{V}_p, \\ \omega(X) &= 0, \quad X \in \mathcal{K}_p, \end{cases}$$

where we use the identification provided by Proposition 2.2.1. It follows that

$$\mathcal{K} = \ker \omega \tag{2.2.3}$$

and condition (2) in the definition of $\mathcal{K}$ gets replaced by

$$R_g{}^*\omega = \mathrm{Ad}_{g^{-1}}\omega, \tag{2.2.4}$$

as the following computation shows:

$$\begin{aligned}
(R_g^*\omega)_p(\hat{X}) &= \omega_{pg}(R_{g_*}(\hat{X})) \\
&= \omega_{pg}\left(R_{g_*}\left(\frac{d}{dt}p\exp(tX)|_{t=0}\right)\right) \\
&= \omega_{pg}\left(\frac{d}{dt}\left(p\exp(tX)g\right)|_{t=0}\right) \\
&= \omega_{pg}\left(\frac{d}{dt}\left(pgg^{-1}\exp(tX)g\right)|_{t=0}\right) \\
&\overset{(1.2.17)}{=} \omega_{pg}\left(\frac{d}{dt}\left(pg\exp(t\mathrm{Ad}_{g^{-1}}X)\right)|_{t=0}\right) \\
&= \omega_{pg}\left(\left(\widehat{\mathrm{Ad}_{g^{-1}}X}\right)_{pg}\right) \\
&= \mathrm{Ad}_{g^{-1}}X \\
&= \mathrm{Ad}_{g^{-1}}\left(\omega(\hat{X})\right).
\end{aligned}$$

We conclude that giving a connection is equivalent to defining a $\mathfrak{g}$-valued 1-form over P satisfying (2.2.4) above, and the relationship between the two definitions is made explicit by (2.2.3). Note that, under the identification $\pi^{-1}(U) = U \times \mathbf{G}$ provided by a trivializing chart, the connection ω gets identified to the Maurer-Cartan form ϑ of $\mathbf{G}$. This is because (2.2.4) becomes $L_{g_*}\omega = \omega$ and the way ω is defined on vertical vector fields.

In order to illustrate the advantages of this more computational way of looking at a connection, let us prove

Proposition 2.2.2 *Let $\pi : P \to \mathbf{M}$ be a principal $\mathbf{G}$-bundle, $\alpha : [0,1] \to \mathbf{M}$ a (piecewise smooth) curve and $p \in P$ such that $\pi(p) = \alpha(0)$. Then there exists a unique horizontal lift $\tilde{\alpha} : [0,1] \to P$ starting at p, i.e. a unique $\tilde{\alpha}$ such that:*

1. $\tilde{\alpha}(0) = p$;
2. $\tilde{\alpha}'(t) \in \mathcal{K}_{\tilde{\alpha}(t)}$, $t \in [0,1]$;
3. $\pi \circ \tilde{\alpha} = \alpha$.

Proof. It suffices to prove this in case $\alpha([0,1]) \subset U$, where U is associated to a trivializing map as in Definition 2.1.1. Pick any lift β of α starting at p (i.e. any β satisfying $\beta(0) = p$ and $\pi \circ \beta = \alpha$) and try to find $\tilde{\alpha}$ by

$$\tilde{\alpha} = \beta g,$$

for some $g : [0,1] \to \mathbf{G}$ with $g(0) = e$. It then follows that

$$\tilde{\alpha}' = \beta' g + \beta g'. \tag{2.2.5}$$

But

$$\beta' g = R_{g_*}(\beta')$$

and

$$\beta g' = \tilde{\alpha} g^{-1} g' = (\widehat{L_{g^{-1}}{}_* g'})_{\tilde{\alpha}},$$

and from this we get

$$\omega(\beta' g) = \omega(R_{g_*}(\beta')) = R_g^* \omega(\beta') = \mathrm{Ad}_{g^{-1}}(\omega(\beta')) \tag{2.2.6}$$

and

$$\omega(\beta g') = \omega\left((\widehat{L_{g^{-1}}{}_*(g')})_{\tilde{\alpha}} \right) = L_{g^{-1}}{}_*(g'). \tag{2.2.7}$$

But $\tilde{\alpha}$ is doomed to be horizontal and then (2.2.5), (2.2.6) and (2.2.7) imply

$$0 = \omega(\tilde{\alpha}') = \mathrm{Ad}_{g^{-1}}(\omega(\beta')) + L_{g^{-1}}{}_*(g'),$$

which can be rewritten as

$$g' = -R_{g_*}\left(\omega(\beta')\right),$$

an ordinary differential equation on $\mathbf{G}$ which can be shown to be uniquely solved for $g : [0,1] \to \mathbf{G}$ with initial condition $g(0) = e$.⌋

We say that $\tilde{\alpha}$ is the *parallel transport* of p along α. Later on, we shall relate this to the more familiar notion of parallel transport in Riemannian Geometry.

Thus we end up with still another way of looking at a connection on P. It is just a $\mathbf{G}$-equivariant way of lifting paths from $\mathbf{M}$ to P, as it follows from the uniqueness in the construction above that if $pg = p'$ then the lift of α starting at p' is $\tilde{\alpha} g$.

Chapter 3

Vector bundles

The standard approach to Spin Geometry can be briefly described as a quest for both canonical vector bundles over a Riemannian manifold $\mathbf{M}$ and canonical differential operators acting on the space of sections of such bundles. In this setting, the definition of the differential operators involves consideration of an extra piece of geometric data, namely, a connection. In this chapter we present the basic material on vectors bundles and connections and relate this to the concepts introduced in the last chapter (principal bundles and corresponding connections).

3.1 The concept of a vector bundle

Roughly speaking, a vector bundle is a family of linear objects (vector spaces) parameterized by a nonlinear object (a manifold). The definition is as follows.

Definition 3.1.1 *A real vector bundle over a manifold $\mathbf{M}$ is given by a manifold $\mathcal{E}$ (the total space) together with a surjection $\lambda : \mathcal{E} \to \mathbf{M}$ satisfying the following local trivializing condition. For each $x \in \mathbf{M}$ there exists an open neighborhood U of x in $\mathbf{M}$ and a diffeomorphism (a trivializing chart) $\eta : \lambda^{-1}(U) \to U \times \mathbb{R}^r$ preserving the fibred structure (i.e. η takes $\mathcal{E}_x = \lambda^{-1}(x)$ to $\{x\} \times \mathbb{R}^r$ diffeomorphically) and inducing a well defined linear structure on the fibers. This last sentence means that if $x \in U_\alpha \cap U_\beta$ for trivializing charts $\eta_\alpha : \lambda^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$ and $\eta_\beta : \lambda^{-1}(U_\beta) \to U_\beta \times \mathbb{R}^r$ then the induced map on fibers $\eta_\alpha \circ \eta_\beta^{-1}(x) : \{x\} \times \mathbb{R}^r \to \{x\} \times \mathbb{R}^r$ is a linear isomorphism. The integer r apearing here is called the rank of $\mathcal{E}$, denoted* $\mathrm{rank}(\mathcal{E})$.

By replacing $\mathbb{R}$ by $\mathbb{C}$ in the above definition, we come up with the important concept of a *complex* vector bundle. In both cases, if $\mathrm{rank}(\mathcal{E}) = 1$ we say that $\mathcal{E}$ is a *line bundle*.

A *bundle map* between vector bundles $\mathcal{E} \to \mathbf{N}$ and $\mathcal{E}' \to \mathbf{M}'$ is a map $f : \mathcal{E} \to \mathcal{E}'$ that preserves the fibred structure and is linear when restricted to the fibers. If $\mathbf{M} = \mathbf{M}'$, $f|_{\mathbf{M}} = \mathrm{Id}_{\mathbf{M}}$ and f is an isomorphism along the fibers we say that $\mathcal{E}$ and $\mathcal{E}'$ are *isomorphic*. In particular, if $\mathcal{E}$ is isomorphic to the product bundle $\mathbf{M} \times \mathbb{R}^r$ we say that $\mathcal{E}$ is *trivial*. If this happens to be the case for $\mathcal{E} = T\mathbf{M}$, then we say that $\mathbf{M}$ is *parallelizable*.

Triviality of a vector bundle can be rephrased as follows. A *section*[1] of $\mathcal{E}$ is a map $\sigma : \mathbf{M} \to \mathcal{E}$ such that $\lambda \circ \sigma = \mathrm{Id}_{\mathbf{M}}$. Then we have that $\mathcal{E}$ is trivial if and only if $\mathcal{E}$ admits r sections linearly independent at each point of $\mathbf{M}$. It is immediate from this that any Lie group is parallelizable. As we shall see in Chapter 7, the theory of characteristic classes provides a systematic way of detecting how far from trivial a given vector bundle is.

As in the case of principal bundles, vector bundles can also be described in terms of transition functions. For an overlap of trivializing charts as in the definition, we conceive $\tau_{\alpha\beta}(x)$ for $x \in U_\alpha \cap U_\beta$ as being the linear transformation induced by $\eta_\alpha \circ \eta_\beta^{-1}$, so that $\tau_{\alpha\beta}$ takes values in $\mathrm{GL}_r(\mathbb{R})$. These transition functions also satisfy corresponding cocycle conditions and determine the vector bundle up to isomorphism. We can also restrict a given vector bundle $\mathcal{E} \to \mathbf{M}$ to a submanifold $\mathbf{M}' \subset \mathbf{M}$ so that the new bundle is denoted by $\mathcal{E}|_{\mathbf{M}'}$. On the other hand, if there exists a Lie group $\mathbf{G} \subset \mathrm{GL}_r(\mathbb{R})$ such that the transition functions of $\mathcal{E}$ can be chosen to take value in $\mathbf{G}$, we say that $\mathcal{E}$ admits a *reduction* to $\mathbf{G}$. For example, by definition $\mathcal{E}$ is *orientable* if it admits a reduction to $\mathrm{GL}_r^+(\mathbb{R})$, the group of $r \times r$ real matrices with *positive* determinant. From a geometric viewpoint, a more interesting example of group reduction is given by

Definition 3.1.2 *A Riemannian metric on $\mathcal{E}$ is a rule that to each $x \in \mathbf{M}$ assigns an inner product $\langle\,,\,\rangle_x$ on the fiber $\mathcal{E}_x$ depending smoothly on x. A vector bundle endowed with a specific Riemannian metric is termed a Riemannian bundle. In case $\mathcal{E}$ is a complex vector bundle and each $\langle\,,\,\rangle_x$ is a hermitian product on $\mathcal{E}_x$ we will say that $\mathcal{E}$ is a hermitian bundle.*

Riemmanian or hermitian metrics can always be constructed (just first do it locally by using a trivializing chart and then splice together the results via a partition of unity). In the real case, it then follows that any vector bundle can be reduced to O_r. Assuming further that $\mathcal{E}$ is orientable, we get

[1] As usual, the space of sections of $\mathcal{E}$ is represented by $\Gamma(\mathcal{E})$.

a reduction to SO_r. In general, if a Riemannian metric is introduced on $T\mathbf{M}$, the tangent bundle of $\mathbf{M}$, we can integrate the pointwise inner product of sections $\eta, \eta' \in \Gamma(\mathcal{E})$ with respect to the volume element $d\mathbf{M}$ induced by the metric in order to obtain the L^2 *product* of sections

$$(\eta, \eta') = \int_{\mathbf{M}} \langle \eta, \eta' \rangle \, d\mathbf{M}. \tag{3.1.1}$$

In general, we will represent by $\| \ \|$ the corresponding seminorm on $\Gamma(\mathcal{E})$.

We remark that the usual linear algebra constructions for vector spaces carry over naturally to the category of vector bundles (both real and complex). In this way, if vector bundles $\mathcal{E}$ and $\mathcal{E}'$ over $\mathbf{M}$ are given, we can form for instance their *direct sum* $\mathcal{E} \oplus \mathcal{E}'$, their *tensor product* $\mathcal{E} \otimes \mathcal{E}'$, the *homomorphism bundle* $\mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ and the *determinant bundle* $\Lambda^r \mathcal{E}$, $r = \mathrm{rank}(\mathcal{E})$. The canonical isomorphisms associated to the pointwise constructions translate into corresponding isomorphisms for vector bundles. Thus, $\mathrm{Hom}(\mathcal{E}, \mathcal{E}') = \mathcal{E}^* \otimes \mathcal{E}'$, where $\mathcal{E}^* = \mathrm{Hom}(\mathcal{E}, \underline{\mathbb{R}})$ is the *dual bundle*.[2] If we start, for instance, with the tangent bundle $T\mathbf{M}$ we can dualize to obtain $T^*\mathbf{M}$, the cotangent bundle of $\mathbf{M}$. If we take the tensor product of r copies of $T\mathbf{M}$ and s copies of $T^*\mathbf{M}$, we end up with the bundle $\otimes^{(r,s)}(\mathbf{M})$, whose sections are tensors of type (r, s) over $\mathbf{M}$. And by the use of the antisymmetrization operation, we construct, for $p = 0, 1, \ldots, n$, the *exterior bundle* $\Lambda^p(\mathbf{M})$, whose sections are precisely the differential p-forms over $\mathbf{M}$. More generally, in the presence of an arbitrary vector bundle, we can consider $\Lambda^p(\mathbf{M}) \otimes \mathcal{E}$, the *bundle of p-forms with coefficients in* $\mathcal{E}$. The space of sections of $\Lambda^p(\mathbf{M}) \otimes \mathcal{E}$ is represented by $\mathcal{A}^p(\mathbf{M}; \mathcal{E})$, and notice that we can identify $\mathcal{A}^0(\mathbf{M}; \mathcal{E})$ to $\Gamma(\mathcal{E})$. In case $\mathcal{E} = \underline{V}$ for a fixed space V, we write $\mathcal{A}^p(\mathbf{M}; V) = \mathcal{A}^p(\mathbf{M}; \mathcal{E})$. For example, the $\mathfrak{g}$-valued 1-form defining a connection on a principal $\mathbf{G}$-bundle belongs to $\mathcal{A}^1(P; \mathfrak{g})$.

We complement this brief discussion on vector bundles by indicating yet another way of constructing new vector bundles out of old ones: the *pullback bundle* $f^*\mathcal{E}$ of a vector bundle $\mathcal{E} \to \mathbf{M}$ under a map $f : \mathbf{M}' \to \mathbf{M}$. Roughly speaking, this is the vector bundle over $\mathbf{M}'$ whose fiber over $x \in \mathbf{M}'$ is precisely $\mathcal{E}_{f(x)}$. This kind of construction plays a central role in the theory of characteristic classes and, as a matter of fact, in the classification theory of vector bundles.

A rather instructive exercise here is to reformulate all the above constructions in terms of transition functions. Finally, we remark that it is not hard to introduce natural metrics on all derived bundles above once metrics on $\mathcal{E}$, $\mathcal{E}'$, etc., are given.

[2] In general, if V is a vector space, we denote by $\underline{V}$ the trivial bundle over $\mathbf{M}$ with typical fiber equal to V.

3.2 Connections on vector bundles

Unless a vector bundle $\mathcal{E} \to \mathbf{M}$ is trivial, there is in general no preferred way to identify the various fibers of $\mathcal{E}$. In order to overcome this, some sort of extra structure is required. This is usually accomplished by first prescribing a way to differentiate sections of $\mathcal{E}$ and then performing the identification along curves in $\mathbf{M}$ by the use of sections with null derivative. This gives rise both to the geometric notion of parallel transport and its analytical counterpart described in the definition below.

Definition 3.2.1 *A connection on a vector bundle $\mathcal{E}$ over $\mathbf{M}$ is a linear first order differential operator $\nabla : \mathcal{A}^0(\mathbf{M};\mathcal{E}) \to \mathcal{A}^1(\mathbf{M};\mathcal{E})$ satisfying the Leibniz rule*

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma, \tag{3.2.2}$$

where $\sigma \in \Gamma(\mathcal{E})$ and f is a function on $\mathbf{M}$.

Recalling that $\mathcal{A}^1(\mathbf{M};\mathcal{E}) = \Gamma(T^*\mathbf{M} \otimes \mathcal{E}) = \Gamma(\mathrm{Hom}(T\mathbf{M},\mathcal{E}))$, we obtain for any $X \in \mathcal{X}(\mathbf{M})$ the *covariant derivative* $\nabla_X : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ with (3.2.2) replaced by

$$\nabla_X(f\sigma) = X(f)\sigma + f\nabla_X\sigma, \quad \nabla_{fX}\sigma = f\nabla_X\sigma.$$

Either way of looking at a connection has its own advantages and we switch from one to another as convenience demands.

Definition 3.2.2 *If $\mathcal{E}$ is Riemannian, then we say that ∇ is compatible with the metric if*

$$X\langle\sigma_1,\sigma_2\rangle = \langle\nabla_X\sigma_1,\sigma_2\rangle + \langle\sigma_1,\nabla_X\sigma_2\rangle, \ \ X \in \mathcal{X}(\mathbf{M}), \sigma_1,\sigma_2 \in \Gamma(\mathcal{E}). \tag{3.2.3}$$

Assume from this point on that $\mathcal{E}$ is an oriented Riemannian vector bundle endowed with a compatible connection ∇. Let $\mathrm{e} = (\mathrm{e}_1,\ldots,\mathrm{e}_r)$ be a local pointwise positive orthonormal basis (or simply a *frame*) defined over $U \subset \mathbf{M}$. If we take into account the principal SO_r-bundle $P_{\mathcal{E}}^{\mathrm{SO}}$ over $\mathbf{M}$ formed by the set of all frames over $\mathbf{M}$ (we call this the *frame bundle* of $\mathcal{E}$) then we can regard e as a local section (and hence a local trivialization) of $P_{\mathcal{E}}^{\mathrm{SO}}$. Now, any local section σ of $\mathcal{E}$ over U can be uniquely written as

$$\sigma = \sum_{i=1}^{r} \sigma_i \mathrm{e}_i, \tag{3.2.4}$$

for some functions σ_i. Defining the *connection* 1*-form* $\theta = \{\theta_{ij}\}$ (with respect to e) by

$$\nabla \mathrm{e}_i = \sum_{j=1}^{r} \theta_{ij} \otimes \mathrm{e}_j, \tag{3.2.5}$$

it follows from (3.2.4) and the Leibniz rule that

$$\nabla\sigma = \sum_{i=1}^{r}(d\sigma_j + \sum_{j=1}^{r}\sigma_i\theta_{ij}) \otimes \mathrm{e}_j. \tag{3.2.6}$$

In other words, the connection ∇ is locally completely determined by the connection form θ. Notice that as a result of the compatibility, we have the additional relations

$$\theta_{ij} = -\theta_{ji}. \tag{3.2.7}$$

As an outcome of (3.2.6) and (3.2.7) we have

Proposition 3.2.1 *Locally, any connection ∇ on a vector bundle $\mathcal{E}$ can be written in the form*

$$\nabla = d + A, \tag{3.2.8}$$

where d is the usual derivative (acting on r-tuples of functions) and A is an $\mathfrak{so}_r$-valued endomorphism acting on the fibers of $\mathcal{E}$.

The relations (3.2.7) just mean that $\theta \in \mathcal{A}^1(U, \mathfrak{so}_r)$ and this suggests a way to define a connection ω on $P_{\mathcal{E}}^{\mathrm{SO}}$ (in the sense of Definition 2.2.1) starting from θ. The next proposition addresses this point and clarifies the relationship between the two notions of connections introduced so far.

Proposition 3.2.2 *Given a connection ∇ on a vector bundle $\mathcal{E}$ over $\mathbf{M}$, there exists a unique connection ω on $P_{\mathcal{E}}^{\mathrm{SO}}$ such that for any local choice of frame $\mathrm{e} = (\mathrm{e}_1, \ldots, \mathrm{e}_r)$ for $\mathcal{E}$ as above we have*

$$\theta = \mathrm{e}^*\omega,$$

where we regard $\omega \in \mathcal{A}^1(P; \mathfrak{so}_r)$ and θ is defined by (3.2.5).

Proof. Since obviously ω is locally defined, this is just a matter of comparing the transformation rules for θ and (the sought for) ω under a change of frames (see [LM]). We start with θ. Let $\bar{\mathrm{e}} = (\bar{\mathrm{e}}_1, \ldots, \bar{\mathrm{e}}_r)$ be another local frame over U related to e by the transformation rule

$$\mathrm{e} = \bar{\mathrm{e}}\,\xi, \tag{3.2.9}$$

for some map $\xi : U \to \mathrm{SO}_r$. As the notation makes it transparent, ξ is precisely the transition function of $P_{\mathcal{E}}^{\mathrm{SO}}$ associated to the change of frames (3.2.9). For the sake of brevity, we rewrite (3.2.5) as $\nabla\mathrm{e} = \theta\mathrm{e}$, and taking into account that $\bar{\mathrm{e}}$ defines a new connection form $\overline{\theta}$ by $\nabla\bar{\mathrm{e}} = \overline{\theta}\bar{\mathrm{e}}$ we compute

$$\theta\mathrm{e} = \nabla\mathrm{e} = \nabla(\bar{\mathrm{e}}\,\xi) = \nabla\bar{\mathrm{e}}\,\xi + \bar{\mathrm{e}}\,d\xi = \overline{\theta}\bar{\mathrm{e}}\,\xi + \bar{\mathrm{e}}\,d\xi.$$

After left multiplication by ξ^{-1} this gives

$$\theta = \xi^{-1}\overline{\theta}\xi + \xi^{-1}d\xi, \tag{3.2.10}$$

and we just have to check that ω transforms exactly as (3.2.10) under (3.2.9).

Now, recall that e defines a trivialization $\phi : \pi^{-1}(U) \to U \times \mathbf{G}$ by the rule $\phi(\mathrm{e}(x)g) = (x,g)$ where $g = \varphi : U \to \mathbf{G}$. We claim that, under this identification,

$$\omega = \mathrm{Ad}_{\varphi^{-1}}(\mathrm{e}^*\omega) + \varphi^{-1}d\varphi, \tag{3.2.11}$$

where, in the righthand side, $\mathrm{e}^*\omega$ denotes the restriction of ω to $\{(x,e); x \in U\}$. This follows by breaking ω into two pieces, say $\omega = \omega_0 + \omega_1$, according to the product structure of $U \times \mathbf{G}$. Since ω equals the Maurer-Cartan form of $\mathbf{G}$ along the fibers, we have $\omega_1 = \varphi^{-1}d\varphi$ (cf. 1.2.20). One the other hand, $\omega_0 = R_g^*(\mathrm{e}^*\omega) = \mathrm{Ad}_{\varphi^{-1}}(\mathrm{e}^*\omega)$ by (2.2.4). Similarly, using the local trivialization induced by $\bar{\mathrm{e}}$ (and self-explanatory notation) we have

$$\omega = \mathrm{Ad}_{\overline{\varphi}^{-1}}(\bar{\mathrm{e}}^*\omega) + \overline{\varphi}^{-1}d\overline{\varphi}. \tag{3.2.12}$$

By equating the right hand sides of (3.2.11) and (3.2.12), applying Ad_φ to the resulting equation and simplifying, it follows at last that

$$\mathrm{e}^*\omega = \mathrm{Ad}_{\xi^{-1}}(\bar{\mathrm{e}}^*\omega) + \xi^{-1}d\xi, \tag{3.2.13}$$

since $\xi = \overline{\varphi}\varphi^{-1}$. Now compare this with (3.2.10).⌋

Notice that the non-homogeneous term $\xi^{-1}d\xi$ in (3.2.10) matches with the fact that connections are not tensors. However, as soon as transformations rules are concerned, it is relatively straightforward to build up a more manageable object out of ω: we simply put

$$\Omega = d\omega + \omega \wedge \omega, \tag{3.2.14}$$

where matrix multiplication of 1-forms is implicit. One checks without difficulty that $\Omega \in \mathcal{A}^2(P; \mathfrak{so}_r)$ so defined transforms under (3.2.9) in a homogeneous way:

$$\Omega = \xi^{-1}\overline{\Omega}\xi. \tag{3.2.15}$$

We call $\Omega = \{\Omega_{ij}\}$ the *curvature 2-form* associated to ω. Pulling this back to $\mathbf{M}$ by the use of frames we obtain a family of 2-forms

$$\Theta = \mathrm{e}^*\Omega \in \mathcal{A}^2(U, \mathfrak{so}_r), \tag{3.2.16}$$

satisfying the corresponding compatibility condition

$$\Theta = \xi^{-1}\overline{\Theta}\xi. \tag{3.2.17}$$

In order to relate this to the connection, prolong ∇ to a two-step map

$$\mathcal{A}^0(\mathbf{M};\mathcal{E}) \overset{\nabla}{\to} \mathcal{A}^1(\mathbf{M};\mathcal{E}) \overset{\tilde{\nabla}}{\to} \mathcal{A}^2(\mathbf{M};\mathcal{E}), \tag{3.2.18}$$

where

$$\tilde{\nabla}(\alpha \otimes \sigma) = d\alpha \otimes \sigma - \alpha \wedge \nabla\sigma, \;\; \alpha \otimes \sigma \in \mathcal{A}^1(\mathbf{M};\mathcal{E}).$$

Denoting the composition in (3.2.18) by R, we easily compute that

$$R(f\sigma) = fR(\sigma),$$

for a function f, so that R is a tensor, the so-called *curvature tensor* associated to ∇. As expected, we have

Proposition 3.2.3 *One has* $R\mathrm{e}_i = \sum_{j=1}^r \Theta_{ij} \otimes \mathrm{e}_j$, *where* $\Theta_{ij} = \mathrm{e}^*\Omega_{ij}$. *Moreover, if for* $X, Y \in \mathcal{X}(M)$, $R_{X,Y}$ *denotes the skew-symmetric transformation on fibers defined by*

$$R_{X,Y}\mathrm{e}_i = \sum_{j=1}^r \Theta_{ij}(X,Y)\mathrm{e}_j$$

then

$$R_{X,Y}\sigma = \nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma - \nabla_{[X,Y]}\sigma, \;\; \sigma \in \Gamma(\mathcal{E}).$$

The proof is a simple computation. Notice that this shows that R coincides with the usual notion of curvature of ∇, which measures the deviation from commutativity of the covariant derivative. Also, we can regard $R \in \mathcal{A}^2(\mathbf{M}, \mathfrak{so}(\mathcal{E}))$, where $\mathfrak{so}(\mathcal{E})$ is the vector bundle whose fiber over $x \in \mathbf{M}$ is the space of skew-symmetric linear transformations of $\mathcal{E}_x$.

We now use the formalism above in order to establish a preliminary link between geometric and topological aspects of a vector bundle. This is just a glance at deeper results considered in Chapter 7, where a more systematic study of the profound relationship between topological and geometric properties of a vector bundle is presented. But first we need to reinterpret the concept of parallel transport on $P_{\mathcal{E}}^{\mathrm{SO}}$ (with respect to ω) in terms of the more down-to-earth notion of parallel transport on $\mathcal{E}$. Start with a curve α on $\mathbf{M}$ and let $p \in P_{\mathcal{E}}^{\mathrm{SO}}$ be such that $\pi(p) = \alpha(0)$. Let $\tilde{\alpha}$ be the parallel transport (with respect to ω) of p along α. Equivalently, $\tilde{\alpha}$ defines a frame along α. Without loss of generality, we may assume the existence of a frame e defined in a neighborhood of α and extending $\tilde{\alpha}$ there. We then compute (connection forms with respect to e)

$$\begin{aligned} \nabla_{\alpha'}\mathrm{e}_i &= \sum_j \theta_{ij}(\alpha') \otimes \mathrm{e}_j = \sum_j \mathrm{e}^*\omega_{ij}(\alpha') \otimes \mathrm{e}_j \\ &= \sum_j \omega_{ij}(\mathrm{e}_*\alpha') \otimes \mathrm{e}_j = \sum_j \omega_{ij}(\tilde{\alpha}') \otimes \mathrm{e}_j = 0, \end{aligned}$$

and we conclude that the frame $\tilde{\alpha} = \mathrm{e}$ is parallel along α in the sense that $\nabla_{\alpha'}\mathrm{e} = 0$. This clearly induces a notion of *parallel transport* along α and settles the problem of identifying fibers at different points of $\mathcal{E}$, at least in case a curve joining the points is given. Alternatively, for a section $\sigma = \sum \sigma_i \mathrm{e}_i$ along α, parallel transport can be defined by solving the linear system of ordinary differential equations

$$d\sigma_i(\alpha') + \sum_j \theta_{ij}(\alpha')\sigma_j = 0, \;\; j = 1, \ldots, r,$$

for $\sigma = \{\sigma_i\}$ with given initial conditions. Reciprocally, the parallel transport determines the connection uniquely for if $\alpha : [0,1] \to \mathbf{M}$ is given and $\sigma \in \Gamma(\mathcal{E})$ we clearly have

$$\nabla_{\alpha'}\sigma = \frac{d}{dt}\Pi_t(\sigma_{\alpha(t)})|_{t=0}, \tag{3.2.19}$$

where Π_t denotes parallel transport along α from $\alpha(t)$ to $\alpha(0)$.

With these preliminaries out of the way, we say that $\mathcal{E}$ is *flat* if $R = 0$ identically. By (3.2.16) and Proposition 3.2.3 this is equivalent to $\Omega \equiv 0$ since, as can be easily checked, Ω is *horizontal* in the sense that it vanishes in case at least one of its entries is vertical. By (3.2.14) and Frobenius theorem, this means that the horizontal distribution $\mathcal{K}$ on $P_{\mathcal{E}}^{\mathrm{SO}}$ is integrable. By uniqueness, we see that parallel transport takes place along the integral manifolds of $\mathcal{K}$ and then an easy monodromy argument implies that parallel transport does *not* depend on the curve joining two points at $\mathbf{M}$, as far as $\mathbf{M}$ is simply connected. From this it is immediate how to define a global parallel frame[3] along $\mathbf{M}$ and we have obtained

Proposition 3.2.4 *Any flat vector bundle over a simply connected manifold is trivial.*

Notice that being trivial is a purely topological question (it just boils down to checking whether $\mathcal{E}$ admits a reduction to the trivial group) while flatness is definitely a geometric issue. More importantly, this suggests that the 'total amount' of curvature should somehow measure the degree of nontriviality of $\mathcal{E}$. That this is the case indeed is confirmed by the theory of characteristic classes developed in Chapter 7.

Applying Proposition 3.2.4 to $\mathcal{E} = T\mathbf{M}$, with $\mathbf{M}$ Riemannian and carrying the corresponding Levi-Civita connection (see Chapter 4), we conclude that $\mathbf{M}$ is locally isometric to Euclidean space. We just have to compute the Riemannian metric in terms of the parallel frame of vector fields constructed above so as to verify that it is precisely the standard Euclidean metric.

[3] We say that $\eta \in \Gamma(\mathcal{E})$ is *parallel* if $\nabla\eta = 0$.

We conclude this section by mentioning yet another elementary application of parallel transport, which makes it possible to normalize the *gauge potential* A arising in (3.2.8) at an arbitrary $x \in \mathbf{M}$. This will be crucial when doing computations later.

Proposition 3.2.5 *Given $x \in \mathbf{M}$ there exists a frame $\mathrm{e} = (\mathrm{e}_1, \ldots, \mathrm{e}_r)$ for $\mathcal{E}$ defined in a neighborhood of x such that*

$$\nabla_X \mathrm{e}_i(x) = 0, \;\; i = 1, \ldots, r,$$

for any $X \in T_x\mathbf{M}$. In other words, $A(x) = 0$

Proof. Pick any local radial system of curves issuing from x and parallel translate along the curves a given frame at x.⌋

A frame like this one is said to be *normalized* at x.

3.3 The associated bundle construction

We now explain how vector bundles over $\mathbf{M}$ can be manufactured from representations of a Lie group. Let $\pi : P \to \mathbf{M}$ be a principal $\mathbf{G}$-bundle and let $\rho : \mathbf{G} \to \mathrm{Aut}(V)$ be a representation (real or complex) of $\mathbf{G}$ on V. The action of $g \in \mathbf{G}$ on V is denoted by $\rho(g)$. We first form the product $P \times V$ and let $\mathbf{G}$ act on this from the right by the rule

$$((p, v), g) \mapsto (pg, \rho(g^{-1})v), \;\; (p, v) \in P \times V.$$

It turns out that the corresponding orbit space $\mathcal{E} = P \times_\rho V$ is naturally a vector bundle over $\mathbf{M}$ (in the appropriate category), the *associated vector bundle*. It inherits a linear structure on fibers as follows. If $\{p, v\} \in \mathcal{E}_{\pi(p)}$ is the orbit of an element $(p, v) \in P \times V$, we define $c\{p, v\} = \{p, cv\}, \in \mathbb{R}$. If some other $\{p', v'\} \in \mathcal{E}_{\pi(p)}$ is given we have $p'g = p$ for some $g \in \mathbf{G}$ and we then set $\{p, v\} + \{p', v'\} = \{p, v + \rho(g^{-1})v'\}$. From this we see that the transition functions of $P \times_\rho V$ are compositions of the transition functions of P with ρ.

It turns out that any vector bundle $\mathcal{E}$ is of the form $P \times_\rho V$. For example, if $\mathcal{E}$ is oriented and Riemannian, we just have to take $P = P_{\mathcal{E}}^{\mathrm{SO}}$, the principal SO_r-bundle of all frames of $\mathcal{E}$. It follows at once that

$$\mathcal{E} = P_{\mathcal{E}}^{\mathrm{SO}} \times_{\mu_r} \mathbb{R}^r,$$

where $\mu_r : \mathrm{SO}_r \to \mathrm{GL}_r(\mathbb{R})$ is the standard representation of SO_r.

The question now remains: how can one implement geometry on the associated bundle $\mathcal{E} = P \times_\rho V$? From the metric viewpoint, we see easily that if ρ is *orthogonal* (or *unitary*, if V is complex) in the sense that there exists an inner product on V such that the $\mathbf{G}$-representation on V is by isometries (i.e. $\langle \rho(g)v, \rho(g)v' \rangle = \langle v, v' \rangle$) then $\mathcal{E}$ inherits a natural metric: we just put $\langle \{p, v\}, \{p', v'\} \rangle = \langle v, v' \rangle$.

Moreover, one should be able to canonically define a connection ∇ starting from a connection ω on P. To this effect we isolate the following useful fact.

Proposition 3.3.1 *There is a natural one-to-one correspondence between sections of $\mathcal{E} = P \times_\rho V$ and maps $f : P \to V$ which are equivariant in the sense that $f(pg) = \rho(g^{-1})(f(p))$, $p \in P$, $g \in \mathbf{G}$.*

Proof. This is given by $\sigma \in \Gamma(\mathcal{E}) \mapsto f = f(\sigma)$, where $\sigma = \{p, f(p)\}$.⌋

We now take $\alpha : [0, 1] \to \mathbf{M}$. Given $\sigma_0 = \{p, v\} \in \mathcal{E}_{\alpha(0)}$, for each t we want to define $\sigma_t \in \mathcal{E}_{\alpha(t)}$, its parallel transport along α. Once this is done, we can then prescribe ∇ by the recipe (3.2.19). Now, if $\tilde{\alpha}$ is the parallel transport (with respect to ω) starting at $p \in P$ we simply put $\sigma_t = \{\tilde{\alpha}(t), v\}$. If $\sigma \in \Gamma(\mathcal{E})$ is given, one then has

$$\Pi_t \sigma(\alpha(t)) = \Pi_t \{\tilde{\alpha}(t), f(\tilde{\alpha}(t))\} = \{p, f(\tilde{\alpha}(t))\},$$

where $f = f(\sigma)$. Using (3.2.19) one finally has

$$\nabla_{\alpha'(0)} \sigma = \{p, \mathcal{L}_{\tilde{\alpha}'(0)} f(p)\}, \tag{3.3.20}$$

where $\mathcal{L}$ denotes Lie derivative. This shows at once that the covariant derivative does *not* depend on α but only on its tangent vector at $t = 0$. Clearly, the Leibniz property follows from the corresponding property for $\mathcal{L}$.

As a by-product of the constructions above we get

Proposition 3.3.2 *The connection ∇ is compatible with the metric $\langle\, , \rangle$.*

Proof. This is equivalent to checking that parallel transport acts on the fibers by isometries, but this is clear: in the notation above, $|\sigma_t| = |v| = |\sigma_0|$.⌋

One might ask as well how the curvature R of the associated bundle $\mathcal{E} = P \times_\rho V$ is related to the curvature Ω of P. Here is the relevant computation. Take $x \in \mathbf{M}$ and $p \in P$ such that $\pi(p) = x$. Choose $X, Y \in \mathcal{X}(\mathbf{M})$ with $[X, Y] = 0$ around x and let $\tilde{X}$ and $\tilde{Y}$ be their horizontal lifts.[4] Finally, let $f : P \to V$ correspond to $\sigma \in \Gamma(\mathcal{E})$ according to Proposition 3.3.1. We have

[4]Given $X \in \mathcal{X}(\mathbf{M})$, there exists a unique $\mathbf{G}$-invariant $\tilde{X} \in \mathcal{X}(P)$ such that $\omega(\tilde{X}) = 0$ and $\pi_*(\tilde{X}) = X$. This is the *horizontal lift* of X.

$$\begin{aligned} R_{X,Y}\sigma &= \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma \\ &= \{p, \tilde{X}(\tilde{Y}(f))(p) - \tilde{Y}(\tilde{X}(f))(p)\} \\ &= \{p, [\tilde{X}, \tilde{Y}](f)(p)\}. \end{aligned}$$

Now, $\pi_*[\tilde{X}, \tilde{Y}] = [X, Y] = 0$ and then $[\tilde{X}, \tilde{Y}]$ is vertical. Thus, there exists a fundamental vertical vector field $\hat{A} \in \mathcal{X}(P)$ (corresponding to some $A \in \mathfrak{g}$) such that $\hat{A}(p) = [\tilde{X}, \tilde{Y}](p)$. On the other hand, since $f(gp) = \rho(g^{-1})f(p)$ we obtain $[\tilde{X}, \tilde{Y}]f(p) = -(\rho_* A)f(p)$, where $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$ is the derivative map at $e \in \mathbf{G}$. It then follows that

$$R_{X,Y}\sigma = \{p, -(\rho_* A)f(p)\}.$$

We now recall the well-known formula for the exterior derivative of ω:

$$d\omega(\tilde{X}, \tilde{Y}) = \tilde{X}(\omega(\tilde{Y})) - \tilde{Y}(\omega(\tilde{X})) - \omega([\tilde{X}, \tilde{Y}]).$$

In our case, $\omega(\tilde{X}) = \omega(\tilde{Y}) = 0$ and we get

$$\begin{aligned} A &= \omega_p([\tilde{X}, \tilde{Y}]) \\ &= -d\omega_p(\tilde{X}, \tilde{Y}) \\ &\overset{(3.2.14)}{=} -\Omega(\tilde{X}, \tilde{Y}). \end{aligned}$$

After cleaning up a bit the notation we summarize this in

Proposition 3.3.3 *The associated bundle construction yields the relation*

$$R_{X,Y} = \rho_* \Omega(\tilde{X}, \tilde{Y}). \tag{3.3.21}$$

Since Ω is horizontal, in (3.3.21) we can replace $\tilde{X}$ and $\tilde{Y}$ by any vector fields whose horizontal components are X and Y, respectively.

Chapter 4

The rotation group and Hodge-de Rham theory

In the last chapter we have shown how a connection on a principal **G**-bundle P canonically induces a covariant derivative on any associated bundle $\mathcal{E} = P \times_\rho V$. Conversely, we have also indicated how a connection (or covariant derivative) ∇ on a vector bundle $\mathcal{E}$ determines a connection ω on the frame bundle $P^{\mathrm{SO}}_{\mathcal{E}}$. The upshot was Proposition 3.2.4 relating geometric and topological aspects of $\mathcal{E}$. In this chapter we go one step further and explore the principle according to which the associated bundle construction, when applied to judiciously chosen representations of the rotation group SO_n, allows us to establish a solid arch between the geometry and topology in the case $\mathcal{E} = T\mathbf{M}$, where $\mathbf{M}$ is an oriented Riemannian manifold of dimension n. A variant of this procedure applies in case $\mathbf{M}$ is a spin manifold, a more sophisticated concept we consider in Chapter 6. This chapter is intended to have a pedagogical flavor (a preparation for things to come) and here we make use of some basic analytical material a more detailed discussion of which will be postponed to Chapter 9.

4.1 The Hodge-de Rham set-up

If $\mathbf{M}$ is Riemannian we recall that $T\mathbf{M}$ is endowed with a canonical connection ∇, the so-called *Levi-Civita connection*. This is completely determined by the following conditions:

1. ∇ is compatible with the Riemannian metric $\langle\, , \,\rangle$ on $T\mathbf{M}$;

2. ∇ is *torsion free*, which means that

$$\nabla_X Y - \nabla_Y X = [X, Y], \ \ X, Y \in \mathcal{X}(\mathbf{M}).$$

Whenever we refer to a connection on $T\mathbf{M}$, this is the one we shall pick.

Assuming $\mathbf{M}$ oriented, from the considerations in Section 3.3 we know that

$$T\mathbf{M} = P^{\mathrm{SO}}_{\mathbf{M}} \times_{\mu_n} \mathbb{R}^n,$$

where for simplicity we write $P^{\mathrm{SO}}_{\mathbf{M}} = P^{\mathrm{SO}}_{T\mathbf{M}}$ for the principal frame bundle of $T\mathbf{M}$ and $\mu_n : \mathrm{SO}_n \to \mathrm{GL}_n(\mathbb{R})$ is the standard representation of SO_n. By considering the dual representation $\mu_n^* : \mathrm{SO}_n \to \mathrm{GL}_n(\mathbb{R}^*)$, we have

$$T^*\mathbf{M} = P^{\mathrm{SO}}_{\mathbf{M}} \times_{\mu_n^*} (\mathbb{R}^n)^* .$$

In fact, in the presence of the metric, $\mu_n = \mu_n^*$ and this induces the usual isomorphism

$$T\mathbf{M} = T^*\mathbf{M}. \tag{4.1.1}$$

This allows us to suppress the asterisks whenever convenient in the following discussion.

Now, μ_n induces a series of representations $\wedge^p \mu_n : \mathrm{SO}_n \to \mathrm{Aut}\,(\Lambda^p \mathbb{R}^n)$, $p = 0, 1, \ldots, n$, where as usual Λ means exterior powers. These are defined by

$$\wedge^p \mu_n(g)(v_1 \wedge \ldots \wedge v_p) = \mu_n(g)(v_1) \wedge \ldots \wedge \mu_n(g)(v_p)$$

on simple elements and then extended by linearity. Applying the associated bundle construction we clearly have

$$\Lambda^p(\mathbf{M}) = P^{\mathrm{SO}}_{\mathbf{M}} \times_{\wedge^p \mu_n} \Lambda^p \mathbb{R}^n,$$

$\Lambda^p(\mathbf{M})$ being the vector bundle of differential p-forms over $\mathbf{M}$ so that

$$\Gamma(\Lambda^p(\mathbf{M})) = \mathcal{A}^p(\mathbf{M}).$$

The representation $\wedge^p \mu_n$ is obviously orthogonal with respect to the natural inner product on $\Lambda^p \mathbb{R}^n$ given by

$$\langle v_1 \wedge \ldots \wedge v_p, w_1 \wedge \ldots \wedge w_p \rangle = \det(\langle v_i, w_j \rangle),$$

It follows that $\Lambda^p(\mathbf{M})$ inherits a natural metric, still denoted by $\langle\, , \rangle$. By its turn this induces an L^2 inner product $(\, ,)$ on $\mathcal{A}^p(\mathbf{M})$ according to (3.1.1):

$$(\eta, \eta') = \int_{\mathbf{M}} \langle \eta, \eta' \rangle \, d\mathbf{M}. \tag{4.1.2}$$

In addition, the Levi-Civita connection on $T\mathbf{M}$ (or more precisely its connection 1-form θ) can be uniquely lifted to a connection ω on $P^{\mathrm{SO}}_{\mathbf{M}}$ and this is further pushed forward to canonical connections (still denoted by ∇) on the associated bundles $\Lambda^p(\mathbf{M})$. An important observation is that the canonical connection and metric so constructed are compatible (see Proposition 3.3.2).

At this point, we explain how the exterior derivative $d : \mathcal{A}^p(\mathbf{M}) \to \mathcal{A}^{p+1}(\mathbf{M})$ defined in (1.1.2) can be recovered in this context. Consider the sequence of arrows

$$\mathcal{A}^p(\mathbf{M}) \xrightarrow{\nabla} \mathcal{A}^{p+1}(\mathbf{M}) = \Gamma(T^*\mathbf{M} \otimes \Lambda^p(\mathbf{M})) \xrightarrow{\langle,\rangle} \Gamma(T\mathbf{M} \otimes \Lambda^p(\mathbf{M})) \xrightarrow{\wedge} \mathcal{A}(\mathbf{M}), \tag{4.1.3}$$

so that in terms of a local frame the composite operator is given by

$$\hat{d} = \sum_{i=1}^{n} \mathrm{e}_i \wedge \nabla_{\mathrm{e}_i}. \tag{4.1.4}$$

It turns out that $\hat{d} = d$, the exterior differential. In fact, choose by Proposition 3.2.5 a normalized frame e for $T\mathbf{M}$ at $x \in \mathbf{M}$ and compute (at x) assuming that $\eta = f\mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_p$ for some function f:

$$\begin{aligned}
\hat{d}\eta &= \sum_i \mathrm{e}_i \wedge \nabla_{\mathrm{e}_i}(f\mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_p) \\
&= \sum_i \mathrm{e}_i \wedge (\mathrm{e}_i(f))\, \mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_p \\
&= \left(\sum_i \mathrm{e}_i(f)\mathrm{e}_i\right) \mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_p \\
&= df \wedge \mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_p,
\end{aligned}$$

and we recover the usual definition of d as in (1.1.2).

We now claim that the adjoint operator[1] d^* of d with respect to the L^2 inner product is

$$d^* = -\sum_i \mathrm{e}_i \wedge^* \nabla_{\mathrm{e}_i}.$$

Here,

$$v \wedge^* (\mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_p) = \sum_{i=1}^{p} (-1)^{i+1} \langle v, \mathrm{e}_i \rangle \mathrm{e}_1 \wedge \ldots \wedge \hat{\mathrm{e}}_i \wedge \ldots \wedge \mathrm{e}_p$$

is the adjoint of $v\wedge$, exterior multiplication by v. This is usually referred to as *contraction* with v.

[1]This means that $(d\eta, \eta) = (\eta, d^*\eta')$ for any $\eta \in \mathcal{A}^{p-1}(\mathbf{M})$ and $\eta' \in \mathcal{A}^p(\mathbf{M})$. Notice that, as a rule, in the definition of adjoint operators we always assume that $\mathbf{M}$ is closed.

In order to check the formula for d^* above we rely upon the *Hodge star operator*, the linear map

$$* : \mathcal{A}^p(\mathbf{M}) \to \mathcal{A}^{n-p}(\mathbf{M})$$

defined on simple elements by

$$*\left(\mathrm{e}_{i_1} \wedge \ldots \wedge \mathrm{e}_{i_p}\right) = \epsilon \mathrm{e}_{j_1} \wedge \ldots \wedge \mathrm{e}_{j_{n-p}},$$

with $\{i_1, \ldots, i_p, j_1, \ldots, j_{n-p}\}$ being a permutation of $\{1, \ldots, n\}$ of sign ϵ. Here, $\{\mathrm{e}_1, \ldots, \mathrm{e}_n\}$ is a *positive* orthonormal frame. Notice that $*^2 = (-1)^{p(n-p)}$ on $\mathcal{A}^p(\mathbf{M})$.

Proposition 4.1.1 $d^* = (-1)^{np+n+1} * d* : \mathcal{A}^p(\mathbf{M}) \to \mathcal{A}^{p-1}(\mathbf{M})$.

Proof. As above, we choose a normalized frame at $x \in \mathbf{M}$ and assume $\eta = f\mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_p$ so that $*\eta = f\mathrm{e}_{p+1} \wedge \ldots \wedge \mathrm{e}_n$. We then have

$$\begin{aligned} d(*\eta) &= \sum_{i=1}^n \mathrm{e}_i \wedge \nabla_{\mathrm{e}_i}(f\mathrm{e}_{p+1} \wedge \ldots \wedge \mathrm{e}_n) \\ &= \sum_{i=1}^n \mathrm{e}_i(f)\, \mathrm{e}_i \wedge \mathrm{e}_{p+1} \wedge \ldots \wedge \mathrm{e}_n \end{aligned}$$

and hence

$$*d(*\eta) = \sum_{i=1}^n \mathrm{e}_i(f) * (\mathrm{e}_i \wedge \mathrm{e}_{p+1} \wedge \ldots \wedge \mathrm{e}_n).$$

Now, the ordered basis $\{\mathrm{e}_i, \mathrm{e}_{p+1}, \ldots, \mathrm{e}_n, \mathrm{e}_1, \ldots, \mathrm{e}_{i-1}, \mathrm{e}_{i+1}, \ldots, \mathrm{e}_p\}$ can be brought back to the standard positive basis after $(p-1)(n-p+1) + p - i$ $(= np + n + i + 1 \pmod 2)$ permutations so that

$$*\left(\mathrm{e}_i \wedge \mathrm{e}_{p+1} \wedge \ldots \wedge \mathrm{e}_n)\right) = (-1)^{np+n+i+1} \mathrm{e}_1 \wedge \ldots \wedge \hat{\mathrm{e}}_i \wedge \ldots \wedge \mathrm{e}_p$$

and we proceed as follows:

$$\begin{aligned} *d(*\eta) &= (-1)^{np+n+1}\left(-\sum_{i=1}^n \mathrm{e}_i(f)(-1)^{i+1}\mathrm{e}_1 \wedge \ldots \wedge \hat{\mathrm{e}}_i \wedge \ldots \wedge \mathrm{e}_p\right) \\ &= (-1)^{np+n+1}\left(-\sum_{i=1}^n \mathrm{e}_i(f)\mathrm{e}_i \wedge^* (\mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_p)\right) \\ &= (-1)^{np+n+1}\left(-\sum_{i=1}^n \mathrm{e}_i \wedge^* (\nabla_{\mathrm{e}_i}(f\mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_p))\right) \\ &= (-1)^{np+n+1} d^*\eta, \end{aligned}$$

as desired.⌋

Finally we can prove the key property relating d and d^*.

Proposition 4.1.2 *Assuming* $\mathbf{M}$ *closed,* d *and* d^* *are adjoints to each other in the sense that*

$$(d\eta, \eta') = (\eta, d^*\eta'), \ \ \eta \in \mathcal{A}^{p-1}(\mathbf{M}), \eta' \in \mathcal{A}^p(\mathbf{M}).$$

Proof. We start by observing that if $\eta, \eta' \in \mathcal{A}^p(\mathbf{M})$ then $\langle \eta, \eta' \rangle \, d\mathbf{M} = *\langle \eta, \eta' \rangle = \eta \wedge *\eta'$. Thus,

$$\begin{aligned}
(d\eta, \eta') - (\eta, d^*\eta') &= \int_{\mathbf{M}} \langle d\eta, \eta' \rangle \, d\mathbf{M} - \int_{\mathbf{M}} \langle \eta, d^*\eta' \rangle \, d\mathbf{M} \\
&= \int_{\mathbf{M}} d\eta \wedge *\eta' - \int_{\mathbf{M}} \eta \wedge *d^*\eta' \\
&= \int_{\mathbf{M}} d\eta \wedge *\eta' - (-1)^{np+n+1} \int_{\mathbf{M}} \eta \wedge * * d * \eta'.
\end{aligned}$$

But $d * \eta' \in \mathcal{A}^{n-p+1}(\mathbf{M})$ and this gives $*^2(d * \eta) = (-1)^{(n-p+1)(p-1)}$. Inserting this into our computation we get

$$\begin{aligned}
(d\eta, \eta') - (\eta, d^*\eta') &= \int_{\mathbf{M}} d\eta \wedge *\eta' - (-1)^{-p^2} \int_{\mathbf{M}} \eta \wedge d * \eta' \\
&= \int_{\mathbf{M}} \left(d\eta \wedge *\eta' + (-1)^{p-1} \eta \wedge d * \eta' \right) \\
&= \int_{\mathbf{M}} d(\eta \wedge *\eta') = 0,
\end{aligned}$$

by (1.1.6).⌋

We now explain how taking suitable compositions of d and d^* generates a natural *second* order differential operator. For example, if $\eta \in \mathcal{A}^1(\mathbf{M})$ we have

$$\begin{aligned}
d^*\eta &= -\sum_i \mathrm{e}_i \wedge^* \nabla_{\mathrm{e}_i} \eta = -\sum_i \langle \nabla_{\mathrm{e}_i} \eta, \mathrm{e}_i \rangle \\
&= -\mathrm{div}\, \eta,
\end{aligned}$$

where $\mathrm{div}\, \eta = \mathrm{tr}\, (X \mapsto \nabla_X \eta)$ is the usual divergence operator. In particular, if $\eta = df$ for some $f \in \mathcal{A}^0(\mathbf{M})$ we get

$$d^* df = -\mathrm{div}\, (\mathrm{grad}\, f) = \Delta f,$$

where Δ is the classical *Laplacian operator*.[2]

[2]Note that the sign in our definition of Δ has been chosen in such a way that the integration by parts formula for Δ is deprived of the minus sign: $\int f \Delta f d\mathbf{M} = \int |\nabla f|^2 d\mathbf{M}$.

We can also make a similar computation on the other extremity of the de Rham complex. If $\eta = f\mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_n$ is the local expression for an element in $\mathcal{A}^n(\mathbf{M})$, we get without difficulty that

$$dd^*\eta = -\left(\sum_i \mathrm{e}_i\left(\mathrm{e}_i(f)\right)\right)\mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_n = (\Delta f)\,\mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_n.$$

This suggests that one should define in general the *Hodge Laplacian* $\Delta : \mathcal{A}^p(\mathbf{M}) \to \mathcal{A}^p(\mathbf{M})$ by the recipe

$$\Delta_p = dd^* + d^*d,$$

and we have finally met one of the main characters of our story.

The Hodge Laplacian is the prototype of a natural second order elliptic operator on a closed Riemannian manifold. The corresponding spectral theory for this kind of operator (see Section 9.2) assures, among other things, that the kernel spaces

$$\mathcal{H}_p(\mathbf{M}) = \{\eta \in \mathcal{A}^p(\mathbf{M}); \Delta\eta = 0\}$$

have finite dimension and are naturally isomorphic to the de Rham groups introduced in Section 1.1 and hence, by the de Rham isomorphism theorem, to the real cohomology groups of $\mathbf{M}$. Elements in $\mathcal{H}_p(\mathbf{M})$ are called *harmonic p-forms* and the mentioned isomorphism says that each de Rham cohomology class has a unique harmonic representative. As a consequence, for each p, $\dim \mathcal{H}_p(\mathbf{M})$ is a topological invariant of $\mathbf{M}$ and in fact equals $b_p(\mathbf{M})$, the p^{th} Betti number of $\mathbf{M}$. This is (part of) the basic theorem in Hodge theory (see Theorem 9.2.1 below).

Thus we end up with the following amazing picture. From general constructions in Algebraic Topology we can assign to a smooth closed manifold $\mathbf{M}$ a series of topological invariants (the Betti numbers of $\mathbf{M}$) which, in the presence of the differentiable structure, can be recovered by inspecting how the exterior differential acts on differential forms of various degrees. If we further introduce a Riemannian structure on $\mathbf{M}$ then we can consider the Hodge Laplacian and Hodge theory says that $b_p(\mathbf{M})$ equals the dimension of the solution space of the elliptic equation $\Delta\eta = 0$, $\eta \in \mathcal{A}^p(\mathbf{M})$. In fact, combining the two theories we see that the singular cohomology of $\mathbf{M}$ can be represented in terms of harmonic forms, and this opens up the exciting possibility of reducing topological questions to analytical (or geometrical) ones.

4.2 Clifford multiplication and the Weitzenböck formula

It is possible to contemplate the circle of ideas expounded above in action by a relatively elementary example and at the same time to illustrate the fundamental steps in the applications of the Clifford algebra formalism in Riemannian Geometry.[3] To this effect, we introduce the *Dirac operator* $\mathcal{D} = d + d^*$ locally expressed as

$$\mathcal{D} = \sum_{i=1}^{n} \mathrm{e}_i \cdot \nabla_{\mathrm{e}_i},$$

where $\cdot = \wedge - \wedge^*$ is *Clifford multiplication* by tangent vectors. We remark that $\mathcal{D}$ can alternatively be expressed as the composition of arrows

$$\mathcal{A}^p(\mathbf{M}) \xrightarrow{\nabla} \mathcal{A}^{p+1}(\mathbf{M}) = \Gamma(T^*\mathbf{M} \otimes \Lambda^p(\mathbf{M})) \xrightarrow{\langle,\rangle} \Gamma(T\mathbf{M} \otimes \Lambda^p(\mathbf{M})) \xrightarrow{\cdot} \mathcal{A}(\mathbf{M}), \tag{4.2.5}$$

where the dot means Clifford multiplication.

It is convenient here to think of $\mathcal{D}$ as a selfadjoint first order linear operator acting on $\mathcal{A}(\mathbf{M}) = \oplus_p \mathcal{A}^p(\mathbf{M})$ since it does *not* preserve the natural grading of differential forms. Now, $\mathcal{D}$ is defined so as to materialize the square root of Δ:

$$\mathcal{D}^2 = \Delta : \mathcal{A}(\mathbf{M}) \to \mathcal{A}(\mathbf{M})$$

since $d^2 = 0$ implies $d^{*2} = 0$. In particular, $(\Delta\eta, \eta) = \|\mathcal{D}\eta\|^2$ (recall we suppose that $\mathbf{M}$ is closed) and we get

$$\ker \mathcal{D} = \ker \Delta. \tag{4.2.6}$$

This is further explored if we realize that $\mathrm{e}_i \cdot \mathrm{e}_j = \mathrm{e}_i \wedge \mathrm{e}_j - \mathrm{e}_i \wedge^* \mathrm{e}_j = \mathrm{e}_i \wedge \mathrm{e}_j - \langle \mathrm{e}_i, \mathrm{e}_j \rangle$, so we get the *Clifford relations*

$$\mathrm{e}_i^2 = -1, \;\; \mathrm{e}_i \cdot \mathrm{e}_j = -\mathrm{e}_j \cdot \mathrm{e}_i, \;\; i \neq j, \tag{4.2.7}$$

and this will allow us to compute $\mathcal{D}^2 = \Delta$ at a point $x \in \mathbf{M}$ by means of a normalized frame:[4]

$$\Delta \;\; = \;\; \sum_i \mathrm{e}_i \nabla_{\mathrm{e}_i} \left(\sum_j \mathrm{e}_j \nabla_{\mathrm{e}_j} \right)$$

[3]Here we follow [LM] closely.

[4]Whenever convenient, we omit the point denoting Clifford multiplication and use throughout that the connection ∇ on $\mathcal{A}(\mathbf{M})$ is a derivation with respect to Clifford multiplication.

$$
\begin{aligned}
&= \sum_{ij} e_i e_j \nabla_{e_i} \nabla_{e_j} \\
&= -\sum_i \nabla_{e_i} \nabla_{e_i} + \sum_{i \neq j} e_i e_j \nabla_{e_i} \nabla_{e_j} \\
&= -\sum_i \nabla_{e_i} \nabla_{e_i} + \sum_{i<j} e_i e_j (\nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i}) \\
&= -\sum_i \nabla_{e_i} \nabla_{e_i} + \frac{1}{2} \sum_{i,j} e_i e_j R_{e_i, e_j},
\end{aligned}
$$

where R is the curvature tensor of $\Lambda(\mathbf{M}) = \oplus_{p=0}^n \Lambda^p(\mathbf{M})$ (see Proposition 3.2.3).

The terms arising in the right hand side of this identity have a rather distinct nature. The second one clearly defines an algebraic selfadjoint operator $\mathcal{R}$ acting on the fibers of $\Lambda(\mathbf{M}) = \oplus_{p=0}^n \Lambda^p(\mathbf{M})$ preserving the graded structure. To stress this, let us represent $\mathcal{R}_p = \mathcal{R}|_{\mathcal{A}^p(\mathbf{M})}$. On the other hand, the first term is a second order differential operator admitting an invariant definition and here it is appropriate to shift to a more general setting than above.

Let $\mathcal{E}$ be a Riemannian vector bundle with metric $\langle\, , \rangle$ and compatible connection ∇ over a Riemannian manifold $\mathbf{M}$. For $X, Y \in \mathcal{X}(\mathbf{M})$ consider the map $\nabla^2_{X,Y} : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ given by $\nabla^2_{X,Y}(\eta) = \nabla_X \nabla_Y \eta - \nabla_{\nabla_X Y}\eta$. After taking trace, the *Bochner Laplacian* $\nabla^*\nabla : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ is given by

$$\nabla^*\nabla(\eta) = -\mathrm{tr}\, \nabla^2_{(\cdot,\cdot)}\mathrm{J}. \tag{4.2.8}$$

In terms of a local frame for $T\mathbf{M}$ normalized at $x \in \mathbf{M}$ we clearly have

$$\nabla^*\nabla = -\sum_i \nabla_{e_i} \nabla_{e_i},$$

and since x is arbitrary, we recover our operator when $\mathcal{E} = T\mathbf{M}$. In general, $\nabla^*\nabla$ is a second order selfadjoint linear operator satisfying the integration by parts formula:

$$\int_{\mathbf{M}} \langle \nabla^*\nabla\eta, \eta\rangle \, d\mathbf{M} = \int_{\mathbf{M}} |\nabla\eta|^2 \, d\mathbf{M}. \tag{4.2.9}$$

To see this, pick $\eta \in \mathcal{A}(\mathbf{M})$ and note that $e_i\langle \nabla_{e_i}\eta, \eta\rangle = \langle \nabla_{e_i}\nabla_{e_i}\eta, \eta\rangle + |\nabla_{e_i}\eta|^2$. Summing over i we get $\operatorname{div} X + \langle \nabla^*\nabla\eta, \eta\rangle = |\nabla\eta|^2$, where $X \in \mathcal{X}(\mathbf{M})$ is given by $\langle X, Y\rangle = \langle \nabla_Y \eta, \eta\rangle$, $Y \in \mathcal{X}(\mathbf{M})$, and (4.2.9) follows after integration.

As a consequence of (4.2.9) we see at once that $\nabla^*\nabla$ is nonnegative and $\ker \nabla^*\nabla$ is formed by *parallel forms*, i.e. forms satisfying $\nabla\eta = 0$. We also mention that $\nabla^*\nabla$ can alternatively be defined as the composition

$$\Gamma(\mathcal{E}) \stackrel{\nabla}{\to} \Gamma(T^*\mathbf{M} \otimes \mathcal{E}) \stackrel{\nabla^*}{\to} \Gamma(\mathcal{E}),$$

where ∇^* is the adjoint of ∇ with respect to the obvious L^2 inner products, and this accounts for the rather clumsy notation for the Bochner Laplacian.

We thus arrive at the *Weitzenböck formula* for the Hodge operator acting on p-forms:

$$\Delta = \nabla^*\nabla + \mathcal{R}_p, \tag{4.2.10}$$

and we now use this to establish a vanishing theorem based on a quasi-positivity assumption on $\mathcal{R}_p$.

Proposition 4.2.1 *Assume (as always) that* $\mathbf{M}$ *is closed and that for* $1 \leq p \leq n-1$, $\mathcal{R}_p$ *is quasi-positive in the sense that* $\mathcal{R}_p \geq 0$ *everywhere and* $\mathcal{R}_p(x) > 0$ *for some* $x \in \mathbf{M}$. *Then* $\mathbf{M}$ *does not carry any nontrivial harmonic p-form. In particular,* $b_p(\mathbf{M}) = 0$.

Proof. Assume first that $\mathcal{R}_p \geq 0$ everywhere and let $\eta \in \mathcal{H}_p(\mathbf{M})$. By (4.2.10) we have

$$0 = \langle \Delta\eta, \eta \rangle = \langle \nabla^*\nabla\eta, \eta \rangle + \langle \mathcal{R}_p\eta, \eta \rangle$$

and integrating this over $\mathbf{M}$ we obtain

$$0 = \int_{\mathbf{M}} |\nabla\eta|^2 d\mathbf{M} + \int_{\mathbf{M}} \langle \mathcal{R}_p\eta, \eta \rangle d\mathbf{M} \geq 0. \tag{4.2.11}$$

Thus $\nabla\eta = 0$ and we have proved in this case that any harmonic form is parallel. Now assume in addition that $\mathcal{R}_p$ is quasi-positive and for the sake of absurd pick a nonzero $\eta \in \mathcal{H}_p(\mathbf{M})$. Since by the above η is parallel we see that η is nonzero everywhere. But then $\int_{\mathbf{M}} \langle \mathcal{R}_p\eta, \eta \rangle d\mathbf{M} > 0$ and we have

$$0 = \int_{\mathbf{M}} |\nabla\eta|^2 d\mathbf{M} + \int_{\mathbf{M}} \langle \mathcal{R}_p\eta, \eta \rangle d\mathbf{M} > \int_{\mathbf{M}} |\nabla\eta|^2 d\mathbf{M} = 0,$$

thus reaching a contradiction.⌋

We are thus left with the rather involved task of finding natural geometric conditions in order to assure quasi-positivity for $\mathcal{R}_p$. In case $p = 1$ the situation is perfectly well understood and we meet a result first proved by Bochner [B].

Theorem 4.2.1 *At the level of* 1*-forms we have*

$$\mathcal{R}_1 = \mathrm{Ric},$$

where Ric *is the Ricci tensor of* $\mathbf{M}$. *In particular, if* $\mathbf{M}$ *carries a metric with* $\mathrm{Ric} \geq 0$ *everywhere then* $b_1(\mathbf{M}) \leq \dim \mathbf{M}$ *and if it carries a metric with quasi-positive Ricci tensor then* $b_1(\mathbf{M}) = 0$.

Proof. We just have to compute $\mathcal{R}_1$ exploring the Clifford relations (4.2.7). For $\eta \in \mathcal{A}^1(\mathbf{M})$ we have

$$\begin{aligned}
\mathcal{R}_1\eta &= \frac{1}{2}\sum_{ij} \mathrm{e}_i\mathrm{e}_j R_{\mathrm{e}_i,\mathrm{e}_j}(\eta) \\
&= \frac{1}{2}\sum_{ij} \mathrm{e}_i\mathrm{e}_j \sum_k \langle R_{\mathrm{e}_i,\mathrm{e}_j}\eta, \mathrm{e}_k\rangle \mathrm{e}_k \\
&= -\frac{1}{6}\sum_{i\neq j\neq k\neq i} \langle R_{\mathrm{e}_i,\mathrm{e}_j}\mathrm{e}_k + R_{\mathrm{e}_j,\mathrm{e}_k}\mathrm{e}_i + R_{\mathrm{e}_k,\mathrm{e}_i}\mathrm{e}_j, \eta\rangle \mathrm{e}_i\mathrm{e}_j\mathrm{e}_k \\
&\quad +\frac{1}{2}\sum_{ij} \mathrm{e}_i\mathrm{e}_j \langle R_{\mathrm{e}_i,\mathrm{e}_j}\eta, \mathrm{e}_j\rangle \mathrm{e}_j + \frac{1}{2}\sum_{ij} \mathrm{e}_i\mathrm{e}_j \langle R_{\mathrm{e}_i,\mathrm{e}_j}\eta, \mathrm{e}_i\rangle \mathrm{e}_i.
\end{aligned}$$

But

$$\mathrm{e}_i\mathrm{e}_j\mathrm{e}_j = -\mathrm{e}_i$$

and

$$\mathrm{e}_i\mathrm{e}_j\mathrm{e}_i = -\mathrm{e}_i\mathrm{e}_i\mathrm{e}_j = \mathrm{e}_j.$$

Thus

$$\frac{1}{2}\sum_{ij} \mathrm{e}_i\mathrm{e}_j \langle R_{\mathrm{e}_i,\mathrm{e}_j}\eta, \mathrm{e}_j\rangle \mathrm{e}_j = -\frac{1}{2}\sum_{ij} \langle R_{\mathrm{e}_i,\mathrm{e}_j}\eta, \mathrm{e}_j\rangle \mathrm{e}_i \tag{4.2.12}$$

and

$$\frac{1}{2}\sum_{ij} \mathrm{e}_i\mathrm{e}_j \langle R_{\mathrm{e}_i,\mathrm{e}_j}\eta, \mathrm{e}_i\rangle \mathrm{e}_i = \frac{1}{2}\sum_{ij} \langle R_{\mathrm{e}_i,\mathrm{e}_j}\eta, \mathrm{e}_i\rangle \mathrm{e}_j = -\frac{1}{2}\sum_{ij} \langle R_{\mathrm{e}_j,\mathrm{e}_i}\eta, \mathrm{e}_i\rangle \mathrm{e}_j,$$

and this equals the expression in (4.2.12). We conclude that the last two terms in the expression for $\mathcal{R}_1\eta$ coincide and the first one vanishes because it is a 3-form sitting on an identity with 1-forms all around.[5] So we have

$$\mathcal{R}_1\eta = -\sum_{ij} \langle R_{\mathrm{e}_i,\mathrm{e}_j}\eta, \mathrm{e}_j\rangle \mathrm{e}_i$$

[5] Note that incidentally we have proved the *first Bianchi identity*:

$$R_{X,Y}Z + R_{Y,Z}X + R_{Z,X}Y = 0, \quad X, Y, Z \in \mathcal{X}(\mathbf{M}). \tag{4.2.13}$$

$$
\begin{aligned}
&= -\sum_{ij}\langle R_{\eta,\mathrm{e}_j}\mathrm{e}_i, \mathrm{e}_j\rangle \mathrm{e}_i \\
&= \sum_{ij}\langle R_{\eta,\mathrm{e}_j}\mathrm{e}_j, \mathrm{e}_i\rangle \mathrm{e}_i \\
&= \sum_{j} R_{\eta,\mathrm{e}_j}\mathrm{e}_j \stackrel{\text{def.}}{=} \mathrm{Ric}(\eta),
\end{aligned}
$$

as promised.⌋

Bochner's theorem is the prototype of a vanishing theorem in Riemannian Geometry. We remark that the result, at least in case Ric is positive everywhere, can also be obtained by variational methods. We just have to appeal to Bonnet-Myers theorem (see [dC], for example) which says, among other things, that the Riemannian universal cover $\tilde{\mathbf{M}}$ of a closed Riemannian manifold $\mathbf{M}$ with $\mathrm{Ric} > 0$ everywhere is compact. It follows that $\pi_1(\mathbf{M})$ is finite and this implies by topology that $b_1(\mathbf{M}) = 0$, as desired. One should emphasize however that in the Bochner's argument above the important input is the method of proof, as it can be greatly generalized to other situations where variational techniques either are not available or their use is comparatively harder to implement. For example, Bochner's theorem implies that a torus T^n cannot carry a metric with quasi-positive Ricci curvature. It is also true that T^n does not admit a metric with quasi-positive scalar curvature (a much weaker condition) and we shall derive this later as a consequence of a vanishing theorem for a twisted Dirac operator following an argument of Gromov and Lawson (see Theorem 10.3.2). On the other hand, Schoen and Yau [SY] first proved this same result (in low dimensions at least) by using minimal submanifolds in a way that is reminiscent of the use of geodesics in the proof of Bonnet-Myers theorem. In fact, one of the big puzzles in Riemannian Geometry is precisely to understand why many results in the classification theory of manifolds admitting metrics with positive scalar curvature can be proved by using either spin or minimal submanifolds techniques.

Chapter 5

Clifford algebras, spin groups and their representations

Clifford algebras are, as we shall see, essential when dealing with Dirac type operators. Moreover, they constitute the natural framework in any discussion of the spin group Spin_n, the universal cover of the rotation group SO_n in dimension $n \geq 3$. The purpose of this chapter is to present the basic facts on Clifford algebras and their (complex) representations. It turns out that in case $n = 2k$ is even, the corresponding (complexified) Clifford algebra $\mathbb{C}\mathrm{l}_n$ has a unique irreducible representation (Theorem 5.2.1). Since $\mathrm{Spin}_n \subset \mathbb{C}\mathrm{l}_n$, this induces by restriction a fundamental representation on Spin_n (the famous spin representation) and, more interestingly, this representation *cannot* be realized by lifting any representation of SO_n under the universal covering map $\gamma_n : \mathrm{Spin}_n \to \mathrm{SO}_n$. When coupled with the topological condition of a Riemannian manifold $\mathbf{M}$ being *spin* (which we shall meet in Chapter 6 and essentially requires the existence of a principal Spin_n-bundle over $\mathbf{M}$ which double covers $P^{\mathrm{SO}}_{\mathbf{M}}$ and restricts to γ_n along the fibers) this allows us to apply the associated bundle construction to the situation. We then end up with a canonical bundle over $\mathbf{M}$, the spinor bundle, on whose sections (spinors) a natural differential operator, the Dirac operator, acts. Careful analysis of this operator unravels unexpected connections between geometric and topological aspects of $\mathbf{M}$ which go far beyond the strict realm of Riemannian Geometry.

5.1 Clifford algebras and spin groups

Unlike vectors or tensors, spinors do not seem to admit an interpretation in terms of familiar geometric notions. The usual estrategy to probe their nature is then rather indirect: we first study their endomorphism algebra, which happens to have a very simple structure.

Definition 5.1.1 *Given $n \geq 1$, the Clifford algebra Cl_n is the real unital algebra over $\mathbb{R}^n$ defined by the relations*

$$vw + wv = -2\langle v, w\rangle 1, \;\; v, w \in \mathbb{R}^n, \tag{5.1.1}$$

where $1 \in \mathrm{Cl}_n$ denotes the unity and $\langle \, , \, \rangle$ is the standard inner product.

Equivalently, if we fix $\{\mathrm{e}_1, \ldots, \mathrm{e}_n\}$ an *orthonormal* basis of $\mathbb{R}^n$ we can replace (5.1.1) by the *Clifford relations*

$$\mathrm{e}_i^2 = -1, \quad \mathrm{e}_i\mathrm{e}_j = -\mathrm{e}_j\mathrm{e}_i, \; i \neq j. \tag{5.1.2}$$

Notice the strict resemblance with (4.2.7).

Similarly to the tensor and exterior algebras over $\mathbb{R}^n$, Cl_n can be defined and characterized in terms of universal properties (see [LM]) but we shall not pursue this here. We stick instead to this more concrete definition as it has a more practical appeal. For instance, it follows easily from the presentation above that $\dim \mathrm{Cl}_n = 2^n$. Moreover, as a vector space, Cl_n is naturally isomorphic to the exterior algebra $\Lambda\mathbb{R}^n$: we just have to check that the assignment

$$\mathrm{e}_{i_1} \ldots \mathrm{e}_{i_p} \leftrightarrow \mathrm{e}_{i_1} \wedge \ldots \wedge \mathrm{e}_{i_p} \tag{5.1.3}$$

does the job since the elements $\mathrm{e}_I = \mathrm{e}_{i_1} \ldots \mathrm{e}_{i_p}$, with $I = \{1 \leq i_1 < \ldots < i_p \leq n\}$, $1 \leq p \leq n$, generate Cl_n. Here of course we set $\mathrm{e}_\emptyset = 1$.

It is easily verified for instance that $\mathrm{Cl}_0 = \mathbb{R}$, $\mathrm{Cl}_1 = \mathbb{C}$ and $\mathrm{Cl}_2 = \mathbb{H}$, the quaternions, which looks promising. A systematic understanding of Cl_n for general n starts with the following proposition.

Proposition 5.1.1 Cl_n *is a $\mathbb{Z}_2$-graded algebra in the sense that there exists a decomposition*

$$\mathrm{Cl}_n = \mathrm{Cl}_n^0 \oplus \mathrm{Cl}_n^1 \tag{5.1.4}$$

with

$$\mathrm{Cl}_n^0\mathrm{Cl}_n^0 \subset \mathrm{Cl}_n^0, \; \mathrm{Cl}_n^0\mathrm{Cl}_n^1 \subset \mathrm{Cl}_n^1, \; \mathrm{Cl}_n^1\mathrm{Cl}_n^1 \subset \mathrm{Cl}_n^0. \tag{5.1.5}$$

Proof. Just define Cl_n^0 (respect. Cl_n^1) to be the subspace generated by the simple elements e_I with $\sharp I$ even (respect. odd).$\rfloor$

We call the terms in (5.1.4) the *even* and *odd* parts of Cl_n, respectively.

Corollary 5.1.1 Cl_n^0 *is a subalgebra of* Cl_n *of dimension* 2^{n-1}. *In fact,* Cl_n^0 *is isomorphic (as an algebra) to* Cl_{n-1}.

Proof. The isomorphism $\Phi : \mathrm{Cl}_{n-1} \to \mathrm{Cl}_n^0$ is given by $\Phi(a) = a_0 + a_1 \mathrm{e}_n$, where $a = a_0 + a_1$ is the $\mathbb{Z}_2$-decomposition of a.⌋

From this we see the possibility of approaching the classification of Clifford algebras by induction and this is the route we shall take.

We shall find it useful later to determine the *center* $\mathcal{Z}(\mathrm{Cl}_n) = \{a \in \mathrm{Cl}_n; aa' = a'a, a' \in \mathrm{Cl}_n\}$ of Cl_n. We shall formulate this in terms of the $\mathbb{Z}$-grading

$$\mathrm{Cl}_n = \mathrm{Cl}_n^{(0)} \oplus \mathrm{Cl}_n^{(1)} \oplus \ldots \oplus \mathrm{Cl}_n^{(n)} \tag{5.1.6}$$

coming from (5.1.3). In other words, under the canonical vector space identification $\mathrm{Cl}_n = \Lambda\mathbb{R}^n$ we have $\mathrm{Cl}_n^{(p)} = \Lambda^p\mathbb{R}^n$. In particular, $\mathrm{Cl}_n^{(1)} = \mathbb{R}^n$, considered as a subspace of Cl_n in the obvious way.

Proposition 5.1.2 *If n is even,* $\mathcal{Z}(\mathrm{Cl}_n) = \mathrm{Cl}_n^{(0)}$, *and if n is odd,* $\mathcal{Z}(\mathrm{Cl}_n) = \mathrm{Cl}_n^{(0)} \oplus \mathrm{Cl}_n^{(1)}$. *Thus, in any case,* $\mathcal{Z}(\mathrm{Cl}_n) \cap \mathrm{Cl}_n^0 = \mathrm{Cl}_n^{(0)}$.

Proof. Clearly, $\mathcal{Z}(\mathrm{Cl}_n) = \{a \in \mathrm{Cl}_n; av = va, v \in \mathbb{R}^n\}$. Now, if $\mathrm{e}_I = \mathrm{e}_{i_1} \ldots \mathrm{e}_{i_p}$ and $j \notin I$ we have

$$\mathrm{e}_I \mathrm{e}_j = (-1)^p \mathrm{e}_j \mathrm{e}_I,$$

and if $j \in I$,

$$\mathrm{e}_I \mathrm{e}_j = (-1)^{p-1} \mathrm{e}_j \mathrm{e}_I.$$

The result follows.⌋

We now shift our attention to Cl_n^*, the group of multiplicative invertible elements in Cl_n. If $v \in \mathbb{R}^n$, $v \neq 0$, we have $v^{-1} = -v/|v|^2$ and $\mathbb{R}^n - \{0\} \subset \mathrm{Cl}_n^*$. In particular, $S^{n-1} \subset \mathrm{Cl}_n^*$.

Definition 5.1.2 *The pin group* Pin_n *is the subgroup of* Cl_n^* *generated by the unit sphere* S^{n-1}, *and the spin group is the even part of* Pin_n: $\mathrm{Spin}_n = \mathrm{Pin}_n \cap \mathrm{Cl}_n^0$.

The pin group will have an ancillary interest here, as we shall be mainly concerned with Spin_n. We are about to show that Spin_n is a compact connected Lie group closely related to SO_n, but for this we need a more workable definition. We start by noticing that the operation on basic elements

$$\mathrm{e}_{i_1} \ldots \mathrm{e}_{i_p} \mapsto \mathrm{e}_{i_p} \ldots \mathrm{e}_{i_1}$$

extends uniquely to an anti-isomorphism $a \mapsto a^t$ of Cl_n. This is the 'transpose' map.

Proposition 5.1.3 *We have*

$$\mathrm{Spin}_n = \{a \in \mathrm{Pin}_n; aa^t = 1\} = \{v_1 \dots v_p; v_i \in S^{n-1},\ p\ even\}.$$

In particular, $\mathrm{Spin}_n \subset \mathrm{Cl}_n^0$.

Proof. By definition, $a \in \mathrm{Pin}_n$ if and only if $a = v_1 \dots v_p$, with $v_i \in S^{n-1}$. Thus

$$aa^t = v_1 \dots v_p v_p \dots v_1 = (-1)^p,$$

and $aa^t = 1$ if and only if p is even.⌋

We now consider the basic homomorphism $\gamma : \mathrm{Pin}_n \to \mathrm{GL}_n(\mathbb{R})$ given by

$$\gamma(a)v = ava^t,\ \ a \in \mathrm{Pin}_n,\ v \in \mathbb{R}^n.$$

Proposition 5.1.4 $\gamma(\mathrm{Pin}_n) = \mathrm{O}_n$.

Proof. We first look at the image of S^{n-1} under γ. If $a \in S^{n-1}$ and $v \in \mathbb{R}^n$ we decompose $v = \lambda a + a'$ with $\lambda \in \mathbb{R}$ and $\langle a, a' \rangle = 0$. We then compute

$$\begin{aligned}
\gamma(a)v &= ava^t \\
&= a(\lambda a + a')a \quad (a^t = a \ \text{ since } \ a \in S^{n-1}) \\
&= \lambda a^3 + aa'a \\
&= -\lambda a + a'
\end{aligned}$$

since $aa' = -a'a - 2\langle a, a' \rangle = -a'a$. Hence, $\gamma(a)$ is the reflection with respect to the hyperplane perpendicular to a and the result follows from the classical fact that any element in O_n is a product of such reflections.[1]⌋

Corollary 5.1.2 $\gamma_n(\mathrm{Spin}_n) = \mathrm{SO}_n$, *where* $\gamma_n = \gamma|_{\mathrm{Spin}_n}$.

Proof. This is indeed a consequence of the proof and Proposition 5.1.3, since any element in SO_n is a product of an *even* number of reflections.⌋

Proposition 5.1.5 *If* $n \geq 3$, Spin_n *is compact, connected, simply connected and* $\gamma_n : \mathrm{Spin}_n \to \mathrm{SO}_n$ *is the universal double covering map.*

[1]Notice that this proof actually justifies the definition of γ i.e. that $\gamma(a) \in \mathrm{GL}_n(\mathbb{R})$ for $a \in \mathrm{Pin}_n$.

Proof. We have $a \in \ker \gamma_n$ if and only if $va = av$ for any $v \in \mathbb{R}^n$ since $a^t = a^{-1}$ by Proposition 5.1.3. But this means by Proposition 5.1.2 that $a \in \mathrm{Spin}_n \cap \mathcal{Z}(\mathrm{Cl}_n) \subset \mathrm{Cl}_n^0 \cap \mathcal{Z}(\mathrm{Cl}_n) = \mathrm{Cl}_n^{(0)} = \mathbb{R}$ and hence $\ker \gamma_n = \{1, -1\}$. This already shows that Spin_n is compact and γ_n is a two-sheeted covering map. But the arc $\alpha(t) = \cos t + \sin t\, \mathrm{e}_1\mathrm{e}_2$, $0 \leq t \leq \pi$, lies entirely in Spin_n and connects the two elements in $\ker \gamma_0$. Since $\pi_1(\mathrm{SO}_n) = \mathbb{Z}_2$ this implies that Spin_n is connected and γ_n is the (nontrivial) universal covering map.⌋

We have thus managed to explicitly describe the universal covering map $\gamma_n : \mathrm{Spin}_n \to \mathrm{SO}_n$, $n \geq 3$. For later reference, we insert here a few comments on Lie algebras. First recall that, since $a^t = a^{-1}$ for $a \in \mathrm{Spin}_n$, we have $\gamma_n(v) = ava^{-1}$, $v \in \mathbb{R}^n$. Taking the derivative at $e \in \mathrm{Spin}_n$, we obtain a Lie algebra isomorphism

$$\gamma_{n*} : \mathfrak{spin}_n \to \mathfrak{so}_n.$$

We want to express this in terms of appropriate basis. Fix a frame $\{\mathrm{e}_1, \ldots, \mathrm{e}_n\}$ and recall from the discussion soon before (1.2.13) that $\{\mathrm{e}_i \wedge \mathrm{e}_j\}_{i<j}$ is a basis for $\mathfrak{so}_n$. As for $\mathfrak{spin}_n$ we have

Proposition 5.1.6 $\{\mathrm{e}_i\mathrm{e}_j\}_{i<j}$ *is a basis for* $\mathfrak{spin}_n \cong \mathbb{C}\mathrm{l}_n^{(2)}$.

Proof. Consider $\delta_{ij}(t) = (\mathrm{e}_i \cos t + \mathrm{e}_j \sin t)(-\mathrm{e}_i \cos t + \mathrm{e}_j \sin t) = \cos 2t + \sin 2t\, \mathrm{e}_i\mathrm{e}_j$. Clearly, $\delta_{ij}(\mathbb{R}) \subset \mathrm{Spin}_n$, $\delta_{ij}(0) = e$ and $\delta'_{ij}(0) = 2\mathrm{e}_i\mathrm{e}_j$. Hence, $\mathfrak{spin}_n$ contains the subspace generated by $\{\mathrm{e}_i\mathrm{e}_j\}_{i<j}$, whose dimension equals $n(n-1)/2 = \dim \mathfrak{so}_n = \dim \mathfrak{spin}_n$.⌋

Proposition 5.1.7 $\gamma_{n*}(\mathrm{e}_i\mathrm{e}_j) = 2\mathrm{e}_i \wedge \mathrm{e}_j$.

Proof. This time we consider $\delta_{ij}(t) = \cos t + \sin t\, \mathrm{e}_i\mathrm{e}_j$. Then $\delta_{ij}(\mathbb{R}) \subset \mathrm{Spin}_n$, $\delta_{ij}(0) = e$, $\delta'_{ij}(0) = \mathrm{e}_i\mathrm{e}_j$ and $(\delta_{ij}^{-1})'(0) = -\mathrm{e}_i\mathrm{e}_j$. On the other hand, for $v \in \mathbb{R}^n$, we have $\gamma_n\left(\delta_{ij}(t)\right) v = \delta_{ij}(t) v \delta_{ij}(t)^{-1}$ and hence

$$\begin{aligned}
\gamma_{n*}(\mathrm{e}_i\mathrm{e}_j)v &= \frac{d}{dt}\gamma_n\left(\delta_{ij}(t)\right) v|_{t=0} \\
&= \mathrm{e}_i\mathrm{e}_j v - v\mathrm{e}_i\mathrm{e}_j \\
&= \mathrm{e}_i\mathrm{e}_j v + (\mathrm{e}_i v + 2\langle \mathrm{e}_i, v\rangle)\mathrm{e}_j \\
&= \mathrm{e}_i\mathrm{e}_j v - \mathrm{e}_i\mathrm{e}_j v - 2\langle \mathrm{e}_j, v\rangle \mathrm{e}_i + 2\langle \mathrm{e}_i, v\rangle \mathrm{e}_j \\
&= 2(\mathrm{e}_i \wedge \mathrm{e}_j)(v),
\end{aligned}$$

as desired.⌋

5.2 Representations of Clifford algebras and spin groups

As explained in the Introduction to this chapter, we are looking for a fundamental representation $\tau : \mathrm{Spin}_n \to \mathrm{Aut}(V)$ which is not the lift of a representation of SO_n (notice that this will happen if and only if $\tau(-1) = -\mathrm{Id}_V$). Since $\mathrm{Spin}_n \subset \mathrm{Cl}_n$, it is natural to obtain τ as the restriction of a suitable representation of Cl_n. As a matter of fact, it is convenient here (and it suffices for our purposes) to complexify the whole picture.

Definition 5.2.1 *The complex Clifford algebra is obtained after tensoring with $\mathbb{C}$ the real Clifford algebra Cl_n:*

$$\mathbb{C}\mathrm{l}_n = \mathrm{Cl}_n \otimes \mathbb{C},$$

the tensor product being taken over $\mathbb{R}$.

In other words, $\mathbb{C}\mathrm{l}_n$ is defined exactly as Cl_n in terms of a frame (in particular, the Clifford relations (5.1.2) remain untouched) except that now we allow complex coefficients. We have the natural inclusion $\mathrm{Cl}_n \subset \mathbb{C}\mathrm{l}_n$ and the gradings (5.1.5) and (5.1.6) have direct counterparts in the complex case. In particular, with obvious notation, we have $\mathrm{Spin}_n \subset \mathbb{C}\mathrm{l}_n^0 \subset \mathbb{C}\mathrm{l}_n$ and Corollary 5.1.1 translates into

$$\mathbb{C}\mathrm{l}_n^0 = \mathbb{C}\mathrm{l}_{n-1}. \tag{5.2.7}$$

Moreover, the complexified version of Proposition 5.1.2 remains equally true.

We now turn to *complex* representations of $\mathbb{C}\mathrm{l}_n$.

Definition 5.2.2 *A complex representation of a complex unital algebra $\mathcal{A}$ is an algebra homomorphism $\zeta : \mathcal{A} \to \mathrm{Hom}(V)$ where V is a finite dimension complex vector space.*

We also say that V is a (left) $\mathcal{A}$-module (or simply a module if no confusion arises). A representation ζ as above is *irreducible* if there exists no nontrivial proper subspace $V_0 \subset V$ such that $\zeta(a)(V_0) \subset V_0$ for any $a \in \mathcal{A}$. Otherwise, ζ is said to be *reducible*.[2] Finally, a representation ζ is called *completely reducible* if there exists a direct sum decomposition

$$V = V_1 \oplus \ldots \oplus V_r$$

of V into irreducible submodules. There is a priori no reason why a representation of an arbitrary algebra should be completely reducible, but of course the situation for Clifford algebras is different.

[2] In this context, the terminology irreducible (or reducible) module should be clear. Moreover, we often represent $\zeta(a)v$ simply by av. And of course we will be interested in classifying representations up to the obvious equivalence relation.

Proposition 5.2.1 *Any representation of* $\mathbb{C}\mathrm{l}_n$ *is completely reducible.*

Proof. Consider the *Clifford group* Cliff_n formed by all the elements of the type $\pm \mathrm{e}_{i_1} \ldots \mathrm{e}_{i_p}$, $0 \leq p \leq n$. This is a finite multiplicative group of order 2^{n+1} and clearly there exists a one-to-one correspondence between algebra representations $\zeta : \mathbb{C}\mathrm{l}_n \to \mathrm{Hom}(V)$ and group representations $\overline{\zeta} : \mathrm{Cliff}_n \to \mathrm{Aut}(V)$ with $\overline{\zeta}(-1) = -\mathrm{Id}_V$. This means that we just have to check that any representation of Cliff_n is completely reducible. In effect, let $\overline{\zeta}$ be such a representation and let $\langle \, , \rangle$ be any hermitian product on V. Define

$$\langle\langle v, w \rangle\rangle = \frac{1}{2^{n+1}} \sum_{g \in \mathrm{Cliff}_n} \langle \overline{\zeta}(g) v, \overline{\zeta}(g) w \rangle, \ \ v, w \in V.$$

It follows that $\overline{\zeta}$ acts by isometries with respect to $\langle\langle \, , \rangle\rangle$. In particular, if V_1 is a submodule then $V_1^{\perp} = \{ v \in V ; \langle\langle v, w \rangle\rangle = 0, w \in V_1 \}$ is a submodule as well. The result follows by splitting off one irreducible module at a time.$\rfloor$

The passage from $\langle \, , \rangle$ to $\langle\langle \, , \rangle\rangle$ above is an example of the *averaging method* in representation theory. In this guise, it says that any representation of a finite group $\mathbf{G}$ can be made unitary and consequently is completely reducible. Both assertions also hold true more generally if $\mathbf{G}$ is compact: we just define

$$\langle\langle v, w \rangle\rangle = \int_{\mathbf{G}} \langle g v, g w \rangle dg, \ \ v, w \in V,$$

where dg is the Haar measure of $\mathbf{G}$. For us, an important application of this is

Proposition 5.2.2 *Let* $\zeta : \mathrm{Spin}_n \to \mathrm{Aut}(V)$ *be a representation induced from a representation of* $\overline{\zeta} : \mathbb{C}\mathrm{l}_n^0 \to \mathrm{End}(V)$ *via the inclusion* $\mathrm{Spin}_n \subset \mathbb{C}\mathrm{l}_n^0$. *Then* ζ *is irreducible if and only if* $\overline{\zeta}$ *is irreducible.*

Proof. Clearly, ζ irreducible implies $\overline{\zeta}$ irreducible. For the converse, assume $\overline{\zeta}$ irreducible and ζ reducible. Since Spin_n is compact, by the remarks above we get a direct sum decomposition

$$V = V_1 \oplus \ldots \oplus V_r, \ \ r \geq 2, \tag{5.2.8}$$

into irreducible Spin_n-modules. But each V_i is a $\mathbb{C}\mathrm{l}_n^0$-module because Spin_n generates $\mathbb{C}\mathrm{l}_n^0$. This means that (5.2.8) is also a decomposition into $\mathbb{C}\mathrm{l}_n^0$-modules, a contradiction.$\rfloor$

We are now ready to present the main theorem in the representation theory of complex Clifford algebras. Its proof uses induction and a crucial ingredient is the *complex volume element*

$$\Gamma_n = i^{\left[\frac{n+1}{2}\right]} \mathrm{e}_1 \dots \mathrm{e}_n. \tag{5.2.9}$$

This does not depend on the choice of frame and its definition makes essential use of the complexification. After a simple computation we get

Proposition 5.2.3 $\Gamma_n^2 = 1$ *and* $\Gamma_n v = (-1)^{n+1} v \Gamma_n$ *for any* $v \in \mathbb{R}^n$. *In particular,* $\Gamma_n \in \mathcal{Z}(\mathbb{C}\mathrm{l}_n)$ *if n is odd and* $\Gamma_n \in \mathcal{Z}(\mathbb{C}\mathrm{l}_n^0)$ *if n is even.*

This is going to play a key role in the proof of the following classification theorem (see [Wu]).

Theorem 5.2.1 *If* $n = 2k$ *there exists a unique irreducible module of dimension* 2^k *over* $\mathbb{C}\mathrm{l}_n$. *If* $n = 2k+1$, *there exist exactly two inequivalent irreducible modules of dimension* 2^k *over* $\mathbb{C}\mathrm{l}_n$.

Proof. We have $\mathbb{C}\mathrm{l}_0 = \mathrm{Cl}_0 \otimes \mathbb{C} = \mathbb{R} \otimes \mathbb{C} = \mathbb{C}$ and this settles the theorem in case $n = 0$. Now assume the theorem holds for $n = 0, 1, \dots, 2k-2$, and let V be a $\mathbb{C}\mathrm{l}_{2k-1}$-module. By Proposition 5.2.3, $\Gamma_{2k-1} = i^{k-1}\mathrm{e}_1 \dots \mathrm{e}_{2k-1} \in \mathcal{Z}(\mathbb{C}\mathrm{l}_{2k-1})$ and $\Gamma_{2k-1}^2 = 1$. This implies a decomposition

$$V = V^+ \oplus V^-$$

into $\mathbb{C}\mathrm{l}_{2k-1}$-modules with $V^\pm = \{x \in V; \Gamma_{2k-1}x = \pm x\}$. Since $\mathbb{C}\mathrm{l}_{2k-2} \subset \mathbb{C}\mathrm{l}_{2k-1}$, both V^+ and V^- are $\mathbb{C}\mathrm{l}_{2k-2}$-modules and the induction hypothesis gives us decompositions

$$V^\pm = \bigoplus_j W_j^\pm \tag{5.2.10}$$

into irreducible $\mathbb{C}\mathrm{l}_{2k-2}$-submodules of dimension 2^{k-1}. But $x \in W_j^\pm$ implies

$$\mathrm{e}_{2k-1}x = \pm \mathrm{e}_{2k-1}\Gamma_{2k-1}x = \mp i^{k-1}\mathrm{e}_1 \dots \mathrm{e}_{2k-2}x \in W_j^\pm,$$

and this means that each $W_j^\pm$ is a $\mathbb{C}\mathrm{l}_{2k-1}$-module. Moreover, as $\mathbb{C}\mathrm{l}_{2k-1}$-modules, $W_j^+ \neq W_j^-$ (since left multiplication by e_{2k-1} gives distinct results) and the W_j^+'s (respectively, W_j^-'s) are mutually equivalent (since left multiplication by e_{2k-1} gives the same result). This proves the theorem in this case.

Now assume the theorem holds for $n = 0, 1, \ldots, 2k-1$ and let V be a $\mathbb{C}\mathrm{l}_{2k}$-module. As before, since $\mathbb{C}\mathrm{l}_{2k-1} \subset \mathbb{C}\mathrm{l}_{2k}$, V is a $\mathbb{C}\mathrm{l}_{2k-1}$-module as well and the induction hypothesis splits V into a sum of $\mathbb{C}\mathrm{l}_{2k-1}$-modules

$$V = V^+ \oplus V^-.$$

Here, $V^\pm$ is a sum of irreducible $\mathbb{C}\mathrm{l}_{2k-1}$-modules $W_j^\pm$ as in (5.2.10). Now, since $\Gamma_{2k-1}\mathrm{e}_{2k} = -\mathrm{e}_{2k}\Gamma_{2k-1}$, we have

$$\mu(V^\pm) \subset V^\mp,$$

where μ denotes left multiplication by e_{2k}. Setting $W_j = W_j^+ \oplus \mu(W_j^+)$ we finally get a decomposition

$$V = \bigoplus_j W_j$$

into mutually equivalent $\mathbb{C}\mathrm{l}_{2k}$-modules of dimension 2^k.⌋

After this, we are ready to introduce the class of representations of Spin_n we have been searching for.

Definition 5.2.3 *The spin representation* $\tau : \mathrm{Spin}_n \to \mathrm{Aut}(\Psi)$ *is obtained by restricting any irreducible representation of* $\mathbb{C}\mathrm{l}_n$ *under the inclusion* $\mathrm{Spin}_n \subset \mathbb{C}\mathrm{l}_n$.

This should be promptly appended by

Proposition 5.2.4 *If* $n = 2k+1$*, the equivalence class of* τ *does not depend on which irreducible representation of* $\mathbb{C}\mathrm{l}_n$ *we take. The corresponding module is irreducible and of dimension* 2^k*. If* $n = 2k$*, the spin representation splits as a sum*

$$\Psi = \Psi^+ \oplus \Psi^- \tag{5.2.11}$$

of two inequivalent irreducible modules of dimension 2^{k-1}*. We have* $\Psi^\pm = \{x \in \Psi; \Gamma_{2k}x = \pm x\}$ *and the 'exchange relations'*

$$\mathrm{e}_i \cdot \Psi^\pm \subset \Psi^\mp, \ \ i = 1, \ldots, n, \tag{5.2.12}$$

hold. Moreover, none of the representations above descends to a representation of SO_n.

Proof. Let us examine the case $n = 2k+1$ first. Here, according to Theorem 5.2.1, we have two inequivalent representations $\overline{\tau}^\pm : \mathbb{C}\mathrm{l}_{2k+1} \to \mathrm{Hom}(\mathrm{W}^\pm)$, each of dimension 2^k. Taking into account the inclusions $\mathrm{Spin}_n \subset \mathbb{C}\mathrm{l}^0_{2k+1} \subset \mathbb{C}\mathrm{l}_{2k+1}$ we consider $\tilde{\tau}^\pm = \overline{\tau}^\pm|_{\mathbb{C}\mathrm{l}^0_{2k+1}}$. But $\mathbb{C}\mathrm{l}^0_{2k+1} = \mathbb{C}\mathrm{l}_{2k}$ (Proposition 5.1.1) and Theorem 5.2.1 implies that, as $\mathbb{C}\mathrm{l}^0_{2k+1}$-modules, $\tilde{\tau}^+ = \tilde{\tau}^-$. If we restrict this to Spin_n, following the prescription in Definition 5.2.3, then by Proposition 5.2.2 no further decomposition arises and then $\tau = \tilde{\tau}^+ = \tilde{\tau}^-$ as Spin_n-representations. This proves the result in this case.

For the case $n = 2k$, we have by Theorem 5.2.1 a unique irreducible representation $\underline{\tau} : \mathbb{C}\mathrm{l}_{2k} \to \mathrm{Hom}(\Psi)$ with $\dim \Psi = 2^k$. Restricting this to $\mathbb{C}\mathrm{l}^0_{2k} = \mathbb{C}\mathrm{l}_{2k-1}$, we get a splitting $\underline{\tau} = \tau^+ \oplus \tau^-$ for irreducible representations $\tau^\pm : \mathbb{C}\mathrm{l}^0_{2k} \to \mathrm{Hom}(\Psi^\pm)$, $\dim \Psi^\pm = 2^{k-1}$. On the other hand, we have $\Gamma^2_{2k} = 1$ and $\Gamma_{2k} \in \mathcal{Z}(\mathbb{C}\mathrm{l}^0_{2k})$ by Proposition 5.2.3. Thus, as a $\mathbb{C}\mathrm{l}^0_{2k}$-module, Ψ splits as a sum of irreducible inequivalent modules corresponding to the ± 1-eigenspaces of (left multiplication by) Γ_{2k}. Clearly, these eigenspaces can be identified to the modules $\Psi^\pm$ above. Again, restricting this to Spin_n no further decomposition arises and $\tau = \tau^+ \oplus \tau^-$ as Spin_n-representations, as desired.

Finally, (5.2.12) follows from Proposition 5.2.3 as Γ_{2k} anti-commutes with any $v \in \mathbb{R}^n$, and the last assertion follows from the fact that the various representations we started off send -1 to $-\mathrm{Id}$.⌋

The representations $\tau^\pm : \mathrm{Spin}_n \to \mathrm{Aut}(\Psi^\pm)$ arising in the proposition are called the *half-spin representations* and elements in $\Psi = \Psi^+ \oplus \Psi^-$ are usually termed *spinors*.

From now on we assume $n = 2k$ as this is the only case considered in these notes. It is important here to complement the abstract representation theory above with a more concrete description. For this notice that $\mathbb{C}\mathrm{l}_n$ has a natural representation $\Xi : \mathbb{C}\mathrm{l}_n \to \mathrm{End}(\mathbb{C}\mathrm{l}_n)$ given by multiplication on the left. This is called the *regular representation* and enables us to consider $\mathbb{C}\mathrm{l}_n$ as a left module over itself. Clearly, irreducible submodules of $\mathbb{C}\mathrm{l}_n$ under the regular representation correspond to minimal left ideals in $\mathbb{C}\mathrm{l}_n$. The theory above then implies a decomposition

$$\mathbb{C}\mathrm{l}_n = \underbrace{\Psi \oplus \ldots \oplus \Psi}_{2^k \text{ times}}$$

into mutually equivalent minimal left ideals of dimension 2^k. It is often convenient to use the minimal ideal Ψ above as a model for the spin representation since it is specially well suited for computations. For example, it allows us to introduce a natural inner product on Ψ. We first introduce a natural inner product $\langle\,,\rangle$ on Cl_n by declaring that the monomials $\mathrm{e}_{i_1} \ldots \mathrm{e}_{i_p}$, $0 \le p \le n$, constitute an orthonormal basis. This is by its turn extended in the obvious way as a hermitian product to $\mathbb{C}\mathrm{l}_n = \mathrm{Cl}_n \otimes \mathbb{C}$. In particular, the representation $\mathrm{cl}(\mu_n) : \mathrm{SO}_n \to \mathrm{Aut}(\mathbb{C}\mathrm{l}_n)$ given on simple elements by

$$\mathrm{cl}(\mu_n)(g)(v_1 \ldots v_p) = \mu_n(g)v_1 \ldots \mu_n(g)v_p$$

is *unitary*. Moreover, since $\Psi \subset \mathbb{C}\mathrm{l}_n$ as a minimal left ideal this induces a natural hermitian product on Ψ and $\tau : \mathrm{Spin}_n \to \mathrm{Aut}(\Psi)$ is unitary with respect to this product. To prove this we just have to observe that elements in Spin_n are products of elements in S^{n-1} and this leaves us with the checking that left multiplication by unit elements is an isometry:

$$\langle ua, ua' \rangle = \langle a, a' \rangle, \;\; u \in S^{n-1}, \; a, a' \in \Psi, \tag{5.2.13}$$

and this is immediate.

But one can not finish our discussion of spin representations without mentioning an even more explicit description of the spinor space Ψ. Set $V = \mathbb{R}^n$ and introduce an orthogonal complex structure on V. This is a real linear orthogonal map J satisfying $J^2 = -\mathrm{Id}_V$. Extending this to $V \otimes \mathbb{C}$ in the obvious way we get a decomposition

$$V \otimes \mathbb{C} = W \oplus \overline{W} \tag{5.2.14}$$

into the $\pm i$-eigenspaces of J. If the inner product $\langle \, , \rangle$ on V is bilinearly extended to $V \otimes \mathbb{C}$ we see easily that $\langle b_1, b_2 \rangle = \langle c_1, c_2 \rangle = 0$, $b_i \in W$, $c_i \in \overline{W}$. Now, if we decompose $a = b + c \in V \otimes \mathbb{C}$ according to (5.2.14) we can let this act on $V \otimes \mathbb{C}$ by $av = \sqrt{2}(b \wedge v + c \wedge^* v)$. Since $b^2 = c^2 = 0$ and $bc + cb = -2\langle b, c \rangle$, a standard argument using universality implies that this extends uniquely to a representation of $\mathbb{C}\mathrm{l}_n$ on ΛW which must be equivalent to Ψ by dimensional reasons. In fact, $\mathbb{C}\mathrm{l}_n = \mathrm{End}(\Lambda W)$ and this realizes $\mathbb{C}\mathrm{l}_n$ as a full matrix algebra: $\mathbb{C}\mathrm{l}_n \cong \mathrm{M}_{2^k}(\mathbb{C})$. Actually, we have

$$\mathbb{C}\mathrm{l}_n = \mathrm{End}(\Psi) = \Psi^* \otimes \Psi = \Psi \otimes \Psi, \tag{5.2.15}$$

where the asterisk means duality and the last isomorphism is given by the metric. Notice that (5.2.15) justifies the claim made above that the Clifford algebra is the endomorphism algebra of the space of spinors.

Given this state of affairs, it is natural to ask whether the above picture can be globalized somehow. More precisely, given a Riemannian manifold $\mathbf{M}$ of dimension $n = 2k$, we can form the *Clifford algebra bundle*

$$\mathbb{C}\mathrm{l}(\mathbf{M}) = P^{\mathrm{SO}}_{\mathbf{M}} \times_{\mathrm{cl}(\mu_n)} \mathbb{C}\mathrm{l}_n, \tag{5.2.16}$$

which is a bundle of Clifford algebras in the sense that, for each $x \in \mathbf{M}$, the fiber $\mathbb{C}\mathrm{l}(\mathbf{M})_x$ is the complex Clifford algebra over $T_x\mathbf{M} \cong \mathbb{R}^n$. We can then ask whether there exists a bundle $\mathcal{S}$ over $\mathbf{M}$ such that, for each $x \in \mathbf{M}$, $\mathcal{S}_x$ is an irreducible module over $\mathbb{C}\mathrm{l}(\mathbf{M})_x$ and $\mathbb{C}\mathrm{l}(\mathbf{M}) = \mathrm{End}(\mathcal{S})$.[3] The funny thing

[3]Recall that $\mathrm{End}(\mathcal{E}) = \mathrm{Hom}(\mathcal{E}, \mathcal{E})$.

here is that there exist topological obstructions to the existence of $\mathcal{S}$. In the next chapter we shall meet the class of spin manifolds, for which the bundle $\mathcal{S}$ is always available.

For further reference, we remark that since the representation $\mathrm{cl}(\mu_n)$ is unitary with respect to the natural hermitian inner product on $\mathbb{C}\mathrm{l}_n$ introduced above we have by Proposition 3.3.2 that $\mathbb{C}\mathrm{l}(\mathbf{M})$ comes equipped with a natural metric $\langle\, ,\, \rangle$ and compatible connection ∇^c. Moreover, pointwise Clifford multiplication gives an algebra structure to $\Gamma(\mathbb{C}\mathrm{l}(\mathbf{M}))$. Finally, Clifford multiplication $\mathbb{C}\mathrm{l}_n \times \mathbb{C}\mathrm{l}_n \to \mathbb{C}\mathrm{l}_n$ is a map of SO_n-modules and the usual Leibniz rule for the derivative of a bilinear form together with (3.3.20) allow us to conclude

Proposition 5.2.5 *The connection ∇^c is a derivation on $\Gamma(\mathbb{C}\mathrm{l}(\mathbf{M}))$, which means that*

$$\nabla^c(a \cdot a') = (\nabla^c a) \cdot a' + a \cdot \nabla^c a', \quad a, a' \in \Gamma(\mathbb{C}\mathrm{l}(\mathbf{M})). \tag{5.2.17}$$

Chapter 6

Spin bundles, spin manifolds and the Dirac operator

In this chapter we shall introduce the important concept of a spin bundle. When applied to the case $\mathcal{E} = T\mathbf{M}$, this is a topological condition which makes it possible to globally define the Dirac operator on a Riemannian manifold starting from the spin representation $\tau : \mathrm{Spin}_n \to \mathrm{Aut}(\Psi)$ considered in the last section. Our presentation is an offspring of classical obstruction theory and consequently some familiarity with the basic constructions in Algebraic Topology (notably Čech cohomology) is assumed.

6.1 Spin bundles and spin manifolds

We start by recalling the classification theory of principal $\mathbf{G}$-bundles over a manifold $\mathbf{M}$ (see [St]). If $\mathbf{M}$ is covered by a family of open sets $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ such that a given principal $\mathbf{G}$-bundle P is trivial when restricted to each U_α (we say that P is *adapted* to $\mathcal{U}$) then whenever $U_\alpha \cap U_\beta \neq \emptyset$ we can define the corresponding transition function $\xi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbf{G}$ by (2.1.1) and these satisfy the cocycle conditions (2.1.2). Technically speaking, the set of data $(\mathcal{U}, \xi_{\alpha\beta})$ defines a Čech 1-cocycle with coefficients in $\mathbf{G}$ and completely determines P up to equivalence. This means that if some other principal $\mathbf{G}$-bundle P' adapted to $\mathcal{U}$ is given then P is equivalent to P' if and only if there exists a family of maps $\xi_\alpha : U_\alpha \to \mathbf{G}$ satisfying (self-explanatory notation)

$$\xi'_{\alpha\beta} = \xi_\alpha^{-1}\xi_{\alpha\beta}\xi_\alpha \tag{6.1.1}$$

on $U_\alpha \cap U_\beta$. The equivalence relation imposed by (6.1.1) defines a set $H^1(\mathbf{M}, \mathcal{U})$ parametrizing the equivalence classes of principal $\mathbf{G}$-bundles adapted to $\mathcal{U}$.

By taking refinements of $\mathcal{U}$ we eventually succeed in defining

$$H^1(\mathbf{M}, \mathbf{G}) = \lim_{\mathcal{U}} H^1(\mathbf{M}, \mathcal{U}), \tag{6.1.2}$$

where the direct limit is taken over the set of all open coverings of $\mathbf{M}$. If $\mathbf{G}$ is abelian, (6.1.2) defines the first Čech cohomology group of $\mathbf{M}$ with coefficients in $\mathbf{G}$, but in general $H^1(\mathbf{M}, \mathbf{G})$ is just a pointed space with a distinguished point corresponding to the trivial $\mathbf{G}$-bundle. In any case, the argument sketched above leads to the following classification theorem.

Theorem 6.1.1 $H^1(\mathbf{M}, \mathbf{G})$ *parameterizes the set of principal* $\mathbf{G}$*-bundles over* $\mathbf{M}$.

We now turn abruptly to obstruction theory. We assume that our manifold $\mathbf{M}$ has been triangulated and we denote by $\mathbf{M}^{(i)}$, $i = 0, 1, \ldots, n$, the union of the j-skeletons of the given triangulation for $j = 0, 1, \ldots i$. Let $\mathcal{E}$ be a *real* vector bundle over $\mathbf{M}$ of rank r and assuming that, for a given i, the restriction $\mathcal{E}|_{\mathbf{M}^{(i-1)}}$ is trivial, we pose the important question: under which conditions (if any) is it possible to extend this trivialization to $\mathbf{M}^{(i)}$?

To answer this, we assume that $\mathcal{E}$ admits a reduction to a Lie group $\mathbf{G} \subset \mathrm{Gl}_r(\mathbb{R})$. In fact, after introducing a metric on $\mathcal{E}$ we can assume without loss of generality that $\mathbf{G} \subset \mathrm{O}_r$. We then pass to the frame bundle $P^{\mathrm{O}}_{\mathcal{E}}$ of *orthonormal* frames of $\mathcal{E}$ which, under our assumption, happens to admit a reduction to a principal $\mathbf{G}$-bundle over $\mathbf{M}$, say $P^{\mathbf{G}}_{\mathcal{E}}$. Clearly, for each i, $\mathcal{E}|_{\mathbf{M}^{(i)}}$ is trivial if and only if $P^{\mathbf{G}}_{\mathcal{E}}|_{\mathbf{M}^{(i)}}$ is trivial as well. Returning to our question, the triviality of $P^{\mathbf{G}}_{\mathcal{E}}|_{\mathbf{M}^{(i-1)}}$ translates into the existence of a section $\sigma : \mathbf{M}^{(i-1)} \to P^{\mathbf{G}}_{\mathcal{E}}|_{\mathbf{M}^{(i-1)}}$. We then let $\Delta^i \subset \mathbf{M}^{(i)}$ be an i-simplex and consider its boundary $\partial\Delta^i \simeq S^{i-1}$. Since by assumption σ restricts to a map $\sigma : \partial\Delta^i \to P^{\mathbf{G}}_{\mathcal{E}}|_{\partial\Delta^i}$ and $P^{\mathbf{G}}_{\mathcal{E}}|_{\partial\Delta^i} = \partial\Delta^i \times \mathbf{G}$ we end up with a map $\sigma : \partial\Delta^i \to \mathbf{G}$ defining an element $[\sigma] \in \pi_{i-1}(\mathbf{G})$. Clearly, σ admits an extension to Δ^i if and only if $[\sigma]$ vanishes as an element of $\pi_{i-1}(\mathbf{G})$. More generally, if $C^i(\mathbf{M}, \pi_{i-1}(\mathbf{G}))$ denotes the space of i-cochains with coefficients in $\pi_{i-1}(\mathbf{G})$ then, given a section $\sigma : \mathbf{M}^{i-1} \to P^{\mathbf{G}}_{\mathcal{E}}|_{\mathbf{M}^{(i-1)}}$ corresponding to a trivialization of $P^{\mathbf{G}}_{\mathcal{E}}|_{\mathbf{M}^{(i-1)}}$, the construction above yields an element $[\sigma] \in C^i(\mathbf{M}, \pi_{i-1}(\mathbf{G}))$ which is *always* an i-cocycle in the sense that $\delta^i[\sigma] = 0$ where $\delta^i : C^i(\mathbf{M}, \pi_{i-1}(\mathbf{G})) \to C^{i+1}(\mathbf{M}, \pi_{i-1}(\mathbf{G}))$ is the usual coboundary map. In this context, the basic result in obstruction theory we shall use (cf. [Wh]) is that σ can be extended to $\mathbf{M}^{(i)}$ if and only if $[\sigma]$ is a coboundary, i.e., $[\sigma] = \delta^{i-1}\beta$ for some $\beta \in C^{i-1}(\mathbf{M}, \pi_{i-1}(\mathbf{G}))$. In other words, $\sigma|_{\mathbf{M}^{(i-1)}}$ defines an element $[\sigma] \in H^i(\mathbf{M}, \pi_{i-1}(\mathbf{G}))$, the i^{th} Čech cohomology group of $\mathbf{M}$ with coefficients in $\pi_{i-1}(\mathbf{G})$, which happens to be the obstruction to extending σ to $\mathbf{M}^{(i)}$ in the sense that the extension actually takes place if and only if the 'obstruction element' $[\sigma]$ vanishes in $H^i(\mathbf{M}, \pi_{i-1}(\mathbf{G}))$.[1]

The theory sketched above is the key to understanding the notion of a spin bundle. But first let us see how orientability fits in this context. As before, we introduce a Riemannian metric on $\mathcal{E}$ so that the corresponding frame bundle $P_{\mathcal{E}}^{\mathrm{O}}$ is a principal O_r-bundle over $\mathbf{M}$. Clearly, $P_{\mathcal{E}}^{\mathrm{O}}|_{\mathbf{M}^{(0)}}$ is always trivial and, as we have seen, the obstruction to extending this to $\mathbf{M}^{(1)}$ lies in $H^1(\mathbf{M}, \pi_0(\mathrm{O}_r)) = H^1(\mathbf{M}, \mathbb{Z}_2)$. The corresponding obstruction element is the *first Stiefel-Whitney class* of $\mathcal{E}$, denoted by $w_1(\mathcal{E})$. In order to relate this to orientability, we bring about the short exact sequence of coefficient groups

$$0 \to \mathrm{SO}_r \xrightarrow{i} \mathrm{O}_r \xrightarrow{j} \mathbb{Z}_2 \to 0,$$

which induces a long exact sequence of pointed spaces

$$\ldots \to H^1(\mathbf{M}, \mathrm{SO}_r) \xrightarrow{i_*} H^1(\mathbf{M}, \mathrm{O}_r) \xrightarrow{j_*} H^1(\mathbf{M}, \mathbb{Z}_2) \to \ldots \tag{6.1.3}$$

Recalling that $H^1(\mathbf{M}, \mathrm{O}_r)$ parameterizes the set of principal O_r-bundles over $\mathbf{M}$ so that in particular $P_{\mathcal{E}}^{\mathrm{O}} \in H^1(\mathbf{M}, \mathrm{O}_r)$, we have

Proposition 6.1.1 *$w_1(\mathcal{E}) = j_*(P_{\mathcal{E}}^{\mathrm{O}})$.*

The proof of this (and of Proposition 6.1.2 below) can be carried out by looking at the axiomatic characterization of Stiefel-Whitney classes (see [LM]).

By the exactness of (6.1.3) we see that $w_1(\mathcal{E}) = j_*(P_{\mathcal{E}}^{\mathrm{O}}) = 0$ if and only if $P_{\mathcal{E}}^{\mathrm{O}} = i_*(P')$ for some principal SO_r-bundle P', i.e. if and only if $\mathcal{E}$ is orientable. Thus, w_1 is the obstruction to orientability.

Now assume $\mathcal{E}$ is oriented and Riemannian, so that $P_{\mathcal{E}}^{\mathrm{O}}$ admits a reduction to an SO_r-bundle, say $P_{\mathcal{E}}^{\mathrm{SO}}$, and there exists a section $\sigma : \mathbf{M}^{(1)} \to P_{\mathcal{E}}^{\mathrm{SO}}|_{\mathbf{M}^{(1)}}$. The obstruction to extending this to $\mathbf{M}^{(2)}$ lies in $H^2(\mathbf{M}, \pi_1(\mathrm{SO}_r)) = H^2(\mathbf{M}, \mathbb{Z}_2)$ (here we assume $r \geq 3$) and is the *second Stiefel-Whitney class* of $\mathcal{E}$, denoted $w_2(\mathcal{E})$. As before, we have a short exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}_r \xrightarrow{\gamma_r} \mathrm{SO}_r \to 0$$

[1] Recall that all the homotopy groups of a Lie group are abelian and hence $H^i(\mathbf{M}, \pi_{i-1}(\mathbf{G}))$ is a $\pi_{i-1}(\mathbf{G})$-module indeed. Moreover, in this section we assume for convenience that the local system of coefficients defined by $P_{\mathcal{E}}^{\mathbf{G}} \to \mathbf{M}$ is simple in the sense that $\pi_1(\mathbf{M})$, the fundamental group of $\mathbf{M}$, acts trivially on the fibers of $P_{\mathcal{E}}^{\mathbf{G}} \to \mathbf{M}$, as this implies that the corresponding cohomology with local coefficients may be identified to the usual Čech cohomology (see [Wh] for more on this point).

with corresponding long exact sequence

$$
\begin{aligned}
\ldots \to H^0(\mathbf{M}, \mathrm{SO}_r) \xrightarrow{\partial_*} H^1(\mathbf{M}, \mathbb{Z}_2) \quad &\to H^1(\mathbf{M}, \mathrm{Spin}_r) \xrightarrow{\gamma_{r*}} \\
&\xrightarrow{\gamma_{r*}} H^1(\mathbf{M}, \mathrm{SO}_r) \xrightarrow{\partial_*} H^2(\mathbf{M}, \mathbb{Z}_2) \to \ldots
\end{aligned}
\tag{6.1.4}
$$

where ∂_* is the usual coboundary operator.

Proposition 6.1.2 $w_2(\mathcal{E}) = \partial_*(P_{\mathcal{E}}^{\mathrm{SO}})$.

From the exactness of (6.1.4) we see that $w_2(\mathcal{E}) = \partial_*(P_{\mathcal{E}}^{\mathrm{SO}}) = 0$ if and only if there exists a principal Spin_r-bundle $P_{\mathcal{E}}^{\mathrm{Spin}}$ over $\mathbf{M}$ such that $\gamma_{r*}(P_{\mathcal{E}}^{\mathrm{Spin}}) = P_{\mathcal{E}}^{\mathrm{SO}}$. Since from the construction of (6.1.4) $P_{\mathcal{E}}^{\mathrm{SO}}$ is obtained from $P_{\mathcal{E}}^{\mathrm{Spin}}$ as the quotient by the $\mathbb{Z}_2$-action on fibers we get

Proposition 6.1.3 *Under the above conditions, the following sentences concerning an oriented real vector bundle $\mathcal{E}$ of rank r over $\mathbf{M}$ are equivalent:*

1. $w_2(\mathcal{E}) = 0$;
2. $\mathcal{E}|_{\mathbf{M}^{(2)}}$ *is trivial;*
3. *There exists a* Spin_r*-bundle* $P_{\mathcal{E}}^{\mathrm{Spin}}$ *over* $\mathbf{M}$ *and a nontrivial two-sheeted covering map* $\gamma : P_{\mathcal{E}}^{\mathrm{Spin}} \to P_{\mathcal{E}}^{\mathrm{SO}}$ *such that the diagram*

$$
\begin{array}{ccc}
P_{\mathcal{E}}^{\mathrm{Spin}} & \xrightarrow{\gamma} & P_{\mathcal{E}}^{\mathrm{SO}} \\
& \searrow & \downarrow \\
& & \mathbf{M}
\end{array}
$$

commutes and $\gamma(pg) = \gamma(p)\gamma_r(g)$, *for* $p \in P_{\mathcal{E}}^{\mathrm{Spin}}$ *and* $g \in \mathrm{Spin}_r$.

If any of the conditions above happens we say that $\mathcal{E}$ is a *spin* bundle. In particular, an oriented Riemannian manifold of dimension $n \geq 3$ is said to be *spin* if its tangent bundle $T\mathbf{M}$ is spin in the above sense. We remark that $H^0(\mathbf{M}, \mathrm{SO}_r) = \pi_0(\mathrm{SO}_r)$ is trivial and from (6.1.4) we see that the set of spin structures on $\mathcal{E}$ (corresponding to the various elements in $H^1(\mathbf{M}, \mathrm{Spin}_r)$ mapped onto $P_{\mathcal{E}}^{\mathrm{SO}}$) is parameterized by $H^1(\mathbf{M}, \mathbb{Z}_2)$. We will not go into the question of when a given manifold (or vector bundle) admits a spin structure, except for the following result, whose proof, taking into account the discussion above, is immediate.

Proposition 6.1.4 *Any parallelizable manifold (in particular any Lie group) is spin.*

6.2 The Dirac operator on spin manifolds

We now turn to the geometry of spin manifolds. Our intention is to justify our claim that on a spin manifold $\mathbf{M}$ of dimension $n = 2k$ the Dirac operator is globally defined and enjoys nice properties (recall the discussion at the Introduction to Chapter 5). In case $\mathbf{M}$ is spin we fix a spin structure so we get a Spin_n-bundle $P_{\mathbf{M}}^{\mathrm{Spin}}$ over $\mathbf{M}$ and a two-sheeted covering map $\gamma : P_{\mathbf{M}}^{\mathrm{Spin}} \to P_{\mathbf{M}}^{\mathrm{SO}}$ satisfying the condition in Proposition 6.1.3. If $\omega \in \mathcal{A}^1(P_{\mathbf{M}}^{\mathrm{SO}}, \mathfrak{so}_n)$ is the Levi-Civita connection on the frame bundle $\pi : P_{\mathbf{M}}^{\mathrm{SO}} \to \mathbf{M}$, then we lift this to $P_{\mathbf{M}}^{\mathrm{Spin}}$ under γ in order to obtain a connection $\omega^s = \gamma^*\omega \in \mathcal{A}(P_{\mathbf{M}}^{\mathrm{Spin}}, \mathfrak{so}_n)$, the so-called *spin connection*.[2] With the spin representation τ at hand, we can apply the associated bundle construction to the situation and form the *spinor bundle* over $\mathbf{M}$:

$$\mathcal{S}(\mathbf{M}) = P_{\mathbf{M}}^{\mathrm{Spin}} \times_\tau \Psi.$$

This is a complex vector bundle of rank 2^k over $\mathbf{M}$. Elements in $\Gamma(\mathcal{S}(\mathbf{M}))$ are usually referred to as *spinors*. Clearly, for each $x \in \mathbf{M}$, $\mathcal{S}(\mathbf{M})_x$ is an irreducible module over $\mathbb{C}\mathrm{l}(\mathbf{M})_x$ and in fact (compare with (5.2.15))

$$\mathbb{C}\mathrm{l}(\mathbf{M}) = \mathrm{End}(\mathcal{S}(\mathbf{M})) = \mathcal{S}^*(\mathbf{M}) \otimes \mathcal{S}(\mathbf{M}) = \mathcal{S}(\mathbf{M}) \otimes \mathcal{S}(\mathbf{M}). \tag{6.2.5}$$

In particular, $\Gamma(\mathcal{S}(\mathbf{M}))$ is a module over $\Gamma(\mathbb{C}\mathrm{l}(\mathbf{M}))$. Since Ψ is unitary, $\mathcal{S}(\mathbf{M})$ comes equipped with a natural metric $\langle\,,\,\rangle$ and compatible connection ∇^s. We highlight here two basic properties of these structures:

1. Clifford multiplication by unit tangent vectors is an isometry:

$$\langle u\varphi, u\varphi' \rangle = \langle \varphi, \varphi' \rangle, \;\; u \in \mathcal{X}(\mathbf{M}), \;\; |u| = 1, \;\; \varphi, \varphi' \in \Gamma(\mathcal{S}(\mathbf{M})). \tag{6.2.6}$$

2. The natural connections on $\mathbb{C}\mathrm{l}(\mathbf{M})$ and $\mathcal{S}(\mathbf{M})$ are compatible with the $\Gamma(\mathbb{C}\mathrm{l}(\mathbf{M}))$-module structure on $\Gamma(\mathcal{S}(\mathbf{M}))$:

$$\nabla^s(a\varphi) = \nabla^c a \cdot \varphi + a\nabla^s \varphi, \;\; a \in \Gamma(\mathbb{C}\mathrm{l}(\mathbf{M})), \;\; \varphi \in \Gamma(\mathcal{S}(\mathbf{M})). \tag{6.2.7}$$

The first property follows from (5.2.13) and the second one from the fact that

$$T\mathbf{M} = P_{\mathbf{M}}^{\mathrm{Spin}} \times_{\mu_n \circ \gamma_n} \mathbb{R}^n$$

and that left Clifford multiplication $\mathbb{R}^n \otimes \Psi \to \Psi$ is a map of Spin_n-modules.

[2] In the discussion below, we rely upon the notation of Chapter 3 and will use the superscript s to refer to the objects (covariant derivative, curvature, etc.) coming from the spin connection.

We are now finally ready to introduce the main character of our story, the *Dirac operator* $\partial\!\!\!/ : \Gamma(\mathcal{S}(\mathbf{M})) \to \Gamma(\mathcal{S}(\mathbf{M}))$, defined by the sequence of arrows

$$\Gamma(\mathcal{S}(\mathbf{M})) \stackrel{\nabla^s}{\to} \Gamma(T^*\mathbf{M} \otimes \mathcal{S}(\mathbf{M})) \stackrel{\langle,\rangle}{\to} \Gamma(T\mathbf{M} \otimes \mathcal{S}(\mathbf{M})) \stackrel{\cdot}{\to} \Gamma(\mathcal{S}((\mathbf{M})),$$

where the dot means Clifford multiplication. Alternatively, in terms of a local frame for $T\mathbf{M}$,

$$\partial\!\!\!/ = \sum_{i=1}^{n} \mathrm{e}_i \cdot \nabla^s_{\mathrm{e}_i} \tag{6.2.8}$$

We see that $\partial\!\!\!/$ is a first order linear differential operator.

Further properties of $\partial\!\!\!/$ can be determined by noticing that, since $u^2 = -1$, (6.2.6) means that

$$\langle u\varphi, \varphi' \rangle = -\langle \varphi, u\varphi' \rangle. \tag{6.2.9}$$

For $\varphi, \varphi' \in \Gamma(\mathcal{S}(\mathbf{M}))$ we now compute at $x \in \mathbf{M}$ (after choosing a normalized frame):

$$\begin{aligned}
\langle \partial\!\!\!/ \varphi, \varphi' \rangle - \langle \varphi, \partial\!\!\!/ \varphi' \rangle &= \sum_i \langle \mathrm{e}_i \nabla^s_{\mathrm{e}_i} \varphi, \varphi' \rangle - \langle \varphi, \mathrm{e}_i \nabla_{\mathrm{e}_i} \varphi' \rangle \\
&\stackrel{(6.2.9)}{=} \sum_i \langle \mathrm{e}_i \nabla^s_{\mathrm{e}_i} \varphi, \varphi' \rangle + \langle \mathrm{e}_i \varphi, \nabla_{\mathrm{e}_i} \varphi' \rangle \\
&\stackrel{(6.2.7)}{=} \sum_i \langle \nabla^s_{\mathrm{e}_i} (\mathrm{e}_i \varphi), \varphi' \rangle + \langle \mathrm{e}_i \varphi, \nabla^s_{\mathrm{e}_i} \varphi' \rangle \\
&= \sum_i \mathrm{e}_i \langle \mathrm{e}_i \varphi, \varphi' \rangle = \operatorname{div} X,
\end{aligned}$$

where compatibility was used in the last identity and $X \in \mathcal{X}(\mathbf{M})$ is defined by $\langle X, Y \rangle = \langle \nabla_Y \varphi, \varphi' \rangle$. Integrating this over $\mathbf{M}$ (assuming closedness) we see that $\partial\!\!\!/$ is formally selfadjoint:

$$(\partial\!\!\!/ \varphi, \varphi') = (\varphi, \partial\!\!\!/ \varphi').$$

However, in order to have an interesting index theory, one has to break this symmetry, and this is accomplished by considering the *half-spinor bundles*

$$\mathcal{S}^{\pm}(\mathbf{M}) = P^{\mathrm{Spin}}_{\mathbf{M}} \times_{\tau^{\pm}} \Psi^{\pm}.$$

From (5.2.11), one clearly has $\mathcal{S}(\mathbf{M}) = \mathcal{S}^{+}(\mathbf{M}) \oplus \mathcal{S}^{-}(\mathbf{M})$, a decomposition preserved by ∇^s and orthogonal with respect to $\langle \, , \rangle$. Coupled with (5.2.12), this implies, in view of (6.2.8),

$$\partial\!\!\!/ \left(\Gamma(\mathcal{S}^{\pm}(\mathbf{M})) \right) \subset \Gamma(\mathcal{S}^{\mp}(\mathbf{M}))$$

and this enables us to consider the *Atiyah-Singer-Dirac operator*

$$\partial\!\!\!/^{+} = \partial\!\!\!/|_{\Gamma(\mathcal{S}^{+}(\mathbf{M}))} : \Gamma(\mathcal{S}^{+}(\mathbf{M})) \to \Gamma(\mathcal{S}^{-}(\mathbf{M})), \tag{6.2.10}$$

whose formal adjoint[3] is

$$\partial\!\!\!/^{-} = \partial\!\!\!/|_{\Gamma(\mathcal{S}^{-}(\mathbf{M}))} : \Gamma(\mathcal{S}^{-}(\mathbf{M})) \to \Gamma(\mathcal{S}^{+}(\mathbf{M})). \tag{6.2.11}$$

We now define

$$\operatorname{ind}\partial\!\!\!/^{+} = \dim\ker\partial\!\!\!/^{+} - \dim\ker\partial\!\!\!/^{-}, \tag{6.2.12}$$

the *index* of $\partial\!\!\!/^{+}$. It is precisely this integer that the Atiyah-Singer formula computes in terms of topological-geometric data.[4]

Let us derive a few preliminary consequences of the above constructions. To this effect, we consider the *Dirac Laplacian*

$$\Delta = \partial\!\!\!/^{2} : \Gamma(\mathcal{S}(\mathbf{M})) \to \Gamma(\mathcal{S}(\mathbf{M})).$$

This a second order selfadjoint differential operator and a computation entirely similar to the one leading to (4.2.10) gives

$$\Delta = \nabla^{s*}\nabla^{s} + \mathcal{R}, \tag{6.2.13}$$

where $\nabla^{s*}\nabla^{s} : \Gamma(\mathcal{S}(\mathbf{M})) \to \Gamma(\mathcal{S}(\mathbf{M}))$ is the Bochner Laplacian on $\mathcal{S}(\mathbf{M})$ (see (4.2.8)) and $\mathcal{R} : \Gamma(\mathcal{S}(\mathbf{M})) \to \Gamma(\mathcal{S}(\mathbf{M}))$ is a selfadjoint algebraic operator acting linearly on fibers by

$$\mathcal{R} = \frac{1}{2}\sum_{i,j=1}^{n} \mathrm{e}_i \mathrm{e}_j R^{s}_{\mathrm{e}_i,\mathrm{e}_j}, \tag{6.2.14}$$

and R^s is the curvature tensor of $\mathcal{S}(\mathbf{M})$ with respect to the spin connection ∇^s. Thus, in order to determine the 0^{th}-order in (6.2.13) one has to compute R^s.

For this one has to express Ω^s in terms of a local frame $\mathrm{e} = (\mathrm{e}_1, \ldots, \mathrm{e}_n)$ for $T\mathbf{M}$. We first observe that by (3.3.21) with $\rho = \mu_n$ we have $R_{X,Y} = \Omega(\tilde{X}, \tilde{Y})$ where $\pi_*(\tilde{X}) = X$ and $\pi_*(\tilde{Y}) = Y$ (see the comment following Proposition 3.3.3). Moreover, we can assume that $\tilde{X}$ (respect. $\tilde{Y}$) is an extension of $\mathrm{e}_*(X)$

[3] This means of course that $(\partial\!\!\!/^{+}\psi^{+}, \psi^{-}) = (\psi^{+}, \partial\!\!\!/^{-}\psi^{-})$ for $\psi^{\pm} \in \Gamma(\mathcal{S}^{\pm}(\mathbf{M}))$.

[4] The closedness of $\mathbf{M}$ implies, via elliptic theory, that both terms on the righthand side of (6.2.12) are finite (see Section 9.2 below). It is clear furthermore that all the dimensions appearing in the definition of the index should be computed over $\mathbb{C}$.

(respect. $e_*(Y)$) around the image of e. In terms of the standard basis for $\mathfrak{so}_n$, all of this means that

$$\begin{aligned}\Omega(\tilde{X},\tilde{Y}) &= \sum_{k<l}\langle R_{X,Y}e_k, e_l\rangle e_k\wedge e_l = \sum_{k<l}\Theta_{kl}(X,Y)e_k\wedge e_l\\ &= \sum_{k<l} e^*\Omega_{kl}(X,Y)e_k\wedge e_l = \sum_{k<l}\Omega_{kl}(e_*X,e_*Y)e_k\wedge e_l\\ &= \sum_{k<l}\Omega_{kl}(\tilde{X},\tilde{Y})e_k\wedge e_l,\end{aligned}$$

which we rewrite simply as

$$\Omega = \sum_{k<l}\Omega_{kl}\otimes e_k\wedge e_l.$$

On the other hand, as for the curvature form Ω^s on $P_{\mathbf{M}}^{\mathrm{Spin}}$ associated to ω^s, we have

$$\Omega^s = d\omega^s + \omega^s\wedge\omega^s = \gamma^*(d\omega+\omega\wedge\omega) = \gamma^*(\Omega),$$

and since $\gamma = \gamma_n$ along the fibers, it follows from Proposition 5.1.7 that

$$\Omega^s = \frac{1}{2}\sum_{k<l}\Theta_{kl}\otimes e_k e_l.$$

We now apply again (3.3.21) with $\rho = \tau$ (recalling that the spin representation $\tau : \mathrm{Spin}_n \to \mathrm{Aut}(\Psi)$ is induced by left Clifford multiplication) to get $R^s_{X,Y} = \Omega^s(\tilde{X},\tilde{Y})$.[5] From this we obtain

$$\begin{aligned}R^s_{X,Y} &= \frac{1}{2}\sum_{k<l}\Theta_{k,l}(X,Y)e_k e_l\\ &= \frac{1}{2}\sum_{k<l}\langle R_{e_k,e_l}X,Y\rangle e_k e_l\\ &= \frac{1}{4}\sum_{k,l}\langle R_{e_k,e_l}X,Y\rangle e_k e_l,\end{aligned}$$

and inserting this in (6.2.14) (with $X = e_i$ and $Y = e_j$),

$$\begin{aligned}\mathcal{R} &= \frac{1}{8}\sum_{i,j,k,l}\langle R_{e_i,e_j}e_k, e_l\rangle e_i e_j e_k e_l\\ &= \frac{1}{8}\sum_l\left\{\frac{1}{3}\sum_{i\neq j\neq k\neq i}\langle R_{e_i,e_j}e_k + R_{e_j,e_k}e_i + R_{e_k,e_i}e_j, e_l\rangle e_i e_j e_k\right.\\ &\qquad\left. + \sum_{i,j}\langle R_{e_i,e_j}e_i, e_l\rangle e_i e_j e_i + \sum_{i,j}\langle R_{e_i,e_j}e_j, e_l\rangle e_i e_j e_j\right\} e_l.\end{aligned}$$

[5] Here, for example, $\tilde{X}$ still denotes a local lifting of $\tilde{X}$ under γ.

From (4.2.13), the first term in the l-sum above vanishes and the remaining ones sum up to give

$$\begin{aligned}\mathcal{R} &= \frac{1}{4}\sum_{i,j,l}\langle R_{e_i,e_j}e_i, e_l\rangle e_j e_i \\ &= -\frac{1}{4}\sum_{i,j,l}\langle R_{e_j,e_i}e_i, e_l\rangle e_j e_i \\ &= -\frac{1}{4}\sum_{j,l}\langle \mathrm{Ric}(e_j), e_l\rangle e_j e_l.\end{aligned}$$

But for $j \neq l$, the summand is skew-symmetric in j and l and the corresponding sum vanishes. This leaves us with

$$\mathcal{R} = -\frac{1}{4}\sum_{j}\langle \mathrm{Ric}(e_j), e_j\rangle e_j e_j = \frac{1}{4}\sum_{j}\langle \mathrm{Ric}(e_j), e_j\rangle \stackrel{\text{def.}}{=} \frac{1}{4}\kappa,$$

where κ denotes the *scalar curvature* of $\mathbf{M}$. Putting all the pieces of our computation together we finally get a celebrated result by Lichnerowicz [L]:

Theorem 6.2.1 *Let* $\mathbf{M}$ *be a (not necessarily closed) spin manifold of dimension* $n = 2k$ *with a fixed spin structure and let* $\partial\!\!\!/$ *be the corresponding Dirac operator. Then the action of the Dirac Laplacian* $\Delta = \partial\!\!\!/^2$ *on spinors is given by*

$$\Delta = \nabla^{s*}\nabla^{s} + \frac{1}{4}\kappa. \tag{6.2.15}$$

We now mention a far-reaching application of this formula, also due to Lichnerowicz. A spinor $\psi \in \Gamma(\mathcal{S}(\mathbf{M}))$ is said to be *harmonic* if $\Delta\psi = 0$. Assuming that $\mathbf{M}$ is closed, this is equivalent to $\partial\!\!\!/\psi = 0$ since $(\Delta\psi, \psi) = \|\partial\!\!\!/\psi\|^2$ in this case. Using the same argument as in the proof of Proposition 4.2.1 we obtain

Theorem 6.2.2 *Under the conditions above, let* κ *be quasi-positive in the sense that* $\kappa \geq 0$ *everywhere and* $\kappa > 0$ *somewhere in* $\mathbf{M}$*; here use assume that* $\mathbf{M}$ *is closed. Then* $\ker\partial\!\!\!/ = \{0\}$, *i.e.* $\mathbf{M}$ *does not carry any nontrivial harmonic spinor.*

The importance of this vanishing theorem for Riemannian geometry should not be overlooked. In effect, $\ker\partial\!\!\!/ = \{0\}$ certainly implies $\ker\partial\!\!\!/^+ = \ker\partial\!\!\!/^- = \{0\}$ and then $\operatorname{ind}\partial\!\!\!/^+ = 0$ by (6.2.12). Now, if we further restrict ourselves to the case $k = 2l$, the Atiyah-Singer index formula (Theorem 8.1.2) expresses $\operatorname{ind}\partial\!\!\!/^+$ as a certain topological invariant of the spin manifold $\mathbf{M}$, the $\hat{A}$-genus, which we denote by $\hat{A}(\mathbf{M})$.[6] So we have

[6]See Section 7.2 for the precise definition of this invariant.

Theorem 6.2.3 *([L]) If $\mathbf{M}$ is a closed spin manifold of dimension $4l$ with quasi-positive scalar curvature then $\hat{A}(\mathbf{M}) = 0$.*

One should think of this as an obstruction to the existence of metrics with quasi-positive scalar curvature on certain spin manifolds. In fact, one can exhibit rather explicit examples of spin manifolds with $\hat{A}(\mathbf{M}) \neq 0$. For instance, a standard computation (see [Y], for example) shows that the quartic complex surface $\mathbf{M} \subset P^3$, the complex projective 3-space, given in homogeneous coordinates by $z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$ is spin and has $\hat{A}(\mathbf{M}) \neq 0$. For manifolds like this one, the existence of metrics with quasi-positive scalar curvature is definitely ruled out by the theorem above. Just to put this outstanding result in its proper perspective, we mention that, by using PDE techniques, Kazdan and Warner [KW] were able to show that any smooth function on a closed manifold $\mathbf{M}$ (not necessarily spin) which *is negative somewhere* is the scalar curvature of some Riemannian metric on $\mathbf{M}$. This probably will furnish a rough idea on how subtle are the questions that the methods of Spin Geometry are able to successfully approach.

Chapter 7

Characteristic classes

As we have already proclaimed, the theory of characteristic classes gives a systematic approach to the fundamental question of how far from trivial a given vector bundle is. Also, characteristic classes are basic in the formulation of the Atiyah-Singer index theorem and its many extensions: for example, the $\hat{A}$-genus $\hat{A}(\mathbf{M})$ appearing in Theorem 6.2.3 is a rational linear combination of certain products of the Pontrjagin characteristic classes of $\mathbf{M}$. The most basic characteristic classes appearing in Differential Geometry can be organized into four types: the Stiefel-Whitney classes $w_i(\mathcal{E}) \in H^i(\mathbf{M}, \mathbb{Z}_2)$ for a real vector bundle $\mathcal{E}$ over $\mathbf{M}$, the Chern classes $c_i(\mathcal{E}) \in H^{2i}(\mathbf{M}, \mathbb{Z})$ for a complex vector bundle over $\mathbf{M}$, the Pontrjagin classes $p_i(\mathcal{F}) \in H^{4i}(\mathbf{M}, \mathbb{Z})$ for a real vector bundle $\mathcal{F}$ over $\mathbf{M}$, and if $\mathcal{F}$ is further required to be oriented and of even rank $r = 2s$ we have the Euler class $\chi(\mathcal{F}) \in H^r(\mathbf{M}, \mathbb{Z})$. We have had some acquaintance with the Stiefel-Whitney classes in Section 6.1 and in this chapter we approach the remaining three types by using Chern-Weil theory which, as we shall see, is an offspring of the theory of connections as developed in Chapter 3.

7.1 The concept of characteristic classes

A *characteristic class* is a rule that to each (real or complex) vector bundle $\mathcal{E}$ over $\mathbf{M}$ assigns an element $\mathcal{P}(\mathcal{E}) \in H^*(\mathbf{M}, \Lambda)$, for some coefficient group Λ. We also demand that the assignment $\mathcal{E} \to \mathcal{P}(\mathcal{E})$ is functorial in the sense that $\mathcal{P}(f^*\mathcal{E}) = f^*(\mathcal{P}(\mathcal{E}))$, where $f^*\mathcal{E} \to \mathbf{M}'$ is the pulled back bundle under a map $f : \mathbf{M}' \to \mathbf{M}$ and f^* is the induced homomorphism in cohomology.

The Chern-Weil approach to characteristic classes is an elaboration of the ideas leading to Proposition 3.2.4, according to which the degree of nontriviality of a vector bundle should be related to the total amount of curvature of a suitable connection defined on it. More precisely, let $\mathcal{E} \to \mathbf{M}$ be a *complex* vector bundle and let ∇ be a connection on $\mathcal{E}$. Recall from Section 3.2 that, given a local frame $\mathrm{e} = (\mathrm{e}_1, \ldots, \mathrm{e}_r)$ we have the corresponding connection 1-form $\theta = \{\theta_{ij}\}$, $i, j = 1, \ldots, r = \operatorname{rank} \mathcal{E}$. We are not assuming that $\mathcal{E}$ is equipped with a hermitian metric so that in principle θ takes values in $\mathfrak{gl}_r(\mathbb{C})$, the Lie algebra of $\mathrm{GL}_r(\mathbb{C})$. Even though θ does not behave properly under a change of frames, we have seen that the corresponding curvature 2-form

$$\Theta = d\theta + \theta \wedge \theta \tag{7.1.1}$$

does it: if $\mathrm{e} = \bar{\mathrm{e}}\xi$, with $\xi : U \subset \mathbf{M} \to \mathrm{GL}_r(\mathbb{C})$, one has

$$\Theta = \xi^{-1} \overline{\Theta} \xi. \tag{7.1.2}$$

Moreover, exterior derivation of (7.1.1) gives the *second Bianchi identity*

$$d\Theta = \Theta \wedge \theta - \theta \wedge \Theta,$$

which we rewrite simply as

$$d\Theta = [\Theta, \theta]. \tag{7.1.3}$$

As we shall see, the whole construction is based on formulas (7.1.2) and (7.1.3). We start by exploring the former one.

Definition 7.1.1 *Let $\mathcal{P} : \mathfrak{gl}_r(\mathbb{C}) \to \mathbb{C}$ be a function which is polynomial in the entries of $A \in \mathfrak{gl}_r(\mathbb{C})$. We say that $\mathcal{P}$ is invariant if it satisfies*

$$\mathcal{P}(gAg^{-1}) = \mathcal{P}(A), \tag{7.1.4}$$

for any $g \in \mathrm{GL}_r(\mathbb{C})$ and $A \in \mathfrak{gl}_r(\mathbb{C})$.

Assume for the moment that $\mathcal{P}$ is a complex homogeneous polynomial function of degree k and let θ be as above with corresponding curvature form Θ. Then, the assignment $\theta \mapsto \mathcal{P}(\Theta) \in \mathcal{A}^{2k}(U, \mathbb{C})$ furnishes a locally defined[1] $2k$-form which, by (7.1.2) and (7.1.4), actually does *not* depend on the local choice of frames and hence extends to a globally defined form $\mathcal{P}(\Theta) \in \mathcal{A}^{2k}(\mathbf{M})$ which certainly depends on the choice of θ. We call $\mathcal{P}(\Theta)$ the *Chern-Weil form* associated to θ. We are going to prove that $\mathcal{P}(\Theta)$ is closed indeed ($d\mathcal{P}(\Theta) = 0$) and its de Rham cohomology class does *not* depend on the

[1] Recall that the wedge product is commutative on even forms so $\mathcal{P}(\Theta) = \mathcal{P}(\Theta_{ij})$ is well defined.

choice of θ, but first we pose ourselves the following basic question: what is the structure of the space $\mathrm{Inv}(\mathfrak{gl}_r(\mathbb{C}))$ of invariant functions as above?

First recall from elementary algebra that if $A \in \mathfrak{gl}_r(\mathbb{C})$ then

$$\det(I + A) = \sum_{i=1}^{r} \phi_k(A), \tag{7.1.5}$$

where $\phi_k(A)$ is the k^{th} elementary symmetric function of the eigenvalues of A. Clearly, each $\phi_k \in \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{C}))$ is homogeneous of degree k.

Proposition 7.1.1 $\mathrm{Inv}(\mathfrak{gl}_r(\mathbb{C})))$ *is a polynomial algebra generated by the* ϕ_k*'s.*

Proof. Recall the following two well-known facts: i) the space of complex matrices with distinct eigenvalues is dense in $\mathfrak{gl}_r(\mathbb{C})$; ii) any complex matrix with distinct eigenvalues is conjugate to a diagonal matrix. From this we see that each $\mathcal{P} \in \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{C}))$ is fully determined by its restriction to the space of diagonal matrices. Moreover, since we can arbitrarily exchange the position of the eigenvalues of a diagonal matrix by suitable conjugations, $\mathcal{P}$ should be a polynomial *symmetric* function of the eigenvalues. The result follows from the fundamental theorem on symmetric polynomial functions [Wa].⌋

We can now present a version of the basic result in Chern-Weil theory.

Theorem 7.1.1 *For any* $\mathcal{P} \in \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{C}))$*,* $\mathcal{P}(\Theta)$ *is a closed differential form whose de Rham cohomology class does not depend on the choice of the connection* θ *whose curvature form is* Θ*.*

Proof. It is convenient to assume that $\mathcal{P}$ is homogeneous of degree k and to introduce a *polarization*[2] $\tilde{\mathcal{P}}$ for $\mathcal{P}$. This is a k-linear map $\tilde{\mathcal{P}} : \mathfrak{gl}_r(\mathbb{C}) \times \ldots \times \mathfrak{gl}_r(\mathbb{C}) \to \mathbb{C}$ such that $\mathcal{P}(A) = \tilde{\mathcal{P}}(A, \ldots, A)$. Clearly, $\tilde{\mathcal{P}}$ is also invariant in the obvious way and we can assume, after a change of frames, that $\theta(x) = 0$ for a given $x \in \mathbf{M}$. Then we compute

$$\begin{aligned} d\mathcal{P}(\Theta) &= d\tilde{\mathcal{P}}(\Theta, \ldots, \Theta) \\ &= \sum \tilde{\mathcal{P}}(\Theta, \ldots, d\Theta, \ldots, \Theta) \\ &\overset{(7.1.3)}{=} \sum \tilde{\mathcal{P}}(\Theta, \ldots, [\Theta, \theta], \ldots, \Theta), \end{aligned}$$

and this vanishes at x, showing that $d\mathcal{P}(\Theta) = 0$.

Now let θ_0 and θ_1 be connections with curvature forms Θ_0 and Θ_1, respectively, so that, by the obvious variant of Proposition 3.2.1, $\eta = \theta_1 - \theta_0$ is a

[2] Polarizations always exist.

$\mathfrak{gl}_r(\mathbb{C})$-valued 1-form. Consider, for $0 \le t \le 1$, the 'homotopy' of connections $\theta_t = \theta_0 + t\eta$, with the corresponding arc of curvature forms Θ_t. We have

$$\frac{d\Theta_t}{dt} = d\eta + [\theta_t, \eta],$$

and hence

$$\begin{aligned}
\frac{d}{dt}\mathcal{P}(\Theta_t) &= \frac{d}{dt}\tilde{\mathcal{P}}\left(\Theta_t, \dots, \Theta_t\right) \\
&= k\tilde{\mathcal{P}}\left(\frac{d\Theta_t}{dt}, \Theta_t, \dots, \Theta_t\right) \\
&= k\tilde{\mathcal{P}}\left(d\eta + [\theta_t, \eta], \Theta_t, \dots, \Theta_t\right)
\end{aligned}$$

On the other hand, again by (7.1.3), now applied to Θ_t,

$$d\tilde{\mathcal{P}}(\eta, \Theta_t, \dots, \Theta_t) = \tilde{\mathcal{P}}(d\eta, \Theta_t, \dots, \Theta_t) + (k-1)\tilde{\mathcal{P}}\left(\eta, [\Theta_t, \theta_t], \Theta_t, \dots, \Theta_t\right),$$

and further exploring (7.1.4) and a variant of the argument leading to (1.2.19), we get

$$\tilde{\mathcal{P}}\left([\theta_t, \eta], \Theta_t, \dots, \Theta_t\right) - (k-1)\tilde{\mathcal{P}}\left(\eta, [\Theta_t, \theta_t], \Theta_t, \dots \Theta_t\right) = 0.$$

It follows that

$$\frac{1}{k}\frac{d}{dt}\mathcal{P}(\Theta_t) = d\tilde{\mathcal{P}}(\eta, \Theta_t, \dots, \Theta_t),$$

and integrating we finally have

$$\frac{1}{k}\left(\mathcal{P}(\Theta_1) - \mathcal{P}(\Theta_0)\right) = d\left(\int_0^1 \tilde{\mathcal{P}}(\eta, \Theta_t, \dots, \Theta_t)\, dt\right),$$

showing that $\mathcal{P}(\Theta_0)$ and $\mathcal{P}(\Theta_1)$ are in the same cohomology class.⌋

7.2 The Chern and Pontrjagin classes

From Theorem 7.1.1 we have learned that for $\mathcal{P} \in \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{C}))$ and any connection ∇ on a complex vector bundle $\mathcal{E} \to \mathbf{M}$ the recipe $\theta \mapsto \mathcal{P}(\Theta)$ furnishes a de Rham cohomology class over $\mathbf{M}$. Clearly, this assignment is a characteristic class since, if $f : \mathbf{M}' \to \mathbf{M}$ is any map, $f^*\Theta$ is the curvature form of the pulled back connection on $f^*\mathcal{E}$. Notice that if $\mathcal{P}$ is not homogeneous, we can apply the construction to each of its homogeneous terms and eventually to get the characteristic class corresponding to $\mathcal{P}$ in the cohomology ring

$H^*(\mathbf{M};\mathbb{C})$. As a rule, we shall denote by $[\mathcal{P}(\Theta)]_k$ the component of $\mathcal{P}(\Theta)$ of degree k and if $\mathbf{M}$ is closed of dimension n we shall represent by

$$\langle \mathcal{P}(\Theta), \mathbf{M}\rangle = \langle [\mathcal{P}(\Theta)]_n, \mathbf{M}\rangle = \int_{\mathbf{M}} [\mathcal{P}(\Theta)]_n$$

the evaluation of $\mathcal{P}(\Theta)$ over the fundamental cycle of $\mathbf{M}$.

Definition 7.2.1 *The characteristic classes associated to the invariant functions*

$$c_k(\Theta) = \phi_k\left(\frac{i}{2\pi}\Theta\right) \in H^{2k}(\mathbf{M},\mathbb{C}), \quad k = 0,1,\dots,r,$$

are the Chern classes of $\mathcal{E}$. We shall denote them by $c_k(\mathcal{E})$. The total Chern class of $\mathcal{E}$ is given by

$$c(\mathcal{E}) = \det\left(I + \frac{i}{2\pi}\Theta\right) = \sum_k c_k(\mathcal{E}) \in H^*(\mathbf{M},\mathbb{C}).$$

We list a few interesting properties of Chern classes.

Proposition 7.2.1 *Given complex vector bundles $\mathcal{E}$ and $\mathcal{E}'$ over $\mathbf{M}$ we have:*

1. *$c_k(\mathcal{E}) \in H^{2k}(\mathbf{M},\mathbb{R})$ for any k;*
2. *If $\mathcal{E}$ is trivial then $c(\mathcal{E}) = 1$;*
3. *$c(\mathcal{E}\oplus\mathcal{E}') = c(\mathcal{E})c(\mathcal{E}')$, where the product on the left is the one of the de Rham cohomology ring;*
4. *If $\mathcal{E} = \mathcal{E}'\oplus\mathcal{E}''$ with $\mathcal{E}''$ trivial then $c_i(\mathcal{E}) = 0$ for $i \geq \mathrm{rank}(\mathcal{E}') + 1$;*
5. *Assume that $\mathcal{E}$ is the complexification of a real vector bundle, i.e. $\mathcal{E} = \mathcal{F}\otimes\underline{\mathbb{C}}$ with $\mathcal{F}\to\mathbf{M}$ real. Then $c_k(\mathcal{E}) = 0$ for k odd.*

Proof. The assertions (2) and (3) are easy consequences of the definition. As for (1), choose a hermitian metric on $\mathcal{E}$ compatible with a given connection so that the corresponding curvature 2-form is skew-hermitian: $\Theta_{ij} = -\overline{\Theta}_{ji}$. We then have

$$\begin{aligned} c(\mathcal{E}) &= \det\left(\delta_{ij} + \frac{i}{\pi}\Theta_{ij}\right) = \det\left(\delta_{ji} - \frac{i}{\pi}\overline{\Theta}_{ji}\right) \\ &= \det\left(\delta_{ji} + \frac{\overline{i\Theta_{ji}}}{2\pi}\right) = \overline{\det\left(\delta_{ji} + \frac{i}{\pi}\Theta_{ji}\right)} = \overline{c(\mathcal{E})}. \end{aligned}$$

Clearly, (4) follows from (2) and (3). Finally, in (5) choose a Riemannian metric on $\mathcal{F}$ compatible with a given connection so that the corresponding curvature 2-form is skew-symmetric: $\Theta_{ij} = -\Theta_{ji}$. After complexification, this gives the Chern-Weil data for $\mathcal{E}$ so that $\phi_k(\Theta) = (-1)^k \phi_k(\Theta)$. The result follows.⌋

Notice that (1) above justifies the use of the complex unit i in the normalization of Θ. On the other hand, the factor 2π is due to the following proposition, whose proof we omit.[3]

Proposition 7.2.2 *The Chern classes are in fact integer cohomology classes, i.e.* $c_k(\mathcal{E}) \in H^{2k}(\mathbf{M}, \mathbb{Z})$.

From (5) in Proposition 7.2.1 above we have

Definition 7.2.2 *If* $\mathcal{F} \to \mathbf{M}$ *is a real vector bundle then its Pontrjagin classes are given by*

$$p_k(\mathcal{F}) = (-1)^k c_{2k}(\mathcal{F} \otimes \underline{\mathbb{C}}) \in H^{4k}(\mathbf{M}, \mathbb{Z}). \tag{7.2.6}$$

Now we shall concentrate on certain computational aspects of characteristic classes. The crucial observation is the following *splitting principle* for complex vector bundles.

Theorem 7.2.1 *Let* $\mathcal{E} \to \mathbf{M}$ *be a complex vector bundle of rank* r. *Then there exist a smooth map* $\pi : X_{\mathcal{E}} \to \mathbf{M}$ *such that:*

1. $\pi^* : H^*(\mathbf{M}; \mathbb{C}) \to H^*(X_{\mathcal{E}}; \mathbb{C})$ *is a monomorphism;*
2. $\pi^* \mathcal{E} = \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_r$, *where each* $\mathcal{E}_i$ *is a complex line bundle.*

The proof of this (and of Theorem 7.2.2 below) can be found in [Sh].

The gist of this result is that in order to establish computational properties of characteristic classes of complex vector bundles, it suffices to consider the case in which $\mathcal{E}$ splits as a direct sum of line bundles. We shall use this observation to define an important characteristic class that appears in our formulation of the Atiyah-Singer index theorem. Assume as before that $\mathcal{E} = \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_r$. From (3) in Proposition 7.2.1 we have

$$c(\mathcal{E}) = C(\mathcal{E}_1) \ldots c(\mathcal{E}_r) = (1 + x_1) \ldots (1 + x_r),$$

after the introduction of the formal variables $x_i = c_1(\mathcal{E}_i)$. In particular,

$$c_k(\mathcal{E}) = \sigma_k(x_1, \ldots, x_r),$$

[3] See [GH].

the k^{th} elementary symmetric function of the formal variables. We then define the *Chern character* of $\mathcal{E}$ by the additive invariant formal power series

$$\mathrm{ch}(\mathcal{E}) = \sum_i e^{x_i} \in H^*(\mathbf{M};\mathbb{Q}). \tag{7.2.7}$$

One has the unique decomposition into homogeneous terms

$$\mathrm{ch}(\mathcal{E}) = \mathrm{ch}_0(\mathcal{E}) + \mathrm{ch}_1(\mathcal{E}) + \mathrm{ch}_2(\mathcal{E}) + \ldots,$$

with $\mathrm{ch}_k(\mathcal{E}) \in H^{2k}(\mathbf{M};\mathbb{Q})$. More precisely,

$$\begin{aligned}
\mathrm{ch}_0(\mathcal{E}) &= \mathrm{rank}\,(\mathcal{E})1 \in H^0(\mathbf{M};\mathbb{Z}),\\
\mathrm{ch}_1(\mathcal{E}) &= \sum_i x_i = c_1(\mathcal{E}) \in H^2(\mathbf{M};\mathbb{Q}),\\
\mathrm{ch}_2(\mathcal{E}) &= \frac{1}{2}\sum_i x_i^2 = \frac{1}{2}\left(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})\right) \in H^4(\mathbf{M};\mathbb{Q}),
\end{aligned}$$

and, more generally, Proposition 7.1.1 assures that each $\mathrm{ch}_k(\mathcal{E})$ is a certain *universal* homogeneous polynomial with rational coefficients in the Chern classes of $\mathcal{E}$ since ch is a symmetric function of the x_i's. We remark the following relations:

$$\mathrm{ch}(\mathcal{E} \oplus \mathcal{E}') = \mathrm{ch}(\mathcal{E}) + \mathrm{ch}(\mathcal{E}'), \quad \mathrm{ch}(\mathcal{E} \otimes \mathcal{E}') = \mathrm{ch}(\mathcal{E})\mathrm{ch}(\mathcal{E}'). \tag{7.2.8}$$

We now turn to a version of the splitting principle for real vector bundles.

Theorem 7.2.2 *Let $\mathcal{F} \to \mathbf{M}$ be a real oriented vector bundle of rank $r = 2s$. Then there exists a smooth map $\pi : X_{\mathcal{F}} \to \mathbf{M}$ such that:*

1. *$\pi^* : H^*(\mathbf{M};\mathbb{C}) \to H^*(X_{\mathcal{F}};\mathbb{C})$ is a monomorphism;*
2. *$\pi^*\mathcal{F} = \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_s$, with $\mathrm{rank}(\mathcal{F}_i) = 2$ and $\mathcal{F}_i \otimes \underline{\mathbb{C}} = \mathcal{G}_i \oplus \overline{\mathcal{G}}_i$, where each $\mathcal{G}_i$ is a complex line bundle and $\overline{\mathcal{G}}_i$ is the dual bundle[4] to $\mathcal{G}_i$.*

For computational purposes we can therefore assume that

$$\mathcal{F} \otimes \underline{\mathbb{C}} = \mathcal{G}_1 \oplus \overline{\mathcal{G}}_1 \oplus \ldots \oplus \mathcal{G}_s \oplus \overline{\mathcal{G}}_s. \tag{7.2.9}$$

Introducing the formal variables $y_i = c_1(\mathcal{G}_i)$ and noticing that $c_1(\overline{\mathcal{G}}_i) = -c_1(\mathcal{G}_i)$, we have

$$c(\mathcal{F} \otimes \underline{\mathbb{C}}) = \prod_{i=1}^{s}(1 + y_i)(1 - y_i) = \prod_{i=1}^{s}(1 - y_i^2).$$

[4]This means that the transition functions of $\mathcal{G}_i$ and $\overline{\mathcal{G}}_i$ are conjugate to each other.

It follows that

$$\begin{aligned} c_{2k}(\mathcal{F} \otimes \underline{\mathbb{C}}) &= \sigma_k(-y_1^2, \ldots, -y_s^2) \\ &= (-1)^k \sigma_k(y_1^2, \ldots, y_s^2), \end{aligned}$$

and comparing with (7.2.6),

$$p_k(\mathcal{F}) = \sigma_k(y_1^2, \ldots, y_s^2).$$

Notice that this computation justifies the sign in the definition of p_k.

We are now finally ready to define the other characteristic class appearing in the formulation of the index theorem. Let $\mathcal{F} \to \mathbf{M}$ be a real vector bundle. Using the formal variables introduced above, define the *$\hat{A}$-class* of $\mathcal{F}$ by the multiplicative invariant formal power series

$$\hat{A}(\mathcal{F}) = \prod_i \frac{y_i/2}{\sinh(y_i/2)} \in H^{4*}(\mathbf{M}; \mathbb{R}). \tag{7.2.10}$$

Using that

$$\frac{z/2}{\sinh(z/2)} = 1 - \frac{1}{24} z^2 + \frac{7}{2^7 \cdot 3^2 \cdot 5} z^4 + \ldots,$$

we get the unique decomposition into homogeneous terms

$$\hat{A}(\mathcal{F}) = \hat{A}_0(\mathcal{F}) + \hat{A}_1(\mathcal{F}) + \hat{A}_2(\mathcal{F}) + \hat{A}_3(\mathcal{F}) + \ldots,$$

where

$$\begin{aligned} \hat{A}_0(\mathcal{F}) &= 1 \in H^0(\mathbf{M}; \mathbb{Z}), \\ \hat{A}_1(\mathcal{F}) &= -\frac{1}{24} p_1(\mathcal{F}) \in H^4(\mathbf{M}; \mathbb{Q}), \\ \hat{A}_2(\mathcal{F}) &= \frac{7}{2^7 \cdot 3^2 \cdot 5} \left(-4p_2(\mathcal{F}) + 7p_1(\mathcal{F})^2 \right) \in H^8(\mathbf{M}; \mathbb{Q}), \text{ etc.} \end{aligned}$$

Once again, the important point here is that each $\hat{A}_k(\mathcal{E})$ is a certain universal homogeneous polynomial with rational coefficients in the Pontrjagin classes of $\mathcal{F}$.

There exists another important characteristic class closely related to $\hat{A}$. This is the *Todd class*, defined for a complex vector bundle $\mathcal{E}$ as

$$\mathrm{Todd}(\mathcal{E}) = \prod_i \frac{x_i}{1 - e^{-x_i}}.$$

Proposition 7.2.3 *If $\mathcal{F}$ is a real vector bundle then*

$$\mathrm{Todd}(\mathcal{F} \otimes \underline{\mathbb{C}}) = \hat{A}(\mathcal{F})^2. \tag{7.2.11}$$

Indeed, using (7.2.9),

$$\begin{aligned}\mathrm{Todd}(\mathcal{F}\otimes\underline{\mathbb{C}}) &= \prod_i \frac{y_i}{1-e^{-y_i}}\frac{-y_i}{1-e^{y_i}}\\ &= \prod_i \frac{y_i}{1-e^{-y_i}}\frac{-y_i}{1-e^{y_i}}\frac{1}{e^{y_i/2}e^{-y_i/2}}\\ &= \prod_i \left[\frac{y_i/2}{\sinh(y_i/2)}\right]^2\\ &= \hat{A}(\mathcal{F})^2.\end{aligned}$$

7.3 The Euler class

We now present the Euler characteristic class which appears in the formulation of the Chern-Gauss-Bonnet theorem (see Theorem 10.1.1). Unlike the characteristic classes considered so far, which have been defined by using the Chern classes, and hence can be computed starting with any connection (not necessarily compatible with any underlying metric!), the Euler class is defined in terms of a connection which is compatible with a Riemannian metric on an orientable real vector bundle $\mathcal{F}\to\mathbf{M}$ with $\mathrm{rank}(\mathcal{F})=r=2s$, as the following pointwise construction makes it clear.

Let $V\cong\mathbb{R}^r$ be a real oriented vector space of dimension $r=2s$ endowed with an inner product. Fix an orthonormal positive basis $\{\mathrm{e}_1,\ldots,\mathrm{e}_r\}$ and let $\vartheta=\mathrm{e}_1\wedge\ldots\wedge\mathrm{e}_r$ be the corresponding volume element. Notice that ϑ generates $\Lambda^r V$. Now take $\alpha:V\to V$ skew-symmetric and let $\tilde{\alpha}\in\Lambda^2V$ be its image under the isomorphism (1.2.14). Hence, the s-fold product $\tilde{\alpha}\wedge\ldots\wedge\tilde{\alpha}\in\Lambda^rV$ is a multiple of ϑ and we define the *Pfaffian* of α by

$$\tilde{\alpha}\wedge\ldots\wedge\tilde{\alpha}=s!\mathrm{Pf}(\alpha)\vartheta. \tag{7.3.12}$$

For example, if $\alpha=(a_{ij})$ then in case $r=2$ we have $\mathrm{Pf}(\alpha)=a_{12}$ and in case $r=4$, $\mathrm{Pf}(\alpha)=a_{12}a_{34}-a_{13}a_{24}+a_{14}a_{23}$.

It is not hard to check that

$$\det(\alpha)=\mathrm{Pf}(\alpha)^2, \tag{7.3.13}$$

and, more importantly, $\mathrm{Pf}(g\alpha g^{-1})=\mathrm{Pf}(\alpha)$ for any $g\in\mathrm{SO}_r$. In other words, Pf is an invariant polynomial function on $\mathfrak{so}_r$ and, if $\mathcal{F}$ is as above, a variant of the Chern-Weil argument gives the *Euler characteristic class* of $\mathcal{F}$ by

$$\mathrm{e}(\mathcal{F})=\mathrm{Pf}\left(\frac{\Theta}{2\pi}\right)\in H^r(\mathbf{M},\mathbb{Z}), \tag{7.3.14}$$

where Θ is the curvature form of the compatible connection on $\mathcal{F}$. Notice that, by (7.3.13), in terms of the variables $\{y_i\}$ introduced above,

$$\mathrm{e}(\mathcal{F}) = y_1 \dots y_s. \tag{7.3.15}$$

An important case occurs when $\mathcal{F} = T\mathbf{M}$, the tangent bundle of a closed oriented manifold of dimension $n = 2k$. It is possible to show that the integer $\langle \mathrm{e}(T\mathbf{M}), \mathbf{M} \rangle$ is the obstruction to the existence of a nowhere vanishing vector field over $\mathbf{M}$ (see [S], for example). The famous Chern-Gauss-Bonnet theorem says that

$$\chi(\mathbf{M}) = \langle \mathrm{e}(T\mathbf{M}), \mathbf{M} \rangle, \tag{7.3.16}$$

where $\chi(\mathbf{M})$, the Euler characteristic of $\mathbf{M}$, is given by (1.1.10). By de Rham theory, this establishes a profound link between the topology and geometry of $\mathbf{M}$.

We illustrate the theory by computing the Euler class of TS^2, the tangent bundle of the unit sphere $S^2 \subset \mathbb{R}^3$. Choose a local frame $\{\mathrm{e}_1, \mathrm{e}_2\}$ for $U \subset S^2$ and let e_0 be the position vector for S^2 so that $\{\mathrm{e}_0, \mathrm{e}_1, \mathrm{e}_2\} \subset \mathbb{R}^3$ is a positive orthonormal basis. We use the index ranges $0 \leq \alpha, \beta \leq 2$ and $1 \leq i, j \leq 2$. Thinking of each e_α as a $\mathbb{R}^3$-valued function on U, there exist $\theta_{\alpha\beta} \in \mathcal{A}^1(U)$ such that

$$d\mathrm{e}_\alpha = \sum_\beta \theta_{\alpha\beta} \mathrm{e}_\beta, \quad \theta_{\alpha\beta} = -\theta_{\beta\alpha}.$$

Setting $\theta_i = \theta_{i0}$ this can be rewritten as

$$d\mathrm{e}_0 = -\theta_1 \mathrm{e}_1 - \theta_2 \mathrm{e}_2 \tag{7.3.17}$$

$$d\mathrm{e}_1 = \theta_1 \mathrm{e}_0 + \theta_{12} \mathrm{e}_2 \tag{7.3.18}$$

$$d\mathrm{e}_2 = \theta_2 \mathrm{e}_0 + \theta_{21} \mathrm{e}_1, \tag{7.3.19}$$

where $\theta = (\theta_{ij})$ is the Levi-Civita connection form on TS^2 induced by the embedding $S^2 \subset \mathbb{R}^3$. Notice that $\theta_1 \wedge \theta_2$ is the standard area element in U.

Using that $d^2\mathrm{e}_0 = 0$ in (7.3.17) we find that $d\theta_1 = \theta_{12} \wedge \theta_2$ and combining this with (7.3.18), (7.3.19) and the fact that $d^2\mathrm{e}_2 = 0$, we get $d\theta_{12} = \theta_1 \wedge \theta_2$. Now, because $\theta \wedge \theta = 0$, $\Theta = (\Theta_{ij})$ with $\Theta_{ij} = d\theta_{ij}$ is the corresponding curvature form and we finally compute

$$\langle \mathrm{e}(TS^2), S^2 \rangle = \frac{1}{2\pi} \int_{S^2} \mathrm{Pf}\,(\Theta) = \frac{1}{2\pi} \int_{S^2} \Theta_{12} = \frac{1}{2\pi} \int_{S^2} \theta_1 \wedge \theta_2 = \frac{4\pi}{2\pi} = 2.$$

In particular, this gives a geometric proof that TS^2 is *not* trivial. But notice that if we use the connection with vector fields mentioned above a sharper result pops out: any tangent vector field over S^2 vanishes somewhere!

Chapter 8

The Atiyah-Singer index theorem

In this chapter we finally present the Atiyah-Singer index theorem for Dirac operators on spin manifolds. With an eye toward further applications to Riemannian Geometry, besides the one leading to Theorem 6.2.3, we consider in fact a more general situation than that described in Section 6.2. The idea is to introduce a construction generating, at least locally, the most common differential operators appearing in Riemannian Geometry. The crucial observation follows from the representation theory developed in Chapter 5: since $\mathbb{C}\mathrm{l}_n$ is a full matrix algebra, any $\mathbb{C}\mathrm{l}_n$-module is of the type $\Psi \otimes W$ with $\mathbb{C}\mathrm{l}_n$ acting trivially on the second factor. This leads to the notion of twisted Dirac operators, which we now pass to investigate.

8.1 Twisted Dirac operators and the index theorem

Let $\mathbf{M}$ be an oriented closed Riemannian manifold. Recall from (5.2.16) the construction of the Clifford algebra bundle $\mathbb{C}\mathrm{l}(\mathbf{M})$. We note that no spin structure is required there. Now let $\mathcal{E} \to \mathbf{M}$ be any complex vector bundle whose typical fiber is a module over $\mathbb{C}\mathrm{l}_n$. This implies of course that $\Gamma(\mathcal{E})$ is a module over $\Gamma(\mathbb{C}\mathrm{l}(\mathbf{M}))$. In particular, since $\mathcal{X}(\mathbf{M}) \subset \Gamma(\mathbb{C}\mathrm{l}(\mathbf{M}))$, Clifford multiplication by tangent vectors is well defined. Assume further that $\mathcal{E}$ is equipped with a hermitian metric $\langle\, , \rangle$ and a compatible connection $\nabla^{\mathcal{E}}$ such that:

1. Clifford multiplication by unit tangent vectors is an isometry:
$$\langle u\varphi, u\varphi' \rangle = \langle \varphi, \varphi' \rangle, \;\; u \in \mathcal{X}(\mathbf{M}), \;\; |u| = 1, \;\; \varphi, \varphi' \in \Gamma(\mathcal{E}). \tag{8.1.1}$$

2. The natural connection ∇^c on $\mathbb{C}\mathrm{l}(\mathbf{M})$ and $\nabla^{\mathcal{E}}$ are compatible with the $\Gamma(\mathbb{C}\mathrm{l}(\mathbf{M}))$-module structure on $\Gamma(\mathcal{E})$:
$$\nabla^{\mathcal{E}}(a\varphi) = \nabla^c a \cdot \varphi + a\nabla^{\mathcal{E}}\varphi, \;\; a \in \Gamma(\mathbb{C}\mathrm{l}(\mathbf{M})), \;\; \varphi \in \Gamma(\mathcal{E}). \tag{8.1.2}$$

If this is the case, we say that $\mathcal{E}$ is a *Clifford bundle*. Notice that the conditions above constitute an obvious generalization of the properties (6.2.6) and (6.2.7) holding for the spin bundle $\mathcal{S}(\mathbf{M})$ in case $\mathbf{M}$ is spin, and we shall see in Chapter 10 that this abstraction, for suitable choices of $\mathcal{E}$, actually encompasses many of the natural differential operators appearing in Riemannian Geometry. In any case, in this general setting we can consider the corresponding *Dirac operator* $D : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$, defined by the sequence of arrows

$$\Gamma(\mathcal{E}) \stackrel{\nabla^{\mathcal{E}}}{\to} \Gamma(T^*\mathbf{M} \otimes \mathcal{E}) \stackrel{\langle,\rangle}{\to} \Gamma(T\mathbf{M} \otimes \mathcal{E}) \stackrel{\cdot}{\to} \Gamma(\mathcal{E}),$$

where as usual the dot means Clifford product. Alternatively, in terms of a local frame for $T\mathbf{M}$,
$$D = \sum_{i=1}^{n} \mathrm{e}_i \cdot \nabla^{\mathcal{E}}_{\mathrm{e}_i} \tag{8.1.3}$$
As in the spin case, D is a selfadjoint first order differential operator and a Weitzenböck type decomposition for the associated Dirac Laplacian $\Delta = D^2 : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ holds:
$$\Delta = \nabla^{\mathcal{E}*}\nabla^{\mathcal{E}} + \mathcal{R}^{\mathcal{E}}, \tag{8.1.4}$$
where $\nabla^{\mathcal{E}*}\nabla^{\mathcal{E}}$ is the Bochner Laplacian of $\nabla^{\mathcal{E}}$ and
$$\mathcal{R}^{\mathcal{E}} = \frac{1}{2}\sum_{i,j=1}^{n} \mathrm{e}_i \cdot \mathrm{e}_j \cdot R^{\mathcal{E}}_{\mathrm{e}_i,\mathrm{e}_j}, \tag{8.1.5}$$

with $R^{\mathcal{E}}$ being the curvature tensor of $\mathcal{E}$.

As in the case of the genuine Dirac operator $\partial\!\!\!/$ acting on the spinor bundle of a spin manifold, one has to break the symmetry of D acting on $\Gamma(\mathcal{E})$.

Definition 8.1.1 *We say that a Clifford bundle $\mathcal{E}$ as above with Dirac operator $D : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ is graded if there exists a decomposition $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ such that*

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

with respect to this decomposition.

Thus, in the presence of the grading, we have the operators

$$D^{\pm} = D|_{\Gamma(\mathcal{E}^{\pm})} : \Gamma(\mathcal{E}^{\pm}) \to \Gamma(\mathcal{E}^{\mp}),$$

which are formal adjoints to each other and have finite dimensional kernels.[1] We now define

$$\operatorname{ind} D^+ = \dim\ker D^+ - \dim\ker D^-, \tag{8.1.6}$$

the *index* of D^+.

Notice that if $\mathcal{E}$ is a Clifford bundle as above and $\mathcal{G}$ is *any* hermitian vector bundle over $\mathbf{M}$ with a compatible connection, we can form the tensor product $\mathcal{E} \otimes \mathcal{G}$ and this is naturally a Clifford bundle (just let Clifford multiplication act trivially on the second factor) and moreover the grading of $\mathcal{E}$ induces a further grading

$$\mathcal{E} \otimes \mathcal{G} = \mathcal{E}^+ \otimes \mathcal{G} \oplus \mathcal{E}^- \otimes \mathcal{G},$$

so that the constructions above apply and the corresponding Dirac operator

$$D : \Gamma(\mathcal{E} \otimes \mathcal{G}) \to \Gamma(\mathcal{E} \otimes \mathcal{G})$$

is usually referred to as a *twisted Dirac operator*.

An important example in this setting is the following one. Let $\mathbf{M}$ be a spin manifold of dimension $n = 2k$ so that the spin bundle of $\mathbf{M}$ has a natural grading

$$\mathcal{S}(\mathbf{M}) = \mathcal{S}^+(\mathbf{M}) \oplus \mathcal{S}^-(\mathbf{M}). \tag{8.1.7}$$

It follows that, for any $\mathcal{G}$ as above, $\mathcal{S}(\mathbf{M}) \otimes \mathcal{G}$ is naturally a graded Clifford bundle

$$\mathcal{S}(\mathbf{M}) \otimes \mathcal{G} = \left(\mathcal{S}^+(\mathbf{M}) \otimes \mathcal{G}\right) \oplus \left(\mathcal{S}^-(\mathbf{M}) \otimes \mathcal{G}\right), \tag{8.1.8}$$

and this allows us to consider the twisted Dirac operator $\partial\!\!\!/_{\mathcal{G}} : \Gamma(\mathcal{S}(\mathbf{M}) \otimes \mathcal{G}) \to \Gamma(\mathcal{S}(\mathbf{M}) \otimes \mathcal{G})$. In this regard, one has

Proposition 8.1.1 *If $\Delta = \partial\!\!\!/_{\mathcal{G}}^2$, there holds*

$$\Delta = \nabla^*\nabla + \frac{1}{4}\kappa + \mathcal{R}^{[\mathcal{G}]}, \tag{8.1.9}$$

where $\nabla^*\nabla$ is the Bochner Laplacian of $\mathcal{S}(\mathbf{M}) \otimes \mathcal{G}$, κ is the scalar curvature of $\mathbf{M}$ and, for $\psi \otimes \eta \in \Gamma(\mathcal{S}(\mathbf{M}) \otimes \mathcal{G})$,

$$\mathcal{R}^{[\mathcal{G}]}(\psi \otimes \eta) = \frac{1}{2}\sum_{ij} \mathrm{e}_i\mathrm{e}_j\psi \otimes R^{\mathcal{G}}_{\mathrm{e}_i,\mathrm{e}_j}\eta, \tag{8.1.10}$$

with $R^{\mathcal{G}}$ being the curvature tensor of $\mathcal{G}$.

[1]This also follows from the elliptic theory developed in Chapter 9.

Proof. Notice that

$$\nabla^{\mathcal{S}(\mathbf{M})\otimes\mathcal{G}}(\psi\otimes\eta) = \nabla^s\psi\otimes\eta + \psi\otimes\nabla^{\mathcal{G}}\eta,$$

thus yielding

$$R^{\mathcal{S}(\mathbf{M})\otimes\mathcal{G}}(\psi\otimes\eta) = R^s\psi\otimes\eta + \psi\otimes R^{\mathcal{G}}\eta,$$

and the result follows from (8.1.5), with $\mathcal{E}$ replaced by $\mathcal{S}(\mathbf{M})\otimes\mathcal{G}$, and the computation leading to Theorem 6.2.1.⌋

As before, $\partial\!\!\!/_{\mathcal{G}}$ breaks into pieces according to (8.1.8):

$$\partial\!\!\!/_{\mathcal{G}}^{\pm} = \partial\!\!\!/_{\mathcal{G}}|_{\Gamma(\mathcal{S}^{\pm}(\mathbf{M})\otimes\mathcal{G})} : \Gamma(\mathcal{S}^{\pm}(\mathbf{M})\otimes\mathcal{G}) \to \Gamma(\mathcal{S}^{\mp}(\mathbf{M})\otimes\mathcal{G}), \tag{8.1.11}$$

and we get a well-defined index by

$$\operatorname{ind}\partial\!\!\!/_{\mathcal{G}}^{+} = \dim\ker\partial\!\!\!/_{\mathcal{G}}^{+} - \dim\ker\partial\!\!\!/_{\mathcal{G}}^{-}. \tag{8.1.12}$$

Many notable results in Differential Geometry follow from considerations involving this integer invariant. The main ingredient here is the celebrated Atiyah-Singer index theorem, which reads as follows.

Theorem 8.1.1 *(Atiyah-Singer) Assuming that* $\mathbf{M}$ *is spin and using the notation above,*

$$\operatorname{ind}\partial\!\!\!/_{\mathcal{G}}^{+} = \langle \operatorname{ch}(\mathcal{G})\hat{A}(T\mathbf{M}), \mathbf{M}\rangle. \tag{8.1.13}$$

Specializing to the case $k = 2l$ (so that $\dim\mathbf{M} = 4l$) and $\mathcal{G} = \underline{\mathbb{C}}$ we get

Theorem 8.1.2 *The index of the Atiyah-Singer-Dirac operator acting on the spinor bundle of a closed spin manifold* $\mathbf{M}$ *with* $\dim\mathbf{M} = 4l$ *is given by*

$$\operatorname{ind}\partial\!\!\!/^{+} = \langle \hat{A}(T\mathbf{M}), \mathbf{M}\rangle. \tag{8.1.14}$$

Notice that by definition $\hat{A}(T\mathbf{M})$ is a rational cohomology class and the *$\hat{A}$-genus* $\hat{A}(\mathbf{M}) = \langle \hat{A}(T\mathbf{M}), \mathbf{M}\rangle$ of $\mathbf{M}$ is a priori a rational number. However, (8.1.14) immediately yields the following integrability result: if $\mathbf{M}$ is spin then $\hat{A}(\mathbf{M})$ is an integer! In particular, if $\dim\mathbf{M} = 4$ then we see from Chapter 7 that $\hat{A}(\mathbf{M}) = -\langle p_1(T\mathbf{M}), \mathbf{M}\rangle/24$ and so we get $\langle p_1(T\mathbf{M}), \mathbf{M}\rangle \in 24\mathbb{Z}$ in case $\mathbf{M}$ is spin. Another deep result following from theorem 8.1.2 and the vanishing criterion in Theorem 6.2.2 is Theorem 6.2.3 to the effect that a closed spin manifold of dimension $4k$ with $\hat{A}(\mathbf{M}) \neq 0$ can not carry a metric with quasi-positive scalar curvature. In Chapter 10 we explore further this circle of ideas in order to obtain some more interesting obstructions to the existence of metrics of positive scalar curvature on certain manifolds.

Chapter 9

The heat equation and the index theorem

The purpose of this chapter is to present the main steps in the so-called heat equation proof of the Atiyah-Singer index formula for Dirac operators (Theorem 8.1.1). This proof is the fortunate outcome of certain 'fantastic cancellations' related to the structure of Clifford algebras. Since the proof requires knowledge of certain aspects of the heat flow we include in Section 9.1 a discussion of this analytical material on the unit circle S^1. The general case is treated in Section 9.2, where a discussion of Hodge theory is presented. The cancellations are explained in Section 9.3 and here we have chosen not to go into the details of the many computations involved but instead we concentrate on following the stream of ideas in order to make the presentation less arid. In any case, the reader is referred to the many excellent accounts (specially [BGV], [R] and [T]) for the details.

9.1 The heat equation on the circle

Let us consider the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$ endowed with the usual angular coordinate $\theta \in [0, 2\pi)$. The heat equation in this case is

$$u_t = -\Delta u = u_{\theta\theta}, \tag{9.1.1}$$

and we are interested in finding a solution $u = u(t, \theta) : [0, +\infty) \times S^1 \to \mathbb{R}$ to this equation. This should be thought of as an initial value problem, i.e. we are given a function $u_0 : S^1 \to \mathbb{R}$ and we want to find a function u as above satisfying (9.1.1) identically for $(t,\theta) \in (0,+\infty) \times S^1$ and such that

$$\lim_{t \to 0} u(t,\theta) = u_0(\theta), \quad \theta \in S^1, \tag{9.1.2}$$

in some reasonable sense. Of course, this problem has a well-known physical interpretation: u_0 is the initial heat distribution on S^1 and solving the problem amounts to understanding how the heat on the circle flows as time goes by.

Fortunately this is a case in which we can give an explicit solution to the problem by the honorable *Fourier method*, or method of separation of variables, whose first step consists of finding solutions of the form $u(t,\theta) = v(t)w(\theta)$. Taking this to (9.1.1) we get

$$v'(t)w(\theta) = v(t)w''(\theta)$$

and assuming for the time being that v and w are nowhere vanishing we find

$$\frac{v'(t)}{v(t)} = \frac{w''(\theta)}{w(\theta)}, \tag{9.1.3}$$

and since the left (respect. right) hand side of (9.1.3) depends only on t (respect. θ) it follows that both sides are *constant*, so we have

$$\frac{v'(t)}{v(t)} = \frac{w''(\theta)}{w(\theta)} = -\lambda \tag{9.1.4}$$

and we are left with the problem of finding nonzero solutions to the equations

$$v' + \lambda v = 0 \tag{9.1.5}$$

and

$$w'' + \lambda w = 0 \tag{9.1.6}$$

for a constant λ to be determined.

Clearly, (9.1.5) poses no constraint on λ since its general solution is given by

$$v(t) = Ce^{-\lambda t}.$$

On the other hand, as we shall see, equation (9.1.6) poses severe restrictions on the possible values attained by λ. To see this, assume w is a nonzero solution of (9.1.6), multiply (9.1.6) by w and use integration by parts to get

$$\int_{S^1} w'^2 \, d\theta = \lambda \int_{S^1} w^2 \, d\theta,$$

where $d\theta$ is the length element in S^1. This implies $\lambda \geq 0$ (justifying the minus sign in (9.1.4)) and $\lambda = 0$ if and only if $u =$ const. For $\lambda > 0$ the general solution of (9.1.6) is

$$w(\theta) = C_1 e^{i\sqrt{\lambda}\theta} + C_2 e^{-i\sqrt{\lambda}\theta}$$

but we are assuming that w is periodic and this leaves us with a countable family of solutions

$$w_n(\theta) = C_1 e^{in\theta} + C_2 e^{-in\theta}, \ \ \lambda_n = n^2.$$

For $n \geq 1$ this generates a two-dimensional eigenpace V_{n^2} for the Laplacian on S^1 (the remaining eigenspace is V_0, the space of constant functions associated to $\lambda_0 = 0$) and it is well-known that there are no further eigenspaces, a fact which is a rephrasing of the completeness of the orthogonal system $\{e^{\pm in\theta}\}$ with respect to the L^2 inner product

$$(f, g) = \int_{S^1} f(\theta)\overline{g(\theta)}\, d\theta$$

on functions.[1]

Now, the heart of the Fourier method is to appeal to the linearity of (9.1.1) (or to the superposition principle in physical language) in order to determine the general solution by combining the special solutions above. So we are naturally led to postulate that the general solution of (9.1.1) is

$$u(t, \theta) = \sum_{n=-\infty}^{+\infty} a_n e^{-n^2 t} e^{in\theta} \tag{9.1.8}$$

where the sequence $\{a_n\}$ is determined via the initial value condition (9.1.2):

$$u_0(\theta) = u(0, \theta) = \sum_{n=-\infty}^{+\infty} a_n e^{in\theta}. \tag{9.1.9}$$

This means that the a_n's are the Fourier coefficients of u_0. In effect, multiplying (9.1.9) by $e^{-im\theta}$, integrating over S^1 and using (9.1.7) we get

$$a_m = \frac{1}{2\pi} \int_{S^1} u_0(\theta) e^{-im\theta}\, d\theta, \quad m \in \mathbb{Z}. \tag{9.1.10}$$

[1] To this effect, recall the classical *orthogonality relations*

$$\int_{S^1} e^{i(n-m)\theta}\, d\theta = 2\pi\delta_{nm}. \tag{9.1.7}$$

Now we can finally explain how we can solve (9.1.1) for a given initial data u_0: first we compute the Fourier coefficients of u_0 via (9.1.10) and then we substitute the result in (9.1.8).

We can obtain information on the heat flow on S^1 as $t \to +\infty$. Indeed, the exponential terms $e^{-n^2 t}$ (for $n \neq 0$) in (9.1.8) die out to zero very fastly so that

$$u^\infty(\theta) \stackrel{\text{def.}}{=} \lim_{t \to +\infty} u(t, \theta) = a_0, \tag{9.1.11}$$

that is, the long term behavior of the flow approaches, by (9.1.10) with $m = 0$, the mean value of the initial distribution of temperature u_0. This is in conformity with the general expectance (the heat flow is dissipative!) and shows that the mathematical model is reliable as far as the long term behavior is concerned.

And how about the behavior of the flow when t is close to zero? A better grasping of this can be obtained if we rewrite (9.1.8) in a more convenient manner. We simply substitute the expression (9.1.10) on (9.1.8) and interchange the integral and the summation[2] in order to get

$$u(t, \theta) = \int_{S^1} \mathcal{K}(t, \theta, \theta') u_0(\theta')\, d\theta', \tag{9.1.12}$$

where

$$\mathcal{K}(t, \theta, \theta') = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{-n^2 t} e^{in(\theta - \theta')} \tag{9.1.13}$$

is the *heat kernel* of S^1. In other words, the solution of (9.1.1) is obtained by integrating the initial data u_0 against the heat kernel $\mathcal{K}$.

There are a few aspects of the solution (9.1.12)-(9.1.13) that should be highlighted here.

1. From (9.1.13) it is clear that the heat flow is highly smoothing. More precisely, even if we start with a very irregular initial data, say $u_0 \in L^2(S^1)$, then it follows that for $t > 0$ the corresponding solution $u(t, .)$ is smooth. This is due to the fact that $\mathcal{K}$ is smooth for $t > 0$ and derivation under the integral sign.

2. It is also easy to use (9.1.13) to check that the heat flow is dissipative. First,

$$\mathcal{K}_t(t, \theta, \theta') = -\frac{1}{2\pi} \sum_{n \neq 0} n^2 e^{-n^2 t} e^{in(\theta - \theta')}$$

and from this we see that $\mathcal{K}_t$ goes to zero as t goes to $+\infty$. Thus, u^∞ as defined in (9.1.11) satisfies $u^\infty_{\theta\theta} = 0$, i.e. $u^\infty = \text{const}$.

[2] All these operations are legal if $t > 0$.

3. The heat kernel can be rewritten as

$$\mathcal{K}(t,\theta,\theta') = \sum_{n=-\infty}^{+\infty} e^{-n^2 t} e_n(\theta)\overline{e_n(\theta')},$$

where $\{e_n(\theta) = e^{in\theta}/\sqrt{2\pi}\}_{n=-\infty}^{+\infty}$ is an orthonormal basis for $L^2(S^1)$ given by the eigenfunctions of Δ. This means that the heat kernel (and the heat flow) is determined by the spectral decomposition of the Laplacian Δ on $L^2(S^1)$.

4. For each $t > 0$ we can define a linear operator $e^{-t\Delta} : L^2(S^1) \to L^2(S^1)$ as follows. For $u_0 \in L^2(S^1)$ we take $e^{-t\Delta}u_0$ to be the righthand side of (9.1.12). It is obvious that $\|e^{-t\Delta}u_0\| \le \|u_0\|$ (L^2 norm) so $e^{-t\Delta}$ is a one-parameter family of uniformly bounded operators in $L^2(S^1)$. Moreover, $e^{-(t+s)\Delta} = e^{-t\Delta}e^{-s\Delta}$ so $e^{-\Delta t}$ is actually a semigroup. This is the *heat flow* on S^1. Also, $e^{-t\Delta}$ is trace class, i.e. it has a trace (even though it acts on an infinite dimensional space) which is given by

$$\operatorname{Tr} e^{-t\Delta} = \sum_{n=-\infty}^{+\infty} e^{-n^2 t}.$$

Clearly, $\operatorname{Tr} e^{-t\Delta}$ is completely determined by the spectrum of Δ.

5. We have the asymptotic formula as t goes to 0:

$$\operatorname{Tr} e^{-t\Delta} \sim \sqrt{\frac{\pi}{t}} = \frac{2\pi}{\sqrt{4\pi t}}. \tag{9.1.14}$$

If we look at properties (2) and (5) above, it follows that the long term behaviour of the heat flow is determined by the topology of S^1 (in the sense that it takes arbitrary initial data into constant functions and the space of such functions is a model for the de Rham comohology of S^1) while the short time behaviour is controlled by the geometry of S^1, more precisely, by the fact that locally S^1 looks like $\mathbb{R}$.[3] This striking connection between geometry and topology will manifest itself, as we shall see, in many other situations where an appropriate elliptic operator like the Laplacian Δ and a corresponding heat flow are available. The basic idea is that if we are able to manufacture some quantity *invariant* under the heat flow, then we can use it in order to establish deep relationships between the underlying geometry and topology. It turns out that the index of a twisted Dirac operator is such an invariant (here is the first cancellation!) and this key idea happens to be the foundational principle upon which the so-called heat equation proof of the Atiyah-Singer theorem for Dirac type operators rely.

[3]Notice that the factor $1/\sqrt{4\pi t}$ comes from the heat flow in $\mathbb{R}$; see [T].

9.2 Dirac complexes and the heat flow

In Chapter 4 we have met the Hodge theorem according to which any de Rham cohomology class on a closed orientable Riemannian manifold **M** carries a unique harmonic representative. In this section, we shall sketch a proof of this fundamental result based on heat equation methods. In fact we shall deal with a version of the Hodge theory in the context of Dirac complexes, a notion we now pass to describe.

Let **M** be a closed oriented Riemannian manifold of dimension n and let $\{\mathcal{E}_i\}$ be a finite sequence of hermitian vector bundles over **M** equiped with compatible connections. Assume the existence of first order linear differential operators $d = d_i : \Gamma(\mathcal{E}_i) \to \Gamma(\mathcal{E}_{i+1})$ such that $d^2 = 0$. We then say that the sequence

$$0 \to \ldots \stackrel{d}{\to} \Gamma(\mathcal{E}_{i-1}) \stackrel{d}{\to} \Gamma(\mathcal{E}_i) \stackrel{d}{\to} \Gamma(\mathcal{E}_{i+1}) \stackrel{d}{\to} \ldots \to 0 \tag{9.2.15}$$

is a *differential complex*. In this case we can consider the *cohomology groups* of $\mathcal{E} = \oplus_i \mathcal{E}_i$,

$$H^i(\mathcal{E}) = \frac{\ker\left(\Gamma(\mathcal{E}_i) \stackrel{d}{\to} \Gamma(\mathcal{E}_{i+1})\right)}{\mathrm{im}\left(\Gamma(\mathcal{E}_{i-1}) \stackrel{d}{\to} \Gamma(\mathcal{E}_i)\right)}, \tag{9.2.16}$$

and one would like to have canonical representatives for the corresponding cohomology classes, but for this some extra structure is required. First, recall that we can define the adjoint operators $d^* = d_i^* : \Gamma(\mathcal{E}_{i+1}) \to \Gamma(\mathcal{E}_i)$ for each i, and clearly we can consider both d and d^* as acting on $\Gamma(\mathcal{E}) = \oplus_i \Gamma(\mathcal{E}_i)$.

Definition 9.2.1 *We say that a differential complex $\Gamma(\mathcal{E})$ as above is a Dirac complex if $\mathcal{E} = \oplus_i \mathcal{E}_i$ is a Clifford bundle whose Dirac operator D equals $d + d^*$.*

This allows us to consider the corresponding *Dirac Laplacian* $\Delta : \mathcal{E} \to \mathcal{E}$ by

$$\Delta = D^2 = dd^* + d^*d. \tag{9.2.17}$$

Obviously, Δ preserves the natural grading of $\mathcal{E}$, i.e. $\Delta(\Gamma(\mathcal{E}_i)) \subset \Gamma(\mathcal{E}_i)$.

We say that $\eta \in \Gamma(\mathcal{E})$ is *harmonic* if $\Delta\eta = 0$. The space of harmonic sections is denoted by $\mathcal{H}^*(\mathcal{E})$. This is a graded vector space because $\mathcal{H}^*(\mathcal{E}) = \oplus_i \mathcal{H}^i(\mathcal{E})$, where $\mathcal{H}^i(\mathcal{E})$ is the space of 'pure' harmonic forms in $\Gamma(\mathcal{E}_i)$. Since

$$(\Delta\eta, \eta) = \|d\eta\|^2 + \|d^*\eta\|^2, \tag{9.2.18}$$

η is harmonic if and only $d\eta = 0$ (in particular, η defines a cohomology class in $H^*(\mathcal{E}) = \oplus_i H^i(\mathcal{E})$) and $d^*\eta = 0$. Conversely, we have

Theorem 9.2.1 *(Hodge) Any cohomology class in $H^*(\mathcal{E})$ carries a unique harmonic representative. In other words, in the presence of the various metrics, there exists a canonical isomorphism*

$$H^*(\mathcal{E}) \cong \mathcal{H}^*(\mathcal{E}). \tag{9.2.19}$$

Equivalently, if we take the gradings into account,

$$H^i(\mathcal{E}) \cong \mathcal{H}^i(\mathcal{E}), \tag{9.2.20}$$

for each i. Moreover, $\dim \mathcal{H}^*(\mathcal{E}) < +\infty$.

We remark that this applies notably in case $\mathcal{E}^i = \Lambda^i(\mathbf{M})$ for $\mathbf{M}$ a Riemannian manifold, thus justifying the use of Hodge theory in Chapter 4. In particular, the obvious identity $*\Delta = \Delta*$ easily yields an analytical proof of Poincaré duality of Section 1.1. Moreover, the finiteness result above implies that the indexes of the various Dirac type operators considered before are well-defined indeed.

As mentioned in the Introduction, we approach this via heat equation methods by following the pioneering work in [MR]. The method depends on the existence of a nice spectral decomposition for Δ and a crucial point here is that D is a Dirac operator and then a Weitzenböck type formula holds:

$$\Delta = \nabla^*\nabla + \mathcal{R}, \tag{9.2.21}$$

where $\nabla^*\nabla$ is the Bochner Laplacian of $\mathcal{E}$ and $\mathcal{R}$ is a selfadjoint map acting linearly on fibers. But in order to explore this one has to recall a few basic facts on Sobolev spaces.

Let $u : \mathbb{R}^n \to \mathbb{C}^r$ be a smooth map. We define its *Fourier transform* $\hat{u} : \mathbb{R}^n \to \mathbb{C}^r$ by

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int e^{-i\langle x,\xi\rangle} u(x)\, dx, \quad \xi \in \mathbb{R}^n,$$

whenever the integrals[4] above converge. A standard convention is to assume that u is a *Schwarz map* (notation: $u \in \mathcal{S}(\mathbb{R}^n)$), which means that u is smooth and for any multi-indice α and any $k > 0$ there exists $C_{\alpha,k} > 0$ such that

$$|\partial^\alpha u(x)| \leq C_{\alpha,k} \left(1 + |x|^2\right)^{-k}, \ \ x \in \mathbb{R}^n.$$

The importance of this functional space is due to

[4] For simplicity, in what follows we write $\int_{\mathbb{R}^n} = \int$.

Proposition 9.2.1 *Fourier transform, if viewed as a map from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$, is an isometry with respect to the L^2 inner product induced by the inclusion $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. More precisely,*

$$(u,v) = (\hat{u},\hat{v}), \quad u,v \in \mathcal{S}(\mathbb{R}^n), \tag{9.2.22}$$

and in fact the following inversion formula holds:

$$u(x) = (2\pi)^{-n/2} \int e^{i\langle x,\xi\rangle} \hat{u}(\xi)\, d\xi, \quad x \in \mathbb{R}^n. \tag{9.2.23}$$

Definition 9.2.2 *Given $s \in \mathbb{R}$ and $u \in \mathcal{S}(\mathbb{R}^n)$, define the Sobolev s-norm of u by*

$$\|u\|_s^2 = \int (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi.$$

The Sobolev space $W^s(\mathbb{R}^n)$ is the completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm $\|\ \|_s$.

We note a few basic facts regarding Sobolev spaces.

1. $W^s(\mathbb{R}^n)$ is a Hilbert space for the inner product

$$(u,v)_s = \int (1+|\xi|^2)^s \hat{u}(\xi)\hat{v}(\xi) d\xi.$$

2. $W^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.

3. Using (9.2.23) it is not hard to show that, for each $s \in \mathbb{N}$, one can find positive constants $c(s)$ and $d(s)$ such that

$$c(s) \sum_{|\alpha|\le s} \int |\partial^\alpha u(x)|^2 dx \le \|u\|_s^2 \le d(s) \sum_{|\alpha|\le s} \int |\partial^\alpha u(x)|^2 dx, \tag{9.2.24}$$

for any $u \in \mathcal{S}(\mathbb{R}^n)$.

4. For each s, the natural pairing $W^s(\mathbb{R}^n) \times W^{-s}(\mathbb{R}^n) \to \mathbb{C}$ given by

$$(u,v) \mapsto \int \hat{u}(\xi)\hat{v}(\xi)\, d\xi \tag{9.2.25}$$

is nondegenerate and places $W^s(\mathbb{R}^n)$ and $W^{-s}(\mathbb{R}^n)$ in duality.

Assume now that $\mathbf{M}$ is an n-dimensional closed oriented manifold and $\mathcal{E}$ is a hermitian vector bundle over $\mathbf{M}$ with a compatible connection. Let

$$\mathbf{M} = \bigcup_{\alpha=1}^{N} \mathcal{U}_\alpha$$

be a finite covering of $\mathbf{M}$ with each $\mathcal{U}_\alpha$ being diffeomorphic to $\mathbb{R}^n$. Pick open sets $\mathcal{V}_\alpha \subset \mathcal{U}_\alpha$ with each $\overline{\mathcal{V}}_\alpha$, the closure of $\mathcal{V}_\alpha$, compact and such that we still have

$$\mathbf{M} = \bigcup_{\alpha=1}^{N} \mathcal{V}_\alpha,$$

and finally let $\{\varphi_\alpha\}$ be a partition of unity subordinated to $\{\mathcal{V}_\alpha\}$. If $\eta \in \Gamma(\mathcal{E})$ we have

$$\eta = \sum_\alpha \eta_\alpha, \quad \eta_\alpha = \varphi_\alpha \eta.$$

We then define the *Sobolev s-norm* of η by

$$\|\eta\|_s^2 = \sum_\alpha \|\eta_\alpha\|_s^2, \tag{9.2.26}$$

where $\|\eta_\alpha\|_s$ is given by Definition 9.2.2 using the identification $\mathcal{U}_\alpha = \mathbb{R}^n$. The corresponding Sobolev space $W^s(\mathcal{E})$ is conceived by taking completion with respect to (9.2.26). Clearly, $W^s(\mathcal{E})$ is a Hilbert space with respect to the obvious inner product.

We now collect without proof the basic facts concerning Sobolev spaces. Recall that we are assuming $\mathbf{M}$ closed and this means that, via a partition of unity argument, the proofs follow straightforwardly from the corresponding ones in the 'flat' situation (see [T], for example).

Theorem 9.2.2 *For each s, $\Gamma(\mathcal{E})$ is dense in $W^s(\mathcal{E})$ with respect to the Sobolev norm.*

Theorem 9.2.3 *(Sobolev embedding) If $s > n/2 + k$ then we have a continuous embedding*

$$W^s(\mathcal{E}) \hookrightarrow C^k(\mathcal{E}),$$

where $\Gamma^k(\mathcal{E}) \supset \Gamma(\mathcal{E})$ is the space of sections of class C^k. In particular,

$$\Gamma(\mathcal{E}) = \bigcap_{s=-\infty}^{+\infty} W^s(\mathcal{E}).$$

Theorem 9.2.4 *(Rellich compactness) If $s' < s$ then the continuous embedding*

$$W^s(\mathcal{E}) \hookrightarrow W^{s'}(\mathcal{E})$$

is compact.

Theorem 9.2.5 *The natural pairing $W^s(\mathcal{E}) \times W^{-s}(\mathcal{E}) \to \mathbb{C}$ induced by (9.2.25) is nondegenerate, so that $W^s(\mathcal{E})^* = W^{-s}(\mathcal{E})$.*

After these preliminaries, we present the basic estimate in the spectral theory of Δ.

Theorem 9.2.6 *(Gärding inequality) Let $\Gamma(\mathcal{E}) = \oplus_i \Gamma(\mathcal{E}_i)$ be a Dirac complex over a closed oriented Riemannian manifold $\mathbf{M}$ with Dirac operator $D = d + d^*$ and Dirac Laplacian Δ. Then there exist positive constants α and β such that*

$$(\Delta\eta, \eta) \geq \alpha\|\eta\|_1^2 - \beta\|\eta\|_0^2, \quad \eta \in \Gamma(\mathcal{E}). \tag{9.2.27}$$

Proof. We may assume that η is real and, by using a partition of unity, that the support of η is contained in an open set U with $\mathcal{E}|_U$ trivial. Also, we can take $U = \mathbb{R}^n$ with coordinates $x = (x_1, \ldots, x_n)$. We denote by ∂_i the partial derivative with respect to x_i. By (9.2.21), we have

$$(\Delta\eta, \eta) = \|\nabla\eta\|_0^2 + (\mathcal{R}\eta, \eta),$$

so that if $-\alpha_1$, $\alpha_1 \geq 0$, is a lower bound for $\mathcal{R}_x$ as x varies over $\mathbf{M}$, we get

$$(\Delta\eta, \eta) + \alpha_1\|\eta\|_0^2 \geq \|\nabla\eta\|_0^2.$$

Now,

$$\begin{aligned} \|\nabla\eta\|_0^2 &= \int_{\mathbf{M}} \langle\nabla\eta, \nabla\eta\rangle\, d\mathbf{M} \\ &= \int_{\mathbf{M}} \langle\sum_i \nabla_i\eta \otimes dx_i, \sum_j \nabla_j\eta \otimes dx_j\rangle\, d\mathbf{M} \\ &= \int_{\mathbf{M}} \sum_{ij} g^{ij}\langle\nabla_i\eta, \nabla_j\eta\rangle\, d\mathbf{M}, \end{aligned}$$

where $g^{ij} = \langle dx_i, dx_j\rangle$ and $\nabla_i = \nabla_{\partial_i}$. We now use (3.2.8) in the form $\nabla_i = \partial_i + A_i$, so that

$$\|\nabla\eta\|_0^2 = \int_{\mathbf{M}} \left(\sum_{ij} g^{ij}\langle\partial_i\eta, \partial_j\eta\rangle\right) d\mathbf{M} + \int_{\mathbf{M}} \left(\sum_{ij} g^{ij}\langle A_i\eta, (2\partial_j + A_j)\,\eta\rangle\right) d\mathbf{M}.$$

Again by (9.2.24) and Cauchy-Schwartz, the integrals above are respectively bounded from below by

$$\alpha_2\left(\|\eta\|_1^2 - \|\eta\|_0^2\right), \quad \alpha_2 > 0$$

and

$$-\alpha_3\|\eta\|_0\|\eta\|_1, \quad \alpha_3 > 0.$$

It follows that

$$(\Delta\eta,\eta)+\alpha_1\|\eta\|_0^2\geq\alpha_2\left(\|\eta\|_1^2-\|\eta\|_0^2\right)-\alpha_3\|\eta\|_0\|\eta\|_1.$$

Now, given $0<\varepsilon<\alpha_2$, choose $\alpha_4>0$ such that

$$\alpha_3\|\eta\|_0\|\eta\|_1\leq\varepsilon\|\eta\|_1^2+\alpha_4\|\eta\|_0^2,$$

and this gives

$$(\Delta\eta,\eta)+(\alpha_1+\alpha_2+\alpha_4)\|\eta\|_0^2\geq(\alpha_2-\varepsilon)\|\eta\|_1^2,$$

as desired.⌋

In the light of Theorem 9.2.5, (9.2.27) can be rewritten as

$$\|\mathcal{L}\eta\|_{-1}\geq\alpha\|\eta\|_1, \tag{9.2.28}$$

so that $\mathcal{L}=\Delta+\beta:W^1(\mathcal{E})\to W^{-1}(\mathcal{E})$ is continuos and injective with closed range. Moreover, any element η orthogonal to the range of $\mathcal{L}$ defines by duality an element in $\ker\mathcal{L}\subset W^1(\mathcal{E})$ which vanishes by (9.2.28). Thus, $\mathcal{L}$ is surjective and admits an inverse

$$\mathcal{M}:W^{-1}(\mathcal{E})\to W^1(\mathcal{E}).$$

Restricting this to $W^0(\mathcal{E})=L^2(\mathcal{E})$ we get a selfadjoint map $\mathcal{M}:L^2(\mathcal{E})\to W^1(\mathcal{E})$ and composing this with the compact embedding $W^1(\mathcal{E})\hookrightarrow L^2(\mathcal{E})$ given by Theorem 9.2.4, it follows that $\mathcal{M}:L^2(\mathcal{E})\to L^2(\mathcal{E})$ is compact and selfadjoint. Moreover,

$$0<(\mathcal{M}\eta,\eta)\leq\beta^{-1}\|\eta\|_0^2,\quad \eta\neq 0.$$

We are now in a position to apply the spectral theorem for compact selfadjoint operators (cf. [Br]) to infer the existence of an orthogonal decomposition into eigenspaces:

$$L^2(\mathcal{E})=\oplus_{i=0}^{+\infty}E_i, \tag{9.2.29}$$

where $\dim E_i<+\infty$ and $\mathcal{M}\eta=\lambda_i\eta$ for $\eta\in E_i$, where the real sequence of eigenvalues satisfies $0<\lambda_i\leq\beta^{-1}$ and $\lambda_i\to 0$ as $i\to+\infty$. It follows that

$$\Delta\eta=\mu_i\eta,$$

where $\mu_i=1/\lambda_i-\beta\geq 0$ so that $\mu_i\to+\infty$ as $i\to+\infty$. We remark in addition that, since **M** is closed, standard local elliptic regularity theory and Theorem 9.2.3 imply that $E_i\subset\Gamma(\mathcal{E})$, i.e. each eigensection of Δ is smooth. If we order the eigenvalues so that $\mu_i < \mu_{i+1}$ then $E_0 = \ker\Delta = \mathcal{H}^*(\mathcal{E})$, the space of harmonic sections. In particular, $\dim\mathcal{H}^*(\mathcal{E}) < +\infty$ and this proves the last assertion in Theorem 9.2.1.

For our purposes, the main consequence of the spectral decomposition (9.2.29) for Δ is the definition of the corresponding heat flow. For each $t > 0$ define $e^{-t\Delta} : L^2(\mathcal{E}) \to L^2(\mathcal{E})$ by $e^{-t\Delta}\eta = e^{-t\mu_i}\eta$ if $\eta \in E_i$ and extend by linearity. Since $|e^{-t\mu}| \leq 1$, this defines a semigroup of uniformly bounded operators on $L^2(\mathcal{E})$, the so-called *heat flow*. The terminology of course comes from the fact that, for $\eta \in \Gamma(\mathcal{E})$, $\Upsilon(t,.) = e^{-t\Delta}\eta$ satisfies the heat equation

$$\partial_t\Upsilon + \Delta\Upsilon = 0. \tag{9.2.30}$$

If we denote by $\{\nu_j\}$ the eigenvalues of Δ taking the multiplicities into account, and if $\{\psi_j\}$ is a corresponding complete orthonormal basis (with respect to the L^2 product), it is not hard to check that

$$e^{-t\Delta}\eta(x) = \int_{\mathbf{M}} \mathcal{K}(t,x,y)\eta(y)\, d\mathbf{M}_y, \tag{9.2.31}$$

where the *heat kernel* is given by

$$\mathcal{K}(t,x,y) = \sum_j e^{-\nu_j t}\psi_j(x) \otimes \overline{\psi}_j(y), \tag{9.2.32}$$

in complete analogy with (9.1.12)-(9.1.13). Here, one should recall the natural isomorphism

$$E \otimes F = \mathrm{Hom}(F,E), \tag{9.2.33}$$

where E and F are finite dimensional hermitian vector spaces so that $\psi_j(x) \otimes \overline{\psi}_j(y)$ should be thought of as a homomorphism $\mathcal{E}_y \to \mathcal{E}_x$. Notice that even if $\eta \in L^2(\mathcal{E})$, $e^{-t\Delta}\eta$ is smooth so that, as in the case of the circle, the heat flow is smoothing. However, much more is true in the sense that $e^{-t\Delta}$ is *trace class*, i.e. it has a trace given by

$$\mathrm{Tr}\, e^{-t\Delta} = \int_{\mathbf{M}} \mathrm{tr}\,\mathcal{K}(t,x,x)\, d\mathbf{M}_x = \sum_j e^{-\nu_j t},$$

where tr is the usual trace of matrices. This can be easily seen from the fact that, under (9.2.33) with $E = F$, one has $\mathrm{tr}\left(\psi_j(x) \otimes \overline{\psi}_j(x)\right) = |\psi_j(x)|^2$.

Pursuing further the analogy with the circle, it follows from the representation above that

$$\lim_{t\to+\infty} \mathcal{K}_t(t,x,y) = 0.$$

In particular, if we put

$$\overline{\eta}(x) = \lim_{t\to+\infty} e^{-t\Delta}\eta(x), \quad x \in \mathbf{M},$$

we get

$$\Delta\overline{\eta} = 0,$$

and the long term behaviour of the heat flow is determined by *harmonic* sections.

We can even be a bit more explicit here. Observe that both d and d^* commute with Δ and thus with $e^{-t\Delta}$, so we compute

$$\begin{aligned}
\eta - e^{-t\Delta}\eta &= e^{-0\Delta}\eta - e^{-t\Delta}\eta \\
&= -\int_0^t \partial_t(e^{-t\Delta}\eta)\,dt \\
&\overset{(9.2.30)}{=} \int_0^t \Delta e^{-t\Delta}\eta\,dt \\
&= \int_0^t dd^* e^{-t\Delta}\eta\,dt + \int_0^t d^* d e^{-t\Delta}\eta\,dt \\
&= d\left[\int_0^t e^{-t\Delta} d^*\eta\,dt\right] + d^*\left[\int_0^t e^{-t\Delta} d\eta\,dt\right].
\end{aligned}$$

Letting $t \to +\infty$ we have

$$\eta = \overline{\eta} + \eta_1 + \eta_2, \tag{9.2.34}$$

where

$$\eta_1 = d\left[\int_0^{+\infty} e^{-t\Delta} d^*\eta\,dt\right] \in d(\Gamma(\mathcal{E}))$$

and

$$\eta_2 = d^*\left[\int_0^{+\infty} e^{-t\Delta} d\eta\,dt\right] \in d^*(\Gamma(\mathcal{E})).$$

This is the famous *Hodge decomposition*. In particular, if η is *closed* in the sense that $d\eta = 0$, the above calculation shows that

$$\eta - e^{-t\Delta}\eta = d\left[\int_0^t e^{-t\Delta} d^*\eta\,dt\right].$$

This means that η and $e^{-t\Delta}\eta$ lie on the same de Rham cohomology class and hence that the heat flow selects, as $t \to +\infty$, a *unique* harmonic representative in each cohomology class. We then get the following sharper version of Theorem 9.2.1.

Theorem 9.2.7 *In the conditions of Theorem 9.2.1, one has*[5]

$$\Pi_{\ker\Delta} = \lim_{t\to+\infty} e^{-t\Delta}, \tag{9.2.36}$$

and this establishes the Hodge isomorphism (9.2.19).

Notice that this shows in particular that $\dim H^i(\mathcal{E}) < +\infty$ for each i.

9.3 The cancellations

With the analytical preliminaries of the last section out of the way, we can finally explain the cancellations leading to the Atiyah-Singer index theorem. Assume that $\mathcal{E}$ is a graded Clifford bundle. It follows that

$$0 \to \Gamma(\mathcal{E}^+) \stackrel{d}{\to} \Gamma(\mathcal{E}^-) \to 0$$

is a Dirac complex if we take $d = D^+$ and $d^* = D^-$. Also, since

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix},$$

we get

$$\Delta = D^2 = \begin{pmatrix} D^-D^+ & 0 \\ 0 & D^+D^- \end{pmatrix} \stackrel{\text{def.}}{=} \begin{pmatrix} \Delta^+ & 0 \\ 0 & \Delta^- \end{pmatrix},$$

where $\Delta^\pm : \Gamma(\mathcal{E}^\pm) \to \Gamma(\mathcal{E}^\pm)$. As a consequence,

$$e^{-t\Delta} = \begin{pmatrix} e^{-t\Delta^+} & 0 \\ 0 & e^{-t\Delta^-} \end{pmatrix},$$

where $e^{-t\Delta^\pm} : L^2(\mathcal{E}^\pm) \to L^2(\mathcal{E}^\pm)$ are trace class. Moreover, since

$$\ker\Delta^\pm = \ker D^\pm, \tag{9.3.37}$$

we have by Hodge theory (in the form of Theorem 9.2.7),

$$\lim_{t\to+\infty} e^{-t\Delta^\pm} = \Pi_{\ker\Delta^\pm} = \Pi_{\ker D^\pm}.$$

[5]If W is a Hilbert space and $V \subset W$ is a closed subspace, we denote by $\Pi_V : W \to W$ the orthogonal projection over V. If $\dim V < +\infty$ then Π_V is trace class and

$$\operatorname{Tr}\Pi_V = \dim V. \tag{9.2.35}$$

Moreover, the limit in (9.2.36) is in the Hilbert-Schmidt norm (cf. [RS]).

On the other hand,

$$\begin{aligned} \operatorname{ind} D^+ &= \dim\ker D^+ - \dim\ker D^- \\ &\overset{(9.2.35)}{=} \operatorname{Tr}\Pi_{\ker\Delta^+} - \operatorname{Tr}\Pi_{\ker\Delta^-}, \end{aligned}$$

so that

$$\operatorname{ind} D^+ = \lim_{t\to+\infty} \operatorname{Tr}\left(e^{-t\Delta^+} - e^{-t\Delta^-}\right), \tag{9.3.38}$$

and this suggests

Definition 9.3.1 *If $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ is a graded Clifford bundle, $T : L^2(\mathcal{E}) \to L^2(\mathcal{E})$ is trace class and decomposes under the grading as*

$$T = \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix},$$

then the supertrace of T is given by

$$\operatorname{Tr_s} T = \operatorname{Tr} T_{00} - \operatorname{Tr} T_{11}.$$

From this we rewrite (9.3.38) as

$$\operatorname{ind} D^+ = \lim_{t\to+\infty} \operatorname{Tr_s} e^{-t\Delta}. \tag{9.3.39}$$

Now we observe that D^- (respect. D^+) carries eigensections of Δ^+ (respect. Δ^-) corresponding to a nonzero eigenvalue, say ν, to eigensections of Δ^- (respect. Δ^+) with the same eigenvalue. Since $D^+D^-\psi = \nu\psi$ for any such eigensection, it follows that Δ^+ and Δ^- have the same *nonzero* eigenvalues (counted with multiplicities!), which means that $\operatorname{Tr_s} e^{-t\Delta}$ actually does *not* depend on t, so we get

$$\operatorname{ind} D^+ = \operatorname{Tr_s} e^{-t\Delta}, \quad t > 0. \tag{9.3.40}$$

Now we pause to contemplate the situation. First, we have seen from (9.3.39) that $\operatorname{ind} D^+$ can be computed in terms of the long term behavior of heat flow of Δ, but then the cancellations leading to (9.3.40) show that actually $\operatorname{ind} D^+$ is a quantity manifestly invariant under this same flow and this suggests that one should try to compute $\operatorname{Tr_s} e^{-t\Delta}$ for $t \sim 0$ as a final step in the calculation of $\operatorname{ind} D^+$. That this can be carried out in a satisfactory manner follows from the (highly nontrivial!) fact that the heat kernel (9.2.32) has, along the diagonal, an asymptotical expansion as $t \sim 0$ given by

$$\mathcal{K}(t,x,x) \sim \frac{1}{(4\pi t)^{n/2}} \sum_{m\geq 0} a_m(x) t^m, \tag{9.3.41}$$

where $a_m(x) : \mathcal{E}_x \to \mathcal{E}_x$ is a certain endomorphism whose entries depend on the underlying geometry in a sense to be explained below.[6] Notice that with respect to the decomposition $\mathcal{E}_x = \mathcal{E}_x^+ \oplus \mathcal{E}_x^-$ one has

$$a_m(x) = \begin{pmatrix} a_m^+(x) & 0 \\ 0 & a_m^-(x) \end{pmatrix},$$

and setting in this context $\mathrm{tr_s}\, a_m(x) = \mathrm{tr}\, a_m^+(x) - \mathrm{tr}\, a_m^-(x)$ we get

$$\mathrm{Tr_s}\, e^{-t\Delta} = \int_{\mathbf{M}} \mathrm{tr_s}\, \mathcal{K}(t,x,x)\, d\mathbf{M} \sim \frac{1}{(4\pi t)^{n/2}} \sum_{m \geq 0} \hat{a}_m t^m,$$

where

$$\hat{a}_m = \int_{\mathbf{M}} \mathrm{tr_s}\, a_m(x)\, d\mathbf{M}.$$

Combined with (9.3.40) this immediately implies that only the term $\hat{a}_{n/2}$ survives in the expression for the index:

$$\mathrm{ind}\, D^+ = \frac{1}{(4\pi)^{n/2}} \int_{\mathbf{M}} \mathrm{tr_s}\, a_{n/2}(x)\, d\mathbf{M}. \tag{9.3.42}$$

Thus we arrive at a point where the computation of $\mathrm{ind}\, D^+$ gets reduced to determining the supertrace of the term $a_{n/2}$ in the asymptotic expansion of $\mathcal{K}$.

At this juncture, we restrict ourselves to the case of Theorem 8.1.2, so that $\mathcal{E} = \mathcal{S}(\mathbf{M})$, the spin bundle of a closed spin manifold of dimension $n = 4l$ with the natural grading given by (8.1.7), and $D = \partial\!\!\!/$, the classical Atiyah-Singer-Dirac operator.[7] Recalling the fundamental isomorphism (5.2.15), one can show that in this case the coefficients in (9.3.41) satisfy

$$a_m(x) \in \mathbb{C}\mathrm{l}(\mathbf{M})_x \cong \mathbb{C}\mathrm{l}_n, \tag{9.3.43}$$

and moreover, under the vector space isomorphism (5.1.3), *each a_m defines a (possibly non-homogeneous) differential form with top degree less that or equal to $2m$.* On the other hand, the coefficients depend locally and universally on the geometry of $\mathbf{M}$, which means that, in terms of a local chart $x = (x_1, \ldots, x_n)$, their entries are given by explicit universal formulas (i.e. the *same* formulas hold for any such manifold of dimension n) involving algebraic expressions in the derivatives up to a certain order (depending on m and n) of the coefficients of the Riemannian metric of $\mathbf{M}$. Hence, from now on we

[6] Naturally, (9.3.41) should be seen as a far-reaching generalization of (9.1.14).

[7] The more general situation considered in Theorem 8.1.1 can be treated by a similar argument.

can work at a neighborhood U of 0 in $\mathbb{R}^n$ endowed with normal geodesic coordinates with respect to the induced metric given by the local chart and such that $\mathbb{C}\mathrm{l}(\mathbf{M})|_U$ has been trivialized by the use of parallel transport.

The proposition below provides the key to the final step in the proof of the Atiyah-Singer index formula. Recall that $\mathbb{C}\mathrm{l}_n$ has a $\mathbb{Z}$-grading given by (5.1.6) and notice that $\mathrm{Cl}_n^{(0)} \equiv \mathbb{C}$ and $\mathrm{Cl}_n^{(n)}$ is generated by the volume element $\mathrm{e}_1 \ldots \mathrm{e}_n$, so that if $a \in \mathbb{C}\mathrm{l}_n$ we represent by $\overline{a} \in \mathbb{C}$ the projection of a onto $\mathrm{Cl}_n^{(0)}$ and by $\underline{a}\mathrm{e}_1 \ldots \mathrm{e}_n$, $\underline{a} \in \mathbb{C}$, the top degree part of a with respect to (5.1.6).

Proposition 9.3.1 *If we think of $a \in \mathbb{C}\mathrm{l}_n$ as an element of* $\mathrm{End}(\Psi)$ *via (5.2.15),*

$$\mathrm{tr}_{\mathrm{s}}\, a = \left(\frac{2}{i}\right)^{n/2} \underline{a}.$$

Proof. Since $\mathrm{tr}_{\mathrm{s}}\, a = \mathrm{tr}\, \Gamma_n a$ one has to check that $\mathrm{tr}\, a = 2^{n/2}\overline{a}$. But $1 \in \mathbb{C}\mathrm{l}_n$ acts as the identity on Ψ and hence $\mathrm{tr}\, 1 = 2^{n/2}$. Moreover, by (5.2.15), for each e_I with $I \neq \emptyset$ there exists a constant c_I such that $\mathrm{tr}\, \mathrm{e}_I = c_I \mathrm{tr}\, \Xi(\mathrm{e}_I)$, where Ξ is the regular representation from Section 5.2, and inspecting the action of $\Xi(\mathrm{e}_I)$ on the standard basis $\{\mathrm{e}_J\}$ of $\mathbb{C}\mathrm{l}_n$ one sees without difficulty that $\mathrm{tr}\, \Xi(\mathrm{e}_I) = 0.\rfloor$

This clearly represents a considerable improvement on the above emphasized statement regarding the coefficients a_m, meaning that only the top degree term of $a_{n/2}$ contributes to the index, and of course it represents another formidable cancellation in our calculation. The next major step is, by using a clever scaling argument due to E. Getzler (cf. [BGV]), to deform Δ into another Laplace type second order elliptic operator Δ^∞ defined in a neighborhood of 0 and with the property that the corresponding heat equation $\partial_t \mathrm{Y} + \Delta^\infty \mathrm{Y} = 0$ has as heat kernel a t-dependent $\mathbb{C}\mathrm{l}_n$-valued function $\mathcal{K}^\infty$ such that, for $t \sim 0$,

$$\mathcal{K}^\infty(t,x,x) \sim \frac{1}{(4\pi t)^{n/2}} \sum_{m \geq 0} a_m^\infty(x) t^m, \tag{9.3.44}$$

with

$$a_{n/2}^\infty(0) = \underline{a}_{n/2}(0). \tag{9.3.45}$$

Specifically,

$$\Delta^\infty = -\sum_{j=1}^n \left(\frac{\partial}{\partial x_j} - \frac{1}{8}\sum_{l=1}^n \Theta_{jl} x_l\right)^2,$$

where Θ is the curvature 2-form associated to the Levi-Civita connection on $T\mathbf{M}$ computed at $x = 0$ and acting on $\mathbb{C}\mathrm{l}_n$ by

$$\Theta_{kl} = \sum_{p,q=1}^{n} \langle R_{\mathrm{e}_k,\mathrm{e}_l}\mathrm{e}_p, \mathrm{e}_q\rangle \mathrm{e}_p\mathrm{e}_q.$$

It turns out that Δ^∞ is a sort of generalized harmonic oscillator operator with a quadratic potential depending on the curvature of $\mathbf{M}$, so that by a classical formula due to Mehler we end up with

$$\mathcal{K}^\infty(t,0,0) = \frac{1}{(4\pi t)^{n/2}}\hat{A}\left(-2\pi i t\Theta\right), \tag{9.3.46}$$

where once again the isomorphism (5.1.3) has been used to identify elements in $\Gamma\left(\mathbb{C}\mathrm{l}(\mathbf{M})|_U\right)$ and differential forms.

Now, the Atiyah-Singer index formula can at last be checked as follows. First, from (9.3.42) and Proposition 9.3.1 we have

$$\mathrm{ind}\,\partial\!\!\!/^+ = \frac{1}{(4\pi)^{n/2}}\left(\frac{2}{i}\right)^{n/2}\int_{\mathbf{M}} a^\infty_{n/2}\, d\mathbf{M}.$$

But from (9.3.44) and (9.3.46),

$$\begin{aligned} a^\infty_{n/2}(0) &= \left(\frac{n}{2}\right)^{\text{th}} \text{ coefficient in } t \text{ of } \hat{A}\left(-2\pi i t\Theta\right) \\ &= (2\pi i)^{n/2} \times \left(\frac{n}{2}\right)^{\text{th}} \text{ coefficient in } t \text{ of } \hat{A}\left(t\Theta\right) \\ &= (2\pi i)^{n/2}[\hat{A}(\Theta)]_n, \end{aligned}$$

so that

$$\begin{aligned} \mathrm{ind}\,\partial\!\!\!/^+ &= \frac{1}{(4\pi)^{n/2}}\left(\frac{2}{i}\right)^{n/2}(2\pi i)^{n/2}\int_{\mathbf{M}}[\hat{A}(\Theta)]_n\, d\mathbf{M} \\ &= \int_{\mathbf{M}}[\hat{A}(\Theta)]_n\, d\mathbf{M} \\ &= \langle \hat{A}(T\mathbf{M}), \mathbf{M}\rangle, \end{aligned}$$

and this completes our sketch of the proof of Theorem 8.1.14.

There are two points one would like to emphasize here. First, in the calculation above no previous knowledge of characteristic classes is needed, as the $\hat{A}$-class appears naturally in the end of the computation as a result of applying Mehler's formula. This should be compared to the first generation proofs of the index formula by heat equations methods [G], where the characteristic integrand was determined by the heat asymptotics up to finitely many universal constants (and for this a great deal of information on characteristic classes was required!) and its final shape was only made precise after evaluation on explicit examples. Second, if we examine the proof above in hindsight, we see that (9.3.42) suggests that, if properly normalized, the local density $\mathrm{tr_s}\, a_{n/2}$ given by the heat flow and the Chern-Weil form $\hat{A}(\Theta)$ should give the same result after integration over $\mathbf{M}$, a point already confirmed by the so-called topological proofs of the index formula. But that these quantities actually coincide pointwisely is certainly a sort of mathematical miracle, specially because, by construction, $\mathrm{tr_s}\, a_{n/2}$ depends apriori on high order derivatives of the metric (with the order growing with n) while $\hat{A}(\Theta)$ surely depends on derivatives only up to second order. This remarkable result and its many generalizations, usually referred to in the literature as 'local index formulas', illustrate the power of the 'fantastic cancellations' involved in the arguments sketched above.

Chapter 10

Some applications of the index formula

The Atiyah-Singer index formula for Dirac operators, proved in the early sixties, contains as particular cases some index results previously established for certain 'classical' operators in Differential Geometry. In this chapter we reverse the historical order of events and show how two of the classical cases, namely the Hirzebruch signature and Chern-Gauss-Bonnet formulas, can be obtained by a routine application of the index formula (8.1.13) to suitable twisted Dirac operators. There is still another result, the Hirzebruch-Riemann-Roch formula for algebraic manifolds, which predated by some years the Atiyah-Singer formula, and which can also be retrieved by the same methods, but we shall not treat this here. We also present a beautiful argument of Gromov and Lawson providing interesting *topological-geometric* obstructions to the existence of metrics of positive scalar curvature on certain Riemannian manifolds. As a final bonus, we give a brief introduction to spin^c structures and the Seiberg-Witten equations.

10.1 The Chern-Gauss-Bonnet formula

In this section we shall indicate how the index formula (8.1.13) can be use to prove

Theorem 10.1.1 *(Chern-Gauss-Bonnet) Let* $\mathbf{M}$ *be a closed oriented Riemannian manifold of dimension* $n = 2k$*. Then the Euler characteristic* $\chi(\mathbf{M})$ *can be expressed* *in terms of the Euler class of $T\mathbf{M}$ as*

$$\chi(\mathbf{M}) = \langle \mathrm{e}(T\mathbf{M}), \mathbf{M} \rangle. \tag{10.1.1}$$

We shall treat this by expressing $\chi(\mathbf{M})$ as the index of a certain twisted Dirac operator. We assume for the moment that $\mathbf{M}$ is spin and introduce some notation. Let $\Lambda^{\text{even}}(\mathbf{M})$ (respect. $\Lambda^{\text{odd}}(\mathbf{M})$) be the bundle of even (respect. odd) degree complex differential forms over $\mathbf{M}$ so that $\mathcal{A}^{\text{even}}(\mathbf{M}) = \Gamma(\Lambda^{\text{even}}(\mathbf{M}))$ and $\mathcal{A}^{\text{odd}}(\mathbf{M}) = \Gamma(\Lambda^{\text{odd}}(\mathbf{M}))$. Observe that if we consider $\mathcal{D} = d + d^* : \mathcal{A}^{\text{even}}(\mathbf{M}) \to \mathcal{A}^{\text{odd}}(\mathbf{M})$ then Hodge-de Rham theory implies that $\operatorname{ind} \mathcal{D} = \chi(\mathbf{M})$, but the grading $\Lambda(\mathbf{M}) = \Lambda^{\text{even}}(\mathbf{M}) \oplus \Lambda^{\text{odd}}(\mathbf{M})$ does *not* make $\mathcal{D}$ into a twisted Dirac operator. To proceed, we note the identifications $\mathbb{C}\mathrm{l}^0(\mathbf{M}) = \Lambda^{\text{even}}(\mathbf{M})$ and $\mathbb{C}\mathrm{l}^1(\mathbf{M}) = \Lambda^{\text{odd}}(\mathbf{M})$, where the decomposition $\mathbb{C}\mathrm{l}(\mathbf{M}) = \mathbb{C}\mathrm{l}^0(\mathbf{M}) \oplus \mathbb{C}\mathrm{l}^1(\mathbf{M})$ follows from (5.1.4) and (5.2.16). In terms of the splitting (5.2.11), one clearly has

$$\mathbb{C}\mathrm{l}_n = \operatorname{End}(\Psi) = \Psi \otimes \Psi = \begin{pmatrix} \operatorname{End}(\Psi^+) & \operatorname{Hom}(\Psi^-, \Psi^+) \\ \operatorname{Hom}(\Psi^+, \Psi^-) & \operatorname{End}(\Psi^-) \end{pmatrix}.$$

Now, the diagonal terms correspond to the isomorphisms

$$\mathbb{C}\mathrm{l}_n^0 = \operatorname{End}(\Psi^+) \oplus \operatorname{End}(\Psi^-) = \left(\Psi^+ \otimes \Psi^+\right) \oplus \left(\Psi^- \otimes \Psi^-\right)$$

and the off-diagonal terms correspond to the isomorphisms

$$\mathbb{C}\mathrm{l}_n^1 = \operatorname{Hom}(\Psi^+, \Psi^-) \oplus \operatorname{Hom}(\Psi^-, \Psi^+) = \left(\Psi^+ \otimes \Psi^-\right) \oplus \left(\Psi^- \otimes \Psi^+\right)$$

so that (assuming $\mathbf{M}$ spin) $\operatorname{ind} \mathcal{D} = \operatorname{ind} \partial\!\!\!/^{+}_{(-1)^k \hat{\mathcal{S}}(\mathbf{M})}$, where $\partial\!\!\!/^{+}_{(-1)^k \hat{\mathcal{S}}(\mathbf{M})}$ is the Atiyah-Singer-Dirac operator twisted by the 'virtual' bundle $(-1)^k \hat{\mathcal{S}}(\mathbf{M})$, where $\hat{\mathcal{S}}(\mathbf{M}) = \mathcal{S}^+(\mathbf{M}) - \mathcal{S}^-(\mathbf{M})$ is the *spinor difference element*. In terms of the variables y_i introduced soon after Theorem 7.2.2, with $\mathcal{F} = T\mathbf{M}$, it is not hard to check that (see [LM]):

$$\operatorname{ch}(\hat{\mathcal{S}}(\mathbf{M})) = \prod_{i=1}^{k} \left(e^{-y_i/2} - e^{+y_i/2} \right) \tag{10.1.2}$$

so that

$$\begin{aligned} \operatorname{ch}(\hat{\mathcal{S}}(\mathbf{M})) &= (-1)^k y_1 \dots y_k + \text{terms of higher degree} \\ &\overset{(7.3.15)}{=} (-1)^k \mathrm{e}(T\mathbf{M}) + \text{terms of higher degree.} \end{aligned}$$

Since

$$\hat{A}(T\mathbf{M}) = 1 + \text{terms of higher degree,}$$

from (8.1.13) we finally get

$$\chi(\mathbf{M}) = \langle \mathrm{ch}((-1)^k \hat{\mathcal{S}}(\mathbf{M}))\hat{A}(T\mathbf{M}), \mathbf{M}\rangle = \langle \mathrm{e}(T\mathbf{M}), \mathbf{M}\rangle.$$

It remains to remove the spin condition on $\mathbf{M}$. But we know from Section 9.3 that $\chi(\mathbf{M})$, as the index of a twisted Dirac operator, can be computed by integrating over $\mathbf{M}$ the supertrace of the appropriate coefficient in the asymptotic expansion of the corresponding heat kernel, and this is a local invariant in the geometry of $\mathbf{M}$. On the other hand, the argument above has identified, in the presence of a spin structure, this invariant to the local expression defining the Euler class via Chern-Weil theory. Since locally any manifold is spin, this completes the proof of Theorem 10.1.1.

10.2 The Hirzebruch signature theorem

Let $\mathbf{M}$ be a closed oriented manifold of dimension $n = 2k$ and let $H^k_{\mathrm{dR}}(\mathbf{M})$ be the middle dimensional de Rham cohomology group of $\mathbf{M}$. We can define a bilinear form Q on $H^k_{\mathrm{dR}}(\mathbf{M})$ by

$$Q([\eta], [\eta']) = \int_{\mathbf{M}} \eta \wedge \eta',$$

for closed forms η and η'. Clearly, this is well-defined in the sense that it does not depend on the chosen representatives in each class. Moreover, one has $Q([\eta], [\eta']) = (-1)^{k^2} Q([\eta'], [\eta])$ so that if we assume further that $k = 2l$ so that $\dim \mathbf{M} = 2k = 4l$, one sees that Q is *symmetric*. Poincaré duality implies that Q is non-degenerate and hence the corresponding quadratic form has no null eigenvalue. We then define the *signature* of $\mathbf{M}$ by

$$\mathrm{sign}\,(\mathbf{M}) = b^+ - b^-,$$

where b^+ (respect. b^-) is the number of positive (respect. negative) eigenvalues counted with multiplicities. By de Rham theory, this is an oriented homotopy invariant of $\mathbf{M}$ and, surprisingly enough, it can be computed in terms of local geometric data in $\mathbf{M}$ by integration of a certain curvature dependent expression associated to a Riemannian metric on $\mathbf{M}$. This was first discovered in the fifties by Hirzebruch via topological methods (cobordism theory) and is one of the classical index formulas predating the general Atiyah-Singer index formula. In the following we show how the Hirzebruch signature formula follows from Theorem 8.1.1. As in the Chern-Gauss-Bonnet theorem, the idea is to identity $\mathrm{sign}\,(\mathbf{M})$ to the index of a certain twisted Dirac operator, as we now pass to explain.

First we remark that the grading condition in Definition 8.1.1 is equivalent to the existence of a self-adjoint bundle map $\Gamma : \mathcal{E} \to \mathcal{E}$ satisfying the conditions $\Gamma^2 = 1$ and $D\Gamma = -\Gamma D$. By Proposition 5.2.3, it follows that if we take $\Gamma = \Gamma_n$, the left multiplication by the complex volume element associated to some Riemannian metric on $\mathbf{M}$, then the conditions above are satisfied if

$$\mathcal{E} = \mathbb{C}\mathrm{l}(\mathbf{M}) = \Lambda(\mathbf{M}) \otimes \underline{\mathbb{C}}, \tag{10.2.3}$$

where $\Lambda(\mathbf{M}) \otimes \underline{\mathbb{C}}$ is the bundle of complex valued differential forms over $\mathbf{M}$, and $D = \mathcal{D} = d + d^*$, the Dirac type operator considered in Chapter 4. We now claim that

$$\mathrm{sign}\,(\mathbf{M}) = \mathrm{ind}\,\mathcal{D}^+, \tag{10.2.4}$$

where as usual the righthand side is defined by computing dimensions over $\mathbb{C}$.

To see this, notice that $\Gamma_n(\ker \mathcal{D}) \subset \ker \mathcal{D}$ and Hodge theory imply that Γ_n descends to an involution on $\mathcal{H} = \mathcal{H}^*(\mathbf{M}) \otimes \mathbb{C}$, the space of complex valued harmonic forms with respect to the chosen Riemannian metric. If we set $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ with $\mathcal{H}^\pm = \{\eta \in \mathcal{H}; \Gamma_n\eta = \pm\eta\}$, it is immediate that

$$\ker \mathcal{D}^\pm = \mathcal{H}^\pm \cap \mathcal{H}^k \oplus \bigoplus_{0 \le i < k} \mathcal{H}^\pm \cap \left(\mathcal{H}^i \oplus \mathcal{H}^{n-i}\right),$$

where we have set $\mathcal{H}^j = \mathcal{H}^j(\mathbf{M}) \otimes \mathbb{C}$, $0 \le j \le n$. Notice that, for $0 \le i < k$, there holds $\Gamma_n = i^{k+i(i-1)}*$ on $\Lambda^i(\mathbf{M}) \otimes \mathbb{C}$, and hence if we consider $\varphi_i \in \mathrm{End}(\mathcal{H}^i \oplus \mathcal{H}^{n-i})$ given by $\varphi_i(\eta + \Gamma_n\eta) = \eta - \Gamma_n\eta$, this turns out to be an isomorphism, so that in particular $\ker \mathcal{D}^+|_{\mathcal{H}^i \oplus \mathcal{H}^{n-i}} = \ker \mathcal{D}^-|_{\mathcal{H}^i \oplus \mathcal{H}^{n-i}}$ and the contributions of these subspaces to the index cancel out. As a result,

$$\mathrm{ind}\,\mathcal{D}^+ = \dim \mathcal{H}^+ \cap \mathcal{H}^k - \dim \mathcal{H}^- \cap \mathcal{H}^k.$$

Now, on $\mathcal{H}^k$ one has $\Gamma_n = *$ and from this we have first of all the direct sum decomposition $\mathcal{H}^k = \mathcal{H}^+ \cap \mathcal{H}^k \oplus \mathcal{H}^- \cap \mathcal{H}^k$ and then that $Q|_{\mathcal{H}^+ \cap \mathcal{H}^k}$ is positive definite[1] since, for $\eta \neq 0$ in $\mathcal{H}^+ \cap \mathcal{H}^k$,

$$Q(\eta, \eta) = \int_{\mathbf{M}} \eta \wedge \overline{\eta} = \int_{\mathbf{M}} \eta \wedge *\overline{\eta} = \|\eta\|^2 > 0,$$

and similarly $Q|_{\mathcal{H}^- \cap \mathcal{H}^k}$ is negative definite. This means that $b^\pm = \dim \mathcal{H}^\pm \cap \mathcal{H}^k$, which completes the proof of (10.2.4).

[1] In fact, here Q denotes the obvious hermitian extension of the previously defined quadratic form.

Now, recall from (6.2.5) that

$$\mathbb{C}\mathrm{l}(\mathbf{M}) = \mathcal{S}(\mathbf{M}) \otimes \mathcal{S}(\mathbf{M}), \tag{10.2.5}$$

and combined with (10.2.3) this says that, in the notation of Chapter 8 (and of course assuming that $\mathbf{M}$ is spin), we can take $\mathcal{G} = \mathcal{S}(\mathbf{M})$ and the next step is to check that

$$\operatorname{ind} \mathcal{D}^+ = \operatorname{ind} \partial\!\!\!/^{+}_{\mathcal{S}(\mathbf{M})}. \tag{10.2.6}$$

In words, the Dirac operator $\mathcal{D}$ coming from the grading given by Γ_n has the same index as the classical Atiyah-Singer-Dirac operator $\partial\!\!\!/^+$ twisted by the spinor bundle of $\mathbf{M}$. Now, it is not hard to check that the index of a Dirac type operator only depends on its first order term and on the corresponding grading, which are clearly determined by Clifford multiplication. So we just have to see that these data are preserved under the pointwise isomorphism $\mathbb{C}\mathrm{l}_n = \Psi \otimes \Psi$ underlying (10.2.5), and this is a direct checking left to the reader.

So far we have noted that, for a spin manifold $\mathbf{M}$ as above, $\operatorname{sign}(\mathbf{M}) = \operatorname{ind} \partial\!\!\!/^{+}_{\mathcal{S}(\mathbf{M})}$ and we are in a position to apply Theorem 8.1.1 to verify that $\operatorname{sign}(\mathbf{M}) = \langle \mathrm{ch}(\mathcal{S}(\mathbf{M}))\hat{A}(T\mathbf{M}), \mathbf{M}\rangle$. Similarly to (10.1.2), we have

$$\mathrm{ch}(\mathcal{S}(\mathbf{M})) = \prod_{i=1}^{k} \left(e^{y_i/2} + e^{-y_i/2} \right) = 2^k \prod_{i=1}^{k} \cosh \frac{y_i}{2},$$

so we compute using (7.2.10),

$$\mathrm{ch}(\mathcal{S}(\mathbf{M}))\hat{A}(T\mathbf{M}) = 2^k \prod_{i=1}^{k} \cosh \frac{y_i}{2} \frac{y_i/2}{\sinh(y_i/2)} = \prod_{i=1}^{k} \frac{y_i}{\tanh(y_i/2)}.$$

At this point we use the recipe of Section 7.2 to define the *Hirzebruch $\mathcal{L}$-class* of $\mathbf{M}$ by the multiplicative invariant formal power series

$$\mathcal{L}(T\mathbf{M}) = \prod_{i=1}^{k} \frac{y_i/2}{\tanh(y_i/2)},$$

and we finally have

Theorem 10.2.1 *(Hirzebruch) For a closed oriented manifold of dimension $n = 4l$,*

$$\operatorname{sign}(\mathbf{M}) = \langle \mathcal{L}(T\mathbf{M}), \mathbf{M} \rangle. \tag{10.2.7}$$

Proof. There are two points that remain to be checked. First, when we compare the expressions for $\mathrm{ch}(\mathcal{S}(\mathbf{M}))\hat{A}(T\mathbf{M})$ and $\mathcal{L}(T\mathbf{M})$ above, we see that they do not quite match due to an extra factor of 2. However, it is easy to see that the degree n term in these characteristic classes agree, showing that the computation is all right. The other point is to remove the spin condition in the analysis above and this uses the sane argument as in the proof of Theorem 10.1.1.

The signature formula is a central result in Differential Topology. For example, it has been used in Milnor's celebrated argument [Mi] leading to the existence of exotic differentiable structures on S^7.

10.3 The Gromov-Lawson obstruction

We have already seen how, by combining the Atiyah-Singer formula (8.1.14) with the vanishing criterion given by Theorem 6.2.2, Lichnerowicz was able to detect the first obstruction result for the existence of metrics with quasi-positive scalar curvature. The method proved to be very fruitful and in fact it has been largely perfected by Hitchin [H], who proved for example a striking theorem according to which half of the exotic spheres in dimension 1 and 2 (mod 8) can not carry a metric with positive scalar curvature. In this section, we present still another variant of Lichnerowicz argument explored by Gromov and Lawson in a remarkable series of articles ([GL_1], [GL_2], [GL_3]). The idea here is to bring the fundamental group (more precisely, the way the fundamental group acts on covering spaces by deck transformations) to the core of the discussion and a typical result obtained by their methods is

Theorem 10.3.1 *The n-torus $T^n = S^1 \times \ldots \times S^1$ does not carry any metric with everywhere positive scalar curvature.*

Before proceeding with the proof of this result, some comments are in order. First, in fact a much sharper result holds: any metric with non-negative scalar curvature on T^n is actually flat! Second, in the dimensional range $n \leq 7$, the obstruction was first proved by Schoen and Yau [SY] by using minimal hypersurfaces methods. And third, the argument actually holds for a large class of manifolds, the so-called enlargeable manifolds, from which T^n seems to be a natural representative.

And what is so special about T^n? For each $m \geq 0$, let $C^n_m \subset \mathbb{R}^n$ be the n-cube given by $0 \leq x_i \leq 2^m$, $1 \leq i \leq n$. If we identify points in the boundary of C^n_m in the usual way, we get a n-torus, say T^n_m, equipped with the flat Euclidean metric and the map $C^n_m \to C^n_1$ induced by the natural $\mathbb{Z}^n$-action on $\mathbb{R}^n$ gives a $(2^m n)$-sheeted Riemannian covering maps $T^n_m \to T^n_0 = T^n$. Moreover, if C^n_m is uniformly stretched over the standard unit sphere $S^n \subset \mathbb{R}^{n+1}$ in such a way that the center $P_m = (2^{m-1}, \ldots, 2^{m-1})$ of the open ball $B^n_m = \{x \in C^n_m; |x - P_m| < 2^{m-1}\}$ goes into the north pole and the complement of B^n_m in C^n_m is collapsed onto the south pole, this provides a degree one map $f : T^n_m \to S^n$ satisfying $|f_* v| \leq (\pi/2^{m-1})|v|$ for any v tangent to T^n_m. Thus we have been able to establish the following fundamental property of T^n endowed with *any* Riemannian metric: for every $\varepsilon > 0$ there exists a finite Riemannian covering space $\tilde{T}$ of T^n and a map $f : \tilde{T} \to S^n$ which is ε-contractible in the sense that $|f_* v| \leq \varepsilon |v|$, for any v tangent to $\tilde{T}$, and has $\deg f = 1$. More generally, and taking Proposition 6.1.4 into account, we have

Definition 10.3.1 *A closed Riemannian manifold* $\mathbf{M}$ *of dimension* n *is said to be compactly enlargeable if for every* $\varepsilon > 0$ *there exists a finite Riemannian covering space* $\tilde{\mathbf{M}}$ *of* $\mathbf{M}$ *which is spin and a map* $f : \tilde{\mathbf{M}} \to S^n$ *which is* ε*-contractible in the sense that* $|f_* v| \leq \varepsilon |v|$, *for any* v *tangent to* $\tilde{\mathbf{M}}$, *and has* $\deg f \neq 0$.

With this terminology at hand, we have the following extension of Theorem 10.3.1.

Theorem 10.3.2 *A compactly enlargeable manifold* $\mathbf{M}$ *cannot carry a metric with everywhere positive scalar curvature.*

Proof. After possibly taking product with S^1, we can assume $n = 2k$. Now suppose for the sake of absurd that $\mathbf{M}$ carries a metric with $\kappa > 0$. For each $\varepsilon > 0$ recall that we have a map $f : \tilde{\mathbf{M}} \to S^n$ meeting the conditions of Definition 10.3.1. From results in [AH], we know the existence of a complex vector bundle $\mathcal{E} \to S^n$ (which we may assume endowed with a hermitian metric and compatible connection) with $\langle c_k(\mathcal{E}), S^n \rangle \neq 0$ and we are thus led to consider the twisted bundle $\mathcal{S}(\tilde{\mathbf{M}}) \otimes f^*\mathcal{E} \to \tilde{\mathbf{M}}$, where the pulled back bundle $f^*\mathcal{E} \to \tilde{\mathbf{M}}$ has the induced metric and connection. Since $\tilde{\mathbf{M}}$ is closed we can apply (8.1.13) to conclude that the index of the twisted Dirac operator

$$\partial\!\!\!/^{+}_{f^*\mathcal{E}} : \Gamma(\mathcal{S}^+(\tilde{\mathbf{M}}) \otimes f^*\mathcal{E}) \to \Gamma(\mathcal{S}^-(\tilde{\mathbf{M}}) \otimes f^*\mathcal{E})$$

is given by

$$\operatorname{ind} \partial\!\!\!/^{+}_{f^*\mathcal{E}} = \langle \operatorname{ch}(f^*\mathcal{E}) \hat{A}(T\tilde{\mathbf{M}}), \tilde{\mathbf{M}} \rangle. \tag{10.3.8}$$

The proof is completed by computing this integer in two different ways and showing that, under the assumption $\kappa > 0$, the outcomes are different indeed, thus reaching a contradiction.

First, one recall from Propositon 8.1.1 that

$$\partial\!\!\!/^{2}_{f^*\mathcal{E}} = \nabla^*\nabla + \frac{\tilde{\kappa}}{4} + \mathcal{R}^{[f^*\mathcal{E}]},$$

where $\tilde{\kappa}$ is the scalar curvature of the metric on $\tilde{\mathbf{M}}$ induced by the Riemannian covering $\tilde{\mathbf{M}} \to \mathbf{M}$ (so that in particular $\tilde{\kappa} > 0$). An easy estimation using (8.1.10) gives the pointwise bound $|\mathcal{R}^{[f^*\mathcal{E}]}| \leq \varepsilon^2 c_n |R^{\mathcal{E}}|$ where $c_n > 0$ is a universal constant depending only on n, so that by taking ε small enough we will have $\tilde{\kappa}/4 + \mathcal{R}^{[f^*\mathcal{E}]} > 0$ and a vanishing argument as in the proof of Theorem 6.2.2 shows that $\ker \partial\!\!\!/_{f^*\mathcal{E}} = \{0\}$ and hence $\operatorname{ind} \partial\!\!\!/^+_{f^*\mathcal{E}} = 0$.

We now show that $\operatorname{ind} \partial\!\!\!/^+_{f^*\mathcal{E}} \neq 0$ by using (10.3.8). First, one has $\mathrm{ch}(f^*\mathcal{E}) = f^*\left(\mathrm{ch}(\mathcal{E})\right)$ and since S^n has trivial cohomology in degree $1 \leq i \leq n-1$,

$$\mathrm{ch}(\mathcal{E}) = \mathrm{rank}(\mathcal{E})1 + \frac{1}{(k-1)!} c_k(\mathcal{E}).$$

This allows us to compute

$$\begin{aligned} \langle \mathrm{ch}(f^*\mathcal{E})\hat{A}(T\tilde{\mathbf{M}}), \tilde{\mathbf{M}}\rangle &= \mathrm{rank}(\mathcal{E})\hat{A}(\tilde{\mathbf{M}}) + \frac{1}{(k-1)!}\langle f^*(c_k(\mathcal{E})), \tilde{\mathbf{M}}\rangle \\ &= \frac{1}{(k-1)!}\langle f^*(c_k(\mathcal{E})), \tilde{\mathbf{M}}\rangle, \end{aligned}$$

where we have applied Theorem 6.2.3 to $\tilde{\mathbf{M}}$, so that finally

$$\begin{aligned} \operatorname{ind} \partial\!\!\!/^+_{f^*\mathcal{E}} &= \frac{1}{(k-1)!}\langle f^*(c_k(\mathcal{E})), \tilde{\mathbf{M}}\rangle \\ &= \frac{1}{(k-1)!}\int_{\tilde{M}} f^*(c_k(\mathcal{E})) \\ &\overset{(1.1.12)}{=} \frac{1}{(k-1)!}\deg f \int_{S^n} c_k(\mathcal{E}) \neq 0, \end{aligned}$$

as desired.⌋

This beautiful argument can actually be generalized to the case in which some element in the tower of coverings of $\mathbf{M}$ is merely complete, but for this one needs a sort of relative index theorem for pairs of Dirac operators on a noncompact spin manifold 'agreeing' at infinity. The interested reader should consult [GL$_3$] and [LM] for the details.

10.4 The Seiberg-Witten equations

Spin geometry is particularly effective in low dimensions where, due to the existence of certain 'exceptional isomorphisms', it usually admits a concrete description in terms of more familiar geometric data. In dimension four, a fundamental breakthrough was achieved by Witten [Wi], where the moduli space of solutions of a certain monopole type equation (the Seiberg-Witten equation) was used to address many important questions on the topology and geometry of smooth 4-manifolds. The purpose of this section is to sketch some points of this recent development, notably because the index formula (8.1.13) plays a prominent role in the theory.

In what follows, unless otherwise stated, $\mathbf{M}$ will denote a closed oriented 4-manifold with a fixed Riemannian metric. The relevance of dimension four is that it is the only dimension for which the rotation group SO_n is *not* a simple Lie group. More precisely, recall from (1.2.14) that one has in general the isomorphism $\mathfrak{so}_n = \Lambda^2\mathbb{R}^n$. However, if $n = 4$, the Hodge star operator $* : \Lambda^2\mathbb{R}^4 \to \Lambda^2\mathbb{R}^4$ satisfies $*^2 = \mathrm{Id}$, and this yields an orthogonal splitting

$$\Lambda^2\mathbb{R}^4 = \Lambda^2_+\mathbb{R}^4 \oplus \Lambda^2_-\mathbb{R}^4, \tag{10.4.9}$$

where $\Lambda^2_\pm\mathbb{R}^4 = \{\eta \in \Lambda^2\mathbb{R}^4; *\eta = \pm\eta\}$. Notice that, in view of Propositions 5.1.6 and 5.1.7, $\mathfrak{spin}_4$ has a similar splitting

$$\mathfrak{spin}_4 = \mathfrak{spin}_4^+ \oplus \mathfrak{spin}_4^-.$$

Notice also the existence of homomorphisms $\wedge^2\mu_4^\pm : \mathrm{SO}_4 \to \mathrm{SO}_3$ given by the composition of the standard representation $\wedge^2\mu_4$ of SO_4 on $\Lambda^2\mathbb{R}^4$ with the orthogonal projections over the factors of (10.4.9).

Now, since Spin_4 is simply connected, one would expect a corresponding direct product decomposition

$$\mathrm{Spin}_4 = \mathrm{Spin}_4^+ \times \mathrm{Spin}_4^-, \tag{10.4.10}$$

for certain 3-dimensional Lie groups $\mathrm{Spin}_4^\pm$. Fortunately, it is easy to provide an explicit description of the factors entering in (10.4.10). Indeed, distinguish two copies, say SU_2^+ and SU_2^-, of the special unitary group SU_2, and let $(g_+, g_-) \in \mathrm{SU}_2^+ \times \mathrm{SU}_2^-$ act on $\mathbb{R}^4$ by

$$(g_+, g_-)x = g_+x(g_-)^{-1}, \tag{10.4.11}$$

where

$$x = (x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 + ix_4 & -x_2 + ix_3 \\ x_2 + ix_3 & x_1 - ix_4 \end{pmatrix}.$$

Since $|x|^2 = \det x$, this gives a Lie group homomorphism $\gamma : \mathrm{SU}_2^+ \times \mathrm{SU}_2^- \to \mathrm{SO}_4$ with $\ker\gamma = \mathbb{Z}_2$, meaning that $\mathrm{Spin}_4 = \mathrm{SU}_2^+ \times \mathrm{SU}_2^-$ and $\gamma = \gamma_4$, the standard covering map in Proposition 5.1.5. Incidentally, notice that if we restrict γ to SU_2 via the diagonal embedding $\mathrm{SU}_2 \subset \mathrm{SU}_2^+ \times \mathrm{SU}_2^-$, we get a map $\tilde{\gamma} : \mathrm{SU}_2 \to \mathrm{SO}_4$. However, $\tilde{\gamma}$ fixes pointwisely the real axis $x_2 = x_3 = x_4 = 0$ and hence defines a homomorphism $\tilde{\gamma} : \mathrm{SU}_2 \to \mathrm{SO}_3$ with $\ker \tilde{\gamma} = \mathbb{Z}_2$, so that $\mathrm{Spin}_3 = \mathrm{SU}_2$ and $\tilde{\gamma} = \gamma_3$.

The above computations show that a spin structure on a Riemannian 4-manifold $\mathbf{M}$ amounts to the existence of a pair $(\mathcal{S}^+, \mathcal{S}^-)$ of complex plane bundles over $\mathbf{M}$ such that:

1. Both $\mathcal{S}^+$ and $\mathcal{S}^-$ have SU_2 as structural group or, equivalently, by Chern-Weil theory, $c_1(\mathcal{S}^+) = c_1(\mathcal{S}^-) = 0$. Of course, $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ is the spinor bundle of Section 6.2.

2. Clifford multiplication defines a bundle isomorphism

$$T\mathbf{M} \otimes \underline{\mathbb{C}} = \mathrm{Hom}(\mathcal{S}^+, \mathcal{S}^-) = \mathcal{S}^+ \otimes \mathcal{S}^-.$$

We will now see that these are rather stringent conditions on $\mathbf{M}$. First, recall from Proposition 7.2.1 that $c_1(T\mathbf{M} \otimes \underline{\mathbb{C}}) = 0$. In order to compute $c_2(T\mathbf{M} \otimes \underline{\mathbb{C}})$, which by (7.2.6) amounts to computing $p_1(\mathbf{M})$, one uses (7.2.8) in the form $\mathrm{ch}(\mathcal{S}^+ \otimes \mathcal{S}^-) = \mathrm{ch}(\mathcal{S}^+)\mathrm{ch}(\mathcal{S}^-)$ to conclude that $p_1(\mathbf{M}) = -c_2(T\mathbf{M} \otimes \underline{\mathbb{C}}) = -2\left(c_2(\mathcal{S}^+) + c_2(\mathcal{S}^-)\right)$, so that a necessary condition for $\mathbf{M}$ to be spin is the Pontrjagin number $\langle p_1(\mathbf{M}), \mathbf{M}\rangle$ to be an *even* integer. On the other hand, one knows from Theorem 10.2.1 that $\langle p_1(\mathbf{M}), \mathbf{M}\rangle = 3\,\mathrm{sign}\,(\mathbf{M})$ and we end up with the following criterion: *if* $\mathrm{sign}\,(\mathbf{M})$ *is odd then* $\mathbf{M}$ *does not carry a spin structure*. This allows us to exhibit many examples of nonspin manifolds. For example, if nP^2 denotes the connected sum of n copies of P^2, the complex projective plane, then $\mathrm{sign}\,(nP^2) = n$ so that nP^2 is not spin for n odd. In particular, the construction in Section 6.2 does not apply but, and this is crucial in what follows, spinors can still be globally defined on those manifolds (and, as a matter of fact, on any closed oriented 4-manifold). The relevant point is that Spin_4 lies inside the real Clifford algebra Cl_4 while spinors have been constructed in terms of representations of the complex Clifford algebra $\mathbb{C}\mathrm{l}_4$, and the proper inclusion $\mathrm{Cl}_4 \subset \mathbb{C}\mathrm{l}_4$ leaves us some room to explore. In terms of the concrete models above, we simply bring about a complex parameter $z \in S^1 = \mathrm{SO}_2 = \mathrm{U}_1$ and replace (10.4.11) by

$$(g_+, g_-, z)x = (zg_+)x(zg_-)^{-1};$$

this gives a representation $\gamma_4^c : \mathrm{Spin}_4^c \to \mathrm{SO4}$, $\gamma_4^c([g_+, g_-, z]) = \gamma_4(g_+, g_-)$, where we have set

$$\mathrm{Spin}_4^c = \mathrm{Spin}_4 \times \mathrm{U}_1 / \sim . \qquad (10.4.12)$$

Here, $\sim$ represents the identification induced by the obvious $\mathbb{Z}_2$-action. Another 'exceptional' isomorphism comes from the standard identification $\mathrm{U}_2 = \mathrm{SU}_2 \times \mathrm{U}_1/\sim$, which means that

$$\mathrm{Spin}_3^c \stackrel{\text{def.}}{=} \mathrm{Spin}_3 \times \mathrm{U}_1/\sim = \mathrm{U}_2.$$

From this we have well defined complex representations $\chi_\pm : \mathrm{Spin}_4^c \to \mathrm{Spin}_3^c = \mathrm{U}_2$, $\chi_\pm([g_+, g_-, z]) = [g_\pm, z]$, so that the whole discussion can be summarized in the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Spin}_3^c = \mathrm{U}_2 & \xleftarrow{\chi_+} & \mathrm{Spin}_4^c & \xrightarrow{\chi_-} & \mathrm{Spin}_3^c = \mathrm{U}_2 \\
\phi \downarrow & & \gamma_4^c \downarrow & & \downarrow \phi \\
\mathrm{SO}_3 & \xleftarrow{\wedge^2\mu_4^+} & \mathrm{SO}_4 & \xrightarrow{\wedge^2\mu_4^-} & \mathrm{SO}_3
\end{array}$$

with $\phi([g, z]) = \gamma_3(g)$.

We can now mimic the definition of spin manifolds (cf. Proposition 6.1.3) and say that an oriented Riemannian 4-manifold carries a spin^c *structure* if its orthonormal frame bundle P^{Spin} lifts to a principal Spin_4^c-bundle P^{Spin^c} in such a way that γ_4^c is fiberwisely reproduced. An argument of Hirzebruch and Hopf [HH] shows that any closed oriented 4-manifold carries a spin^c structure. Thus in this case we can form:

- A pair of complex plane bundles over **M** by

  $$\mathcal{S}_\pm^c = P^{\mathrm{Spin}^c} \times_{\chi_\pm} \mathbb{C}^2.$$

  These are the *half-spinor bundles* of the spin^c structure.

- The complex line bundle

  $$\mathcal{L} = P^{\mathrm{Spin}^c} \times_\rho \mathbb{C},$$

  with $\rho : \mathrm{Spin}^c \to \mathrm{U}_1$, $\rho([g_+, g_-, z]) = z^2$. Since $\mathcal{L} = \Lambda^2\mathcal{S}_+^c = \Lambda^2\mathcal{S}_-^c$, $\mathcal{L}$ is said to be the *determinant bundle* of the given spin^c structure.

Notice that these bundles come equipped with natural hermitian metrics. Moreover, it is not hard to check that the integral cohomology class $c_1(\mathcal{L})$ completely determines the spin^c structure up to isomorphism.

As an example, assume that **M** is spin (with spin structure given by the principal bundle P^{Spin}) and fix a complex line bundle L over **M** with a Hermitian metric. If P^L is the corresponding principal U_1-bundle, form the principal Spin_4^c-bundle over **M** by the fibered product

$$P^{\mathrm{Spin}^c} = P^{\mathrm{Spin}} \times_{\mathbf{M}} P^L / \sim . \tag{10.4.13}$$

This clearly defines a spin^c structure over $\mathbf{M}$ with $\mathcal{L} = L \otimes L$ and $\mathcal{S}^c_\pm = \mathcal{S}^\pm \otimes L$. Thus, in the presence of a spin structure, the determinant bundle associated to a spin^c structure admits a *global* square root $L = \mathcal{L}^{1/2}$. Now, in general, we can cover $\mathbf{M}$ by a finite collection $\{U_\alpha\}_{\alpha=1}^N$ of contractible open subsets so that each U_α is spin (in a unique way!) with spinor bundle $\mathcal{S}_\alpha = \mathcal{S}^+_\alpha \oplus \mathcal{S}^-_\alpha$. Thus we get locally defined line bundles L_α such that $L_\alpha = \mathcal{L}^{1/2}$ and $\mathcal{S}^c_\pm = \mathcal{S}^\pm_\alpha \otimes L_\alpha$ over U_α. This validates the computation

$$\begin{array}{rcl} T\mathbf{M} \otimes \underline{\mathbb{C}}|_{U_\alpha} & = & \mathrm{Hom}\left(\mathcal{S}^+_{\mathrm{ff}}, \mathcal{S}^-_{\mathrm{ff}}\right) \\ & = & \mathcal{S}^+_\alpha \otimes \mathcal{S}^-_\alpha \\ & \overset{(*)}{=} & \left(\mathcal{S}^+_\alpha \otimes L_\alpha\right) \otimes \left(\mathcal{S}^-_\alpha \otimes L_\alpha\right) \\ & = & \mathrm{Hom}\left(\mathcal{S}^+_\alpha \otimes L_\alpha, \mathcal{S}^-_\alpha \otimes L_\alpha\right), \end{array}$$

where (*) is perhaps best explained by the fact that the transitions functions for L_α cancel out. If we assume further that the nonempty intersections $U_\alpha \cap U_\beta$ are contractible as well, this globalizes to

$$T\mathbf{M} \otimes \underline{\mathbb{C}} = \mathrm{Hom}\left(\mathcal{S}^c_+, \mathcal{S}^c_-\right),$$

thus showing that Clifford multiplication is well defined in the spin^c setting.

At this point one is tempted to define a Dirac operator but a word of caution is in order. Recall that in the genuine spin case, the construction in Section 6.2 yields a connection ∇^s on $\mathcal{S}$ which depends *canonically* on the metric on $\mathbf{M}$, but in the more general spin^c case no such connection is available. To understand this, notice that the spin^c condition can be rephrased in terms of the existence of a 2-sheeted covering map

$$\begin{array}{c} P^{\mathrm{Spin}^c} \\ \downarrow \tilde{\gamma}^c_4 \\ P^{\mathrm{Spin}} \times P^{\mathcal{L}} \end{array}$$

which fiberwisely reproduces $\xi^c = \gamma^c_4 \times \rho : \mathrm{Spin}^4_c \to \mathrm{SO}_4 \times \mathrm{U}_1$, and this shows that in order to define a connection on P^{Spin^c} one needs an extra piece of data, namely, a compatible connection, say A, on $\mathcal{L}$. By the standard construction, this then leads to a connection $\nabla^A : \Gamma(\mathcal{S}^c) \to \Gamma(\mathcal{S}^c)$ which is compatible with Clifford multiplication and the Hermitian metric, and moreover preserves the orthogonal decomposition $\mathcal{S}^c = \mathcal{S}^c_+ \oplus \mathcal{S}^c_-$. We remark that even though this freedom to choose A may seem awkward at first sight, it actually represents, as we shall see, a crucial input in Seiberg-Witten theory. Anyway, we can define a Dirac operator (depending upon A) by the familiar rule:

$$D_A = \sum_{i=1}^{4} \mathrm{e}_i \cdot \nabla^A_{\mathrm{e}_i},$$

where as usual the dot means Clifford multiplication. Notice that, at least locally, D_A is a twisted Dirac operator: in the notation of Section 8.1, $D_A = \partial\!\!\!/_{\mathcal{L}^{1/2}}$ and this of course leads, via (8.1.9), to a Weitzenböck decomposition

$$D_A^2 = \nabla^{A^*}\nabla^A + \frac{\kappa}{4} + \frac{1}{2}F^A, \tag{10.4.14}$$

where F^A is the curvature[2] of A and the factor $1/2$ comes from the local expression for D_A above.

As before, D_A is a formally selfadjoint operator and, in order to have an interesting index, we consider $D_A^+ : \Gamma(S_+^c) \to \Gamma(\mathcal{S}_-^c)$, the restriction of D_A to positive half-spinors.

Theorem 10.4.1 *In the notation above,*

$$\operatorname{ind} D_A^+ = \frac{1}{8}\Big(\langle c_1(\mathcal{L})^2, \mathbf{M}\rangle - \operatorname{sign}(\mathbf{M})\Big). \tag{10.4.15}$$

Proof. By the argument in the proof of Theorem 10.2.1 and the comments above, we can assume $\mathbf{M}$ spin so that (8.1.13) applies with $\mathcal{G} = \mathcal{L}^{1/2}$. Now, it follows from (7.2.8) (with $\mathcal{E} = \mathcal{E}' = \mathcal{L}^{1/2}$) that $c_1(\mathcal{L}) = 2c_1(\mathcal{L}^{1/2})$, so that

$$\mathrm{ch}(\mathcal{L}^{1/2}) = 1 + \frac{1}{2}c_1(\mathcal{L}) + \frac{1}{8}c_1(\mathcal{L})^2.$$

Moreover,

$$\hat{A}(T\mathbf{M}) = 1 - \frac{1}{24}p_1(\mathbf{M}),$$

and the result follows in view of the already mentioned identity $\langle p_1(\mathbf{M}), \mathbf{M}\rangle = 3\operatorname{sign}(\mathbf{M})$).$\rfloor$

Remark. We point out that $\tilde{\gamma}_4^c$ above can be used to express the spin^c condition in terms of characteristic classes. First note that one has the natural

[2] One should stress that, since A is a compatible connection on a hermitian line bundle, $F^A = dA$ is a genuine purely imaginary 2-form over $\mathbf{M}$. This allows F^A to act on spinors by Clifford multiplication via the canonical isomorphim (5.1.3).

commutative diagram

$$\begin{array}{ccc} \mathrm{Spin}_4^c & \hookrightarrow & \mathrm{Spin}_6 \\ \gamma_4^c \downarrow & & \downarrow \gamma_6 \\ \mathrm{SO}_4 \times \mathrm{U}_1 & \hookrightarrow & \mathrm{SO}_6 \end{array}$$

thus showing that $\mathbf{M}$ is spin^c if and only if $T\mathbf{M} \oplus \mathcal{L}$ is a spin bundle in the sense of Definition 6.1.3. This means of course that $0 = w_2(T\mathbf{M} \oplus \mathcal{L}) = w_2(T\mathbf{M}) + w_2(\mathcal{L}) + w_1(T\mathbf{M})w_1(\mathcal{L})$ or equivalently $w_2(T\mathbf{M}) = w_2(\mathcal{L})$ since both $T\mathbf{M}$ and $\mathcal{L}$ are orientable. However, it follows from general principles that $w_2(\mathcal{L})$ is the so-called mod 2 reduction of an integral cohomology class, namely, $c_1(\mathcal{L})$. This means that $w_2(T\mathbf{M}) = j^*(c_1(\mathcal{L}))$, where $j^* : H^2(\mathbf{M}, \mathbb{Z}) \to H^2(\mathbf{M}, \mathbb{Z}_2)$ is induced by the standard mod 2 reduction $j : \mathbb{Z} \to \mathbb{Z}_2$. Since any integral cohomology class is the Chern class of some line bundle, the conclusion is that $\mathbf{M}$ is spin^c if and only if $w_2(T\mathbf{M})$ is the mod 2 reduction of some integral cohomology class. This should be compared to the much stronger spin condition $w_2(T\mathbf{M}) = 0$.⌋

We proceed by noticing that the splitting (10.4.9) induces a decomposition

$$\mathcal{A}^2(\mathbf{M}) = \mathcal{A}^2_+ \oplus \mathcal{A}^2_- \tag{10.4.16}$$

of the space of real valued 2-forms into the subspaces of *selfdual* and *antiselfdual* forms. Each closed $\eta \in \mathcal{A}^2_\pm$ satisfies $d^*\eta = 0$ and hence is harmonic. On the other hand, since $*\Delta = \Delta *$, any harmonic 2-form uniquely decomposes as $\eta = \eta_+ + \eta_-$ with $\Delta\eta_\pm = 0$. It follows from Hodge-de Rham theory that (10.4.16) induces a decomposition at the cohomology level:

$$H^2_{\mathrm{dR}}(\mathbf{M}) \otimes \mathbb{C} = H^2_+ \oplus H^2_-.$$

It is immediate that the quadratic form Q of Section 10.2 satisfies $Q(\eta, \eta) > 0$ (respect. $Q(\eta, \eta) < 0$) for $\eta \neq 0$, $\eta \in H^2_+$ (respect. $\eta \in H^2_-$), so that if we set $b^\pm = \dim H^2_\pm$, these integers coincide with the invariants defined in Section 10.2 if $n = 4$. Finally, if $\psi \in \Gamma(\mathcal{S}^c_+)$ is given, define

$$\sigma(\psi)(X, Y) = \langle X \cdot Y \cdot \psi, \psi \rangle + \langle X, Y \rangle |\psi|^2.$$

It is easy to check that $\sigma(\psi)$ is a purely imaginary valued 2-form such that $i\sigma(\psi)$ is selfdual.

Now we can at last to write down the famous *Seiberg-Witten equations*. These are equations for a pair $(\psi, A) \in \Gamma(\mathcal{S}^c_+) \times \mathcal{C}(\mathcal{L})$, where $\mathcal{C}(\mathcal{L})$ is the space of compatible connections on the determinant bundle of a fixed spin^c structure over $\mathbf{M}$:

$$\left\{ \begin{array}{rcl} D_A\psi & = & 0 \\ F_+^A & = & \sigma(\psi) + i\eta, \end{array} \right. \tag{10.4.17}$$

where the real selfdual 2-form η should be thought of as a deformation parameter. Notice that the above equations, and hence the corresponding space of solutions $\mathcal{Z}_{\mathcal{L},g,\eta}$, depend essentially on the underlying metric, say g, on $\mathbf{M}$.

From this point on the discussion becomes too technical to be included here and we merely indicate the line of action. At least if $b^+ \geq 2$, the idea is to show that after dividing out $\mathcal{Z}_{\mathcal{L},g,\eta}$ by the natural action of the *gauge group* of maps $f : \mathbf{M} \to S^1$ so as to obtain the *Seiberg-Witten moduli space* $\mathcal{M}_{\mathcal{L},g,\eta}$, we end up, for a generic choice of the pair (g, η), with a closed oriented smooth manifold which, if not empty, satisfies[3]

$$\dim_{\mathbb{R}} \mathcal{M}_{\mathcal{L},g,\eta} = \frac{1}{4}\Big(\langle c_1(\mathcal{L})^2, \mathbf{M}\rangle - (2\chi(\mathbf{M}) + 3\,\mathrm{sign}\,(\mathbf{M}))\Big). \tag{10.4.18}$$

Moreover, the oriented cobordism type of this moduli space is preserved under generic variations of the data (g, η), thus providing an invariant of the spin^c structure depending only on the *diffeomorphism* type of $\mathbf{M}$. The method is particularly effective for manifolds supporting an almost complex structure, for then a canonical spin^c structure exists for which $\dim \mathcal{M}_{\mathcal{L},g,\eta} = 0$, so that the cobordism type furnishes an integer invariant obtained by counting signs over a finite collection of points. With this invariant (and its many refined forms) at hand, one can successfully address many profound questions in the geometry and topology of 4-manifolds, including the construction of many new examples of 4-manifolds carrying no metric with positive scalar curvature (as evidenced by the occurrence of κ in (10.4.14)), the problem of distinguishing smooth structures on the same topological 4-manifold (thus simplifying and extending the previous technology introduced by Donaldson [DK]), the problem of finding obstructions to the existence of symplectic structures on certain 4-manifolds, among others. We strongly recommend the excellent monograph [N] for anyone interested in pursuing this line of study.

[3]Needless to say, (10.4.18) is obtained by a clever application of the index formula in the guise (10.4.15).

Chapter 11

The index of general elliptic operators

In this chapter we briefly indicate how the index formula for general elliptic operators follows from the Dirac case described in Chapter 8. The essential ingredient here is K-theory, an extraordinary cohomology theory based on vector bundles whose central result, Bott's periodicity theorem, provides the link between the index of elliptic operators and the Clifford product acting on the (half) spin representations.

11.1 Fredholm maps

Let H be a complex separable Hilbert space of infinite dimension. We shall denote by $\mathcal{B}(H)$ the space of bounded linear operators on H and by $\mathcal{K}(H) \subset \mathcal{B}(H)$ the two-sided ideal of compact operators. There exists the canonical projection $\varrho : \mathcal{B}(H) \to \mathcal{C}(H)$ over $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$, the *Calkin algebra*, and we define $\mathcal{F}(H) = \varrho^{-1}(\mathcal{C}(H)^*)$, the *space of Fredholm maps*, where $\mathcal{C}(H)^*$ is the group of units in $\mathcal{C}(H)$. Thus, $T \in \mathcal{F}(H)$ if and only if there exists $T' \in \mathcal{B}(H)$ such that

$$TT' = I + K, \qquad T'T = I + K', \tag{11.1.1}$$

where $K, K' \in \mathcal{K}(H)$. T' is called a *parametrix* for T and clearly $T' \in \mathcal{F}(H)$ as well.

Another way of characterizing $T \in \mathcal{F}(H)$ is to require that $\dim \ker T < +\infty$, $\operatorname{im} T$ is closed and $\dim \operatorname{coker} T < +\infty$, where $\operatorname{coker} T = H/\operatorname{im} T$. Thus to each $T \in \mathcal{F}(H$ we can associate the integer

$$\operatorname{ind} T = \dim \ker T - \dim \operatorname{coker} T,$$

the *index* of T.

Fredholm maps (and their indexes) have some nice properties that we now recall:

1. $\mathcal{F}(H)$ is closed under composition and taking adjoints. Moreover,
$$\operatorname{ind} TT' = \operatorname{ind} T + \operatorname{ind} T';$$
2. If $K \in \mathcal{K}(H)$ then $I + K \in \mathcal{F}(H)$ and $\operatorname{ind}(I+K) = 0$. In particular, $\operatorname{ind} T = -\operatorname{ind} T'$ if T' is a parametrix for T;
3. $\mathcal{F}(H) \subset \mathcal{B}(H)$ is open in the sup norm and $\operatorname{ind} : \mathcal{F}(H) \to \mathbb{Z}$ is locally constant.

The last item says that the index labels the connected components of $\mathcal{F}(H)$. In words, the index is *the* topological invariant of Fredholm maps.

Example 11.1.1 *Consider $H = \{x = (x_0, x_1, \ldots, x_n, \ldots); \, x_i \in \mathbb{C}, \sum_i |x_i|^2 < +\infty\}$ and for $n \geq 1$ define $T_n : H \to H$ by*

$$T_n x = (\underbrace{0, \ldots, 0}, x_0, x_1, \ldots),$$

with n zeros in the braces. Then $\ker T_n = \{0\}$ *and* $\operatorname{coker} T_n = [e_0, \ldots, e_n]$*, where $e_i = (0, \ldots, 0, 1, 0, \ldots)$, so that* $\operatorname{ind} T_n = -(n+1)$.

Example 11.1.2 *This is a more sophisticated example that plays a key role in the so-called analytical proof of the Bott periodicity theorem[A2]. Recall that each $f \in L^2(S^1)$ can be expanded as*

$$f = \sum_{-\infty}^{+\infty} a_n e^{in\theta},$$

where $a_n \in \mathbb{C}$ is the n^{th} Fourier coefficient of f. Let $e_n = e^{in\theta}$ and consider

$$H = [e_0, e_1, \ldots].$$

Fix $g : S^1 \to \mathbb{C}^*$ continuous and define the Toeplitz operator $T_g : H \to H$ *by $T_g = \Pi_+ \circ \mathcal{M}_f$, where $\mathcal{M}_g$ is pointwise multiplication by g and $\Pi_+ : L^2(S^1) \to H$ is orthogonal projection. Then $T_g \in \mathcal{F}(H)$ and a computation shows that*

$$\operatorname{ind} T_g = -\text{winding number of } g \text{ around } 0.$$

11.2 Elliptic operators as Fredholm maps

Let $\mathbf{M}^n$ be a smooth manifold and $\mathcal{E}$, $\mathcal{E}'$ complex vector bundles over $\mathbf{M}$. A linear map $P : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E}')$ is a linear differential operator (l.d.o.) of order m if locally it can be written as

$$(P\sigma)_i = \sum_{|\alpha|\leq m} \sum_{j=1}^{r=\operatorname{rank}\mathcal{E}} a_\alpha^{ij}(x) D^\alpha \sigma_j, \qquad i = 1, \ldots, r' = \operatorname{rank}\mathcal{E}',$$

where x is the coordinate relative to a coordinate patch $U \subset \mathbf{M}$ such that both $\mathcal{E}$ and $\mathcal{E}'$ are trivial restricted to U, a_α^{ij} are smooth functions and $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$. The space of all l.d.o of order m is denoted by $\mathrm{LDO}_m(\mathcal{E}, \mathcal{E}')$.

Given the local expansion for P as above we can define for $x \in U$ and $\xi \in \mathbb{R}^n$ the linear map $\sigma_P(x,\xi) : \mathbb{C}^r \to \mathbb{C}^{r'}$, by

$$(\sigma_P(x,\xi)(v))_i = \sum_{|\alpha|=m} \sum_{j=1}^{r=\operatorname{rank}\mathcal{E}} a_\alpha^{ij}(x)\xi^\alpha v_j, \qquad i = 1, \ldots, r'.$$

If we view (x,ξ) as coordinates in $T^*\mathbf{M}$ this can be globalized to a bundle morphism $\sigma_P : \pi^*\mathcal{E} \to \pi^*\mathcal{E}'$, where $\pi : T^*\mathbf{M} \to \mathbf{M}$ is the standard projection. We then say that P is *elliptic* if σ_P, the *symbol*, is a bundle isomorphism outside of the zero section $\mathbf{M} \subset T^*\mathbf{M}$. The space of all elliptic l.d.o of order m is denoted by $\mathrm{Ell}_m(E, E')$.

Given a Riemannian metric on $\mathbf{M}$ (assumed to be closed) and hermitian metrics on $\mathcal{E}$ and $\mathcal{E}'$ we see that elliptic operators come in pairs. More precisely, if $P \in \mathrm{Ell}_m(\mathcal{E}, \mathcal{E}')$ then there exists another $P^* \in \mathrm{Ell}_m(\mathcal{E}', \mathcal{E})$ such that

$$(P\sigma, \sigma) = (\sigma, P^*\sigma'), \qquad \sigma \in \Gamma(\mathcal{E}),\ \sigma' \in \Gamma(\mathcal{E}'),$$

where as usual $(\,,)$ is the L^2 inner product on sections induced by the metrics. Moreover, there holds the pointwise identity $\sigma_{P^*} = \sigma_P^*$.

Remark. Given the choices above, P^* is completely determined by P. Moreover, if $\mathcal{E} = \mathcal{E}'$ and $P = P^*$ then P is said to be *self-adjoint*.

Example 11.2.1 *Let $\mathcal{E} \to \mathbf{M}$ be a hermitian bundle and let $\nabla : \Gamma(\mathcal{E}) \to \Gamma(T^*\mathbf{M} \otimes \mathcal{E})$ be a compatible connection. Then ∇ is a first order l.d.o with $\sigma_\nabla(x,\xi)(v) = i\xi \otimes v$. Thus, ∇ is not elliptic but notice that its symbol is injective. If $\mathbf{M}$ is given a Riemannian metric we can consider the adjoint $\nabla^* : \Gamma(T^*X \otimes \mathcal{E}) \to \Gamma(\mathcal{E})$ and the composition $\nabla^*\nabla : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$, the Bochner Laplacian associated to ∇. This is a self-adjoint second order elliptic l.d.o because $\sigma_{\nabla^*\nabla}(x,\xi)(v) = -|\xi|^2 v$.*

For each $s \in \mathbb{R}$ let $W^s(\mathcal{E})$ be the s^{th} Sobolev space of sections of $\mathcal{E}$; see Section 9.2. Then each $P \in \mathrm{LDO}_m(\mathcal{E}, \mathcal{E}')$ extends uniquely to a map

$$P_s : W^s(\mathcal{E}) \to W^{s-m}(\mathcal{E}').$$

Moreover, if P is elliptic, we can use pseudo-differential operators to invert each P_s up to smoothing operators [LM]. More precisely, say that $P : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E}')$ is *Sobolev of order* $k \in \mathbb{R}$ (Notation: $P \in \mathrm{Sob}_k(\mathcal{E}, \mathcal{E}')$) if it extends continuously to $P_s : H^s(\mathcal{E}) \to H^{s-k}(\mathcal{E}')$ for any s. The inversion theorem referred to above says that for each $P \in \mathrm{Ell}_m(\mathcal{E}, \mathcal{E}') \subset \mathrm{Sob}_m(\mathcal{E}, \mathcal{E}')$ there exists $Q \in \mathrm{Sob}_{-m}(\mathcal{E}', \mathcal{E})$ such that

$$PQ = I + K', \quad QP = I + K,$$

where $K \in \cap_k \mathrm{Sob}_k(\mathcal{E})$ and similarly $K' \in \cap_k \mathrm{Sob}_k(\mathcal{E}')$.

By Theorem 9.2.4, K and K' are compact operators and P_s is Fredholm for each s. By Theorem 9.2.3, $\operatorname{im} K \subset \Gamma(\mathcal{E})$, so that if $\sigma \in \ker P_s$ then $\sigma = -K\sigma$ and hence $\sigma \in \Gamma(\mathcal{E})$. Thus, $\ker P|_{W^s}$ is formed by smooth sections and does not depend upon the parameter s. This common subspace will be denoted simply by $\ker P$. In the presence of metric structures we have an isomorphism $\operatorname{coker} P = \ker P^*$ and so

$$\operatorname{ind} P = \dim \ker P - \dim \ker P^*,$$

the *index* of P, is well defined (independently of s).

Standard arguments show that $\operatorname{ind} P$ is a rather stable object only depending upon the homotopy class of σ_P (as a bundle isomorphism) and hopefully it would be expressed in terms of topological data associated to σ_P and $\mathbf{M}$. For $P = \partial^+_{\mathcal{E}}$, the twisted Atiyah-Singer-Dirac operator, this is the case indeed as we have shown in Chapter 8 how this integer is computed in terms of characteristic classes associated to $\mathcal{E}$ and $\mathbf{M}$. It turns out that in the spin case the apparently special formula (8.1.13) is completely equivalent to the index formula for a general elliptic operators. As remarked in the Introduction, the key ingredient here is K-theory, a (generalized) cohomology theory that we now pass to describe.

11.3 K-theory

Let X be a compact Hausdorff space (not necessarily connected) and let $\mathrm{Vect}(X)$ be the abelian semigroup of complex vector bundles over X (the addition is induced by Witney sum $\oplus$). We can build an abelian group $K(X)$ out of $\mathrm{Vect}(X)$ by considering formal differences $\mathcal{E} - \mathcal{F}$ of elements in $\mathrm{Vect}(X)$ and identifying $\mathcal{E} - \mathcal{F}$ to $\mathcal{E}' - \mathcal{F}'$ if there exists $\mathcal{G} \in \mathrm{Vect}(X)$ such that $\mathcal{E} \oplus \mathcal{F}' \oplus \mathcal{G} = \mathcal{E}' \oplus \mathcal{F} \oplus \mathcal{G}$. There exists a natural homomorphism $\iota : \mathrm{Vect}(X) \to K(X)$ and the pair $(K(X), \iota)$ admits the standard universal characterization with respect to semigroup homomorphisms $\mathrm{Vect}(X) \to A$, with A an abelian group. We shall denote $[\mathcal{E}] = \iota(\mathcal{E})$, $\mathcal{E} \in \mathrm{Vect}(X)$. It is easy to check that $X \to K(X)$ is functorial in the sense that any $f : X \to Y$ continuous induces a homomorphism $f^* : K(Y) \to K(X)$ which only depends on the homotopy class of f. In particular, if $i : \{\mathrm{pt}\} \hookrightarrow X$ is a base point inclusion then $i^* : K(X) \to K(\{\mathrm{pt}\}) \cong \mathbb{Z}$ is surjective and

$$K(X) = \tilde{K}(X) \oplus \mathbb{Z}, \tag{11.3.2}$$

where $\tilde{K}(X) = \ker i^*$. Notice that the splitting above depends on the base point. The contravariant functor K is a homotopy type invariant; details can be found in [A].

The reduced group $\tilde{K}(X)$ plays an important role in what follows and it is convenient to rework its definition. For this define $\mathcal{E} \sim \mathcal{E}'$ if there exists nonnegative integers m and n such that $\mathcal{E} \oplus \underline{\mathbb{C}}^m = \mathcal{E}' \oplus \underline{\mathbb{C}}^n$, and consider the quotient space $\mathrm{Vect}(X)/\sim$. If $\langle \mathcal{E} \rangle$ denotes the class of $\mathcal{E}$, then $\mathrm{Vect}(X)/\sim$ is an abelian group with identity $\underline{\mathbb{C}}^0$ and $-\langle \mathcal{E} \rangle = \langle \mathcal{E}' \rangle$, where $\mathcal{E} \oplus \mathcal{E}'$ is trivial. By the same device of adding a bundle so as to reach triviality, we see that any $\alpha \in K(X)$ can be written as $\alpha = E - \underline{\mathbb{C}}^n$. Now consider the map $\beta : K(X) \to \mathrm{Vect}(X)/\sim$, $\alpha = \mathcal{E} - \underline{\mathbb{C}}^n \mapsto \langle \mathcal{E} \rangle$. This is a well-defined, surjective homomorphism and $\alpha = \mathcal{E} - \underline{\mathbb{C}}^n \in \ker \beta$ iff $\langle \mathcal{E} \rangle = \langle \underline{\mathbb{C}}^0 \rangle$ iff there exist j and k such that $\mathcal{E} \oplus \underline{\mathbb{C}}^j = \underline{\mathbb{C}}^k$, and thus $\alpha = \mathcal{E} \oplus \underline{\mathbb{C}}^j - \underline{\mathbb{C}}^n \oplus \underline{\mathbb{C}}^j = \underline{\mathbb{C}}^k - \underline{\mathbb{C}}^l$, $l = n + j$. This means that $\ker \beta = \mathbb{Z}$, so that

$$\tilde{K}(X) = \mathrm{Vect}(X)/\sim .$$

Finally, if X is connected, the rank map rank : $\mathrm{Vect}(X) \to \mathbb{Z}^+$ induces a homomorphism $\mathrm{rank} : K(X) \to \mathbb{Z}$, $\mathrm{rank}(\mathcal{E} - \mathcal{F}) = \mathrm{rank}\,\mathcal{E} - \mathrm{rank}\,\mathcal{F}$. On the other hand, $\xi : \mathrm{Vect}(X) \to \tilde{K}(X) \oplus \mathbb{Z}$, $\xi(\mathcal{E}) = (\mathcal{E} - \underline{\mathbb{C}}^{\mathrm{rank}\,\mathcal{E}}, \mathrm{rank}\,\mathcal{E})$ extends to the isomorphism (11.3.2). Thus, in this case, $\tilde{K}(X) = \ker \mathrm{rank}$.

Example 11.3.1 *Since any complex vector bundle over S^1 is trivial we have $\tilde{K}(S^1) = \{0\}$ and hence $K(S^1) = \mathbb{Z}$. Any vector bundle over S^2 is of the form $H^{(k)} \oplus \underline{\mathbb{C}}^k$, where H is the Hopf line bundle and $H^{(k)} = H \otimes \ldots \otimes H$ k times. Hence*

$$\begin{aligned}
\tilde{K}(S^2) &= \{H^{(k)} \oplus \underline{\mathbb{C}}^{k'} - H^{(l)} \oplus \underline{\mathbb{C}}^{l'}; \mathrm{rank}(H^{(k)} \oplus \underline{\mathbb{C}}^{k'}) = \mathrm{rank}(H^{(l)} \oplus \underline{\mathbb{C}}^{l'})\} \\
&= \{H^{(k)} \oplus \underline{\mathbb{C}}^{k'} - H^{(l)} \oplus \underline{\mathbb{C}}^{l'}; k' = l'\} \\
&= \{(H^{(k)} - H^{(l)}) \oplus \underline{\mathbb{C}}^{l'}\} \\
&= \mathbb{Z},
\end{aligned}$$

and $K(S^2) = \mathbb{Z} \oplus \mathbb{Z}$.

We remark that the tensor product $\otimes$ on bundles induces a ring structure on $K(X)$ with unit element given by the class of $\underline{\mathbb{C}}$.

Now, a version of the celebrated Bott periodicity theorem [A] asserts that $\tilde{K}(S^{n+2}) = \tilde{K}(S^n)$ for $n \geq 0$. Using the computations above we then have

$$\tilde{K}(S^n) = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{11.3.3}$$

or equivalently,

$$K(S^n) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n \text{ even} \\ \mathbb{Z} & n \text{ odd} \end{cases} \tag{11.3.4}$$

At this point an extension of the functor K to locally compact Hausdorff spaces is needed: we set $K(X) = \tilde{K}(X^\times)$, where $X^\times$ is the one-point compactification of X. This is called K-theory with compact supports, denoted $K_c(X)$. Notice that Bott periodicity can be rewritten as

$$K_c(\mathbb{C}^n) = \tilde{K}(S^{2n}) = \mathbb{Z} = K(\mathrm{pt}). \tag{11.3.5}$$

In this form it relates the K-groups of the total and base space of the (rather trivial) vector bundle $\mathbb{C}^n \to \{\mathrm{pt}\}$.

We end up this section by remarking that there exists an equivalent, useful way to define $K_c(X)$. We consider triples $[\mathcal{E}, \mathcal{F}; \alpha]$ where $\mathcal{E}, \mathcal{F} \in \mathrm{Vect}(X)$ and $\alpha : \mathcal{E} \to \mathcal{F}$ is a bundle homomorphism with compact support (Note: $\operatorname{supp} \alpha = \{x \in X; \alpha_x : \mathcal{E}_x \to \mathcal{F}_x \text{ is not an isomorphism}\}$). Triples $[\mathcal{E}, \mathcal{F}; \alpha]$ and $[\mathcal{E}', \mathcal{F}'; \alpha']$ are equivalent if there exists a compactly supported path of triples $[\mathcal{E}_t, \mathcal{F}_t; \alpha_t]$ with $\alpha_0 = \alpha$ and $\alpha_1 = \alpha'$. Then

$$K_c(X) = \frac{\{\text{equivalence classes of triples}\}}{\{\text{equivalence classes of elementary triples}\}},$$

where a triple is elementary if its support is empty.

11.4 The general index formula

In this final section we briefly indicate how the index formula for Dirac operators (8.1.13) yields, after a simple computation involving characteristic classes, the general index formula for elliptic manifolds in a spin manifold. As remarked in the Introduction, the key ingredient is a refined version of Bott periodicity generalizing (11.3.5), which happens to relate the symbol of a general eliptic operator to Clifford multiplication.

Recall from (5.2.11) that if $n = 2k$ the spin representation splits as a sum

$$\Psi = \Psi^+ \oplus \Psi^-$$

of two inequivalent representations of dimension 2^{k-1}. Indeed, if $0 \neq \xi \in \mathbb{R}^n$ then Clifford multiplication by ξ, $\mu_\xi : \Psi^+ \to \Psi^-$, $\mu_\xi(\varphi) = \xi \cdot \varphi$, is an isomorphism.

Now consider the diagram

$$\begin{tikzcd} \mathbb{R}^n \times \Psi + \arrow[rr, "\mu"] \arrow[dr] & & \mathbb{R}^n \times \Psi^- \arrow[dl] \\ & \mathbb{R}^n & \end{tikzcd}$$

with $\mu(\xi, \varphi) = (\xi, \mu_\xi(\varphi))$. Then $\operatorname{supp} \mu = \{0\}$ and the triple $\sigma = [\Psi^+, \Psi^-; \mu]$ defines a nonzero element in $K_c(\mathbb{R}^n)$. In fact, this matches nicely with Bott periodicity in the sense that $K_c(\mathbb{R}^n) = \mathbb{Z}$ is generated by σ. More generally, let $U \subset \mathbb{R}^n$ be a (closed) disk and consider

$$\begin{tikzcd} TU \times \Psi^+ \arrow[rr, "\mu_U"] \arrow[dr] & & TU \times \Psi^- \arrow[dl] \\ & TU = U \times \mathbb{R}^n \arrow[d] & \\ & U & \end{tikzcd}$$

with $\mu_U(x, \xi, \varphi) = (x, \xi, \mu_\xi(\varphi))$. We see that $\operatorname{supp} \mu_U = U \times \{0\} = U$ and the triple $\sigma_U = [TU \times \Psi^+, TU \times \Psi^-; \mu_U]$ defines a nonzero element in $K_c(TU)$.

The point now is that there exist topological obstructions to globalizing the previous diagram (i.e. replacing U by a general manifold $\mathbf{M}$). However, the obstruction vanishes if $\mathbf{M}^n$ is spin and in this case we have the global diagram

$$\begin{tikzcd} \pi^*\mathcal{S}^+(\mathbf{M}) \arrow[rr, "\mu_{\mathbf{M}}"] \arrow[dr] & & \pi^*\mathcal{S}^-(\mathbf{M}) \arrow[dl] \\ \mathcal{S}^+(\mathbf{M}) \arrow[dr] & T\mathbf{M} \arrow[d] & \mathcal{S}^-(\mathbf{M}) \arrow[dl] \\ & \mathbf{M} & \end{tikzcd}$$

In particular, if $\mathbf{M}$ is closed, $\operatorname{supp} \mu_{\mathbf{M}} = \mathbf{M} \subset T\mathbf{M}$ is compact and the triple $\sigma_{\mathbf{M}} = [\pi^*\mathcal{S}^+(\mathbf{M}), \pi^*\mathcal{S}^-(\mathbf{M}); \mu_{\mathbf{M}}]$, where $\pi : T\mathbf{M} \to \mathbf{M}$ is the standard projection, defines an element in $K_c(T\mathbf{M})$. Moreover, it is clear that the Atiyah-Singer-Dirac operator

$$\partial\!\!\!/^+ : \Gamma(\mathcal{S}^+(\mathbf{M})) \to \Gamma(\mathcal{S}^-(\mathbf{M}))$$

is an elliptic operator with

$$\sigma_{\partial\!\!\!/^+} = \sigma_{\mathbf{M}}. \tag{11.4.6}$$

Now notice that the projection $\pi : T\mathbf{M} \to \mathbf{M}$ makes $K_c(T\mathbf{M})$ into a $K(\mathbf{M})$-module. In this setting, the following global version of Bott periodicity describes this $K(\mathbf{M})$-module structure.

Theorem 11.4.1 *The triple $\sigma_{\mathbf{M}} = \sigma_{\partial\!\!\!/^+}$ generates $K_c(T\mathbf{M})$ as a free $K(\mathbf{M})$-module. Thus, if $\sigma \in K_c(T\mathbf{M})$ then $\sigma = \sigma_{\partial\!\!\!/^+}\pi^* u$ for some $u \in K(\mathbf{M})$.*

Now, if $P : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E}')$ is a general elliptic operator on $\mathbf{M}$, the index construction in Section 11.2 yields the diagram

$$\begin{array}{ccccc}
\pi^*\mathcal{E} & & \xrightarrow{\sigma_P} & & \pi^*\mathcal{E}' \\
& \searrow & & \swarrow & \\
\mathcal{E} & & T\mathbf{M} & & \mathcal{E}' \\
& \searrow & \downarrow & \swarrow & \\
& & \mathbf{M} & &
\end{array}$$

Ellipticity means precisely that $\operatorname{supp}\sigma_P = \mathbf{M}$, so that the triple $[\pi^*\mathcal{E}, \pi^*\mathcal{E}'; \sigma_P]$, still denoted by σ_p, defines an element in $K_c(T\mathbf{M})$. By Theorem 11.4.1, $\sigma_P = \sigma_{\partial\!\!\!/^+}\pi^* u$, $u \in K(\mathbf{M})$. On the other hand, the following obvious generalization of (11.4.6) holds: if $\mathcal{E} \in \operatorname{Vect}(\mathbf{M})$ then the twisted Atiyah-Singer-Dirac operator

$$\partial\!\!\!/^+_{\mathcal{E}} : \Gamma(\mathcal{S}^+(\mathbf{M}) \otimes \mathcal{E}) \to \Gamma(\mathcal{S}^-(\mathbf{M}) \otimes \mathcal{E})$$

is elliptic with

$$\sigma_{\partial\!\!\!/^+_{\mathcal{E}}} = \sigma_{\partial\!\!\!/^+}[\mathcal{E}], \quad [\mathcal{E}] \in K(\mathbf{M}).$$

Recalling that $K(\mathbf{M})$ is additively generated by $\operatorname{Vect}(\mathbf{M})$ we finally get

Theorem 11.4.2 *If $\mathbf{M}$ is a closed even dimensional spin manifold of dimension $n = 2k$ and P is an elliptic operator on $\mathbf{M}$ then*

$$\sigma_P = \sigma_{\partial\!\!\!/^+_u},$$

where $u = [\mathcal{E}] - [\mathcal{E}'] \in K(X)$. Thus, at the symbolic level, any elliptic operator is a difference of twisted Atiyah-Singer-Dirac operators.

Thus, using (8.1.13), (7.2.11) and the fact that the index of an elliptic operator only depends on its symbol,

$$\begin{aligned}
\operatorname{ind} P &= \operatorname{ind}\sigma_{\partial\!\!\!/^+_u} \\
&= \langle \operatorname{ch}(u)\hat{A}(T\mathbf{M}), \mathbf{M}\rangle \\
&= \langle \operatorname{ch}(u)\hat{A}(T\mathbf{M})^{-1}\hat{A}(T\mathbf{M})^2, \mathbf{M}\rangle \\
&= \langle \operatorname{ch}(u)\hat{A}(T\mathbf{M})^{-1}\operatorname{Todd}(T\mathbf{M} \otimes \underline{\mathbb{C}}), \mathbf{M}\rangle.
\end{aligned}$$

To relate $\mathrm{ch}(u)\hat{A}(T\mathbf{M})^{-1}$ to the symbol σ_P, we recall that, given an oriented manifold $\mathbf{M}$ of dimension n, there exists a class $\tau \in H^n_c(\mathbf{M};\mathbb{R})$ such that, restrict to each fiber, τ reproduces the orientation class in compactly supported cohomology. This is the so-called *Thom class* and in fact the inclusion $i : \mathbf{M} \to T\mathbf{M}$ actually defines the *Thom isomomorphism* $i_! : H^k(\mathbf{M};\mathbb{R}) \to H^{n+k}_c(T\mathbf{M};\mathbb{R})$, $i_! u = \tau\pi^* u$, whose inverse $\pi_! : H^{n+k}_c(T\mathbf{M};\mathbb{R}) \to H^k(\mathbf{M};\mathbb{R})$ is induced by the projection $\pi : T\mathbf{M} \to \mathbf{M}$. Moreover, this is related to the Euler class of $T\mathbf{M}$ by

$$i^* i_! u = \chi(\mathbf{M}) u, \qquad u \in H^*(\mathbf{M}). \tag{11.4.7}$$

The details of the above constructions can be found in [MS].

Using (7.2.10), (7.3.15) and (10.1.2) one easily gets

$$\mathrm{ch}(\hat{\mathcal{S}}(\mathbf{M})) = (-1)^k \chi(\mathbf{M}) \hat{A}(T\mathbf{M})^{-1}. \tag{11.4.8}$$

Moreover, it is clear that the spinor difference element $\hat{\mathcal{S}}(\mathbf{M}) \in K(\mathbf{M})$ lifts to $\sigma_{\partial\!\!\!/^+} = \sigma_{\mathbf{M}} = [\pi^*\mathcal{S}^+(\mathbf{M}), \pi^*\mathcal{S}^-(\mathbf{M}); \mu_{\mathbf{M}}] \in K_c(T\mathbf{M})$ so that $i^*\sigma_{\partial\!\!\!/^+} = \hat{\mathcal{S}}(\mathbf{M})$ and from this we get

$$\begin{aligned} \chi(\mathbf{M})\pi_!\mathrm{ch}(\sigma_{\partial\!\!\!/^+}) &= i^* i_! \pi_! \mathrm{ch}(\sigma_{\partial\!\!\!/^+}) \\ &= i^*\mathrm{ch}(\sigma_{\partial\!\!\!/^+}) \\ &= \mathrm{ch}(i^*\sigma_{\partial\!\!\!/^+}) \\ &= \mathrm{ch}(\hat{\mathcal{S}}(\mathbf{M})) \\ &= (-1)^k \chi(\mathbf{M}) \hat{A}(T\mathbf{M})^{-1}. \end{aligned}$$

Thus, at least if $\chi(\mathbf{M}) \neq 0$,

$$\pi_!\mathrm{ch}(\sigma_{\partial\!\!\!/^+}) = (-1)^k \hat{A}(T\mathbf{M})^{-1}.$$

More generally one verifies that for $u \in K(\mathbf{M})$,

$$\pi_!\mathrm{ch}(\sigma_{\partial\!\!\!/^+}\pi^* u) = (-1)^k \mathrm{ch}(u) \hat{A}(T\mathbf{M})^{-1}, \tag{11.4.9}$$

since

$$\begin{aligned} \pi_!\mathrm{ch}\left(\sigma_{\partial\!\!\!/^+}\pi^* u\right) &= \pi_!\left(\mathrm{ch}(\sigma_{\partial\!\!\!/^+})\mathrm{ch}(\pi^* u)\right) \\ &= \pi_!\left(\mathrm{ch}(\sigma_{\partial\!\!\!/^+})\pi^*\mathrm{ch}(u)\right) \\ &= \left(\pi_!\mathrm{ch}(\sigma_{\partial\!\!\!/^+})\right)\mathrm{ch}(u). \end{aligned}$$

The computation above then yields the following fundamental result.

Theorem 11.4.3 *(Atiyah-Singer) If P is an elliptic operator on a closed spin manifold of dimension $n = 2k$ then*

$$\operatorname{ind} P = (-1)^k \langle \pi_! \left(\operatorname{ch}(\sigma_P)\right) \operatorname{Todd}(T\mathbf{M} \otimes \underline{\mathbb{C}}), \mathbf{M} \rangle. \tag{11.4.10}$$

The formula above holds true on *any* spin manifold $\mathbf{M}$ since there are alternate ways of deriving (11.4.9) in case $\chi(\mathbf{M}) = 0$; see [LM]. Moreover, even the spin assumption can be removed. In effect, in the general (non-spin) case one similarly proves that at the symbolic level any elliptic operator is a difference of twisted signature operators. The index formula then follows from the obvious twisted signature formula generalizing (10.2.7).

Needless to say, formula (11.4.10) stands as one of the towering results in twentieth century mathematics and one is immediately tempted to illustrate some of its many consequences. However, we refrain to do so since this would take us far beyond the introductory character of these Notes. Nevertheless, we shall demonstrate the versatility of (11.4.10) by merely stating one of its most illustrious consequences, namely, the *Hirzebruch-Riemann-Roch* formula, without even trying to explain in detail its ingredients.

Theorem 11.4.4 *If $\mathbf{M}$ is a compact complex manifold and $\mathcal{E}$ is a holomorphic vector bundle over $\mathbf{M}$ then*

$$\chi(\mathbf{M}, \mathcal{E}) = \langle \operatorname{ch}(\mathcal{E}) \operatorname{Todd}(\mathbf{M}), \mathbf{M} \rangle. \tag{11.4.11}$$

Here, $\chi(\mathbf{M}, \mathcal{E})$ is the Euler characteristic of the Dolbeault complex of differential $(0, q)$-forms with values in $\mathcal{E}$ and $\operatorname{Todd}(\mathbf{M})$ is the Todd class of the (holomorphic) tangent bundle of $\mathbf{M}$. An amazing consequence of (11.4.11) is that $\chi(\mathbf{M}, \mathcal{E})$, in spite of being defined in terms of the holomorphic structure on $\mathcal{E}$, only depends on the underlying complex structure since $\mathcal{E}$ enters the righthand side through its Chern character.

The yoga of characteristic classes needed to derive (11.4.11) from (11.4.10) is explained in [LM], Example 13.14, pg. 258.

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