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DECOMPOSITIONS OF THE SPHERE

BY

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PREFÁCIO

As presentes Notas constituiram a têse de Master of Science que o autor apresentou ao Institute of Mathematical Sciences da New York University. Êsse é o motivo pelo qual elas aparecem em inglês. Trata-se de um trabalho de natureza expositória.

No primeiro capítulo, apresentamos um resultado clássico de Hausdorff, dando uma formulação ligeiramente mais completa que aquela originalmente dada por Hausdorff. Nos capítulos seguintes, seguimos a orientação de um artigo de R.M. Robinson. Introduzimos, porém, uma modificação no sentido de que, na classe de movimentos rígidos, permitimos também reflexões.

Deixamos assinalados os nossos agradecimentos ao Prof. W.M. Hirsch, pela orientação que nos deu durante a leitura dos diversos artigos que conduziu ao presente trabalho.

Também agradecemos ao Prof. Leopoldo Nachbin, por incluir o nosso trabalho nas "Notas de Matemática".

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CONTENTS

INTRODUCTION

In this paper we deal with some striking statements concerning the congruence between a set and a proper subset of it. We begin by presenting some simple examples in order to get acquainted with this sort of ideas.

1) On the unit-circle we consider the set A of points whose angles with the x-axis are $n\alpha$, $n = 0,1,2,\ldots,$ where α is such that α/π is not a rational number. Let B be the subset of $n\alpha$ for which $n = 1,2,\ldots$. If we rotate the set A by an angle α then it will coincide with B .

2) In the complex plane we consider the set A of points given by polynomials in e^i

$$P_n(e^i) = c_n(e^i)^n+c_{n-1}(e^i)^{n-1}+\ldots+c_1e^i+c_o \ , \ n=0,1,\ldots,$$

where the coefficients are non-negative integers.

Let A_1 be the subset of A consisting of all points such that $c_o = 0$. Let $A_2 = A-A_1$. It is easy to see that the set A coincides with A_1 by rotation of one radian around the origin (i.e. multiplication by e^i). On the other hand the set A coincides with A_2 by translation of 1 in the positive direction of the real axis (i.e. by addition of $+1$ to the points in A).

3) The set A is countable in both of the preceding examples. Now we give an example in which the set A is not countable. Again let us consider the complex plane and to each complex number z we associate the set $E(z)$ of complex numbers of the following form

$$R_1(e^i).z + R_2(e^i) \quad , \tag{1}$$

where $R_1(e^i)$ and $R_2(e^i)$ are rational functions of e^i with integral coefficients and $R_1(e^i) \neq 0$. We can easily see that if $z \neq z'$, then either $E(z) = E(z')$ or $E(z) \cap \cap E(z') = \emptyset$. So the complex plane is divided into equivalence classes E . Let E_o be the cass that contains 0 . We form a set B by choosing a point from each of the remaining classes. In doing so we are using the Axiom of Choice. We observe that the set B is not countable.

Let T be the function that associates $z + 1$ to each point z of the complex plane. Let R be the function that associates $z.e^i$ to z . For each z we define a set $C(z)$ as the set of the images of z by all the products of T and R such that the last factor is T . A similar definition for $D(z)$, being R the last factor. Then the following inclusions are true

$$\begin{aligned} C(z) &\subset \{w\colon\ w = P_1(e^i).z + P_2(e^i) \ , \quad P_2(0) \neq 0 \} \\ D(z) &\subset \{w\colon\ w = P_1(e^i).z + P_2(e^i) \ , \quad P_2(0) = 0 \} , \end{aligned} \tag{2}$$

where $P_1(e^i)$ and $P_2(e^i)$ run through all the polynomials with non-negative integral coefficients and $P_1(e^i) \neq 0$.

Now we take

$$C = \bigcup_{z \varepsilon B} C(z) \quad , \quad D = \bigcup_{z \varepsilon B} D(z)$$

and let

$$A = B + C + D .$$

We claim that $C \cap D = \emptyset$. According to (2), both $C(z)$ and $D(z)$ are contained in $E(z)$, for each z. So we have only to show that $C(z) \cap D(z) = \emptyset$. Suppose not. Then

$$P_1(e^i).z + P_2(e^i) = Q_1(e^i).z + Q_2(e^i) ,$$

where $P_2(0) \neq 0$ and $Q_2(0) = 0$. This last condition implies that $P_1(e^i) \neq Q_1(e^i)$. So

$$z = \frac{Q_2(e^i) - P_2(e^i)}{P_1(e^i) - Q_1(e^i)}$$

which implies that $z \varepsilon E_o$. This is not possible because $z \varepsilon B$. It is easy to see that

$$T(A) = C$$

$$R(A) = D .$$

Summarizing we have presented an example in which the set A is not countable and contains two disjoint proper subsets each one being congruent with A itself. We observe that both sets C and D are not bounded in the usual R^2-metric.

A historic question is the so-called "measure problem at large". That is, whether it is possible or not to define a real function f for all bounded sets A of an Euclidean space R^n such that:

i) $f(A) \geqslant 0$ for all A .

ii) $f(A_o) > 0$ for some set A_o in R^n .

iii) $f(A+B) = f(A) + f(B)$ if $A \cap B = \emptyset$.

iv) $f(A) = f(B)$ if $A \simeq B$.

$\simeq$ - means congruence (*). Such a function f is called a finite measure. Property iii) is known as finite additivity.

In 1923 Banach [2] gave an affirmative answer to the problem for the case $n = 2$.

In 1914 Hausdorff [6] obtained a decomposition of the surface S of the sphere into four disjoint subsets A, B, C, and D such that

(I) $S = A + B + C + D$

(II) $A \simeq B \simeq C$, $A \simeq B + C$,

where D is countable. Here the congruence $\simeq$ means only a rotation about the center of the sphere.

Hausdorff then gave a negative answer to the measure problem for the spherical surface S , basing it on

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(*) The concept of congruence will be made precise on pages 30 & 38. For the time being we understand that $A \simeq B$ means that A can be superimposed upon B by rotation and/or translation.

the above decomposition. Indeed it is easy to see that if such a function f exists then f(D) must be equal to zero. This follows from properties iii) and iv) and from the fact that D is countable. Now congruences (II) implies that

$$f(S) = 3f(A) \quad , \quad f(S) = 2f(A) \quad .$$

Then f(S) is equal to 0 and so property ii) cannot be fulfilled.

The impossibility of defining a measure (even finitely additive) for all the subsets of the surface S of the sphere is closely related to the existence of a free non--Abelian subgroup of the group of all rotations of the sphere.

The negative answer to the measure problem for S implies a similar negative answer for R^n, $n > 2$.

The decomposition of the surface of the sphere given by Hausdorff is known as the Hausdorff Paradox. It is the source of subsequent amazing facts presented by Banach and Tarski in 1924. Banach and Tarski [3] using Hausdorff Paradox and an extension of the Schroder-Bernstein Theorem for an equivalence relation that they introduced, i.e., e-quivalence by finite decomposition, showed that any two bounded sets (*) in R^3 are equivalent by finite decomposition.

(*) With interior points. That is, there exists at least one point of the set such that a neighborhood of it is entirely contained in the set.

That is, if A and B are bounded sets in R^3 then there exists an integer n such that

$$A = A_1 + A_2 + \dots + A_n \quad , \quad B = B_1 + B_2 + \dots + B_n$$

$$A_i \cap A_j = \emptyset \quad , \quad i \neq j \quad , \quad B_i \cap B_j = \emptyset \quad , \quad i \neq j$$

and

$$A_i \simeq B_i \quad , \quad i = 1,2,\dots,n \; .$$

The key point of the proof of the above statement is the following theorem: "If S_1 and S_2 are congruent spheres in R^3 then S_1 is equivalent by finite decomposition to S_1+S_2". In other words,

Theorem DS': "There exist three decompositions of the three-dimensional sphere S' into disjoint sets

$$1) \quad S' = A'_1 + A'_2 + \dots + A'_k$$

$$2) \quad S' = B'_1 + B'_2 + \dots + B'_k$$

$$3) \quad S' = B'_{k+1} + \dots + B'_{k+\ell}$$

such that

$$A'_i \simeq B'_i \quad , \quad i = 1,2,\dots,k+\ell \; . \text{"}$$

In their paper Banach and Tarski did not say anything about the values of the integers k and ℓ. Sierpinski 9 in 1945 showed that k equal to 3(or 2) and ℓ equal to 5(or 6) are a solution of the problem. The question received a complete answer in 1947 when Robinson [7] showed that $k = 2$ and $\ell = 3$ are the smallest integers that solve the problem.

The greatest part of our work is to establish the Theorem DS: "There exist three decompositions of the surface S of the three-dimensional sphere into disjoint sets:

$$1) \quad S = A_1 + A_2 + \dots + A_{k+\ell}$$

$$2) \quad S = B_1 + B_2 + \dots + B_k$$

$$3) \quad S = B_{k+1} + \dots + B_{k+\ell}$$

such that

$$A_i \simeq B_i \quad , \qquad i = 1,2,\dots,k+\ell \text{ ."}$$

Then it will be quite easy to prove the theorem DS' .

We follow Robinson's paper [7] . In it congruence means superposition by rotations or translations. Here we admit any rigid motion, that is, reflections are also possible. Although some of our theorems are simpler than those of Robinson we do not improve his values $k = 2$ and $\ell = 3$ when he admitted only rotations and translations.

I. A FREE NON-ABELIAN SUBGROUP OF ROTATIONS

Let G be the group of all rotations about the origin in the three-dimensional Euclidean space. Such a group G can be represented by the multiplicative group of all 3-by-3 orthogonal matrices (*) with determinant equal to +1 .

We will show that there exists a free non-Abelian subgroup H of the group G of rotations. (A group is said to be free if it has a basis, i.e., a set of elements such that any element of the group can be uniquely expressed as a product of elements of the basis with integral exponents.)

Let us consider the rotations φ and ψ given by

$$\Phi = \begin{bmatrix} -\cos\theta & 0 & \sin\theta \\ 0 & -1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \qquad \Psi = \begin{bmatrix} \lambda & -\mu & 0 \\ \mu & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\lambda = \cos 2\pi/3 = -1/2$, $\mu = \sin 2\pi/3 = \sqrt{3}/2$, $\theta = 2$. Looking at these matrices we see that ψ is a rotation of $2\pi/3$ about the z-axis in the positive direction and φ is a rotation of π around an axis E which is in the xz-plane and makes an angle of $\theta/2 = 1$ radian with the z-axis .

(*) A real matrix A is orthogonal if $AA^* = A^*A = 1$, where A^* is the transpose matrix of A .

Lemma 1 - The rotations φ and ψ are independent except for the relations $\varphi^2 = \psi^3 = 1$.

Proof: We want to prove that for every positive integer n the following expressions are different from 1:

$$\alpha = \varphi\psi^{m_1}\varphi\psi^{m_2}\ldots\varphi\psi^{m_n}$$
$$\beta = \psi^{m_1}\varphi\psi^{m_2}\ldots\varphi\psi^{m_n}\varphi$$
$$\gamma = \varphi\psi^{m_1}\varphi\psi^{m_2}\ldots\varphi\psi^{m_n}\varphi$$
$$\delta = \psi^{m_1}\varphi\psi^{m_2}\ldots\varphi\psi^{m_n} ,$$

where $m_1, m_2, \ldots, m_n$, are equal to 1 or 2. (Observe that $\varphi^{-1} = \varphi$, $\psi^{-1} = \psi^2$, $\psi^{-2} = \psi$) .

We need only to show that the products of the α form are different from 1. Because, if a β was equal to 1, then an α would be equal to 1, namely $\varphi\beta\varphi$. If a δ was equal to 1 we can easily see that an α should be equal to 1. Finally if a γ was equal to 1, then a δ would be equal to 1.

Let N be the point (0,0,1) in the surface of the unit-sphere. If we show that for every α , $N\alpha \neq N$, we will be through. In order to prove that, we begin by showing that N is transformed by α into a point P with coordinates

$$\text{(I)}\qquad \begin{aligned} x &= \sin\Theta\,(a_n\cos^{n-1}\Theta + \ldots + a_1) \\ y &= \mu\sin\Theta\,(b_n\cos^{n-1}\Theta + \ldots + b_1) \\ z &= c_n\cos^n\Theta + \ldots + c_0 \end{aligned} ,$$

where $x/\sin\Theta$ and $y/\mu\sin\Theta$ are polynomials of degree $n-1$ in $\cos\Theta$ with rational coefficients, and z is a similar polynomial of degree n. The proof of this will be by induction over the number n of double factors in α which are of the form $\varphi\Psi$ (or $\varphi\Psi^2$).

The statement is true for $n=1$, because the point N will go into $N\varphi\Psi$ (or $N\varphi\Psi^2$) with coordinates

$$\Psi \times \Phi \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda\sin\Theta \\ \mu\sin\Theta \\ \cos\Theta \end{bmatrix}, \quad \Psi^2 \times \Phi \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda\sin\Theta \\ -\mu\sin\Theta \\ \cos\Theta \end{bmatrix} \quad (*)$$

Let us suppose now that after n steps the point N goes into a point $P = (x,y,z)$ where x,y,z are given by (I). Then it is easy to verify that $P\varphi\Psi$ or $P\varphi\Psi^2$ have coordinates

$$x' \quad \sin\Theta\ (a'_{n+1}\cos^n\Theta + \dots + a'_1)$$

$$y' \quad \mu\sin\Theta\ (b'_{n+1}\cos^n\Theta + \dots + b'_1)$$

$$z' \quad c'_{n+1}\cos^{n+1}\Theta + \dots + c'_0$$

where $x'/\sin\Theta$ and $y'/\mu\sin\Theta$ are polynomials of degree n and have rational coefficients and z' is a similar polynomial of degree $n+1$.

Then the point N is transformed by α into a

(*) Observe that

$$\Psi^2 = \begin{bmatrix} \lambda & \mu & 0 \\ -\mu & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

point P with coordinates given by (I).

Now we prove that c_n is different from zero for each integer n. The coefficients a'_{n+1} and c'_{n+1} are given by

$$a'_{n+1} = \lambda c_n - \lambda a_n = \lambda(c_n - a_n)$$

$$c'_{n+1} = c_n - a_n .$$

This implies

$$c'_{n+1} - a'_{n+1} = (1 - \lambda)(c_n - a_n) .$$

Since for $n = 1$ $c_1 - a_1 = 1-\lambda$, we see that $c_n - a_n = (1-\lambda)^n$ and so

$$c_n = (1 - \lambda)^{n-1} = (3/2)^{n-1}$$

Then, for each n, the coordinate z is given by

$$z = (1-\lambda)^{n-1}\cos^n\Theta + c_{n-1}\cos^{n-1}\Theta + \ldots + c_o ,$$

where $c_{n-1}, \ldots, c_o$ are rational numbers. Since cos2 is a transcendental number it cannot be a root of an equation with rational coefficients

$$(1-\lambda)^{n-1}x^n + c_{n-1}x^{n-1} + \ldots + c_o - 1 = 0 .$$

Then N is always different from N, i.e., no α is the identity rotation of the group G. QED

<u>Lemma 2</u> - Let φ and ψ be the two rotations defined before. Then the rotations $\mu = \varphi\psi\varphi\psi$ and $\rho = \varphi\psi^2\varphi\psi^2$ are independent.

Proof: Let us suppose by contradiction that

$$\mu^{i_1}\rho^{i_2} \dots \mu^{i_{r-1}}\rho^{i_r} = 1 \quad ,$$

where $i_1, i_2, \dots, i_r$, are integers.* We assume without loss of generality that $i_2, \dots, i_{r-1}$, are different from zero. Now we replace μ and ρ by their expressions in terms of φ and ψ. We observe that $\mu^{-1} = \psi^2\varphi\psi^2\varphi$ and $\rho^{-1} = \psi\varphi\psi\varphi$. Then we effect all the possible simplifications in terms of φ and ψ. It is clear that there is no simplifications of φ's and ψ's in the following double products: $\mu\mu$, $\rho\rho$, $\mu\rho$, $\rho\mu$, $\mu^{-1}\rho^{-1}$, $\rho^{-1}\mu^{-1}$, $\mu^{-1}\mu^{-1}$, $\rho^{-1}\rho^{-1}$, $\mu\rho^{-1}$.

The other ones have the following simplifications:

$$\mu^{-1}\rho = (\psi^2\varphi\psi^2\varphi)(\varphi\psi^2\varphi\psi^2) = \psi^2\varphi\,\psi\,\varphi\psi^2 \quad ,$$
$$\rho^{-1}\mu = (\psi\varphi\psi\varphi)(\varphi\psi\varphi\psi) = \psi\varphi\psi^2\varphi\psi \quad ,$$
$$\rho\mu^{-1} = (\varphi\psi^2\varphi\psi^2)(\psi^2\varphi\psi^2\varphi) = \varphi\psi^2\varphi\psi\varphi\psi^2\varphi \quad .$$

So we see that in every case the first two factors (ψ^2 is counted as a single factor) and the last two factors remain untouched. Then, if $i_1 \neq 0$, the first two factors in μ^{i_1} remain untouched. If $i_1 = 0$, the first two factors in ρ^{i_2} remain untouched. The same reasoning for the last term. So we obtain an expression in one of the forms α, β, γ or δ (see p.9). But such an expression cannot be equal to 1 as we have already proved (lemma 1). QED

Observation: Lemma 2 assures us that there exist at least two independent rotations in G.

(*) Some of them being different from zero.

THEOREM: Let μ and ρ be two independent rotations. Then so are $\psi_1 = \mu\rho\mu\rho$, $\psi_2 = \mu\rho^2\mu\rho^2, \ldots, \psi_m = \mu\rho^m\mu\rho^m$ for any positive integer m .

Proof: The proof is similar to the proof of lemma 2. We do not give it here.

CONCLUSION: Given any integer m we can find m independent rotations, namely, $\psi_1, \psi_2, \ldots, \psi_m$. Such rotations generate a free non-Abelian subgroup of the group G of rotations.

APPLICATION:

THE HAUSDORFF PARADOX.

Let us consider the subgroup F' of G spanned by φ and ψ . According to lemma 1 φ and ψ are independent except for $\varphi^2 = \psi^3 = 1$. Therefore every element ($\neq 1$) of G' is in one of the four forms α, β, γ or δ . We note that the number of elements in G' is countable.

Lemma - It is possible to divide G' into three disjoint subsets A', B', and C' such that

i) given two rotations α and $\alpha\varphi$ (*) one belongs to A' and the other to $B' + C'$;

ii) given three rotations α, $\alpha\psi$, and $\alpha\psi^2$ (**) one and only one belongs to each of the sets A', B' and C'.

(*) In this product, α cannot simplify with φ .

(**) In this product, α cannot simplify with ψ or ψ^2 .

Proof: Let us start by alloting 1 to A', φ to B', ψ to B', ψ^2 to C', and so on, using the two above rules i) and ii). We proceed by induction. Let us assume we have done this for any element with n factors or less. (ψ^2 is counted as a single factor). An element of n 1 factors is in one of the forms $\alpha\varphi$ (*), $\alpha\psi$, $\alpha\psi^2$ (**), where α is an element of n factors. Then,

if $\alpha \varepsilon A', B'$ or C', $\alpha\varphi$ will be alloted to B', A' or A'(resp.);

if $\alpha \varepsilon A', B'$ or C', $\alpha\psi$ (or $\alpha\psi^2$) will be alloted to B', C' or A' (or C', A' or B')(resp.). QED

<u>Proof or Hausdorff Paradox:</u>

Each rotation around the origin is equivalent to a rotation about an axis through the origin. So in each rotation two points of the surface S of the sphere remain fixed. Let D be the set of such points for rotations ($\neq 1$) in G'. The set D is countable.

Now for each $x \varepsilon S-D$ we define the set

$$E(x) = \{y: y = x\alpha \ , \ \alpha \varepsilon G'\}$$

It is easy to see that if $x, y \varepsilon S-D$ then either $E(x) = E(y)$ or $E(x) \cap E(y) = \emptyset$. Then $S-D$ is divided into equivalence classes. By choosing a point from each class (axiom of choice) we form a set M. It is obvious that

$$S-D = \bigcup_{\alpha \varepsilon G'} M\alpha \ .$$

(*),(**) See foot-notes, p.13

Let us consider the following three sets

$$A = \bigcup_{\alpha \varepsilon A'} M\alpha \;, \quad B = \bigcup_{\alpha \varepsilon B'} M\alpha \;, \quad C = \bigcup_{\alpha \varepsilon C'} M\alpha \;.$$

It is easy to see that the three sets are pairwise disjoint and that

$$A + B + C = S-D \;.$$

It also follows from the construction of A', B' and C' that

$$A\varphi = B+C$$

$$A\psi = B \;, \quad A\psi^2 = C \;.$$

Therefore

$$A \simeq B + C \;, \quad A \simeq B \simeq C \;. \quad \text{QED}$$

Remark: This proof was given by Hausdorff in 1914.

Essentially the division of G' was done in such a way as to obtain the desired congruences. The process was simplified extremely by Robinson. He introduced certain relations corresponding to the given congruences and showed how to obtain a decomposition of the surface of the sphere in parts satisfying those congruences. We will return to this later.

II. A FREE GROUP OF MAPPINGS IN THE SPHERE

1. A free group pf mappings in the sphere.

So far we have been concerned only with rotations. Now we introduce reflections of the surface S of the sphere. A reflection of S with respect to its center is the mapping ω that transforms each point of S into its diametrically opposite point. It is obvious that ω commutes with any rotation of G . In this paper we use the term "mapping" to mean any finite product of rotations and reflections in S . These mappings are isometric. That is, if x and y belong to S and their images, by a mapping φ , are $x\varphi$ and $y\varphi$, then $d(x,y) = d(x\varphi,y\varphi)$.

We have proved in the preceding section that the group G contains any finite number of independent rotations.(*) So let $\psi_0, \psi_1, \ldots, \psi_m$, be independent rotations of G . They generate a free subgroup H .
We define the following mappings:

$$\varphi_0 = \omega\,\psi_0$$

$$\varphi_i = \psi_i \quad , \quad i = 1,\ldots,m \ .$$

(*) Dekker and De Groot recently showed that there exist $\aleph$ independent rotations in G . $\aleph$ is the cardinal number of the real numbers.

THEOREM 1 - The mappings $\varphi_o, \varphi_1, \ldots, \varphi_m$, are independent generators of a free group Φ of mappings.

Proof: Let us assume by contradiction that

$$\varphi_{i_1}^{j_1} \ldots \varphi_{i_s}^{j_s} = 1$$

where $0 \leq i_1, \ldots, i_s \leq m$, and $j_1, \ldots, j_s$ are equal to ± 1. Since ω commutes with each ψ_i it follows that

$$\omega^r \psi_{i_1}^{j_1} \ldots \psi_{i_s}^{j_s} = 1 .$$

This implies that r is even. Therefore $\omega^r = 1$. But

$$\psi_{i_1}^{j_1} \ldots \psi_{i_s}^{j_s} = 1$$

is not possible because we have assumed that $\psi_o, \psi_1, \ldots, \psi_m$ are independent rotations. QED

2. Classes of equivalence on the sphere.

In this paragraph we show how the group Φ determines a partition of S into equivalence classes.

DEFINITION 1 - Two points x and y in S are equivalent if there exists some mapping $\varphi \varepsilon \Phi$ such that $y = x\varphi$.

This defines an equivalence relation between the points of S.

Indeed:

i) x is equivalent to x. $(x = x1)$.

ii) if x is equivalent to y, $(y = x\varphi)$, then y is equivalent to x, $(x = y\varphi^{-1})$.

iii) if x is equivalent to y, $(y = x\varphi)$, and y is equivalent to z, $(z = y\alpha)$, then x is equivalent to z, $(z = x(\varphi\alpha))$.

So the spherical surfface S is divided into equivalence classes E.

<u>DEFINITION 2</u> - A point $x \varepsilon S$ is called a fixed point if there exists a mapping $\varphi \varepsilon \Phi$ such that $x\varphi = x$.

<u>Lemma 1</u> - If a class E has a fixed point then every other point of E is also fixed.

<u>Proof</u> - Let x be a fixed point in E and y other point also in E. So there exist mappings α and φ in Φ such that $x\alpha = x$ and $y = x\varphi$. Now we see that $\varphi^{-1}\alpha\varphi \varepsilon \Phi$ and $y(\varphi^{-1}\alpha\varphi) = y$. QED

<u>Remark</u>: Lemma 1 says that a class E consists either entirely of fixed points or entirely of non-fixed points.

<u>Lemma 2</u> - If $\varphi \varepsilon \Phi$ has a fixed point then φ contains φ_0 an even number of times.

<u>Proof</u> - Let us suppose that φ contains φ_0 an odd number of times. Then $\varphi = \omega\Psi$, where Ψ is a rotation of H. So if x is a fixed point of φ then

$$x = x\varphi = x(\omega\Psi') = (x\omega)\Psi .$$

If a point coincides with its diametrically opposite point by a rotation Ψ then $\Psi^2 = 1$. This is impossible because $\Psi \varepsilon H$.
QED

Remark: So if $\varphi \varepsilon \Phi$ has a fixed point then φ is equivalent to a rotation $\Psi \varepsilon H$. That is,

$$\varphi = \omega^r \Psi ,$$

where r is an even number and $\Psi \varepsilon H$.

3. Representation of elements in each class.

a) Non-fixed points. If E is a class of non-fixed points and x is any element of E, then any point of E may be represented by $x\alpha$, where $\alpha \varepsilon \Phi$. This representation is unique. For let us suppose that $y = x\alpha = x\beta$. Then $x = x(\alpha\beta^{-1})$, which contradicts the fact that x is not a fixed point.

b) Fixed points. Let E be a class of fixed points. We determine a mapping Θ which has the smallest possible number of factors $\varphi_i^{\pm 1}$ and has a fixed point x in E. Θ has the form

$$\Theta = \varphi_{i_1}^{j_1} \ldots \varphi_{i_s}^{j_s} ,$$

where

$$\varphi_{i_1}^{j_1} \neq \varphi_{i_s}^{-j_s}$$

must hold. Otherwise, $\varphi_{i_1}^{-j_1} \Theta \varphi_{i_1}^{j_1}$ would be shorter than Θ and would have a fixed point in E, namely $x\varphi_{i_1}^{j_1}$.

Lemma 3 - Let E be a class of fixed points and Θ the mapping just described. If $x\alpha = x$ then $\alpha = \Theta^n$, where n is an integer.

Proof - If $\alpha = 1$, then $n = 0$. Let us suppose that $\alpha \neq 1$.

Since α and Θ are equivalent to rotations around the same axis then $\alpha\Theta = \Theta\alpha$.

If $\Theta\alpha$ does not simplify in terms of the φ_i we may write

$$\alpha = \Theta\alpha\Theta^{-1}$$

and we conclude α must begin with the bloc Θ. That is, $\alpha = \Theta\alpha'$. This implies that $\alpha'\Theta = \Theta\alpha'$.

If $\Theta\alpha$ does simplify then $\Theta^{-1}\alpha$ does not because $\varphi_{i_1}^{-j_1} \neq \varphi_{i_s}^{j_s}$. So we conclude from $\Theta^{-1}\alpha = \alpha\Theta^{-1}$ that $\alpha = \Theta^{-1}\alpha\Theta$.

Then α begins with the bloc Θ^{-1}. We proceed on that way a finite number of times and we get $\alpha = \Theta^n$. QED

THEOREM 2 - Any point of a class E of fixed points can be represented as $y = x\alpha$, where α does not begin either with the bloc Θ or with $\varphi_{i_s}^{-j_s}$. That representation is unique.

Proof: Let us suppose that a point $y \varepsilon E$ is represented as $y = x\alpha$, where α begins with Θ^n $(n > 0)$. Then y can be represented as $y = x\alpha'$, where $\alpha' = \Theta^{-n}\alpha$. It is obvious that α' begins neither with Θ nor with $\varphi_{i_s}^{-j_s}$.

Now we assume $y = x\alpha$, where α begins with $\varphi_{i_s}^{-j_s}$. Then $y = x\alpha'$, where $\alpha' = \Theta\alpha$ and α' does not begin either with the bloc Θ (observe that $\Theta\alpha$ simplifies) or with $\varphi_{i_s}^{-j_s}$. In order to see that such a representation is unique, let us assume $x\alpha = x\beta$. Then $x(\alpha\beta^{-1}) = x$. According lemma 3 it follows that

$$\alpha\beta^{-1} = \Theta^n$$

and so

$$\alpha = \Theta^n\beta .$$

If $n > 0$, since $\Theta^n\beta$ does not simplify because β does not begin with $\varphi_{i_s}^{-j_s}$ we conclude that α begins with the bloc Θ, which is impossible.

If $n < 0$, since $\Theta^n\beta$ does not simplify because β does not with the bloc Θ, we conclude that α begins with $\varphi_{i_s}^{-j_s}$, which is also impossible.

Therefore $n = 0$, which implies that $\alpha = \beta$. QED

Remark: In each class E of fixed points having chosen the mapping Θ we have a closed cycle

$$x,\ x\varphi_{i_1}^{j_1},\ \dots,\ x\varphi_{i_1}^{j_1} \dots \varphi_{i_s}^{j_s} = x .$$

The theorem 2 implies that there exists no other closed cycle in E.

III. DECOMPOSITION OF S ACCORDING TO RELATIONS

1. Relations.

DEFINITION 1 - Let I be a set of elements. A relation R is a function defined in the Cartesian product IxI with values in a set of two elements, say 0 and 1. If $k \epsilon I$ and $\ell \epsilon I$ are such that $R(k, \ell) = 1$, then we say that k and ℓ are related. If $R(k, \ell) = 0$, they are not related.

Example - Let R be the relation "greater than" and I be the set of real numbers. We have $R(5,3) = 1$, i.e., 5 is greater than 3. But $R(3,5) = 0$ because 3 is not greater than 5.

From now on we consider only cases in which I is the set of integers $1,2,\ldots,n$. Let K and L be two non-empty proper subsets of I. We denote by $\bar{K}$ the complement of K with respect to I.

DEFINITION 2 - Given a pair (K,L) we define a canonical relation, R_{KL}, as being the following relation:

$$R_{KL}(k, \ell) = 1 , \quad \text{if } k \epsilon K \text{ and } \ell \epsilon L$$
$$\text{or } k \epsilon \bar{K} \text{ and } \ell \epsilon L$$
$$R_{KL}(k, \ell) = 0 , \quad \text{otherwise.}$$

Remark: $R_{KL} = R_{\overline{KL}}$.

DEFINITION 3 - The inverse of a relation R is the relation R^{-1} defined as follows:

$$R^{-1}(\ell, k) = R(k, \ell) \ .$$

EXAMPLES: 1) The inverse of "greater than" is the relation "less than".

2) The inverse of a canonical relation R_{KL} is the canonical relation R_{LK} .

DEFINITION 4 - The product of two relations R and R' is the relation RR' such that

$RR'(k, \ell) = 1$, if there exists $s \in I$ such that $R(k,s) = 1$ and $R'(s, \ell) = 1$;

$RR'(k, \ell) = 0$, otherwise .

Remark: 1) The product of relations is not in general commutative.

2) The product of relations is associative. This will allow us to talk about products of more than two factors.

DEFINITION 5 - A relation is said to have a fixed point if there exists $k \in I$ such that $R(k,k) = 1$.

DEFINITION 6 - A relation includes a constant ℓ if $R(k, \ell) = 1$ for all $k \in I$.

Remark: If R includes a constant then R has a fixed point.

Lemma 1 - The product of two canonical relations is either a canonical relation or a relation including a constant.

Proof - Let $R = R_{KL}$ and $S = R_{MN}$ be two canonical relations.

We have the following five cases:

i) $L = M$. Then $RS = R_{KN}$.

ii) $L = \bar{M}$. Then $RS = R_{K\bar{N}}$.

iii) $L \subset M$, $L \neq M$. Then RS includes a constant, namely any point of N .

iv) $L \subset \bar{M}$, $L \neq \bar{M}$. Then RS includes a constant, namely any point of $\bar{N}$.

v) $L \cap M \neq \emptyset$, $L \cap \bar{M} \neq \emptyset$. Then RS includes a constant, namely any point of I . QED

Lemma 2 - The product of a canonical relation by a relation that includes a constant is a relation that includes a constant.

Proof - Let $R = R_{KL}$ be a canonical relation, and S be a relation that includes the constant s . So RS includes the constant s . And SR includes any element of L (or $\bar{L}$) as a constant if $s \varepsilon K$ (or $s \varepsilon \bar{K}$) . QED

Lemma 3 - The product of two relations that include a constant is a relation that also includes a constant.

Proof - Obvious.

Lemma 4 - If $R_0, R_1, \ldots , R_m$, are canonical relations then $R_0R_1\ldots R_m$ is either a canonical relation or a relation including a constant.

Proof - It is the successive application of the preceding three lemmas. QED

Let us consider a partition of the set of all proper non-empty subsets of I into classes, C_1 and C_2, such that if a subset K of I belongs to a class, then $\bar{K}$ belongs to the other class.

Let $R_i = R_{K_iL_i}$, $(i = 1,2,\ldots,m)$, be the m different canonical relations that are formed with couples of sets in C_1. Let $R_0 = R_{K\bar{K}}$, for some $K\varepsilon C_1$. We define corresponding to each element

$$\varphi = \varphi_{i_1}^{j_1} \cdots \varphi_{i_s}^{j_s}$$

of Φ (see p.16) the following relation:

$$R = R_{i_1}^{j_1} \cdots R_{i_s}^{j_s} .$$

Lemma 5 - If $\varphi\varepsilon\Phi$ has a fixed point, then the corresponding R also has a fixed point.

Proof - According to lemma 4, R is either a canonical relation or includes a constant. If R includes a constant, then we will be through because it follows that R has a fixed point. Now let us suppose that R is a canonical relation. Then we must have

$$R_{i_1}^{j_1} = R_{M_0M_1} , \quad R_{i_2}^{j_2} = R_{M_1M_2}, \ldots, R_{i_s}^{j_s} = R_{M_{s-1}M_s} .$$

If $i_k \neq 0$, then M_{k-1} and M_k are in the same class (C_1 or C_2).
If $i_k = 0$, then M_{k-1} and M_k are in different classes.
Since φ has an even number of φ_0 (according to lemma 2,II), it follows that M_0 and M_s must be in the same class.
Thus $M_0 \neq \bar{M}_s$. Therefore $R_{M_0M_s} = R$ has a fixed point. QED

2. A theorem of decomposition.

DEFINITION 1 – A mapping $\varphi \varepsilon \Phi$ is said to be compatible with a relation R , for the subdivision of S into the parts $A_1, \ldots, A_n$, if

$$A_k\varphi \cap A_\ell \neq \emptyset \implies R(k, \ell) = 1 .$$

In short we say: φ is compatible with R .

Lemma 1 – If φ is compatible with R , then φ^{-1} is compatible with R^{-1} .

Proof – Let us suppose that

$$A_\ell \varphi^{-1} \cap A_k \neq \emptyset .$$

Then

$$\emptyset \neq (A_\ell \varphi^{-1} \cap A_k)\varphi \subset A_k\varphi \cap A_\ell .$$

Since φ is compatible with R it follows that

$$R(k, \ell) = 1 .$$

So

$$R^{-1}(\ell, k) = 1 . \quad \text{QED}$$

Lemma 2 - If φ_i is compatible with R_i, $i = 0,1,\ldots,m$, then $\varphi = \varphi_{i_1}^{j_1} \ldots \varphi_{i_s}^{j_s}$ is compatible with $R = R_{i_1}^{j_1} \ldots R_{i_s}^{j_s}$, where the j's are equal to ± 1 and the i's are integers between 0 and m.

Proof - The proof is by induction on the numbers s of factors in φ. For $s = 1$, the statement is precisely lemma 1.

Let us assume that it is true φ with s factors. We claim that $\varphi\varphi_i^j$ is compatible with RR_i^j. Indeed, let us suppose

$$A_k(\varphi\varphi_i^j) \cap A_\ell \neq \emptyset .$$

Then

$$(A_k\varphi)\varphi_i^j \cap A_\ell \neq \emptyset ,$$

which means that there exists an x in some A_p such that $x\varphi_i^j \varepsilon A_\ell$. Therefore $R_i^j(p, \ell) = 1$.
On the other hand, since $x \varepsilon A_p$ and $x \varepsilon A_k\varphi$, it follows that $R(k,p) = 1$.
Then $RR_i^j(k, \ell) = 1$. QED

THEOREM - Let $R_0,\ldots,R_m$, the $m + 1$ canonical relations defined on page 25 and $\varphi_0,\ldots,\varphi_m$, be the $m + 1$ independent mappings that were introduced in § II. (See page

16). Then we can decompose the surface S of the sphere into n disjoint parts $A_1,\dots,A_n$, in such a way that each mapping φ_i is compatible with R_i, for such a decomposition.

Proof - We have proved (§II) that the group Φ determines a decomposition of the surface S of the sphere into equivalence classes E. Now we have to prove that we can allot the points of each class to sets $A_1,\dots,A_n$, in such a way that the mappings φ_i $(i = 0,1,\dots,m)$ are compatible with R_i $(i = 0,1,\dots,m)$, for the decomposition of S into these sets.

Classes of non-fixed points: From each class of non-fixed points choose a point x (axiom of choice). As it was proved in § II, all the elements of a class of non-fixed points can be represented as $x\alpha$, where α runs through all the mappings of Φ. Let us start by allot x to any A_i. By induction we suppose that $x\alpha$ was assigned to A_k. Then $x\alpha\varphi_i^j$ (where φ_i^j does not simplify with α) would be put in A_ℓ so that $R_i^j(k,\ell) = 1$. Classes of fixed points: Let E be a class of fixed points. Let Θ be a shortest mapping of Φ which has a fixed point x in the class E. As before we denote Θ by

$$\Theta = \varphi_{i_1}^{j_1} \dots \varphi_{i_s}^{j_s}$$

and the only closed cycle in E will be

$$x,\ x\varphi_{i_1}^{j_1},\ \dots,\ x\varphi_{i_1}^{j_1}\dots\varphi_{i_s}^{j_s} = x \quad .$$

We have no trouble with the points of E which do not belong to the cycle. They can be alloted to the classes $A_1,\dots,A_n$, in the same way as we did for the non-fixed points. But we must be careful with the points in the cycle. They cannot be put arbitrarily in the sets A_i. Let us see why. Suppose we put x in a class A_k and $x\varphi_{i_1}^{j_1} \dots \varphi_{i_{s-1}}^{j_{s-1}}$ in a class A_ℓ so that

$$(R_{i_1}^{j_1} \dots R_{i_{s-1}}^{j_{s-1}})(k,\ell) = 1 .$$

Now $(x\varphi_{i_1}^{j_1} \dots \varphi_{i_{s-1}}^{j_{s-1}})\varphi_{i_s}^{j_s} = x$ must be in A_k. But if we have $R_{i_s}^{j_s}(\ell,k) = 0$, then $\varphi_{i_s}^{j_s}$ is not compatible with $R_{i_s}^{j_s}$. We avoid this by proceeding as follows.

The relation $R = R_{i_1}^{j_1} \dots R_{i_s}^{j_s}$ has a fixed point k, because it is a relation corresponding to a mapping with a fixed point (lemma 5, p.25). Then there exist integers $k_0 = k$, $k_1,\dots,k_s = k$, between 1 and n such that $R_{i_r}^{j_r}(k_{r-1},k_r) = 1$. If we allot x to A_k and $x\varphi_{i_1}^{j_1}\dots\varphi_{i_r}^{j_r}$ to A_{k_r} $(r = 1,\dots,s-1)$ we will be through. QED

IV. DECOMPOSITION OF S ACCORDING TO CONGRUENCES

DEFINITION: Two sets A and B on the surface S of the sphere are said to be congruent if there exists a mapping φ (in the sense of p.16) which transforms A onto B.

We write $A \simeq B$ for $A\varphi = B$.

The following properties are obvious:

i) $A \simeq A$.

ii) $A \simeq B$ then $B \simeq A$.

iii) $A \simeq B$, $B \simeq C$ then $A \simeq C$.

So $\simeq$ defines an equivalence relation between the subsets of S. Such a relation is called congruence.

Let $A_1, \dots, A_n$, be a decomposition of S into n disjoint parts. A congruence

$$A_{k_1} + \dots + A_{k_r} \simeq A_{\ell_1} \dots A_{\ell_s}$$

(where $1 \leq k_1 < \dots < k_r \leq n$, $1 \leq \ell_1 < \dots < \ell_s \leq n$) is denoted by

$$A_K \simeq A_L ,$$

where $K = (k_1, \dots, k_r)$ and $L = (\ell_1, \dots, \ell_s)$.

It is clear that $A_K \simeq A_L$ is equivalent to $A_{\bar{K}} \simeq A_{\bar{L}}$.

Our aim in this chapter is to prove the following

THEOREM: 1 - It is possible to decompose the surface S of the sphere into n disjoint parts $A_1,\dots,A_n$, in such a way that they satisfy any system of congruences of the form

$$A_K \simeq A_L \quad , \tag{1}$$

where K and L are proper non-empty subsets of I.

2 - Moreover, the sets $A_1,\dots,A_n$, can be connected and locally connected. (*)

Remark: Any system (1) of congruences is a part of the following system

$$\begin{aligned}
&A_1 \simeq A_2 \simeq \dots \simeq A_n \simeq \\
&A_1 + A_2 \simeq \dots \simeq A_{n-1} + A_n \simeq \\
&A_1 + A_2 + A_3 \simeq \dots \simeq A_{n-2} + A_n \simeq \\
&\dots \\
&A_1 + A_2 + A_3 + \dots + A_{n-1} \simeq \dots \simeq A_2 + A_3 + A_4 + \dots + A_n \;.
\end{aligned} \tag{2}$$

If the sets A_i satisfy the general system (2), then they obviously satisfy any given system (1).
Yet, in the system (2) there are some superfluous congruences. Indeed, by using transitivity of congruences and complementations of congruences we can easily see that the system (3) is equivalent to (2).

(*) The second part of this theorem as well as the proof of it is from Dekker and De Groot [5].

$$A_K \simeq A_L \text{ , where } K, L \varepsilon C_1 \qquad (3)$$
$$A_K \simeq A_K \text{ , where } K \text{ is a fixed set in } C_1 .$$

Recall that C_1 and C_2 are the two classes of the partition of all proper non-empty subsets of I in such a way that if $K \varepsilon C_1$ then $\bar{K} \varepsilon C_2$.

Therefore if we prove the theorem for the system (3), it will be true for any system (1) .

Lemma - Let $A_1, \dots, A_n$, be a decomposition of S . A congruence $A_K \simeq A_L$ holds if and only if there exists a mapping φ compatible with the canonical relation R_{KL} .

Proof: 1 - Necessary condition. Let us suppose that $A_K \simeq A_L$. Then there exists a mapping φ such that $A_K\varphi = A_L$. We claim φ is compatible with R_{KL} . In fact

$$A_k\varphi \cap A_\ell \neq \emptyset$$

implies that either $k \varepsilon K$ and $\ell \varepsilon L$, or $k \varepsilon \bar{K}$ and $\ell \varepsilon \bar{L}$.
So $R_{KL}(k, \ell) = 1$.

2 - Sufficient condition. Let us assume there exists φ compatible with R_{KL} . We claim $A_K\varphi = A_L$.

$A_K\varphi \subset A_L$. We suppose there exists an x such that $x \varepsilon A_K\varphi$ and $x \notin A_L$. So $x \varepsilon A_k\varphi$ for some $k \varepsilon K$, and $x \varepsilon A_\ell$ for $\ell \varepsilon \bar{L}$. Then $A_k\varphi \cap A_\ell \neq \emptyset$, which implies that $R_{KL}(k, \ell) = 1$. This is not possible because $k \varepsilon K$ and $\ell \varepsilon \bar{L}$.

$A_K\varphi \supset A_L$ follows similarly. QED

Proof of the Theorem:

1 - Given the congruences (3) we obtain the corresponding canonical relations $R_0, R_1, \dots, R_m$. Let $\varphi_0, \varphi_1, \dots, \varphi_m$, be the independent mappings (see p.16) that generate Φ. Now we apply the theorem of section III and obtain the desired decomposition. Using the above lemma we are through.

2 - We consider all the uncountable compact sets in S to be well ordered

$$M_1, M_2, \dots, M_\omega, \dots, M_\nu, \dots \quad , \quad \nu < c \quad (*) .$$

Let D be the set of fixed points for some mapping ($\neq 1$) of Φ. The points of S-D are considered well ordered

$$(X) \qquad x_1, x_2, \dots, x_\omega, \dots, x_\nu, \dots \quad , \quad \nu < c .$$

By $E(x)$ we denote the equivalence class determined by Φ that contains x.

By transfinite induction we define for every M a sequence of n points from (X).

The sequence in M_1 is defined by induction as follows: p_{11} is the first point of (X) that belongs to M_1 ; p_{1i} is the first point of (X) that belongs to $M_1 - \bigcup_{j<i} [E(p_{1j})]$.

It is obvious that the sets $M_1 - \bigcup_{j<i} [E(p_{1j})]$ are not empty because each class E has a countable number of points.

Let us suppose $p_{\xi i}$ is defined for all $\xi < \nu$ and $i=1,\dots,n$.

(*) c is the ordinal number of the set of real numbers.

is the ordinal number of the set of integers.

Now we define $p_{\nu i}$ as follows: $p_{\nu 1}$ is the first point of (X) that belongs to

$$M_\nu - \bigcup_{\substack{1 \leq j \leq n \\ \xi < \nu}} [E(p_{\xi j})] \quad ;$$

$p_{\nu i}$ is the first point of (X) that belongs to

$$M_\nu - \left[\bigcup_{\substack{1 \leq j \leq n \\ \xi < \nu}} [E(p_{\xi j})] \cup \bigcup_{1 \leq j \leq i} [E(p_{\nu j})] \right] .$$

Now we make a slight modification in the way of allotting the non-fixed points to the sets $A_1, \ldots, A_n$. The change is the following: if a class E contains a point $p_{\nu i}$ then such a point is put in the set A_i . This is possible because a class E contains at most one p ; and moreover, as we have seen, any arbitrary point of a class of non-fixed points can be allotted to any set A_i .

So A_i $(i = 1, \ldots, n)$ does not contain any uncountable compact set. Then each A_i is totally imperfect. Similarly, $S-A_i$ is also totally imperfect. According to a theorem of Sierpinski A_i is connected and locally connected. QED

V. APPLICATIONS

1 - A sharper form of the Hausdorff Paradox.

Using the theorem of §IV we can eliminate the set D that appears in the Hausdorff Paradox (see Introduction). That is, the surface S of the sphere can be decomposed in three disjoint sets A, B and C such that

$$A \simeq B \simeq C$$

$$A \simeq B + C$$

This was possible because we have also allowed reflections. As Robinson [7] observed the set D can not be dropped if by congruece we understand superposition only by rotations around the center of the sphere.

2 - Theorem DS . (*)

We can decompose S into four disjoint parts A_1, A_2, A_3 and A_4 such that

$$A_1 \simeq A_2 \simeq A_1 + A_2$$

$$A_3 \simeq A_4 \simeq A_3 + A_4$$

Then there exist four independent mappings φ_1, φ_2, φ_3 and φ_4 such that

$$S = A_1 \varphi_1 + A_3 \varphi_3$$

$$S = A_2 \varphi_2 + A_4 \varphi_4$$

Let us call $B_i = A_i \varphi_i$, $i = 1,2,3,4$. Then we have found three decompositions of S into disjoint parts

$$S = A_1 + A_2 + A_3 + A_4$$

$$S = B_1 + B_3$$

$$S = B_2 + B_4 \quad , \quad (*)$$

such that

$$A_i \simeq B_i \quad , \quad i = 1,2,3,4 .$$

In this case the numbers k and ℓ are both equal to 2. These are the minimum values for k and ℓ . Otherwise a proper subset of S would be congruent to the whole of S , which is impossible.

3 - A particular decomposition of S .

Now we show that it is possible to decompose S into five disjoint sets A_1, A_2, A_3, A_4 and P (a single point) such that

$$A_1 \simeq A_2 \simeq A_1 + A_2 + P$$

$$A_3 \simeq A_4 \simeq A_3 + A_4 \quad .$$

Moreover we can make this decomposition in such a way that the four independent mappings φ_1, φ_2, φ_3, φ_4 , presented in application 2) are such that

---

(*) Indeed, the two last decompositions are the same.

$$A_1\varphi_1 = A_1 + A_2 + P \qquad A_3\varphi_3 = A_3 + A_4$$

$$A_2\varphi_2 = A_1 + A_2 + P \qquad A_4\varphi_4 = A_3 + A_4$$

All we have to do is slightly to change the way of constructing the sets A_1, A_2, A_3 and A_4. The change will be only in one class of non-fixed points. Any point (u) of such a class will be assigned to P. Then we assign

$u\varphi_1$ to A_3 or A_4, $\quad u\varphi_1^{-1}$ to A_1,

$u\varphi_2$ to A_3 or A_4, $\quad u\varphi_2^{-1}$ to A_2

$u\varphi_3$ to A_1 or A_2, $\quad u\varphi_3^{-1}$ to A_1, A_2 or A_4,

$u\varphi_4$ to A_1 or A_2, $\quad u\varphi_4^{-1}$ to A_1, A_2 or A_3.

The other points of this class will be allotted to the sets A as we did in the theorem of p.27.

4 - Theorem DS'. (*)

We will prove the theorem for the sphere $x^2 + y^2 + z^2 \leq 1$. We denote by $S(r)$ the surface of the sphere $x^2 + y^2 + z^2 \quad r^2$, for $0 \quad r \quad 1$.

According to applications 2) and 3) we can write

$$S(r) = A_1(r) + A_2(r) + A_3(r) + A_4(r), \quad 0 < r < 1,$$

and

$$S(1) = A_1(1) + A_2(1) + A_3(1) + A_4(1) + P .$$

Let us call

$$A_i' = \bigcup_r A_i(r) \quad , \quad 0 < r \leq 1 \quad .$$

(*)

Then the sphere S' is decomposed into six disjoint sets

$$S' = A'_1 + A'_2 + A'_3 + A'_4 + P + O \quad ,$$

(where O is the set containing just the center of the sphere), such that

$$A'_1\varphi_1 = A'_2\varphi_2 = A'_1 + A'_2 + P \; , \quad A'_3\varphi_3 = A'_4\varphi_4 = A'_3 + A'_4 \; .$$

Observe that

$$(A'_3 + O)\varphi_3 = A'_3 + A'_4 + O \; .$$

Let us call

$$B'_1 = B'_2 = A'_1 + A'_2 + P \quad , \qquad B'_3 = A'_3 + A'_4 = O$$

$$B'_4 = A'_3 + A'_4 \quad , \qquad B'_5 = O \; .$$

Then

$$S' = A'_1 + A'_2 + (A'_3 + O) + A'_4 + P \quad ,$$

$$S' = B'_1 + B'_3 \quad , \qquad S' = B'_2 + B'_4 + B'_5 \; . \quad (*)$$

are three decompositions of S' such that

(*) As yet we have not defined congruence of sets in R^3. We say that two sets A and B in R^3 are congruent if there exists an isometric transformation $f: R^3 \longrightarrow R^3$ such that $f(A) = B$. So f can be a translation, a rotation (as defined in §I), a reflection $(f(x,y,z) = (-x,-y,-z))$ or any product of these three transformations.

$$B_1' \simeq A_1'$$

$$B_3' \simeq A_3' + 0$$

$$B_2' \simeq A_2'$$

$$B_4' \simeq A_4'$$

$$B_5' \simeq P \quad .$$

The last congruence is morely a translation.

So we have proved the theorem DS' and even exhibited the values of k and ℓ as 2 and 3. It is possible to prove that these are the smallest values for k and ℓ. A very short proof of this fact is presented by Robinson [7].

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