to our families

Contents

Chapter 1

Introduction

The study of differential equations started with Newton's monumental work on mechanics and differential calculus. Among many other achievements Newton mathematically established the law of gravitation and the laws of motion of point masses. Newton himself was able to solve the equation for two point masses interacting under gravity. We remark that for more than three point masses interacting under gravity the general solution to Newton's equation does not admit an integral expression as in the case of two point masses (Poincaré). The equations of motion of point masses under gravity are still the most studied system of ordinary differential equations (ode).

Newton's second law of motion can be written as

$$\dot{x} = v, \quad M\dot{v} = F(x,v) \qquad \text{(Newton's ode)}$$

where: $x \in \mathbb{R}^n$ represents the positions of the particles of a system, $v = \dot{x}$ represents the velocities of the particles, M is "the inertia matrix", and F represents the interaction force among the particles. The force can be usually decomposed into several parts corresponding to different physical agencies. The gravitational force F_G, for instance, can be written as the gradient of a scalar function, the gravitational potential $U_G(x)$, $F_G(x) = -\nabla U_G(x)$. This implies that the "energy function"

$$E(x,v) = \frac{(v, Mv)}{2} + U_G(x)$$

is invariant under the time evolution determined by the equations of motion $\dot{x} = v$, $M\dot{v} = -\nabla U_G(x)$. For this reason the gravitational force is called "conservative". "Dissipative forces" are those for which an "energy function" is not anymore preserved under time evolution. The great success of classical celestial mechanics, in which gravity is the only force, in describing the real position of celestial bodies immediately raises the question: How much relevant are dissipative forces in celestial mechanics?

From the point of view of fundamental physics "energy" is always conserved. So, any dissipative physical model must be understood as an incomplete description of reality. The energy dissipated by a system under consideration must be absorbed by other parts of the physical world which are neglected in the model. For instance, in the point mass model in celestial mechanics the internal structure of the stars and planets are completely neglected. If the dimensions of stars and planets are taken into account then gravity and the overall motion of the system cause time-dependent deformations on these bodies called "tides". The deformation of real solids and liquids usually dissipate energy in the form of heat. The overall conservation of energy would be restored if the system would be redefined as the set of elementary particles that form the stars and planets plus their respective radiation fields. It is impossible to treat a system with so many degrees of freedom. So, from a macroscopic point of view, tides are the main source of dissipation of energy in planetary systems. Tidal force is the main issue in this book. Among the long term effects of tidal energy dissipation are circularization of orbits and spin-orbit synchronization (as happens to the motion of the moon around the earth). Dissipative forces are crucial for the understanding of the time-asymptotic motion of celestial bodies.

Dissipative forces in celestial mechanics are also important during planetary system formation. It is well accepted that in an early stage of a planetary system the environment around the star is filled with dust and/or gas, just called the star nebula. So, the birthing planets move inside a "fluid". The effect of the fluid upon the planet is a dissipative force: the drag (again energy dissipation occurs as a consequence of the heating of the fluid and the planet). Fluid dynamic drag is also treated in this book.

Another important issue in planetary system formation and also in planetary ring dynamics is the motion of dust particles around a star or a planet. For a one micrometer large dust particle the gravity force may be comparable to some electromagnetic forces. Among these electromagnetic forces the strongest dissipative one is the so called "Poynting-Robertson drag", which arises from the transfer of momentum from star radiation to the dust. While tides, and therefore the dissipative forces associated to them, are caused by gravity, Poynting-Robertson drag and other types of dissipative forces are of electromagnetic origin. Although the physical principles of fluid-dynamical drag and forces of electromagnetic origin are explained in this book, their dynamical consequences are not discussed (see for instance [6], [7], [14]). The book is organized as follows.

In chapter 2 the fundamental equations for fluids and solids are presented. In the case of fluids, we emphasize the role of the Reynolds number (*Re*) in characterizing different drag regimes experienced by rigid bodies steadily placed under an incoming uniform flow. Three different drag formulas are presented and discussed: Form drag (for $Re \gg 1$), Stokes drag (for $Re \approx 1$ or smaller), and Epstein drag (for low *Re* and for particles with diameter less than the gas mean free path). Next, we consider the two dimensional problem of attenuation of gravity waves for highly viscous fluids. This problem is also considered under a special forcing term that is a planar two dimensional analogue of the tidal gravitational force in three dimensions. While the attenuation of gravity waves is a classical subject in hydrodynamics the analysis of the forced system seems to be new. By means of the study of these two problems we are able to present the theoretical foundations of the rheological approach to tidal deformations given in chapter 3. The remaining of chapter 2 is devoted to solids. We present the variational formulation of the equations of motion of a rotating solid under self-gravitational force. The concept of strain and elastic energy is discussed in detail under the small deformation hypothesis. A critical analysis of this hypothesis in the presence of self-gravitational force leads naturally to the definition of incompressible solids, which are then presented and discussed. We stress that our presentation of the mechanics of solids has a strong bias toward the variational approach. This is the starting point for our method of study of tidal deformations given in chapter 4.

Chapters 3 and 4 contain our own recent contribution to the study of tidal forces in celestial mechanics. Chapter 3 is based on the work of Ferraz-Mello and chapter 4 on the work of Grotta-Ragazzo and Ruiz. Some parts of the text in these chapters were copied directly from the original papers. As mentioned above the starting point of Ferraz-Mello is a rheological approach valid for highly viscous fluids. The starting point of Grotta-Ragazzo and Ruiz is the variational formulation of mechanics plus the description of dissipative forces by the so called "Rayleigh dissipation function". The interested reader may consult the review books [69] and [68] for more information on tides and forces on extended celestial bodies.

Chapter 5 is a summary of forces of nongravitational origin in celestial mechanics. These forces are mostly relevant for small bodies: from asteroids of tens of kilometers to sub-micrometric dust particles, being more relevant in the latter. In this chapter we first describe the mechanism of charging of micrometric dust particles, particularly those in planetary magnetospheres, and the consequent Lorenz force. The remainder of the chapter is devoted to radiation forces. We first discuss the radiation pressure and the Poynting-Robertson drag on small particles (micrometric) and then the diurnal and the seasonal Yarkovsky effects on large bodies (from centimetric to kilometric).

We finally remark that this book is focused on dissipative forces in natural celestial bodies. Although some of these forces are also relevant on man-made celestial bodies there are other issues to be considered in the later case. For instance, due to their smallness, tidal forces on man-made bodies are not as relevant as they are for large natural celestial bodies. See [3] for some developments on the dynamics of man-made bodies.

Chapter 2

Dissipative forces on solids and fluids

2.1 Introduction

This chapter contains some basic information on the mechanics of continuous materials. The two main classes of materials to be considered are: solids and fluids. Solids are characterized by their resistance to changes in shape under external stress. Fluids continuously deform under shear stress. Fluids can be subdivided into liquids and gases. Liquids are fluids with a strong resistance to pressure stress. They do not considerably change volume under compression and are called incompressible. Gases are compressible fluids. A real material, as for instance most types of soils, cannot be characterized as solid, liquid or gas. A whole branch of physics deals with the deformation of this type of matter: Rheology. In this chapter we only consider the most standard models for solids and fluids. Due to their simplicity they are the most studied from a theoretical point of view. They are also the most used in experimental physics since they require the knowledge of a minimal number of material-dependent parameters which sometimes are difficult to measure or estimate (specially in astrophysics).

The fundamental principles to be used in the study of fluids and solids are Newton's Laws. As in point particle mechanics, the fundamental equation of motion for the continuum can be written as "Force equals to mass times acceleration". The term "mass times acceleration", or the inertial term, is essentially the same as that for point particle mechanics regardless we are working with solids, fluids, etc. The inertial term depends only on the geometry of the underlying space that here is always the Euclidean geometry. The "force" term is the one that encodes the physics of the continuum. It depends on how the particles of the continuum interact among themselves and with external agencies. The force term is different for solids and liquids.

This chapter is organized as follows. In section 2.2, we analyze the the kinematics of the continuum and the inertial part of the equations of motion. In the usual description of the motion of a solid, which can only experience small deformations, we initially label each particle of the body and track their trajectories as a function of time. This is the so called "Lagrangian description" of the motion. Fluids can undergo large deformations: particles initially close can be far apart after some time. So, for fluids it is more common to use a "Eulerian description" of motion that means to focus on the particles velocities instead of on the particles positions. Both the Lagrangian and the Eulerian description of the motion are discussed in section 2.2. In 2.3 we present the Navier-Stokes equation that is the simplest model for fluid motion that allows for dissipation of energy. Then we introduce the Reynolds number (*Re*) that measures the relative importance of the inertial forces over the dissipative forces. Finally, four problems are addressed: drag on solids at high Reynolds number (form drag), drag on solids at low Reynolds number (Stokes drag), drag on small particles in a low density gas (Epstein drag), and dissipation of energy in overdamped gravity waves. In section 2.4 we present the variational formulation for the equations of motion of an elastic solid taking into account the self-gravitation energy. Viscosity and energy dissipation are introduced by means of a Rayleigh dissipation function. The equations in this section form the starting point for section 4.3.

2.2 The Lagrangian and Eulerian descriptions of the motion

Consider a continuum of particles or points that initially occupy a region $\mathscr{B}$ in $\mathbb{R}^3$. In principle, $\mathscr{B}$ can be the whole of $\mathbb{R}^3$, or a ball, or any other nice open domain. The set $\mathscr{B}$ is the "reference configuration". In order to simplify the explanation we initially suppose that $\mathscr{B} = \mathbb{R}^3$. The case where $\mathscr{B}$ is bounded can be similarly treated with the introduction of boundary conditions. Let $\phi(t,x)$ denote the position of a particle at time t that at $t = 0$ was at the point $x \in \mathscr{B}$. The motion of the particle is given by the function $t \to \phi(t,x)$ and must satisfy Newton's law: $\rho(x)\ddot{\phi}(t,x) = F$, where $\rho(x)$ is the density of the continuum at x and F is the force per unit of volume that acts upon the particle. If the particle velocity $\partial_t \phi(t,x) \overset{\text{def}}{=} \dot{\phi}(t,x)$ changes with time then the force upon it is non null. At a given time t the set of velocities of all particles define a vector-field $x \to v(t,\phi(t,x)) = \dot{\phi}(t,x)$. Notice that the function $(t,x) \to \phi(t,x)$, naturally called "the flow", is an assembly of all solutions to the problem $\dot{z} = v(t,z)$, $z(0) = x$. The particles acceleration also generate a vector-field:

$$\ddot{\phi}(t,x) = \partial_t v(t, \underbrace{\phi(t,x)}_{=z}) + \sum_{k=1}^{3} \underbrace{(\dot{\phi}(t,x))_k}_{=v_k(t,z)} \partial_{z_k} v(t,z) = \partial_t v(t,z) + v \cdot \nabla v(t,z).$$

In Riemannian geometry [55] the particle acceleration, denoted as $\frac{Dv}{dt}$, is the covariant derivative of the time dependent vector $v(t,z)$ along the curve $t \to \phi_t(x)$. The expression $v \cdot \nabla v$, also denoted as $\nabla_v v$, is the covariant derivative of v in the direction of v with respect to the Levi-Civita connection associated with the Euclidean metric. So, using the Riemannian geometry notation we write

$$\ddot{\phi}_t(x) = \frac{Dv}{dt}(t,z)|_{z=\phi(t,x)}$$

In fluid mechanics the covariant derivative $\frac{Dv}{dt}$ is called "material derivative". If f is any real valued function of (t,x) then we define the material derivative of f as:

$$\frac{Df}{dt}(t,x) = \partial_t f(t,x) + v(t,x) \cdot \nabla f(t,x).$$

The approach to continuum mechanics in which the independent spatial variable is the material point $x \in \mathscr{B}$ is called the Lagrangian description (or material description). The alternative approach in which the fixed point in space is the independent spatial variable is called the Eulerian description (or spatial description). In the Lagrangian setting the motion is described by the flow map $(t,x) \to \phi(t,x)$. In the Eulerian setting the motion is described by the velocity field $(t,z) \to v(t,z)$. Notice that the passage from the Eulerian to the Lagrangian setting requires the "integration of a vectorfield", in general a very difficult task. In principle both descriptions are equivalent. The Lagrangian description is the most used for solids and the Eulerian description for fluids.

2.3 Fluids

In order to fulfill Newton's equations $\rho \frac{Dv}{dt} = F$ for fluid motion it is necessary to know the forces F applied upon each fluid particle. These forces can be of two types. The first type is composed by long range forces that penetrate the fluid and act on all particles. The most important of these forces is gravity and it is given by $F = -\rho \nabla V(x)$ (Force per unit of volume), where V is the gravitational potential. The second type is composed by short-range forces of molecular origin. These forces, due to the interaction of neighboring particles, are mathematically modeled by the "stress-tensor".

The stress-tensor $\sigma(t,x)$ at time t is a symmetric bilinear form on the tangent space at x such that σ_{ij} gives the i-component of the force per unit area exerted across an element of plane surface normal to the j-direction. If a small "cubic particle" of the fluid centered at x is isolated and a balance of stress forces acting upon each face is made we conclude that the i-component of the net force per unit volume acting upon this particle is

$$F_i = \sum_j \partial_{x_j} \sigma_{ij}(x) = \operatorname{div}\big(\sum_j \sigma_{ij} \mathbf{e}_j\big)$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal reference frame. In particular, if W is a portion fluid and the i-component of the total force due to neighboring particle interactions is computed we get

$$\int_W \sum_j \partial_{x_j} \sigma_{ij}(x) dx^3 = \int_{\partial W} \sum_j \sigma_{ij} n_j dx^3$$

where $n_j(x)$ is the j-component of the external normal vector to the surface ∂W at the point x. So, the net force upon W due to the stress depends only on its surface and not on its volume. Summing up we can write the equations of motion for a fluid as

$$\rho\left[\partial_t v_i + v \cdot \nabla v_i\right] = -\rho \frac{\partial V}{\partial x_i} + \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} \tag{2.1}$$

It is still necessary to write the stress at a point x as a function of the velocity. At this stage comes into play the physical properties of the fluid.

Since the stress tensor is symmetric there is an orthonormal frame in which it is diagonal with components $\sigma_{11}, \sigma_{22}, \sigma_{33}$. Consider a cubic fluid element with faces orthogonal to the vectors of this frame. Due to the definition of a fluid, if the three components $\sigma_{11}, \sigma_{22}, \sigma_{33}$ are not all equal then the fluid element will deform. So, if a fluid is at rest then necessarily $\sigma_{11} = \sigma_{22} = \sigma_{33} = -p$ and the stress-tensor is a multiple of the identity (in any reference frame). The function p is the "hydrostatic pressure". If the stress-tensor is not the identity we define the pressure as

$$p = -\sum_i \frac{\sigma_{ii}}{3} = -\frac{\mathrm{Tr}\,\sigma}{3},$$

which is the average value of the stress on a fluid element at position x over all directions normal to the surface of the fluid element. Notice that the trace is an invariant of the tensor and so is the pressure. The non-isotropic part of the stress-tensor

$$d_{ij} = p\delta_{ij} + \sigma_{ij} \tag{2.2}$$

is called the "deviatoric stress tensor". Notice that the deviatoric stress tensor must be null for a fluid with uniform velocity. So, since d_{ij} is due to the interaction of adjacent fluid particles, the simplest possible relation between the $d_{ij}(t,x)$ and the fluid velocity at x is linear: $d_{ij} = \sum_{kl} A_{ijkl} \partial_{x_k} v_l$. Symmetry arguments (as those in section 2.4.1) can be used to show that this relation reduces to

$$d_{ij} = 2\eta \left[e_{ij} - \frac{\mathrm{div}\,(v)}{3} \delta_{ij} \right], \tag{2.3}$$

where

$$e_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{2.4}$$

is the "rate of strain tensor" and the constant η is the "viscosity". Fluids for which this relation is satisfied are called "Newtonian Fluids". Water and air are Newtonian fluids with $\eta_{air} = 1.81 \times 10^{-4}$ and $\eta_{water} = 1.002 \times 10^{-2}$ (units are $gr/cm \cdot sec$) at a temperature of 20^oC. Viscosity varies considerably with the temperature. For instance, $\eta_{air} = 2.09 \times 10^{-4}$ and $\eta_{water} = 0.355 \times 10^{-2}$ (units are $gr/cm \cdot sec$) at a temperature of 80^oC. Fluids like butter and mayonnaise are non-Newtonian fluids, for both the viscosity η depends on the velocity itself: for the first it increases with the velocity, for the second it decreases with the velocity. It happens often that the temperature differences in a fluid flow are sufficiently small such that the viscosity can be assumed to be constant. In this case the equation of motion for a Newtonian fluid, the so called "Navier-Stokes equation", becomes:

$$\rho[\partial_t v_i + v \cdot \nabla v_i] = -\rho \frac{\partial V}{\partial x_i} - \frac{\partial p}{\partial x_i} + \eta \left(\sum_j \frac{\partial^2 v_i}{\partial x_j^2} - \frac{1}{3} \frac{\partial \operatorname{div} v}{\partial x_i} \right) \tag{2.5}$$

In the above equation not only the three components of the velocity are unknown but also the density and the pressure. Conservation of mass can be used as an additional scalar equation but this is not enough to determine the five unknowns. So, thermodynamic relations must be used to close the system, more precisely an equation of state and the first law of thermodynamics. This adds one more variable, for instance, the internal energy of the fluid per unit of mass. At the end we get six equations to determine six unknowns. The use of a thermodynamic equation of state raises the question about whether or not the pressure defined by $p = -\sum_i \sigma_{ii}/3$ coincides with the thermodynamic pressure. Moreover, the definition of thermodynamic pressure requires the fluid to be in a state of equilibrium what may not be true for a fluid in motion. So, at first we need to clarify the meaning of thermodynamic pressure for a fluid in motion. The thermodynamic state of a pure substance is determined by two thermodynamic quantities. For a fluid element the definition of its density as the instantaneous ratio mass/volume has no ambiguity, but the instantaneous definition of a second thermodynamic quantity is not that simple. It is convenient to choose the internal energy E per unit of mass as this second thermodynamic quantity. It is defined as follows. The first law of thermodynamics states that the difference between the values of the internal energy per unit mass ΔE of a fluid element in two different equilibrium states is given by $\Delta E = W + Q$, where W is the work done on the fluid element by unit mass and Q is the heat transferred to the fluid element by unit mass. Now both W and Q are, in principle, measurable quantities that do not depend on the existence of an equilibrium. In this way it is possible to define the instantaneous internal energy E of a fluid. The equilibrium state to which E corresponds is obtained by the sudden isolation of the fluid element from the remainder of the fluid in such a way that the equilibrium state is achieved with neither exchange of heat nor work. So, from the instantaneous value of ρ and E all the other thermodynamic properties can be obtained including p_e the thermodynamic pressure of equilibrium. For a fluid at rest or moving with constant velocity $p = p_e$ but for a fluid in motion, in general, $p_e \neq p$. A difference between p_e and p can be understood as being the result of the delay in the achievement of the thermodynamic equilibrium by a fluid element in motion. Supposing that the difference $p - p_e$ depends only on the local gradients of velocity, symmetry arguments like those used to get relation (2.3) lead to the simplest possible relation

$$p - p_e = -\zeta\Lambda \quad \text{where} \quad \Lambda = \operatorname{div} v \tag{2.6}$$

and $\zeta > 0$ is named the "expansion viscosity" (or "bulk viscosity" or "compression viscosity") coefficient of the fluid (η is also called "shear viscosity"). The expansion viscosity is related to the dissipation of energy in the fluid due to isotropic changes of volume. This can be seen from the internal energy balance as follows.

It can be shown ([5] chapter $3 \cdot 4$) that the rate of change of the internal energy per unit mass of a fluid element is given by

$$\frac{DE}{dt} = -\frac{p\Lambda}{\rho} + \frac{2\eta}{\rho}\left(\sum_{ij}[e_{ij}e_{ij}] - \frac{1}{3}\Lambda^2\right) + \frac{1}{\rho}\sum_i \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) \tag{2.7}$$

where T is the fluid temperature and k is the thermal conductivity of the fluid. The right hand side of this equation have the following interpretation [5]. The term $-p\Lambda/\rho$ represents the work done upon the fluid element by the isotropic part of the stress (the pressure) in association with the isotropic rate of change of volume of the fluid element. This term can be either positive or negative. The second term

$$\Phi = \frac{2\eta}{\rho}\left(\sum_{ij}[e_{ij}e_{ij}] - \frac{1}{3}\Lambda^2\right) = \frac{1}{\rho}\sum_{ij}\underbrace{2\eta\left(e_{ij} - \frac{1}{3}\Lambda\delta_{ij}\right)}_{=d_{ij}}\left(e_{ij} - \frac{1}{3}\Lambda\delta_{ij}\right) \tag{2.8}$$

is always nonnegative. It represents the work done by the deviatoric part of the stress in association with the nonisotropic part (or shearing part) of the rate of strain. So, every shear motion is associated to an increasing in the internal energy so that the mechanical energy of the fluid is irreversibly transformed into heat. This is the main mechanism of dissipation of mechanical energy in fluids (and also in solids) and is related to the "shear-friction" between adjacent fluid elements. The last term $\frac{1}{\rho}\sum_i \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)$ represents the rate of heat exchange by molecular conduction between the fluid element and the surrounding fluid. This term can be either positive or negative.

The term $-p\Lambda/\rho$ can be further analyzed using equation (2.6)

$$-\frac{1}{\rho}p\Lambda = -\frac{1}{\rho}p_e\Lambda + \frac{1}{\rho}\zeta\Lambda^2.$$

Notice that the term $\frac{1}{\rho}\zeta\Lambda^2$ is always positive. As said above it represents the dissipation of mechanical energy of the fluid due to isotropic changes of volume. The first term $-\frac{1}{\rho}p_e\Lambda$ represents a reversible transformation of energy associated only to instantaneous equilibrium values of E and ρ. We recall that the difference $p - p_e = -\zeta\Lambda$ is associated to the lag in the adjustment of pressure to the continuing changes of ρ and E. The dissipation of energy related to ζ may be interpreted as the result of this lag. Using the equation $p - p_e = -\zeta\Lambda$ the stress tensor can be written as:

$$\sigma_{ij} = -p\delta_{ij} + d_{ij} = -p_e\delta_{ij} + \underbrace{2\eta e_{ij} + \left[\zeta - \eta\frac{2}{3}\right]\operatorname{div} v}_{\text{dissipative term}} \tag{2.9}$$

In the remainder of this book we will suppose that $p = p_e$, for the following reasons. A change of volume of a fluid domain requires variations of normal stresses that are much larger than the usual shear stresses. So, the rate of expansion Λ of the majority of flows is much smaller than the rates of shear. Moreover, for small Λ, the term $-p_e\Lambda$ is much larger than the dissipation term $\zeta\Lambda^2$, so $-p\Lambda = -p_e\Lambda + \zeta\Lambda^2 \approx -p_e\Lambda$. The expansion viscosity ζ is significant in the attenuation of sound waves and also in the structure of shock waves, two issues that will not be considered in this book.

The compressibility of liquids is so small that it is appropriate to assume the incompressibility condition $\operatorname{div} v = \Lambda = 0$ with $\rho =$constant. The incompressibility is reasonable even for gas flows provided that the maximum velocity of the gas is much smaller than the sound velocity. In this case the Navier-Stokes equation (2.5) becomes

$$\rho[\partial_t v_i + v\cdot\nabla v_i] = -\rho\frac{\partial V}{\partial x_i} - \frac{\partial p}{\partial x_i} + \eta\left(\sum_j \frac{\partial^2 v_i}{\partial x_j^2}\right), \qquad \operatorname{div} v = 0 \quad (2.10)$$

Notice that these equations are enough for the determination of the four unknowns v_1, v_2, v_3, p. If the flow is steady and inviscid, that is $\partial_t v = \partial_t p = \partial_t V = 0$ and $\eta = 0$, then these equations imply the "Bernoulli equation"

$$v\cdot\nabla\left\{\rho\frac{||v||^2}{2} + \rho V + p\right\} = 0,$$

which means that

$$\rho\frac{||v||^2}{2} + \rho V + p = constant \quad (2.11)$$

is constant over streamlines $t \to \phi(t,x)$. This is simply the conservation of energy for the autonomous Newtonian system $\ddot{\phi} = -\nabla[V + p/\rho]$.

2.3.1 Drag

One of the most important problems in fluid mechanics is the determination of the flow generated by a body moving through a fluid at rest at infinity. The most significant question is about the force exerted upon the body by the fluid. The theory on the subject is not enough to provide precise answers in the majority of the situations of practical interest. The theory is mostly useful in setting guidelines for experimental and numerical investigations.

The forces and torques acting upon the body can be decomposed into several parts: (1) If the body does not move with constant velocity then inertial fluid forces act upon the body. These inertial forces, also called **added-mass effect**, appear because any acceleration of the body implies an acceleration of fluid particles neighboring the body. In this way the inertia of the fluid particles adds to the inertia of the body by means of a pressure distribution over the body that resists to the acceleration. (2) The **friction drag** results from the integration of the tangential stress over the surface of the body. This force is approximately opposite in direction to the velocity of the body. This is the reason for the denomination **drag**. (3) The

lift is the component of the total force normal to the direction of motion. This force does not do work and it is very significant in wings. (4) The **pressure drag** is the force opposite in direction to the velocity of the body that results from the integration of the pressure stress (normal stress) over the surface of the body. The pressure drag is usually decomposed into two parts: (4a) The **induced drag** is a type of pressure drag that always appear associated to the lift; (4b) The **form drag** is the part of the pressure drag obtained after the induced drag has been subtracted. If the lift is absent then the pressure stress is the form drag. Since in this book we are interested in energy dissipative forces we only consider the simplest situation of forces on bodies moving steadily with no generation of lift. The conservative force due to the external potential V will also be neglected. So, in this case the only forces acting on the body are the friction drag and the form drag.

The form drag is particularly important for the motion of bluff bodies at "high velocity". In order to give a meaning to high velocity we define the Reynolds number as

$$Re = \frac{\rho U L}{\eta} \tag{2.12}$$

where L is a typical length of the body (for instance, the diameter of a disk or of a sphere) and U is the speed of the body through a fluid at rest at infinity. If the variables in the Navier-Stokes equation (2.10) are expressed in nondimensional form as

$$v' = \frac{v}{U}, \quad t' = \frac{tU}{L}, \quad x' = \frac{x}{L}, \quad p' = \frac{p}{\rho U^2}$$

then the Navier-Stokes equation (with $V = 0$) becomes

$$\begin{aligned} \frac{\partial v_i'}{\partial t'} + \sum_j v_j' \frac{\partial v_i'}{\partial x_j'} &= -\frac{\partial p'}{\partial x_i'} + \frac{1}{Re} \sum_j \frac{\partial^2 v_i'}{\partial x_j'^2}, \\ \sum_i \frac{\partial v_i'}{\partial x_i'} &= 0 \end{aligned} \tag{2.13}$$

Notice that the equation in this form depends only on the nondimensional parameter Re. This implies that if a homothetic transformation on the system $L \to \alpha L$ is followed by the transformation $U \to U/\alpha$ in a way that the Reynolds number remains invariant then the new velocity field is given by $\alpha v'(t'/\alpha^2, x'\alpha)$.

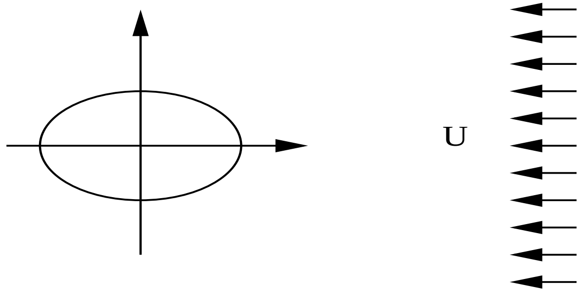

Figure 2.1: Body at rest and incoming stream

The drag on a body that is moving with velocity $(U,0,0)$ is given by

$$F = -\int \sum_j \sigma_{1j} n_j dA$$

where n is the external unit normal vector to the body and dA is the surface element of area. If we write this expression using the nondimensional variables we get

$$F = -\rho U^2 L^2 \int \left[-p' n_1 + \frac{1}{Re} \sum_j \left(\frac{\partial v'_1}{\partial x'_j} + \frac{\partial v'_j}{\partial x'_1} n_j \right) \right] dA'$$

that shows that the dimensionless drag $F/\rho U^2 L^2$ depends only on the Reynolds number.

The most common way to express the drag in nondimensional form is

$$F = C_D(Re) \rho \frac{A}{2} U^2 \quad \text{or} \quad C_D(Re) = \frac{F}{\frac{1}{2}\rho U^2 A} \tag{2.14}$$

where A is the area of the projection of the body on a plane normal to the stream at infinity. The advantage of this normalization is the following. Consider a frame of reference that moves with the same constant velocity as the body, see Figure 2.1. With respect to this frame the body is at rest and the fluid velocity at infinity is $(-U,0,0)$. It is an experimental fact that the incoming velocity field is approximately steady and smooth up to

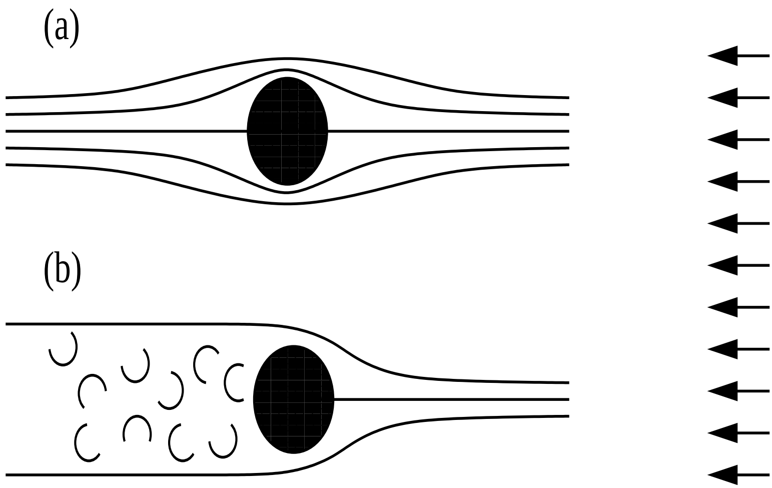

Figure 2.2: (a) Laminar flow at low Reynolds number. (b) Turbulent flow behind the body (turbulent wake) at high Reynolds number.

some small distance ahead of the body. Moreover, if U is small, or better say the Reynolds number (2.12) is small, then the velocity field remains steady and smooth even behind the body, see Figure 2.2 (a). In this case the streamlines smoothly circulate the body and the flow is said laminar. On the other hand, if the Reynolds number is large then there is a region behind the body where the velocity field is unsteady and the flow is highly chaotic, see Figure 2.2 (b). The flow in this region is said turbulent. The form and size of this turbulent region (called turbulent wake) depends a lot on the geometry of the body, it is relatively small for streamlined bodies, like a wing, and large for bluff bodies, like a sphere.

The analysis of the flow around a circular flat disk of radius R placed transversally to the incoming stream illustrates the idea behind the normalization in equation (2.14). If apply Bernoulli's equation (2.11) to the streamline that connects $x = \infty$ to $x = 0_+$, see Figure 2.3, we obtain that

$$p(t, 0_+) = p_{0+} = \rho \frac{U^2}{2} + p_\infty$$

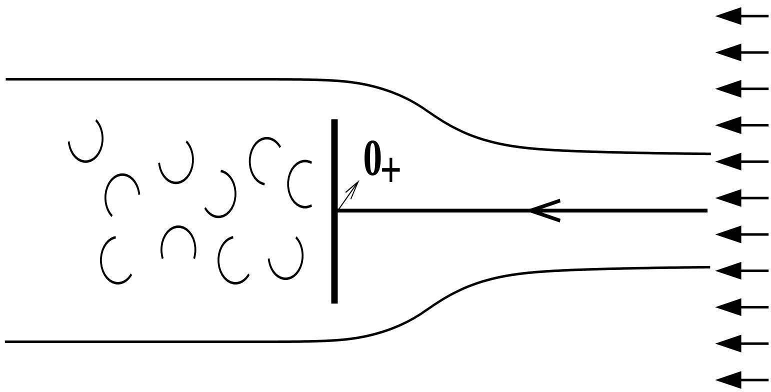

Figure 2.3: Streamline connecting the center of the disk to infinity.

where we used that the flow has a stagnation point $v(t,0_+) = 0$ at $x = 0_+$ and p_∞ is the pressure at infinity that is constant. The face of the disk opposite to the incoming flow, the negative face, is in the turbulent wake region. The pressure in the turbulent wake fluctuates but its average is homogeneous and approximately equal to p_∞ (see [51], pages 39 and 146 for an explanation of this fact). If the pressure on the positive face of the disk were uniform and equal to p_{0+} then the drag on the disk would be

$$F = -A(p_{0+} - p_\infty) = -A\rho\frac{U^2}{2}, \qquad A = \pi R^2$$

which would imply a drag coefficient $C_D = 1$. For all $Re = \rho U 2R/\eta > 5000$, the experimentally measured $C_D(Re)$ for the circular disk is 1.1. The drag coefficient $C_D(Re)$ of a smooth sphere of radius R varies a lot with the Reynolds number $Re = \rho U 2R/\eta$: $C_D(10^5) = 0.47$, $C_D(10^6) = 0.1, \ldots$ (see [5] ch 5.11). For the majority of body shapes the drag coefficient is determined by means of wind tunnel experiments (see, for instance, [43]).

2.3.2 Drag at low Reynolds number: Stokes drag.

For very low velocities the quadratic term $v' \cdot \nabla v'$ in equation (2.13) is much smaller than the terms that depend linearly on the velocity. Neglecting this quadratic term we obtain the Stokes equation (the primes are omitted)

$$\frac{\partial v_i}{\partial t} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\sum_j \frac{\partial^2 v_i}{\partial x_j^2}, \quad \sum_i \frac{\partial v_i}{\partial x_i} = 0 \qquad \textbf{(Stokes equation)}, \quad (2.15)$$

This linear equation is much simpler than the original one. It is possible to show that the force F of the fluid upon the body depends linearly on the stream velocity $(U,0,0)$:

$$F_i = -4\pi\eta L P_i U$$

where P_i, $i = 1,2,3$, are constants that depend on the shape of the body and L is a characteristic linear dimension of the body. If the body is a sphere of radius R then this formula becomes

$$F_1 = -6\pi R\eta U \Longrightarrow C_D(Re) = \frac{24}{Re} \qquad \textbf{(Stokes formula)}, \qquad (2.16)$$

where $Re = \rho U 2R/\eta$. The drag on a sphere is accurately given by this formula for $Re < 1$ (for $Re < 0.5$ the formula is practically exact).

2.3.3 Very small particles: Epstein drag.

An important issue in planetary system formation is the motion of spherical dust particles in a low density gas. The gas is supposed to have thermodynamic properties that approximate closely those of a perfect gas. Let ℓ be the mean free path of the gas molecules (the average distance traveled by a molecule between successive collisions). Let $\overline{v} = \sqrt{8kT/(m\pi)}$ be the mean thermal speed (the mean of the magnitude of the velocity of the molecules), where T is the gas temperature, m is the molecular mass, and k is the Boltzmann constant. If the radius R of the particle is smaller than ℓ and the particle speed U is much smaller than $\overline{v}$ then the drag on the particle can be computed by summing the momentum transferred to the dust particle through collisions with individual gas molecules [30]. The resulting expression is

$$F_1 = -\delta\frac{4\pi}{3}\rho R^2 \overline{v} U \quad \text{with} \quad \delta \approx 1 \qquad \textbf{(Epstein Drag)}, \qquad (2.17)$$

where the factor $\delta \approx 1$ [74] accounts for the microscopic mechanism of the collision between the gas molecule and the surface of the dust particle.

Using the relation $\eta = 0.5\rho\bar{v}\ell$ (mostly written with $\bar{v}$ replaced by the root mean square of the total velocity), we get from equation (2.17) with $\delta = 1$:

$$C_{DEpstein} = \frac{8\bar{v}}{3U} = \frac{16\eta}{3\rho U\ell}.$$

The Stokes $C_D = 24/Re$ is equal to the Epstein C_D when $\ell/R \approx 4/9$. This ratio is assumed to be the transition point between the two laws [74].

2.3.4 Creeping flow and Ferraz-Mello rheophysical approximation

A creeping flow is a type of flow where the inertial forces are small compared with viscous forces, which happens when $Re \ll 1$. In this situation the Navier-Stokes equation can be substituted by the Stokes equation (2.15). This approximation was used by Ferraz-Mello [34] to construct a rheophysical model for tidal waves that is presented in chapter 3. Here by means of two dimensional examples we build up a rheological model from Stokes equation and explain how, and under what conditions, rheological models can be obtained from primitive physical equations.

Overdamped gravity waves

Consider an unbounded amount of incompressible fluid of constant density ρ and viscosity η initially occupying the "lower part" , $y \leq 0$, of $\mathbb{R}^3$ with coordinates (x, y, z). The fluid has a gravitational potential energy $V = \rho g y$ where g is a constant. The upper part of $\mathbb{R}^3$ is occupied with a fluid with negligible density and viscosity. We want to study the motion of the fluid in the lower part of $\mathbb{R}^3$ induced by a small elevation of the interface. We suppose that this interface can be described by a graph $x \to y = \zeta(t, x)$ (see Figure 2.4). We assume that the fluid velocity satisfies the Stokes equation (2.15), with an additional gravitational term, in an a priori unknown domain given by $y \leq \zeta(t, x)$. Since all the analysis will be two dimensional we simplify the notation and write the (x, y) components of the velocity field

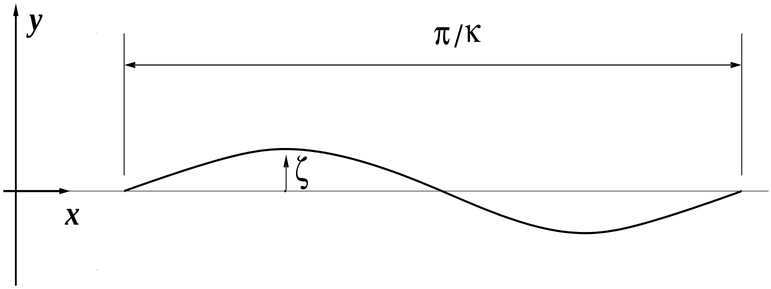

Figure 2.4: Geometrical parameters of the wave

as (u,v) and the Stokes equations as:

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} - \frac{\partial V}{\partial x} + \eta\underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}_{=\Delta u},$$

$$\rho\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} - \frac{\partial V}{\partial y} + \eta\underbrace{\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)}_{=\Delta v},$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.18}$$

These equations must be supplemented by the boundary conditions: the velocity remains finite as $y \to -\infty$ and the normal and the tangential stresses at the surface are null. Supposing that the surface normal deviates infinitesimally from the vertical, the boundary conditions at the surface $\{y = \zeta(t,x)\}$ can be written as

$$-p + 2\eta\frac{\partial v}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0, \tag{2.19}$$

where we used the expression for the stress tensor given in equations (2.9) and (2.4).

Following [48] (paragraph 349) we write the velocity field as (Helmholtz decomposition):

$$u = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x}. \tag{2.20}$$

Then equation (2.18) becomes:

$$\Delta\varphi = 0, \qquad \frac{\partial \psi}{\partial t} = \nu\Delta\psi = 0, \qquad \frac{p}{\rho} = -\frac{\partial \varphi}{\partial t} - gy \tag{2.21}$$

The equations for φ and ψ are linear, therefore the method of superposition is applicable. Consider a typical solution:

$$\varphi = A\mathrm{e}^{i\kappa x - \gamma t + \kappa y}, \quad \psi = B\mathrm{e}^{i\kappa x - \gamma t + \beta y}, \quad \mathrm{Re}\,\gamma > 0, \tag{2.22}$$

that is periodic with respect to x with a prescribed wavelength $2\pi/\kappa$ and such that $\Delta\varphi = 0$. Equation $\partial_t \psi = \nu\Delta\psi$ and the boundary condition at $y = -\infty$ imply that

$$\beta^2 = \kappa^2 - \frac{\gamma}{\nu} \quad \text{with} \quad \mathrm{Re}\,\beta \geq 0. \tag{2.23}$$

At the interface $y = \zeta(t,x)$, we must have $\partial_t \zeta(t,x) = v(t,x,\zeta) \approx v(t,x,0)$, for a wave amplitude assumed infinitely small. Integrating with respect to t we obtain

$$\zeta(t,x) = -\frac{\kappa}{\gamma}(A - iB)\mathrm{e}^{i\kappa x - \gamma t} \tag{2.24}$$

Now, the boundary conditions at the surface (2.19), up to first order on the amplitudes A and B, imply

$$\begin{aligned} 0 &= -\frac{p}{\rho} + 2\nu\frac{\partial v}{\partial y} = \frac{\partial \varphi}{\partial t} + g\zeta + 2\nu\left(\frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x \partial y}\right) \Longrightarrow & (2.25)\\ 0 &= A(\gamma^2 + g\kappa - 2\nu\kappa^2\gamma) + iB(2\gamma\nu\kappa\beta - g\kappa) & \\ 0 &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \Longrightarrow & (2.26)\\ 0 &= iA2\kappa^2\nu + B(-\gamma + 2\nu\kappa^2) & \end{aligned}$$

This system of equations admit nontrivial solutions A and B if, and only if,

$$g\kappa + (2\nu\kappa^2 - \gamma)^2 = 4\kappa^3\nu^2\beta$$

or, using equation (2.23),

$$(g\kappa)^2\left[1+\left(\frac{2\nu\kappa^2-\gamma}{\sqrt{g\kappa}}\right)^2\right]16\kappa^6\nu^3(-\gamma+\nu\kappa^2)$$

Following [48] paragraph 349, if we write

$$\theta=\frac{\nu\kappa^2}{\sqrt{g\kappa}},\quad x=\frac{2\nu\kappa^2-\gamma}{\sqrt{g\kappa}},$$

then the above equation can be written as

$$(1+x^2)^2=16\theta^3(x-\theta) \tag{2.27}$$

The rheological hypothesis of Ferraz-Mello is that the viscosity is large enough so that

$$\theta=\frac{\nu\kappa^2}{\sqrt{g\kappa}}\gg 1 \tag{2.28}$$

Under this hypothesis there are only two real solutions x of equation (2.27) that imply $\mathrm{Re}\,\beta\geq 0$. One of these solutions is $\gamma\approx 0.91\nu\kappa^2$, which decays fast to the equilibrium state $\zeta=0$. The other solution is $x=2\theta-1/2\theta\cdots$ that gives the slow decay

$$\gamma=\frac{g}{2\nu\kappa}=\frac{g\rho}{2\eta\kappa} \tag{2.29}$$

This expression is exactly that in [34] equation (3) p. 114 after substituting $\rho=m3/4\pi R^3$ (m is the mass of a spherical body of radius R) and $2\pi/\kappa=2\pi R$ is the wave length that corresponds to the circumference of the body at the equator. This suggests that κ can be supposed proportional to the inverse of R which gives as a condition for the validity of the rheological approximation:

$$\theta=\frac{\nu}{R\sqrt{gR}}\gg 1\Longrightarrow\frac{\nu}{R^3\sqrt{G\rho}}\gg 1 \tag{2.30}$$

where G is the universal constant of gravitation and $g=Gm/R^2$.

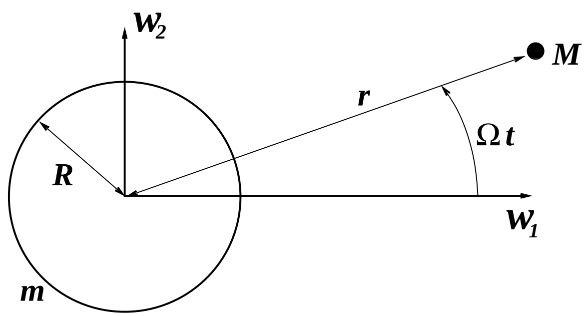

Figure 2.5: Two dimensional arrangement of an extended body of mass m and radius R centered at the origin and a point mass M with uniform circular motion given by $re^{i\Omega t}$. Positions are described by complex numbers $w = w_1 + iw_2$.

Tidal overdamped waves

In this section we want to consider a forcing term to the Stokes equation (2.18) that resembles the tidal forcing of a body upon another. A simplified planar situation is illustrated in Figure 2.5. The two dimensional gravitational potential due to the point mass M is given by $\frac{GM}{2\pi}\log|w - re^{i\Omega t}|$. If $R \ll r$ then in the region $|w| \le R$ the gravitational potential can be expanded as

$$\frac{Gm}{2\pi}\log\sqrt{|w - re^{i\Omega t}|^2} = \frac{Gm}{4\pi}\left\{\log r^2 - \frac{2}{r}\mathrm{Re}\,(e^{-i\Omega t}w) - \frac{1}{r^2}\mathrm{Re}\,(e^{-i\Omega t}w)^2 \cdots\right\}$$

The first term in this expression is constant and can be neglected. The second term gives rise to a constant rotating force that represents the force that the mass M exerts on a point body at the origin. Finally, the third term

$$H(w) = -\frac{Gm}{4\pi r^2}\mathrm{Re}\,(e^{-i\Omega t}w)^2 \tag{2.31}$$

gives rise to the tidal harmonic potential. Now, consider the conformal mapping $z \to w$

$$w = R\mathrm{e}^{ik\bar{z}} = R\mathrm{e}^{ikx+ky}, \quad \text{where} \quad k = \frac{1}{R}$$

This mapping sends the region $\{z : 0 \leq x < 2\pi R, y \leq 0\}$ to the punctured ball $\{|w| \leq R, w \neq 0\}$. Moreover the pull-back of the tidal potential

$$H(w(z)) = -\frac{GmR^2}{4\pi r^2}\mathrm{Re}\left[\mathrm{e}^{-i2\Omega t+i2kx+2ky}\right]$$

is a harmonic function periodic in time and in x.

The above reasoning suggests that we must consider the problem of finding the motion of a fluid occupying the region $\{(x,y) : y \leq \zeta(t,x)\}$ with the velocity field satisfying Stokes equations (2.18) with the time-dependent complex potential

$$V(t,x,y) = \rho g y - (\rho f R)\mathrm{e}^{-i2\Omega t+i2kx+2ky}. \tag{2.32}$$

In this case $\rho g y$ represents the self-gravitational potential, as in section 2.3.4, and the constant f measures the strength of the tidal potential. The relative magnitude of the several constants k, f, g, and Ω are suggested by the analogy with a three-dimensional two-body system as depicted in Figure 2.5

$$k = \frac{1}{R}, \quad f = \frac{GMR}{r^3}, \quad g = \frac{Gm}{R^2}, \quad \Omega^2 = \frac{G(M+m)}{r^3}, \tag{2.33}$$

where the last identity is the Kepler's third law. The tidal force is supposed to be much smaller then the self-gravitation force:

$$\frac{f}{g} = \frac{M}{m}\frac{R^3}{r^3} \ll 1 \tag{2.34}$$

As in the previous section Stokes equations (2.18) must be supplemented by the condition of the velocity being finite at $y = -\infty$ and by the free-stress boundary condition (2.19) at the interface. Again it is convenient to use the Helmholtz decomposition of the velocity field in equation (2.20) that leads to the linear equations (2.21) with a different

$$\frac{p}{\rho} = -\frac{\partial \varphi}{\partial t} - gy + (fR)\mathrm{e}^{-i2\Omega t+i2kx+2ky}.$$

In this case the time-dependent part of the potential energy suggests looking for a solution of the form

$$\varphi = A\mathrm{e}^{i2kx-i2\Omega t+2ky}, \quad \psi = B\mathrm{e}^{i2kx-i2\Omega t+\beta y},$$

which is equal to that in equation (2.22) with $\kappa = 2k$ and with a prescribed $\gamma = i2\Omega$. The complex amplitudes A and B are supposed to be small (A/R and B/R are supposed proportional to $f/g \ll 1$). Notice that $\Delta\varphi = 0$ and that $\partial_t \psi = \nu\Delta\psi$ and the boundary condition at $y = -\infty$ determine

$$\beta^2 = 4k^2 - \frac{i2\Omega}{\nu} \quad \text{with} \quad \mathrm{Re}\,\beta \geq 0.$$

Defining

$$\delta = \frac{\beta}{2k} \quad \text{and} \quad \alpha = \frac{\Omega}{2k^2\nu}$$

we obtain

$$\delta^2 = 1 - i\alpha \tag{2.35}$$

As in equation (2.24), the interface $y = \zeta(t,x)$ is given by

$$\zeta(t,x) = C\mathrm{e}^{i2kx-i2\Omega t} \quad \text{with} \quad C = \frac{k}{\Omega}(B + iA)$$

Now, the boundary conditions at the surface (2.19), up to first order on A, B, and f imply

$$\begin{aligned} 0 &= -\frac{p}{\rho} + 2\nu\frac{\partial v}{\partial y} = \frac{\partial\varphi}{\partial t} + \frac{V}{\rho} + 2\nu\left(\frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\psi}{\partial x\partial y}\right) \Longrightarrow \\ fR &= Cg + A4\nu k^2(2 - i\alpha) - iB24\nu k^2\delta \\ 0 &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} \Longrightarrow \\ 0 &= i2A + (2 - i\alpha)B \end{aligned} \tag{2.36}$$

This last equation and (2.35) imply

$$iB = 4k\nu C, \qquad A = 2k\nu(2 - i\alpha)C$$

and substituting these relations in equation (2.36) we obtain

$$fR = C\left(g + 8\nu^2k^3(2 - i\alpha)^2 - 32\nu^2k^3\delta\right) \tag{2.37}$$

The right hand side of equation (2.37) can be further simplified using the hypothesis (2.28) with $\kappa = 2k$, namely

$$\theta = \frac{4\nu k^2}{\sqrt{2gk}} \gg 1 \tag{2.38}$$

Then

$$\alpha = \frac{\Omega}{2k^2\nu} = \frac{1}{\theta}\sqrt{\frac{2\Omega^2}{gk}} = \frac{1}{\theta}\sqrt{2\frac{(M+m)}{m}\frac{R^3}{r^3}} = \frac{\sqrt{2}}{\theta}\sqrt{\frac{f}{g}+\frac{R^3}{r^3}} \ll \frac{1}{\theta}$$

where we used equations (2.33), (2.34) and $R/r \ll 1$. Therefore $\alpha \ll 1$ and from equation (2.35) we get

$$\delta = \sqrt{1-i\alpha} = 1 - i\frac{\alpha}{2} + \frac{\alpha^2}{8} + \cdots$$

Substituting δ into equation (2.37) and keeping only the leading order terms we obtain:

$$\frac{C}{R} = \frac{f}{g - 8\nu k\Omega i} = \frac{f}{\sqrt{g^2 + (8\nu k\Omega)^2}}\mathrm{e}^{i\sigma} = \frac{\frac{f}{g}\gamma}{\sqrt{\gamma^2 + (2\Omega)^2}}\mathrm{e}^{i\sigma} \tag{2.39}$$

where

$$\gamma = \frac{g}{4\nu k} \tag{2.40}$$

is the constant in equation (2.29) for the unforced problem with $\kappa = 2k$ and

$$\sigma = \arctan\left(\frac{8\nu k\Omega}{g}\right) = \arctan\left(\frac{2\Omega}{\gamma}\right) \in (0, \pi/2) \tag{2.41}$$

is the so called "tidal lag". Equation (2.41) and (2.39) are similar to equations (8) and (9) obtained by Ferraz-Mello in [34]. So, the motion of the surface is given by

$$\frac{\zeta}{R} = \frac{f}{\sqrt{g^2 + (8\nu k\Omega)^2}}\mathrm{e}^{i2(kx - \Omega t + \sigma/2)} \tag{2.42}$$

It has a phase-lag $\sigma/2$ with respect to the forcing term phase, see Figure 2.6.

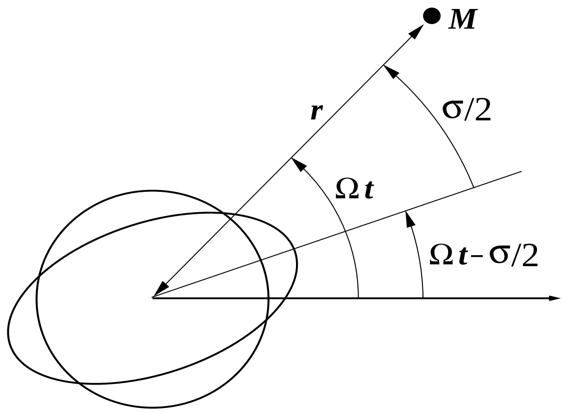

Figure 2.6: Tidal phase-lag σ with respect to the phase of the tidal force.

Finally, we remark that the dynamic behavior of the surface in equation (2.39) is the same as that of a one-dimensional oscillator with negligible mass $\mu = 0$, damping coefficient $4\nu k$, and elastic constant g:

$$0\ddot{\psi} + 4\nu k\dot{\psi} + g\psi = f\mathrm{e}^{-i2\Omega t}$$

If $f = 0$ then $\dot{\psi} = -\gamma\psi$ where $\gamma = g/4\nu k$ is the coefficient of free decay in equation (2.40). If $f \neq 0$ then the periodic forced solution is

$$\psi = \frac{f}{\sqrt{g^2 + (8\nu k\Omega)^2}}\mathrm{e}^{i(-2\Omega t+\sigma)}$$

that is exactly the solution in equation (2.42) with $\sigma = \arctan\frac{2\Omega}{\gamma}$ as in equation (2.41).

2.4 Solids

The motion of a solid is almost always studied in the Lagrangian setting. As before, let $\mathscr{B} \subset \mathbb{R}^3$ be the "reference configuration" of the body and $\phi(t,\cdot)$:

$\mathscr{B} \to \mathbb{R}^3$ be the configuration of the body at time t. The equation of motion is the usual Newton's law: $\rho(x)\ddot{\phi}(t,x) = F$, where ρ is the mass density in the reference configuration and F is the force per unit of volume that acts upon the body particle. If the force F is conservative then F can be written as the derivative of a potential function V with respect to the configurations. If F has an additional dissipative term that depends linearly on the velocities then F can be written as the sum of the derivative of V plus the derivative with respect to the velocities of a Rayleigh dissipation function. We will only consider solid materials for which the corresponding F satisfy these hypotheses. Indeed, these are the most studied type of materials.

The state space of continuum mechanics is infinite dimensional: the particles of the continuum are labeled by their positions $x \in \mathscr{B}$. As a result the equations of motion for a solid, are partial differential equations. In this book we do not use these partial differential equations but only a particular finite-dimensional "approximation" to it obtained from the Lagrangian and the dissipation functions. In order to motivate the results in the following sections it is convenient to briefly explain how this approximation is obtained.

In principle a configuration $\phi(t,\cdot) : \mathscr{B} \longrightarrow \mathbb{R}^3$ can be given by an arbitrary diffeomorphism onto its image. The set of all these diffeomorphisms endowed with the composition operation form a group denoted by $\mathscr{C}$. The crucial step, which allows the reduction from infinite to finite dimensions, is to constrain the configurations of the body to be of the form:

$$\phi(t,x) = G(t)x, \tag{2.43}$$

where $G(t)$ is an invertible matrix with positive determinant. Notice that the group GL_+ of orientation preserving linear transformations $x \to G(t)x$ is a subgroup of $\mathscr{C}$. Moreover, GL_+ is a nine-dimensional manifold contained in $\mathscr{C}$. We can further restrict G to be in the group $SL_+ \subset GL_+$ of volume preserving transformations, $\det G = 1$. The group SL_+ is an eight-dimensional manifold contained in $\mathscr{C}$. The equations of motion for a solid for which the possible configurations are restricted to a finite-dimensional manifold $\mathscr{M} \subset \mathscr{C}$ are obtained using the variational principle plus the D'Alembert's principle as it is usually done in the mechanics of point particles. The dissipation is introduced by means of a Rayleigh dissipation function.

The assumption (2.43) was first proposed by Dirichlet. It was used by Dirichlet and Riemann to study the equilibrium shapes of a rotating isolated and incompressible fluid under self-gravity (see, for example, chapter 4 of [17]). The hypothesis (2.43) in the context of solid mechanics is often called "pseudo-rigid body" assumption. The pseudo-rigid body approach raises the question: Are the main dynamical properties of the infinite-degree-of-freedom system preserved under the pseudo-rigid body constraint (2.43)? There are at least two works, [56] and [57], where these questions are analyzed from a mathematical point of view. In these papers it is proved the relationship between *fine* and *coarse* theories, where the former represents a "complete" theory and the later an approximation in which persist the "mean" characteristics of the fine one. In the second paper it is shown that the pseudo-rigid body is a coarse theory for the continuum theory of solids. In this sense, we suppose that results obtained from (2.43) represent a good approximation to the behavior in the continuous problem. Another comparison between the pseudo-rigid body approach and the *Cosserat point* theory is presented in [59].

2.4.1 The Lagrangian and the dissipation functions.

In this section we informally present a particular variational formulation of continuum mechanics that is suitable for the study of small deformations of celestial bodies. Our main references on this classical subject are the two papers [2] and [3] by Levi and Baillieul.

A family of configurations of the body $t \to \phi(t,x)$ can always be written as $\phi(t,x) = Y(t)u(t,x)$, where $Y \in \mathrm{SO}(3)$ represents a "rotation" of the body and $u : \mathscr{B} \longrightarrow \mathbb{R}^3$ the particles positions relative to a "body frame" at time t. Notice that, at this point, the family of rotations $t \to Y(t)$ is supposed to be obtained from $\phi(t,x)$, which implies $u(t,x) = Y^T(t)\phi(t,x)$. In [3], the space of all $u : \mathscr{B} \longrightarrow \mathbb{R}^3$ is denoted by $\mathscr{C}(\mathscr{B},\mathbb{R}^3)$ and is the set of diffeomorphisms from $\mathscr{B}$ onto their images containing the identity in its interior. Since here we are only interested on isolated bodies the center of mass of the body can be considered at rest. Our configuration space is $\mathrm{SO}(3) \times \mathscr{C}$.

The Lagrangian function is given by $\mathscr{L}(Y,\dot{Y},u,u_t) := T(Y,\dot{Y},u,u_t) - V(u)$, where $V : \mathscr{C} \longrightarrow \mathbb{R}$ is the potential energy of the deformed body and T is the kinetic energy of the body. The potential V does not depend on Y because we are neglecting interactions of the body with external agencies. So, V is invariant under rotations. The potential V is obtained from the addition of an elastic potential term V_{el} and a gravitational term V_{gr}.

The dissipation function $\mathscr{D}$ is supposed to depend only on u and its time derivative u_t and neither on the body attitude Y nor on its time derivative $\dot{Y}$. The usual derivation of the equations of motion from the Lagrangian and the dissipation functions is given, for instance, in [2]. Our goal in the following is to write the four functions: T, V_{el}, V_g, and $\mathscr{D}$.

The strain tensor, the small deformation hypothesis, and the elastic energy

The definition of elastic energy requires a preliminary definition of the "strain" associated to a deformation of the body. The definition of strain will be done in detail. So, consider a smooth curve parametrized by arclength in the reference configuration of the body, $s \to \gamma_0(s) \in \mathscr{B}$, with $\|\gamma_0'(s)\| = 1$. Under a change in the body configuration the curve is transformed to $s \to \phi(t,\gamma(s)) = Y(t)\gamma_t(s)$, where $\gamma_t(s) = u(t,\gamma_0(s))$. The strain associated to $\phi(t,\cdot)$ at the point $\gamma_0(s) \in \mathscr{B}$ and in the direction $\gamma_0'(s)$ is: $\|Y(t)\gamma_t'(s)\| - 1 = \|\gamma_t'(s)\| - 1$. In order to write the strain in terms of the derivative of u it is convenient to define the deformation matrix Q:

$$du = (\,\mathrm{Id} + Q)dx, \quad \text{with} \quad Q_{ij} = \frac{\partial w_i}{\partial x_j},$$

where

$$w(x) = u(x) - x \tag{2.44}$$

defines a vector-field on $\mathscr{B}$ that represents the body deformation. Then

$$du^T du = dx^T(\,\mathrm{Id} + Q^T)(\,\mathrm{Id} + Q)dx = dx^T dx + 2dx^T\left(\frac{Q^T + Q}{2} + \frac{Q^T Q}{2}\right)dx$$

Notice that $\|\gamma_t'(s)\|^2 = \|du[\gamma_0'(s)]\|^2$. Suppose that the deformations are small, that means $\|Q\| \ll 1$, then

$$du^T du \approx dx^T dx + 2dx^T\frac{Q^T + Q}{2}dx.$$

This leads to the definition of the symmetric (linear) strain matrix

$$\mathscr{E} = \frac{Q^T + Q}{2}, \tag{2.45}$$

that implies

$$\sqrt{\|\gamma_t'(s)\|^2} - 1 \approx \sqrt{1 + 2\gamma_0'(s) \cdot \mathscr{E}\gamma_0'(s)} - 1 \approx \gamma_0'(s) \cdot \mathscr{E}\,\gamma_0'(s)$$

The symmetric tensor defined by $\mathscr{E}$, and denoted by the same letter, is the strain tensor. If the material of the body is supposed incompressible, $\det(du) = 1$, then $\operatorname{Tr}\mathscr{E} = \sum \partial_{x_i} w_i = 0$.

The strain tensor at a point $x \in \mathscr{B}$ describes locally the magnitude and the geometry of the deformation. With a convenient orthogonal transformation the strain tensor at x can be diagonalized $\mathscr{E}_{ij} = \lambda_i \delta_{ij}$, with $\lambda_1 \geq \lambda_2 \geq \lambda_3$. The eigenvalues λ_i give the strain along the principal directions of the strain tensor. The change of volume of the original element is given by $\det(du) - 1 \approx \operatorname{Tr} Q = \operatorname{Tr}\mathscr{E} = \lambda_1 + \lambda_2 + \lambda_3 = \Lambda$. Therefore we can write $\mathscr{E} = (\Lambda/3)\,\mathrm{Id} + D$ where D is the deviatoric strain and $\operatorname{Tr} D = 0$. Finally there is an orthogonal transformation that transforms D to a matrix with all diagonal elements equal to zero. In this last reference frame the matrix $\mathscr{E}$ can be written as:

$$\mathscr{E} = \begin{pmatrix} \frac{\Lambda}{3} & 0 & 0 \\ 0 & \frac{\Lambda}{3} & 0 \\ 0 & 0 & \frac{\Lambda}{3} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & \theta_3 & \theta_2 \\ \theta_3 & 0 & \theta_1 \\ \theta_2 & \theta_1 & 0 \end{pmatrix}}_{D}.$$

In this equation the isotropic strain term $(\Lambda/3)\,\mathrm{Id}$ corresponds to a deformation in which the volume of a material element is changed but not its shape. We recall that the stress-tensor $\sigma(t,x)$ at time t and at the point $x \in \mathscr{B}$ is a symmetric bilinear form on the tangent space at x such that σ_{ij} gives the i-component of the force per unit area exerted across a plane surface element normal to the j-direction. An isotropic deformation is caused by an isotropic stress $p\,\mathrm{Id}$ (hydrostatic pressure). If the material of the body is isotropic and elastic, then the small deformation hypothesis implies that Hooke's law is valid. So, there exists an elastic constant K (bulk modulus of compression or modulus of compression) such that

$$p = -K\Lambda, \qquad \text{or} \qquad V\frac{dp}{dV} = -K$$

where V is the volume of a material element. The elastic energy associated to a pure compression is $E_{comp} = K\Lambda^2/2$. In the decomposition of $\mathscr{E}$, the

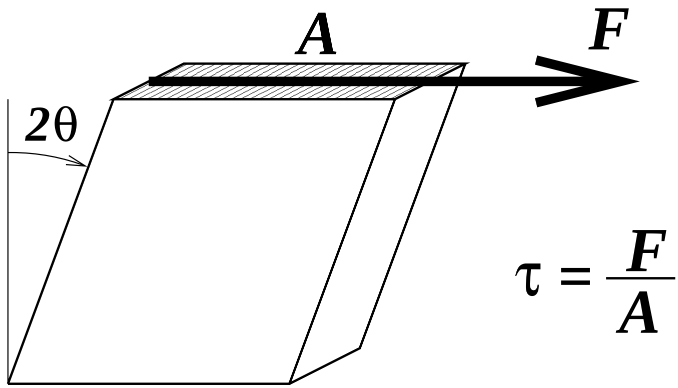

Figure 2.7: A pure-shear deformation that illustrates how μ can be experimentally measured.

strain term

$$D_3 = \begin{pmatrix} 0 & \theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

corresponds to a deformation in which the shape of a material element is changed but not its volume. For instance, the deformation

$$du = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2\theta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} \Longrightarrow \mathscr{E} = \begin{pmatrix} 0 & \theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

is caused by a pure shear stress τ as shown in Figure 2.7. For a pure shear deformation, Hooke's law implies the existence of an elastic constant μ (shear modulus or modulus of rigidity) such that $\tau = 2\mu\theta$. The elastic energy associated to a pure shear deformation is $E_{shear} = \mu(2\theta)^2/2$. Therefore if the strain matrix $\mathscr{E}$ is decomposed into a pure-compression term and three pure-shear terms and the energy of the pure-deformations are added we obtain the elastic energy

$$E_{el}(\mathscr{E}) = \frac{K}{2}\Lambda^2 + 2\mu(\theta_1^2 + \theta_2^2 + \theta_3^2) = \mu\,\mathrm{Tr}\,(\mathscr{E}^2) + \left(\frac{K}{2} - \frac{\mu}{3}\right)(\,\mathrm{Tr}\,\mathscr{E})^2 \tag{2.46}$$

This last expression is invariant under rotations of the strain tensor and is taken as the definition of the strain energy. Therefore integrating over the reference configuration we obtain the elastic energy of the body

$$V_{el} = \int_{\mathscr{B}} \mu \operatorname{Tr}(\mathscr{E}^2) + \left(\frac{K}{2} - \frac{\mu}{3}\right) \operatorname{Tr}(\mathscr{E})^2 dx, \tag{2.47}$$

Notice that the moduli K and μ may depend on (t,x) due to temperature variations or lack of homogeneity of the body material.

Differentiating the elastic energy given in equation (2.46) we obtain a linear relation between the stress and the strain tensors

$$\sigma_{ij} = K\Lambda\delta_{ij} + 2\mu D_{ij} = K\Lambda\delta_{ij} + 2\mu\left(\mathscr{E}_{ij} - \frac{\Lambda}{3}\delta_{ij}\right). \tag{2.48}$$

Gravitational energy

Since we are interested in isolated bodies, the gravitational energy is given only by the self-gravitational potential

$$V_g(u) = -\frac{G}{2}\int_{\mathscr{B}}\int_{\mathscr{B}} \frac{\rho(x)\rho(y)}{\|u(x)-u(y)\|} dxdy. \tag{2.49}$$

This expression can be simplified by means of the substitution $u(x) = x + w(x)$ and further use of the hypothesis of small deformations ($||w(x)||$ is small).

Rayleigh dissipation function

For an isotropic material the generalization of the usual linear damping for elastic springs gives the local Rayleigh dissipation function (compare to equation (2.46) for the elastic energy)[50]:

$$\mathscr{D}_{loc}(\dot{\mathscr{E}}) = \eta \operatorname{Tr}(\dot{\mathscr{E}}^2) + \left(\frac{\zeta}{2} - \frac{\eta}{3}\right) \operatorname{Tr}(\dot{\mathscr{E}})^2, \tag{2.50}$$

where η and ζ are the viscous shear modulus and the viscous compression modulus of the material. The dissipative stress tensor σ'_{ij} associated to this dissipative function is (compare to the elastic stress in equation (2.48))

$$\sigma'_{ij} = \frac{\partial \mathscr{D}_{loc}}{\partial \dot{\mathscr{E}}_{ij}} = \zeta(\operatorname{Tr}\dot{\mathscr{E}})\delta_{ij} + 2\eta\left(\dot{\mathscr{E}}_{ij} - \frac{\operatorname{Tr}\dot{\mathscr{E}}}{3}\delta_{ij}\right). \tag{2.51}$$

The components of σ'_{ij} are the same as those in the dissipative term of the stress tensor of a fluid in equation (2.9). The viscosity coefficients in equations (2.51) and (2.9) are denoted with the same letter because they have the same physical nature. A solid material satisfying the constitutive relations given in equations (2.48)) and (2.51) with the total stress given by $\sigma_{ij} + \sigma'_{ij}$ is called a "Maxwell-Voigt material" [26].

Integrating over the reference configuration we obtain the dissipation function of the body

$$\mathscr{D} = \int_{\mathscr{B}} \eta \, \mathrm{Tr}\,(\dot{\mathscr{E}}^2) + \left(\frac{\zeta}{2} - \frac{\eta}{3}\right) \mathrm{Tr}\,(\dot{\mathscr{E}})^2 dx, \tag{2.52}$$

The kinetic energy

The kinetic energy of the body at time t is given by

$$T(Y, \dot{Y}, u, u_t) := \frac{1}{2} \int_{\mathscr{B}} \left\| \frac{\partial}{\partial t} \phi(t,x) \right\|^2 \rho(x) dx = \frac{1}{2} \int_{\mathscr{B}} \|\Omega u + u_t\|^2 \rho(x) dx, \tag{2.53}$$

where $\Omega = Y^T \dot{Y}$ is the skew-symmetric matrix representing the angular velocity in the body frame. This expression can be simplified by means of the substitution $u(x) = x + w(x)$ and further use of the hypothesis of small deformations ($||w(x)||$ is small).

2.4.2 The equations of motion for a solid with dissipation

The equations of motion for a solid that satisfies the Maxwell-Voigt material hypothesis can be obtained from the Lagrangian and the dissipation functions as

$$\frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \dot{Q}} - \frac{\partial \mathscr{L}}{\partial Q} + \frac{\partial \mathscr{D}}{\partial \dot{Q}} = 0$$

where the generalized coordinate Q is either a component of the rotation matrix Y or "a component" of the deformation w. In order to obtain the equations of motion it is still necessary to impose boundary conditions on the deformation, namely, that $\sigma \cdot n(x) = 0$, $\forall x \in \partial \mathscr{B}$, where $n(x)$ is the exterior normal vector at the point x on the boundary $\partial \mathscr{B}$ of the body. The derivation of the equations of motion is given in [2]. The equation for the deformation is:

$$\rho\left(u_{tt}+2\omega\times u_t+\dot{\omega}\times u+\omega\times(\omega\times u)\right)=-\frac{\delta V}{\delta u}-\frac{\delta\mathscr{D}}{\delta u_t}, \tag{2.54}$$

where $V=V_{el}+V_g$,

$$\frac{\delta V_{el}}{\delta u} = -\left(K+\frac{\mu}{3}\right)\nabla(\operatorname{div}u)-\mu\Delta u, \tag{2.55}$$

$$\frac{\delta\mathscr{D}}{\delta u_t} = -\left(\zeta+\frac{\eta}{3}\right)\nabla(\operatorname{div}u_t)-\eta\Delta u_t, \tag{2.56}$$

$$\frac{\delta V_g}{\delta u} = G\rho(x)\int_{\mathscr{B}}\frac{u(x)-u(y)}{\|u(x)-u(y)\|^3}\rho(y)dy, \tag{2.57}$$

and $\omega=S(\Omega)$ is the vector given by the isomorphism S that associates to each skew-symmetric matrix Ω, a vector $\omega\in\mathbb{R}^3$,

$$S\left[\begin{pmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{pmatrix}\right]=\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}. \tag{2.58}$$

Equation (2.54) can be simplified by means of the substitution $u(x)=x+w(x)$ and further use of the hypothesis of small deformations ($||w(x)||$ is small). In the left hand side of equations (2.55) and (2.56) we can change u for w without modifying the result. The gravitational term in equation (2.57) can be written up to terms of order $\|w\|^2$ as

$$\frac{\delta V_g}{\delta u}=G\rho(x)\int_{\mathscr{B}}\frac{x-y}{\|x-y\|^3}\rho(y)dy \tag{2.59}$$

$$-3G\rho(x)\int_{\mathscr{B}}\frac{x-y}{\|x-y\|^3}\rho(y)\left[\frac{x-y}{\|x-y\|}\cdot\frac{w(x)-w(y)}{\|x-y\|}\right]dy \tag{2.60}$$

Finally, the left hand side of equation (2.54) becomes

$$\rho\left(\dot{\omega}\times x+\omega\times(\omega\times x)\right)+\rho\left(w_{tt}+2\omega\times w_t+\dot{\omega}\times w+\omega\times(\omega\times w)\right) \tag{2.61}$$

Using the above equations we can rewrite equation (2.54) up to terms of order $\|w\|^2$ and from it we immediately realize that the small deformation hypothesis is neither compatible with the gravitational force nor with rotations, unless the body under consideration is small. More precisely, suppose that $\mathscr{B}$ is a ball. Consider the equation for the relative equilibria of the rotating body, namely, equation (2.54) with $\partial_t u = \partial_t w = 0$ and $\dot{\omega} = 0$:

$$\omega \times (\omega \times x) = G \int_{\mathscr{B}} \frac{x-y}{\|x-y\|^3} \rho(y) dy \tag{2.62}$$

$$+ \frac{1}{\rho(x)} \left(K + \frac{\mu}{3} \right) \nabla(\operatorname{div} w) + \frac{\mu}{\rho(x)} \Delta w \tag{2.63}$$

Notice that if the radius of the reference configuration $\mathscr{B}$ is large then the terms in line (2.62) are not small while those in line (2.63) are. So, either we must abandon the small deformation hypothesis or the constants K or μ in line (2.63) must be supposed big enough to compensate the smallness of w. Similarly to what happens for fluids (see discussion below equation (2.9)), a change of volume of a solid domain often requires variations of normal stresses that are much larger than the usual shear stresses. So, the mathematical hypothesis, which in many cases is physically reasonable, is that

$$K \to \infty, \quad \operatorname{div} w \to 0, \quad \text{such that} \quad K \operatorname{div} w = -p \quad \text{and} \quad \mu \operatorname{div} w \to 0,$$

where p, the pressure, is a smooth function that is similar to the hydrodynamic pressure. The mathematical formalization of the above limit is more easily done by means of the imposition of the constraint $\operatorname{div} w = 0$, as done in the next section.

2.4.3 Incompressible solid

If the body material is supposed incompressible, then $\operatorname{Tr}(\mathscr{E}) = \operatorname{div} w = 0$ and the elastic energy and the dissipation function of the body become

$$V_{el,inc} = \int_{\mathscr{B}} \mu \operatorname{Tr}(\mathscr{E}^2) dx, \tag{2.64}$$

$$\mathscr{D}_{inc} = \int_{\mathscr{B}} \eta \operatorname{Tr}(\dot{\mathscr{E}}^2) dx, \tag{2.65}$$

In this case the equations of motion are obtained using $V = V_{el,inc} + V_g$ and $\mathscr{D}_{inc}$ plus the constraint $\operatorname{Tr} \mathscr{E} = 0$. In order to realize the constraint we introduce the Lagrange multiplier $p(x,t)$ (pressure) and extend the definition of $V_{el,inc}$ to

$$V_{el,p} = \int_{\mathscr{B}} \mu \operatorname{Tr}(\mathscr{E}^2) - p \operatorname{Tr} \mathscr{E} dx$$

This implies

$$\frac{\delta V_{el,p}}{\delta u} = \nabla p - \mu \Delta u,$$

where the pressure p represents the isotropic part of the elastic stress tensor (see article 176 of [52]). In this case the elastic stress tensor (2.48) becomes

$$\sigma_{ij} = -p\delta_{ij} + 2\mu \mathscr{E}_{ij}$$

and the dissipative stress tensor (2.51) becomes

$$\sigma'_{ij} = 2\eta \dot{\mathscr{E}}_{ij}$$

The solution to the equation for the relative equilibria for an incompressible rotating ball of constant density and for small angular velocity is given in [52] articles 176 and 177. In the following chapters we will not use the partial differential equations for the motion of a solid but only the Lagrangian and the dissipation functions associated to them.

Chapter 3

The elastic and anelastic tides

3.1 Introduction

In this lecture, we consider a system of two close bodies and the consequences for the energy and angular momentum of the system of the tides raised in one of the bodies by the gravitational attraction of the other. One of these bodies, specifically called *body*, has mass m, mean equatorial radius R_e, rotation angular velocity $\Omega = |\mathbf{\Omega}|$; the other, specifically called *companion*, has mass M, is at a distance $r(t)$ from the body, moves around it in a Keplerian orbit in the equatorial plane of the body and is responsible for the gravitational force that is tidally deforming the body. No hypotheses are done about the relative value of their masses and to know the consequences of the tides raised in the companion, it is enough to invert the role played by the two bodies. In general, it is necessary to consider the two possibilities and add them to get the complete result.

The deformation (tide) of a celestial body due to the gravitational attraction of a companion is usually divided into two components studied separately. The main component, in size, is the *elastic* tide, the other, smaller, but responsible for the dissipation of energy in the system and for the torques acting on the body is the *anelastic* tide.

3.2 The elastic tide

The elastic tide, also called static tide, is the deformation of the body in the limit case where it does not offer any resistance to the deformation. The body behaves as a perfect fluid and takes instantaneously the shape corresponding to the equilibrium of the forces acting on it. The total force acting on the points on the surface of the body must be, in each point, perpendicular to the surface. The only forces acting on those points are the gravitational attraction of the two bodies, and, if the body is rotating, the inertial (centrifugal) force due to the rotation, i.e.

$$-\nabla U_{\text{self}} - \nabla U_{\text{tid}} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{d}) \tag{3.1}$$

(per unit mass)[1]. **d** is the position vector of the considered surface point, U_{self} is the potential of the gravitational forces of the body, U_{tid} is the tidal potential, i.e. the potential of the forces generated in the interior of the body by the gravitational attraction of the companion referred to the center of the body, that is

$$U_{\text{tid}} = \frac{GM}{R} \sum_{n=2}^{\infty} \left(\frac{d}{r}\right)^n P_n(\cos\Psi) \tag{3.2}$$

where $d = |\mathbf{d}|$ and Ψ is the angle formed by the radius vector of the considered point in the surface of the body and the position vector of the companion. P_n are the Legendre polynomials and G is the gravitational constant.

In general, we neglect the terms with $n > 2$ which is equivalent to adopt one homogeneous ellipsoid. This ellipsoid is characterized by the relationships between its three axes: $a > b > c$ (c is directed along the rotation axis and a is directed towards the companion). They are the *polar oblateness*

$$\varepsilon_z = 1 - \frac{c}{R_e} \tag{3.3}$$

where $R_e = \sqrt{ab}$ is the mean equatorial radius of the body, and the *equatorial prolateness*

$$\varepsilon_\rho = \frac{a}{b} - 1. \tag{3.4}$$

This triaxial ellipsoid is often referred to as a Roche ellipsoid, but more rigorously, this denomination refers to the equilibrium triaxial ellipsoid that

[1] The minus signs mean that we are adopting the exact Physics convention: force is equal to minus the gradient of the potential

results when the rotation of the body is synchronized with the orbital motion of the companion. When the body is not rotating, the ellipsoid becomes an ellipsoid of revolution (i.e. a spheroid) with a as axis of revolution. In this case, the name Jeans ellipsoid is often used. The study of the figures of revolution dates from the nineteenth century and is found in many classical references, e.g. [17, 72].

In the more general case in which the ellipsoid is assumed to be composed of homogeneous ellipsoidal layers, the equatorial prolateness is

$$\varepsilon_\rho = \frac{15}{4}\mathscr{H}_n\left(\frac{M}{m}\right)\left(\frac{R_e}{r}\right)^3 \tag{3.5}$$

where $\mathscr{H}_n$ is the reduction factor of the prolateness due to the non-uniform density distribution (see [38]). Since this quantity comes from the solution of the Clairaut equation for the actual density distribution of the body, by analogy with similar quantities introduced by Love, we call it the Clairaut number. If the distribution is such that the moment of inertia of the body with respect to the rotation axis, C, is not smaller than $0.2mR_e^2$, we may use the approximation [70]

$$\mathscr{H}_n \simeq 2\left[1+\left(\frac{5}{2}-\frac{15C}{4mR_e^2}\right)^2\right]^{-1}. \tag{3.6}$$

If the body is homogeneous, $C = 0.4mR_e^2$ and, therefore, $\mathscr{H}_n = 1$. In this case, ε_ρ is the prolateness of the Jeans ellipsoid.

On its turn, the polar oblateness is

$$\varepsilon_z = \mathscr{H}_n\frac{5cR_e^2\Omega^2}{4mG} + \frac{1}{2}\varepsilon_\rho. \tag{3.7}$$

The main term in ε_z is the oblateness of a rotating fluid spheroid [70]; in the homogeneous case, it is the oblateness of the Maclaurin spheroid. We note that the polar oblateness is also affected by the tidal deformation. Indeed, if the body is stretched along the axis a, the conservation of volume forces it to shrink in the directions orthogonal to that axis, thus decreasing both b and c and increasing the polar oblateness of the body.

One may keep in mind that when the companion is very close to the body, terms with $n \geq 3$ must be considered and the symmetry with respect to the orthogonal plane through the origin is lost. The resulting figure can no longer be represented by an ellipsoid. It becomes an ovoid (also said to be pear-shaped).

3.3 Tidal evolution associated with the elastic tide

The deformation of the body will modify the gravitational potential felt by bodies in its neighborhood. The simplest form of this potential is obtained when we use a reference system whose axes are the principal axes of inertia of the ellipsoid. It is

$$V = -\frac{Gm}{r^*} - \frac{G}{2r^{*3}}(A+B+C) + \frac{3G}{2r^{*5}}(AX^2+BY^2+CZ^2) + \cdots \tag{3.8}$$

where A,B,C are the moments of inertia w.r.t. to the three axes, X,Y,Z are the coordinates of one generic point and $r^* = \sqrt{X^2+Y^2+Z^2}$.

$$\begin{aligned} X &= r^* \sin\theta^* \cos(\varphi^* - \varpi - v) \\ Y &= r^* \sin\theta^* \sin(\varphi^* - \varpi - v) \\ Z &= r^* \cos\theta^* \end{aligned} \tag{3.9}$$

where θ^*, φ^* are the co-latitude and longitude of the point in a system of reference whose fundamental plane lies in the equator of the body, but whose axes are fixed (i.e., not rotating with the body), and $\varpi + v$ is the true longitude of the companion.

The acceleration of the considered point is minus the gradient of V. In a right-handed orthogonal set of unit vectors along the positive direction of the increments of $(r^*, \theta^*, \varphi^*)$, the components of the acceleration are

$$a_1 = -\frac{\partial V}{\partial r^*} = -\frac{Gm}{r^{*2}} - \frac{3G}{2r^{*4}}(A+B+C) + \frac{9G}{2r^{*6}}(AX^2+BY^2+CZ^2)$$

$$a_2 = -\frac{1}{r^*}\frac{\partial V}{\partial \theta^*} = -\frac{3G}{r^{*6}}(AX^2\cot\theta^* + BY^2\cot\theta^* - CZ^2\tan\theta^*)$$

$$a_3 = -\frac{1}{r^*\sin\theta^*}\frac{\partial V}{\partial \varphi^*} = \frac{3G}{r^{*6}\sin\theta^*}(A-B)XY$$

In particular, the force acting on the companion, whose coordinates are $X = r, Y = 0, Z = 0$ ($\theta^* = \frac{\pi}{2}$, $\varphi^* = \varpi + v$) is

$$\begin{array}{rcl} F_1 & = & -\frac{GmM}{r^2} - \frac{3GM}{2r^4}(A+B+C) + \frac{9GMA}{2r^4} \\ F_2 & = & 0 \\ F_3 & = & 0 \end{array} \qquad (3.10)$$

The force is radial, its torque is equal to zero and, as a consequence, the rotation of the body is not affected by the elastic tide. There is no exchange of angular momentum between the rotation of the body and the orbiting companion. The rate of work done (power) by the elastic tide force is $\mathbf{F}\cdot\mathbf{v}$ where $\mathbf{v}$ is the velocity of the companion. The sequence $\dot{W} = \mathbf{F}\cdot\mathbf{v} = \mathscr{F}(r)\mathbf{r}\cdot\mathbf{v} = \frac{1}{2}\mathscr{F}(r)d(\mathbf{r}^2)/dt = \frac{1}{2}\mathscr{F}(r)d(r^2)/dt = \mathscr{F}(r)rdr/dt$ shows that the work is an exact differential and, therefore, the total mechanical energy of the system remains constant in a cycle. There is no dissipation due to the elastic tide. Another important consequence is that the variation of the eccentricity, which is a function of the variations of the energy and the angular momentum also averages to zero in an orbit. The only effects not averaged to zero are the precessions of the pericenter and of the longitude at the epoch (the third Kepler law needs a correction).

3.4 The anelastic tide

The elastic tide is the bulk tidal deformation of an inviscid body under the gravitational attraction of one close companion. The anelastic tide is the component of the tidal deformation expected because of the viscoelasticity of the real body. It was first studied by G.H.Darwin [23]. The theory developed by Darwin considers the potential of the elastic tide given in the previous section, assumes that the companion has a Keplerian elliptical motion, and expands the potential in a Fourier series of the two variable angles involved: the mean longitude of the companion and the rotation angle of the body[2]. The delay due to the viscoelasticity was introduced in these equations, by Darwin, by hand. He assumed that the action of every term is

[2] In the actual calculation, it is necessary to keep apart the coordinates of the companion responsible for the tidal deformation of the body, r, φ, θ, and the coordinates of the generic point where the potential is being computed: X, Y, Z (or r^*, φ^*, θ^*). Even if we are interested in knowing the forces acting on the companion and if the two points will be later identified, the gradient of the potential giving the force involves only the derivatives with respect to the

delayed, an operation that introduces lags in the trigonometric arguments of each term of the series. Darwin assumed that these lags are small and proportional to the frequency of the argument. One second hypothesis adopted by Darwin is that the amplitudes of the terms in the series are attenuated by a coefficient proportional to the cosine of the corresponding lag. However, since the lags were assumed small, their cosines are very close to 1, and these factors were neglected in many of the versions of Darwin's theory elaborated during the XX-th century.

The general procedure is then to expand to first order in the lags. An original term whose argument is Φ_i is changed into a similar term with argument $(\Phi_i - \varepsilon_i)$ and the trigonometric function is expanded as

$$\cos(\Phi_i - \varepsilon_i) = \cos\Phi_i + \varepsilon_i \sin\Phi_i. \tag{3.11}$$

Thus, each term is separated into two parts. The first part reproduces back the term of the potential of the elastic tide. The second part, proportional to the lag, is the anelastic tide. One elementary consequence never stressed because unimportant up to the recent discussions involving the shape of the anelastic tide in rheophysical theories, is that the maximum of the anelastic tide does not coincide with the maximum of the elastic tide. Indeed, because of the derivative done to obtain the first Taylor approximation, the arguments corresponding to maxima and minima of one component of the elastic tide have zero derivative and will correspond to zeros of the corresponding component in the anelastic tide. To clearly illustrate this fact, it is enough to write the above expression as

$$\cos\Phi_i + \varepsilon_i \cos(\Phi_i - 90^\circ).$$

This means that the main components of the elastic and inelastic tide, the semi-diurnal tides of frequency $\nu = 2\Omega - 2n$, will have their major axes displaced by 45° one w.r.t. the other [3].

Darwin theory was modified by several authors. The most important modification, which became popular because of its simplicity and widely adopted during the second half of the XX-th century was the introduction

coordinates r^*, φ^*, θ^*, and does not involve the coordinates r, φ, θ of the companion responsible for the tidal deformation. The identification of the two points can only be done after the calculation of the derivatives.

[3]The maximum height of the elastic and inelastic tides will occur at the points $\widehat{\varphi}$ where the arguments of the cosine are $2\widehat{\varphi} - 2\varphi = 0$ and $2\widehat{\varphi} - 2\varphi - 90^\circ = 0$, respectively.

by McDonald of a constant geometric lag [53]. In that theory, the whole ellipsoid is rotated of a fixed angle δ. This is much simpler than Darwin's procedure, what is, maybe, responsible for the popularity of that theory, but the analysis of the expansions shows some unphysical behavior: When the potential of the rotated ellipsoid is Fourier analyzed, we find in the result terms with the same argument, but different lags, and terms with different arguments, but the same lags [33]. It is impossible to associate a rheology to McDonald's model.

A more important modification was introduced by Efroimsky and Lainey [28]. These authors followed the procedures introduced by Darwin, but taking into account some results from Earth's seismology, assumed that the lags follow an inverse power law. An inverse power law brings with itself the problem of the crossing of the zero. With a pure inverse power law, the lag would tend to infinity when the frequency tends to zero. A more complex rheology is necessary and Efroimsky later adopted a rheology in which an inverse power law is followed for high frequencies, but the old Darwin model prevails at lower frequencies [27].

We may, at last, cite the review of Darwin theory by Ferraz-Mello et al.[37] where the theory was develloped keeping the lags free, just assuming that to equal frequencies correspond equal lags. This allows one to introduce the rheology at a later stage. That review also introduced some dynamical response factors k_d which corresponds to attenuate the amplitude of the tide components.

3.5 The rheophysical theory

In the rheophysical theory of Ferraz-Mello [34], the anelastic tide is assumed to derive from a simple rheophysical model, the basis of which is shown in fig. 1. We assume that, at a given time t, the surface of the body is a function $\zeta = \zeta(\widehat{\theta}, \widehat{\varphi}, t)$ where ζ is the height of the point over a fixed reference surface and $\widehat{\theta}, \widehat{\varphi}$ are its co-latitude and longitude with respect to a reference system rotating with the body.

Would the body be inviscid, it would immediately change its shape to coincide with the equilibrium figure, which is an ellipsoid whose height over the reference surface is $\rho = \rho(\widehat{\theta}, \widehat{\varphi}, t)$.

Terms of second order with respect to ε_ρ are neglected in this and in the following calculations.

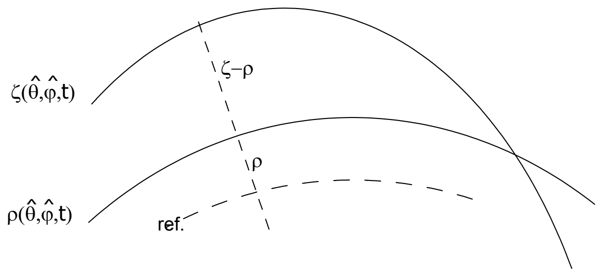

Figure 3.1: Elements of the model: ρ is a section of the surface of the equilibrium ellipsoid at thwe time t and ζ is a section of the surface of the body at the same time.

The adopted rheophysical model is founded on the law

$$\dot{\zeta} = -\gamma(\zeta - \rho). \tag{3.12}$$

Because of the forces acting on the body (self-gravitation, tidal potential and centrifugal), its surface will tend to the equilibrium ellipsoid, but not instantaneously, and its instantaneous response (measured by $\dot{\zeta}$) will be proportional to the height of its surface over the equilibrium ellipsoid, $\rho - \zeta$. Eq. (3.12) is the equation of a *Newtonian creep* (see [61], chap. 5) where the stress was considered as proportional to the distance to the equilibrium. It does not consider inertia or azimuthal motions and is linear.

The relaxation factor γ is a radial deformation rate gradient and has dimension T^{-1}. $\gamma \to 0$ in the solid body limit and $\gamma \to \infty$ in the inviscid fluid limit. Between these two extremes, we have the viscous bodies, which, under stress, relax towards the equilibrium, but not instantaneously.

It is possible to see that Eq. (3.12) is an approximated solution of the Navier-Stokes equation of a radial flow across the two surfaces, for very low Reynolds number (Stokes flow). In this case, the inertia terms can be neglected (see [42]) and the Navier-Stokes equation becomes [67]

$$\mathbf{F}_{\rm ext} - \nabla p + \eta \Delta \mathbf{V} = 0 \tag{3.13}$$

where p is the pressure, η is the uniform viscosity and $\mathbf{V}$ the velocity. We notice that the symbol Δ is operating on a vector, contrary to its usual definition. Actually, in this pseudo-vectorial notation, the formula refers to the

Table 3.1: Typical values of the relaxation factor adopted in applications. See [34]

Body	γ (s^{-1})	$2\pi/\gamma$	η (Pa s)
Moon	$2.0 \pm 0.3 \times 10^{-9}$	36,000 d	$2.3 \pm 0.3 \times 10^{18}$
Titan	$2.9 \pm 0.2 \times 10^{-8}$	2500 d	$1.1 \pm 0.1 \times 10^{17}$
Solid Earth	$0.9 - 3.6 \times 10^{-7}$	200-800 d	$4.5 - 18 \times 10^{17}$
Io	$4.9 \pm 1.0 \times 10^{-7}$	730 d	$1.2 \pm 0.3 \times 10^{16}$
Europa	$1.8 - 8.0 \times 10^{-7}$	90–400 d	$4 - 18 \times 10^{15}$
Neptune	2.7–19	< 2 s	$1.2 - 4.8 \times 10^{10}$
Saturn	> 7.2	< 0.9 s	$< 15 \times 10^{10}$
Jupiter	23 ± 4	~ 0.3 s	$4.7 \pm 0.9 \times 10^{10}$
hot Jupiters	8–50	0.1–0.8 s	$5 \times 10^{10} - 10^{12}$
solar-type stars	> 30	< 0.2 s	$< 2 \times 10^{12}$

components of the vector $\mathbf{V}$ and Δ operates on the individual scalar components of $\mathbf{V}$. We assume that the flow is radial and thus $\mathbf{V}$ is restricted to its radial component $\dot{\zeta}$ and then $\Delta\mathbf{V}$ is reduced to the radial component

$$\Delta\dot{\zeta} = \frac{2}{R+\zeta}\frac{\partial\dot{\zeta}}{\partial\zeta} + \frac{\partial^2\dot{\zeta}}{\partial\zeta^2} - \frac{2\dot{\zeta}}{(R+\zeta)^2}$$

assuming that the height of the points refers to a sphere of radius R. The terms corresponding to the derivatives of $\dot{\zeta}$ w.r.t the longitude and colatitude were not written because these variations are assumed to be equal to zero. We also assume $\mathbf{F}_{\rm ext} = 0$; the stress due to the non-equilibrium may be absorbed into the pressure terms. The pressure due to the body gravitation is given by the weight of the mass which lies above (or is missing below[4]) the equilibrium surface, that is, $-w(\zeta - \rho)$; the modulus of the pressure gradient is the specific weight w. The boundary conditions are $\dot{\zeta} = 0$ at $\zeta = \rho$. Hence

$$w + \frac{2\eta}{R+\zeta}\frac{\partial\dot{\zeta}}{\partial\zeta} - \frac{2\dot{\zeta}}{(R+\zeta)^2} = 0. \tag{3.14}$$

[4]This does not mean that a negative mass is being assigned to void spaces; it means just that the forces included in the calculation of the equilibrium figure need to be subtracted when the masses creating them are no longer there.

The Newtonian creep results from the integration of this equation in the neighborhood of ρ, with the additional assumption $\zeta \ll R$. Hence,

$$\dot{\zeta} = -\frac{wR}{2\eta}(\zeta - \rho) + \mathscr{O}(\zeta^2), \tag{3.15}$$

which shows that the basic equation adopted in Ferraz-Mello's theory is the linearized solution of an approximate version of the Navier-Stokes equation and that the relaxation factor γ is related to the uniform viscosity of the body through

$$\gamma = \frac{wR}{2\eta} = \frac{3gm}{8\pi R^2 \eta}, \tag{3.16}$$

where g is the gravity at the surface of the body and R is its mean radius. Darwin [21] also studied this model. Using a different construction of the Navier-Stokes equations he obtained the numerical factor 3/38 instead of 3/8. His factor was determined by the spheroidal form of the tidal potential, but the intensity of the potential does not appear in his results, which would be the same no matter if the tidal potential is huge or infinitesimal. His approach would deserve a new analysis.

3.6 The Maxwell rheology

Following what is current in the study of mechanical models of rheology, let us add the elastic and anelastic tides. Let us beforehand remember that, because of the visco-elastic nature of the body, the elastic tide will not deploy the whole stretching corresponding to the prolateness ε_ρ. We will assume that the actual prolateness of the body will be $\lambda\varepsilon_\rho$ (where $\lambda < 1$).

In the simplified case in which the polar oblateness of the body due to the rotation is not considered, the surface of the body resulting from the tide is then given by

$$Z = \zeta + \lambda\rho. \tag{3.17}$$

If the creep equation is changed in accordance with this transformation, we obtain

$$\dot{Z} + \gamma Z = (1+\lambda)\gamma\rho + \lambda\dot{\rho}. \tag{3.18}$$

In order to see that this is the equation of a Maxwell model, it is enough to substitute ρ by the stress introduced into the creep equation: $\tau = \rho - \zeta =$

$(1+\lambda)\rho - Z$. Eqn. 3.18 then becomes

$$\dot{Z} = (1+\lambda)\gamma\tau + \lambda\dot{\tau} \tag{3.19}$$

which is the constitutive equation of a Maxwell model (see [73]).

It is worth comparing the above equations with those of the virtually identical Maxwell model later proposed by Correia et al. [20]. These authors equations can be translated into the general equation:

$$\dot{Z} + \gamma Z = \gamma\rho + \lambda\dot{\rho} \tag{3.20}$$

(see [35]). If the elastic part is subtracted from Z using Eqn. 3.17, we obtain the creep equation:

$$\dot{\zeta} = -\gamma\big(\zeta - (1-\lambda)\rho\big). \tag{3.21}$$

This is also a Newtonian creep, but it differs from the Newtonian creep used by Ferraz-Mello [34] by the fact that the stress is taken here as proportional to the distance to a different equilibrium surface, defined by $(1-\lambda)\rho$, instead of the surface of the Jeans ellipsoid, ρ, as in Ferraz-Mello's theory. This different tuning is difficult to explain. Indeed, in this case, the stress is

$$\tau = (1-\lambda)\rho - \zeta \tag{3.22}$$

which is not zero when the surface of the body is the equilibrium surface $\zeta = \rho$.

3.7 The creep equation

Let us, now, develop the rheophysical tidal theory of Ferraz-Mello [34]. We introduce an important simplification. We restrict the calculation of the anelastic tide to the case in which the body is homogeneous. The more realistic case in which the body is differentiated, but formed by homogeneous layers is yet under construction. We also remember that we have assumed that the orbit of the companion lies on the equatorial plane of the body.

To develop the theory, we first have to introduce the locus of the equilibrium surface (ρ) in the creep equation. Let us consider as reference surface a spheroid of radius R_e and polar oblateness $< \epsilon_z >$. The height of the points of the surface of the equilibrium ellipsoid of polar oblateness ε_z and equatorial prolateness ε_ρ directed towards the position of the companion, is

$$\rho = \frac{1}{2} R_e \varepsilon_\rho \sin^2 \widehat{\theta} \cos(2\widehat{\varphi} - 2\varpi - 2v) + R_e(\varepsilon_z - < \varepsilon_z >) \cos^2 \widehat{\theta}. \quad (3.23)$$

where $\widehat{\theta}, \widehat{\varphi}$ are the co-latitude and longitude of a generic point on the surface.

The sequence of the calculations is easy. The function $\rho(t)$ depends on the position of the companion (whose attraction is raising the tide on the body). The true longitude $2\varpi + 2v$ is explicit in the above equation. The other coordinate is the radius vector, which is entering in the expression of ρ through the prolateness of the equator ε_ρ and the polar oblateness ε_z. We assume that the companion is moving around the body on a Keplerian ellipse of equation:

$$r = \frac{a(1-e^2)}{1+e.\cos v}. \quad (3.24)$$

The non-uniform time variation of the angle v is given by a law inversely proportional to the square of r. Its expression involves the resolution of a transcendent equation and the result is given by a series:

$$v = \ell + (2e - \frac{e^3}{4}) \sin \ell + \frac{5e^2}{4} \sin 2\ell + \frac{13e^3}{12} \sin 3\ell + \mathscr{O}(e^4) \quad (3.25)$$

and ℓ is the mean anomaly

$$\ell = \sqrt{\frac{G(M+m)}{a^3}}(t - t_0) \stackrel{\text{def}}{=} n(t - t_0) \quad (3.26)$$

and the origin of time t_0 is taken at one moment in which the companion is moving through the pericenter of its orbit. The mean angular velocity n, is the so-called mean-motion.

After the substitution, the creep differential equation becomes:

$$\dot{\zeta} + \gamma\zeta = \gamma\rho = \gamma R_e \left(\frac{1}{2} \overline{\varepsilon}_\rho \sin^2 \widehat{\theta} \sum_{k\in\mathbb{Z}} E_{2,k} \cos\left(2\widehat{\varphi} + (k-2)\ell - 2\varpi\right) \right.$$

$$\left. - \frac{1}{2} \overline{\varepsilon}_\rho \cos^2 \widehat{\theta} \sum_{k\in\mathbb{Z}, k\neq 0} E_{0,k} \cos k\ell \right) \quad (3.27)$$

where $E_{q,p}$ are the Cayley functions[5]

$$E_{q,p}(e) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3 \cos\big(qv + (p-q)\ell\big)\, d\ell, \tag{3.28}$$

and

$$\overline{\varepsilon}_\rho = \frac{15}{4}\left(\frac{M}{m}\right)\left(\frac{R_e}{a}\right)^3, \qquad \overline{\varepsilon}_z = \frac{5cR_e^2\Omega^2}{4mG}. \tag{3.29}$$

It is worth stressing the fact that Eqn. 3.27 is a non-homogeneous linear ordinary differential equation with constant coefficients whose solution may be calculated by elementary methods. We have to add to the general solution of the associated homogeneous equation: $\zeta = C.e^{-\gamma t}$, the particular solution of the complete equation. Since the equation is linear, each term of the Fourier expansion of the r.h.s. can be considered separately, and the sought particular solution is the sum of the particular solutions obtained with each term of the r.h.s.

It is also necessary to stress that, in the integration, the orbital elements a, e, the rotation velocity Ω and the variation $\dot{\varpi}$ are taken as constants. These quantities are in fact affected by the tide and are variable. However, their resulting variations are of the order $\mathscr{O}(\gamma)$ and their contributions are of second order. Their variation can be neglected, at least for times $t \ll 1/\gamma$.

The integration of the creep differential equation gives[6]

$$\delta\zeta = \frac{1}{2}R_e\overline{\varepsilon}_\rho\left(\sin^2\widehat{\theta}\sum_{k\in\mathbb{Z}} E_{2,k}\cos\sigma_k\cos\big(2\widehat{\varphi} + (k-2)\ell - 2\varpi - \overline{\sigma}_k\big)\right.$$
$$\left. - \cos^2\widehat{\theta}\sum_{k\in\mathbb{Z},k\neq 0} E_{0,k}\cos\sigma'_k\cos(k\ell - \sigma'_k)\right) + Ce^{-\gamma t} \tag{3.30}$$

[5]The Cayley functions introduced here correspond to the degree 3 in a/r – since $\varepsilon_\rho \propto (a/r)^3$. These functions are fully equivalent to the Hansen coefficients preferred by other authors and the equivalence is given by $E_{q,p} = X_{2-p}^{-3,q}$ (see [20]). The expansion of these functions up to degree 7 are found in Cayley's tables [16] and are reproduced in the Online Supplement linked to [36]. In the applications, it is easy to have the integral computed thus overcoming the Tables limitation to the 7-th power of the eccentricities.

[6]We note that terms involving $\overline{\varepsilon}_z$ do not appear in Eqn.3.30. The terms arising from the variation of the polar oblateness are given by the second summation in the equation in which the term $k=0$ does not exist because the polar oblateness remains constant when r is constant (i.e. $e=0$).

where

$$\cos\sigma_k = \frac{\gamma}{\sqrt{(\nu+kn)^2+\gamma^2}} \qquad \sin\sigma_k = \frac{\nu+kn}{\sqrt{(\nu+kn)^2+\gamma^2}}$$

$$\cos\sigma_k' = \frac{\gamma}{\sqrt{k^2n^2+\gamma^2}} \qquad \sin\sigma_k' = \frac{kn}{\sqrt{k^2n^2+\gamma^2}} \tag{3.31}$$

where

$$\nu = 2\Omega - 2n \tag{3.32}$$

is the semi-diurnal frequency, i.e. the frequency of occurrence of the high tide at one fixed point in the body (e.g., if the body is the Earth and the companion is the Sun, the semi-diurnal period is 12 hours and $\nu = \pi/6$ h^{-1}).

The integration constant C depends on $\widehat{\theta}, \widehat{\varphi}$ (the integration was done with respect to t) and may be related to the initial surface $\zeta_0 = \zeta(\widehat{\theta}, \widehat{\varphi})$.

3.8 Semi-diurnal tide

For the sake of simplicity, we restrict the exposition of the theory to the simplest case where the orbit of the companion is circular. In that case, the creep differential equation is reduced to

$$\dot{\zeta} + \gamma\zeta = \frac{1}{2}\gamma R_e \overline{\varepsilon}_\rho \sin^2\widehat{\theta}\cos(2\widehat{\varphi} - 2\varpi - 2\ell) \tag{3.33}$$

and the integration gives

$$\delta\zeta = \frac{1}{2} R_e \overline{\varepsilon}_\rho \sin^2\widehat{\theta}\cos\sigma_0\cos(2\widehat{\varphi} - 2\varpi - 2\ell - \sigma_0) + Ce^{-\gamma t} \tag{3.34}$$

where

$$\cos\sigma_0 = \frac{\gamma}{\sqrt{\nu^2+\gamma^2}} \qquad \sin\sigma_0 = \frac{\nu}{\sqrt{\nu^2+\gamma^2}}. \tag{3.35}$$

The subtracting constant phase σ_0 behaves mathematically as a phase delay, but, physically, it is not a lag and, mainly, it is not an *ad hoc* plugged constant as in Darwinian theories. It is a finite (i.e. not small) well-determined quantity resulting from the integration of the first-order linear differential equation.

In order to know the whole extent of the semi-diurnal tide, we have to add the elastic tide and the semi-diurnal term of the creep tide. For the sake of simplicity, once more, we will restrict ourselves to the case in which the rotation of the body is not taken into account and in which the companion's orbit is circular. In this simple case, we have

$$Z = \lambda\rho + \zeta = \frac{1}{2}\lambda R_e \overline{\varepsilon}_\rho \left(\sin^2\widehat{\theta}\cos\left(2\widehat{\varphi} - 2\varpi - 2\ell\right) - \cos^2\widehat{\theta})\right) \qquad (3.36)$$

$$+\frac{1}{2}R_e\overline{\varepsilon}_\rho \sin^2\widehat{\theta}\cos\sigma_0\cos(2\widehat{\varphi} - 2\varpi - 2\ell - \sigma_0) + Ce^{-\gamma t}$$

where we have referred the equilibrium figure ρ to the spheroid adopted as reference.

3.8.1 The geodetic lag

In this discussion, we introduce the angle $\alpha = \widehat{\varphi} - \varpi - \ell$ which is the longitude of one point on the body reckoned from the sub-companion point. The maximum of the anelastic or creep tide is reached at one point of the equator whose relative longitude is

$$\alpha_0 = \frac{1}{2}\sigma_0 = \arctan\frac{\nu}{\gamma} \qquad (3.37)$$

In the case of a very fluid body (gaseous planets, stars), $\gamma \gg \nu$ and then $\sigma_0 \simeq 0$. However, for rotating stiff bodies (terrestrial planets, satellites), $\gamma \ll \nu$ and then $\sigma_0 \simeq \frac{\pi}{2}$. In this case, the maximum of the creep tide occurs at 45° from the maximum of the tidal stress. However, this is just the maximum of the anelastic component of the tide. The actual maximum of the tide is at

$$\alpha_0 = \frac{1}{2}\arctan\frac{\sin 2\sigma_0}{1 + 2\lambda + \cos 2\sigma_0}. \qquad (3.38)$$

This function is shown in fig. 3.2 (Top). We see that, as far as $\lambda \neq 0$, $\alpha_0 \to 0$ when $\sigma_0 \to \frac{\pi}{2}$, that is, when $\gamma \to 0$. As a bonus, we also have near the rigid limit (i.e. near $\sigma_0 = \frac{\pi}{2}$) α_0 decreasing when σ_0 increases, that is, when the frequency ν increases. This is exactly the behavior that is being advocated by Efroimsky and collaborators [27, 28] for the Earth and the planetary satellites.

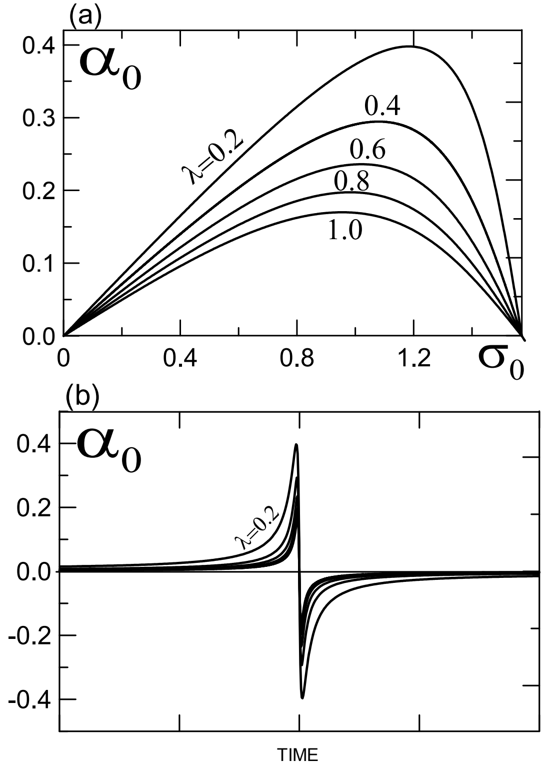

Figure 3.2: *Top:* Geodetic lag α as a function of σ_0. *Bottom:* Time evolution of the geodetic lag when the frequency of the semi-diurnal tide crosses 0 and the tidal bulge changes of side with respect to the sub-companion point.

It is worth recalling that one of the difficulties created by the assumption that the actual tide lag is proportional to a negative power of the frequency happens when the frequency changes of sign. If no additional assumption is done, we just have a singularity with the tide lag tending to infinity or, at least, abrupt jumps between positive and negative values. This is not the case with the solution that results from the superposition of elastic and creep tides. In this case, the transition from one side to another is smooth.

The tide angle increases when the frequency decreases up to reach a maximum; after that point it quickly decreases up to cross zero with a finite derivative. The behavior in the negative side is just symmetrical (see fig3.2 Bottom).

3.9 Force and torque on the companion

At each instant, the body surface resulting from the action of the creep tide (i.e. the anelastic tide) is defined by the radii vectors $R_e - (\overline{\varepsilon}_z + \frac{1}{2}\overline{\varepsilon}_\rho)cos^2\widehat{\theta} + \delta\zeta(\widehat{\theta},\widehat{\varphi},t)$. After the transient phase (i.e. for $\gamma t \gg 1$), only the forced terms of $\delta\zeta$ matter. Since the body is assumed to be homogeneous, the calculation of the force and torque that it exerts on the companion is simple, because the above defined surface is, then, the composition of the bulges of a set of quadrics (which may give positive or negative contributions) [34] superposed to the reference spheroid. Since these bulges are very thin (they are proportional to $\overline{\varepsilon}_\rho, \overline{\varepsilon}_z$), we may calculate the attraction of the companion by the resulting composite, as the sum of the forces due to each ellipsoid bulge. The errors of this superposition are of second order w.r.t the flattenings $\overline{\varepsilon}_\rho, \overline{\varepsilon}_z$.

We may, however, use a more direct approach. We may substitute the bulges by a thin spherical shell of radius R_e and assume for the mass element at the shell coordinates $(\widehat{\theta}, \widehat{\varphi})$, the sum of the masses of the bulges at that point. The generic mass element in the shell is

$$dm(\widehat{\varphi},\widehat{\theta}) = R_e^2 \mu_\mathsf{m} \sin\widehat{\theta} d\widehat{\varphi} d\widehat{\theta} \delta\zeta \tag{3.39}$$

where μ_m is the density of the body. There is a small offset due to the fact that $\delta\zeta$ is the height over an oblate spheroid, not over a sphere, but the differences thus introduced may be neglected [36]. (The offset is of second order w.r.t. the flattenings $\overline{\varepsilon}_\rho, \overline{\varepsilon}_z$).

The contribution of the element dm to the potential in the point $\mathsf{P}(r,\theta,\varphi)$ is

$$dU = -\frac{Gdm}{\Delta} \tag{3.40}$$

where G is the gravitational constant and Δ is the distance from the element dm to the point $\mathsf{P}(r,\theta,\varphi)$; the potential created by the whole shell is given by

$$U = -GR^2\mu_\mathsf{m} \int_0^\pi \sin\widehat{\theta} d\widehat{\theta} \int_0^{2\pi} \frac{\delta\zeta}{\Delta} d\widehat{\varphi} \tag{3.41}$$

The integration is simple and the result is obtained considering separately the contributions of each component of $\delta\zeta$. In the case of a circular motion, the contribution of the semi-diurnal tide is

$$\delta U_0 = -\frac{3GmR_e^2\overline{\varepsilon}_\rho}{10r^3}\cos\sigma_0\sin^2\theta\cos(2\varphi - 2\ell - 2\varpi - \sigma_0). \tag{3.42}$$

It is worth noting that this term is proportional to $1/r^3$; the terms proportional to $1/r^2$ are null because the $\delta\zeta$-shell has the same center of gravity as the body. This fact is of importance since it will be responsible for tidal forces inversely proportional to the 4-th power of the distances.

To obtain the force acting on one mass M located at one point, we have to take the negative gradient of the potential at that point and multiply the result by the mass placed on the point. Hence,

$$\begin{aligned}
F_{1,0} &= -\frac{9GMmR_e{}^2\overline{\varepsilon}_\rho}{10r^4}\cos\overline{\sigma}_0\sin^2\theta\cos(2\varphi - 2\ell - 2\varpi - \sigma_0)\\
F_{2,0} &= \frac{3GMmR_e{}^2\overline{\varepsilon}_\rho}{10r^4}\cos\sigma_0\sin 2\theta\cos(2\varphi - 2\ell - 2\varpi - \sigma_0)\\
F_{3,0} &= -\frac{3GMmR_e{}^2\overline{\varepsilon}_\rho}{5r^4}\cos\overline{\sigma}_0\sin\theta\sin(2\varphi - 2\ell - 2\varpi - \sigma_0)
\end{aligned} \tag{3.43}$$

and the corresponding torques are

$$M_{1,0} = 0, \qquad M_{2,0} = -rF_{3,0}, \qquad M_{3,0} = rF_{2,0}. \tag{3.44}$$

Since we are interested in the force acting on the companion due to the anelastic tidal deformation of the body, once the gradient is calculated we can substitute (θ,φ) by the co-latitude and longitude of the companion. They may be written in terms of the true anomaly of the companion. As the companion was assumed to lie in a circular orbit in the equatorial plane of the body, we have $r = a, \theta = \pi/2$ and $\varphi = \ell + \varpi$. Hence

$$\begin{aligned}
F_{1,0} &= -\frac{9GMmR_e{}^2\overline{\varepsilon}_\rho}{10a^4}\cos^2\sigma_0\\
F_{2,0} &= 0\\
F_{3,0} &= \frac{3GMmR_e{}^2\overline{\varepsilon}_\rho}{10a^4}\sin 2\sigma_0.
\end{aligned} \tag{3.45}$$

The only non-zero component of the torque is

$$M_{2,0} = -\frac{3GMm{R_e}^2\overline{\varepsilon}_\rho}{10a^3}\sin 2\sigma_0 \tag{3.46}$$

3.10 Angular momentum variation and Rotation

We use the equation $C\dot{\Omega} = M_{2,0}\sin\theta$ (the z-component of the torque on the companion is $-M_2\sin\theta$ [37]) where C is the moment of inertia with respect to the rotation axis. Hence, in the circular case,

$$\dot{\Omega} = -\frac{3GM\overline{\varepsilon}_\rho}{4a^3}\sin 2\sigma_0 \tag{3.47}$$

where we have simplified the coefficient by using the homogeneous body value $C \simeq \frac{2}{5}mR_e^2$.

One important characteristic of this equation, due to the invariance of the torque to rotations of the reference system, is that the right-hand side is independent of the rotational attitude of the body. The arguments of the periodic terms do not include the azimuthal angle fixing the position of the rotating body. Therefore, this is a true first-order differential equation and there are no free oscillations. The corresponding physical librations are forced oscillations. This is totally different from the classical spin-orbit dynamics of rigid bodies where a permanent azimuthal asymmetry in the mass distribution of a solid body (potential terms with coefficients J_{22} or J_{31}) gives rise to terms including the azimuthal angle in the arguments and the equation to be considered is a second-order differential equation.

In the circular approximation, $\dot{\Omega} = 0$ when $\nu = 0$. Taking into account the minus sign in front of the r.h.s. of Eqn. 3.47, this means that $\nu = 0$ is a stable equilibrium solution. Tide tends to change the rotation of the body so that it becomes synchronous with the orbital motion. For instance, the rotation of the Earth is being continuously braked by the luni-solar tide. Some fossil corals show evidences that the length of the day was just about 22 hours in the late Paleozoic era, 350 Myr ago, and the predictions for the future are of longer days, reaching 31 hours in 1 Gyr from now.

3.11 Energy variation and Dissipation

The work done by the tidal forces in a displacement $d\mathbf{s}$ is given by $dW = \mathbf{F}\cdot d\mathbf{s}$, or $\dot{W} = \mathbf{F}\cdot\mathbf{v}$ where $\mathbf{v}$ is the velocity vector. This calculation is elementary. The components of $\mathbf{v}$ in the adopted 3D spherical coordinates are

$$\begin{aligned} v_1 &= \frac{nae\sin v}{\sqrt{1-e^2}} \\ v_2 &= 0 \\ v_3 &= \frac{na^2\sqrt{1-e^2}}{r}. \end{aligned} \tag{3.48}$$

The calculations with the forces obtained in the previous section give for the energy variation of the semi-diurnal tide,

$$\dot{W} = \frac{3GMmnR_e^2\overline{\varepsilon}_\rho}{10a^3}\sin 2\sigma_0$$

3.11.1 Dissipation

The energy variation associated with the rotation of the body also plays a role and needs to be taken into account in the energy balance. To the variation of the orbital energy, we have to add $\dot{W}_{\rm rot} = C\Omega\dot{\Omega}$. Hence, using for C the value of the moment of inertia of a homogeneous body, we get the average

$$\dot{W}_{\rm rot} = -\frac{3GMm\Omega R_e^2\overline{\varepsilon}_\rho}{10a^3}\sin 2\sigma_0$$

and, therefore,

$$\dot{W}_{\rm total} = -\frac{3GMm\nu R_e^2\overline{\varepsilon}_\rho}{20a^3}\sin 2\sigma_0 \tag{3.49}$$

which is a function of ν^2 (vanishing when $\nu = 0$) always negative (there is a loss of the total mechanical energy); its modulus is the total energy dissipated inside the body. It is worth reminding that the behavior of $\dot{\Omega}$ in the circular approximation is very simple, but this simplicity is not preserved when the companion is moving in an elliptic orbit (see Section 3.12)

Figure 3.3 shows the dissipation when $\nu/n = 2.5$, in which case the rotation of the body is 2.25 times the orbital mean motion. (A fractionary value was chosen so as to avoid being close to some known stationary solutions).

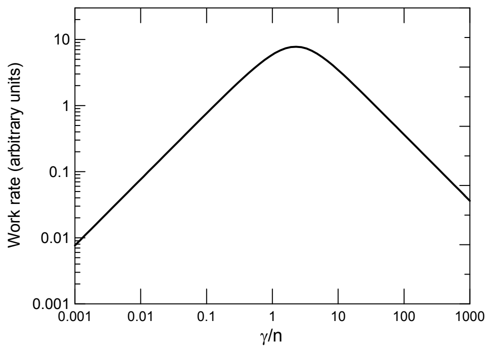

Figure 3.3: Time rate of the energy dissipated in free rotating bodies with a companion in circular orbit when $\nu = 2.5n$. A logarithmic scale is used to show the power laws ruling the dissipation in the two regimes: $\gamma \ll n$ (Efroimsky-Lainey) and $\gamma \gg n$ (Darwin).

It is worth emphasizing the ubiquitous presence of the factor $\sin 2\sigma_0$ in the main formulas for the average variation of the energy and angular momentum. This fact has a major implication. We have

$$\sin 2\sigma_0 = \frac{2\gamma\nu}{\gamma^2 + \nu^2} = 2\left(\frac{\gamma}{\nu} + \frac{\nu}{\gamma}\right)^{-1}. \tag{3.50}$$

The symmetrical form of $\sin 2\sigma_0$ plays a key role in the rheophysical theory of the anelastic tide. It says that, in the circular approximation, the solutions behave exactly in opposite, but symmetrical, ways when $\nu \ll \gamma$ and when $\nu \gg \gamma$. The relaxation factor γ plays the role of one critical frequency dividing two regimes of solution and the solutions have the form of an inverted V in a log-log plot. The most famous is the inverted V of the dissipation plot (see fig. 3.3), which is the most known feature of Maxwell visco-elastic bodies.

Figure 3.3 also shows that , in the case of stiff bodies (telluric planets, planetary satellites), for which we have $\gamma \ll n$ (see Table 3.1), the dissipation is directly proportional to γ/n (the inclination of the Efroimsky-Lainey branch in the log-log plot is +1). At variance, in the case of very fluid bodies (stars, gaseous giant planets), in which case $\gamma \gg n$, the dissipation is inversely proportional to γ/n (the inclination of the Darwin branch in the log-log plot is -1). We may also note that, because of the chosen value of ν/n, the critical value $\gamma = \nu$ corresponds to $\gamma/n = 1/2.5$)

3.11.2 Variation of the semi-major axis

From the two-body energy $E = -GMm/2a$, we obtain

$$\frac{\dot{a}}{a} = -\frac{\dot{E}}{E} \tag{3.51}$$

Since the work of the force acting on the companion increases the energy of the system, $\dot{E} = \dot{W}$ and then

$$\dot{a} = -\frac{a\dot{W}}{E} = \frac{3nR_e^2\overline{\varepsilon}_\rho}{5a}\sin 2\sigma_0 \tag{3.52}$$

Since σ_0 has the same sign as ν, $\dot{a}$ has the same sign as $\nu = 2\Omega - 2n$. For instance, in the case of the lunar tides on the Earth, the Earth rotation is much faster than the orbital motion of the Moon, that is, ν is positive and so is $\dot{a}$. Indeed, we know that the Moon is receding from the Earth with the rate of 3–4 cm/d. This is always the case when the orbit of the companion is above the synchronous orbit, that is, when the orbital period of the companion is larger than the rotation period of the body.

At variance with this example, we may mention the extra-solar planets known as hot Jupiters, which are in orbits very close to the host stars and have orbital periods smaller than the rotation period of the host stars (they are below the synchronous orbit). In the case of these planets, $\dot{a} < 0$ and they are falling on the host stars because of the tides they raise on it.

3.12 The eccentricity

The simplification $e = 0$ done in the previous sections was useful to simplify the exposition of the effects of the elastic and anelastic tides and to study the dependence of the dissipation with the frequency of the semi-diurnal tide. However, it conceals the rich spin-orbit dynamics entailed by the tidal interaction of the two bodies. If the companion is moving in an elliptic orbit around the body, the variation of the rotation angular velocity is given by a more involving equation (see [34]). In the particular case in which the companion orbit lies on the equator of the body, we have

$$\dot{\Omega} = -\frac{3GM\overline{\varepsilon}_\rho}{2a^3}\sum_{k\in\mathbb{Z}} E_{2,k}\cos\sigma_k \sum_{j+k\in\mathbb{Z}} E_{2,k+j}\sin(j\ell+\sigma_k) \tag{3.53}$$

where the summations are done over all terms of order less than or equal to a chosen N.

The average of Eq. (3.53) with respect to ℓ is

$$<\dot{\Omega}>= -\frac{3GM\overline{\varepsilon}_\rho}{4a^3}\sum_{k\in\mathbb{Z}} E_{2,k}^2 \sin 2\sigma_k. \tag{3.54}$$

To truncate at order N, we discard all terms with $|k| > N/2$.

From Eqns. (3.31), we obtain,

$$\sin 2\sigma_k = \frac{2\gamma(\nu+kn)}{\gamma^2+(\nu+kn)^2}. \tag{3.55}$$

One immediate consequence of Eqn. (3.54) is that the synchronous rotation is not a stationary solution of the system if the orbital eccentricity is not zero. Indeed, the inspection of the terms in Eqn. (3.54) show that when $\nu = 0$ they are dominated by the term $k = -1$ (because $E_{2,-1}^2 \sim 50E_{2,1}^2$ and $\sin 2\sigma_{-1} < 0$) and so,

$$<\dot{\Omega}>\Big|_{\nu=0} \geq 0. \tag{3.56}$$

The equality to zero is not possible if $e \neq 0$ (Remember that $E_{2,k}^2 = \mathcal{O}(e^{2k})$). In the synchronous state, the torque is positive, meaning that the rotation is being accelerated by the tidal torque. The stationary solution is reached at a supersynchronous rotation velocity. Indeed, solving the equation $<\dot{\Omega}>= 0$, we obtain

$$\Omega = n + \frac{6n\gamma^2}{n^2+\gamma^2}\,e^2 + \mathcal{O}(e^4). \tag{3.57}$$

The result corresponds to a supersynchronous rotation. However, at variance with the standard Darwinian theories in which the law giving the stationary rotation is independent of the body rheology, here it depends on the viscosity η through the relaxation factor γ.

In the quasi-inviscid limit, $\eta \to 0$, then $\gamma \gg n$ and $\frac{\gamma^2}{\gamma^2+n^2} \simeq 1$. We then obtain

$$\Omega_{\mathrm{lim}} \simeq n(1+6e^2+\frac{3}{8}e^4+\cdots). \tag{3.58}$$

Thus, in the quasi-inviscid limit, the result is the same obtained with the standard Darwinian theories in which it is assumed that the geodetic lags are proportional to the frequencies.

In the quasi-solid limit, however, $\gamma \ll n$ and so

$$\Omega = n + \frac{6\gamma^2}{n}\, e^2 + \mathscr{O}(e^4) \tag{3.59}$$

The solution is also supersynchronous but the excess of angular velocity is of order $\mathscr{O}(\gamma^2)$ and is thus much smaller than the excess obtained in the previous case, when $\gamma \gg n$.

3.12.1 Circularization

Finally, we may consider the variation of the ecccentricity. The quickest way to get it uses the energy and angular momentum definitions to obtain, after derivation and elimination of the other variable parameters,

$$\frac{e\dot{e}}{1-e^2} = \frac{\dot{\mathscr{L}}}{\mathscr{L}} + \frac{\dot{W}}{2E}. \tag{3.60}$$

After some algebraic manipulation and averaging of the short period variations, we obtain

$$< \dot{e} >= -\frac{3GMR_e^2\overline{\varepsilon}_\rho}{20na^5e}\sum_{k\in\mathbb{Z}}\Big(2\sqrt{1-e^2}-(2-k)(1-e^2)\Big)E_{2,k}^2 \sin\, 2\sigma_k$$

$$+\frac{GMR_e^2\overline{\varepsilon}_\rho}{20na^5e}\sum_{k\in\mathbb{Z}}(1-e^2)kE_{0,k}^2\sin 2\sigma_k'. \tag{3.61}$$

Chapter 4

Finite dimensional models for deformable bodies

Part of this chapter was copied from a paper by Grotta-Ragazzo and Ruiz [63].

4.1 Introduction

The problem of the equilibrium shapes that a rotating isolated, incompressible, ideal fluid can attain goes back to Newton in the *Principia Mathematica*. Generations of important scientists contributed to the understanding of this theme, which remains as a fruitful source of questions. For a brief historical review and general exposition, see [17]. Classical treatments can be found in [48], [22], [62] and [52]. The dynamics of the fluid is determined by a set of partial differential equations. Solutions to these equations that are steady in a rotating reference frame are called relative equilibria. They are important in the shape modeling of celestial bodies. Questions on stability of the known equilibria are still open in spite of the celebrated Poincaré's work on the subject [62]. The mathematical complexity of the equations is a challenge to numerical analysts and physicists. As discussed in Chapter 2, more realistic models for the shape of stars and planets may include compressibility, strain-forces, inhomogeneities, etc, which further increase the difficulty of the problem.

From the perspective of celestial mechanics, while a planet or a star is physically perceived as an object of finite size it is usually modeled as a point mass characterized only by its center of mass position. Although the point-mass assumption has been very successful in the study of planetary motion it precludes the analysis of some important phenomena like, for instance, dissipation of energy due to tides, already analyzed in Chapters 2 and 3. We recall that one of the attempts to overcome the limitations of the point-mass model without introducing the infinitely many degrees of freedom of an extended body is provided by the so-called pseudo-rigid body presented in Chapter 2: "a point to which is attached a measure of orientation and deformation", see [18]. Formally a pseudo-rigid body model is obtained in the following way.

Suppose that a body at rest has the shape of a ball $\mathscr{B} \subset \mathbb{R}^3$ with radius $R > 0$ (reference configuration). Let $x \in \mathscr{B}$ denote the initial position of a point in the body and $\phi(t,x) \in \mathbb{R}^3$ denote the position at time t of that point, as described in Section 2.4. We reinforce that the description of continuum mechanics we are using is called the Lagrangian description (or material description). In principle a configuration can be given by an arbitrary diffeomorphism $\phi : \mathscr{B} \longrightarrow \mathbb{R}^3$. Recovering the discussion around the equation 2.43, the crucial hypothesis in the pseudo-rigid body formulation is that any configuration of the body is constrained to

$$\phi(t,x) = G(t)x, \tag{4.1}$$

where $G(t)$ is an invertible matrix, and further assuming incompressibility we impose $\det G(t) = 1$. For simplicity, we suppose that the center of mass is fixed at the origin.

Our objectives in this chapter are: to introduce energy dissipation due to internal viscosity under the pseudo-rigid body hypothesis (4.1) and to further simplify the pseudo-rigid body model under the extra hypothesis of small deformations $G(t)^T G(t) \approx \mathrm{Id} = Identity$.

By imposing the pseudo-rigid body constraint (4.1), the conserved quantities will be obtained from the symmetries of both the Lagrangian and the dissipation functions using a suitable generalization of Noether's theorem.

The remainder of the chapter is organized as follows. In section 4.3 the pseudo-rigid body hypothesis (4.1) is used to constraint the continuum-Lagrangian function to a finite number of degrees of freedom. Then the usual polar decomposition $G = YA$, where Y is a rotation matrix and A is a symmetric positive matrix with $\det(A) = 1$, is used to obtain the Lagrangian as a function of A, Y, and their time derivatives. The same is done for the Rayleigh dissipation function. Finally the Euler-Lagrange equations for the constrained motion are obtained in terms of Y and A. Most of the results in Section 4.3 are similar to those found in the pseudo-rigid body literature (see, for instance, [18]), except for the introduction of the Rayleigh dissipation function and the qualitative analysis of the dynamics of the system. The main contribution of the paper [63] is given in Section 4.4 and it is presented in the following.

In Section 4.4 we introduce the small-ellipticity hypothesis that stems from the almost round shape of most of the observed rotating celestial bodies. More precisely, let ε denote the ellipticity or flattening of the deformed body defined as:

$$\varepsilon = \frac{\text{equatorial - polar radius}}{\text{equatorial radius}}, \tag{4.2}$$

where the instantaneous polar radius is defined as the smallest semi-major axis of the ellipsoid $\{Ax : \|x\| \leq R\}$ and the instantaneous equatorial radius is defined as the arithmetic mean of the two remaining semi-major axis. The hypothesis is that ε is much smaller than one. Then $A = \exp(B) \approx \text{Id} + B$ where B is a symmetric traceless matrix such that

$$\|B\| = \sqrt{B_{11}^2 + B_{12}^2 \ldots} = \sqrt{\text{Tr}\,(BB^T)} \quad \text{is of the order of} \quad \varepsilon$$

Let $\Omega = Y^T \dot{Y}$ denote the instantaneous angular velocity of the body. The analysis of the relative equilibria solutions to the equations of motion given in Section 4.3 shows that $\varepsilon \ll 1$ requires that $\|\Omega\|$ is of the order of $\sqrt{\varepsilon}$. Using these scalings we obtain our simplified Lagrangian function truncating the pseudo-rigid body Lagrangian function given in Section 4.3 at order ε^2. We remark that a truncation using the equations of motion instead of the Lagrangian function leads to a different result! The advantage of the Lagrangian truncation is that the symmetries of the original Lagrangian function are naturally preserved. The novelty of this work relies on this Lagrangian truncation. We must stress that in the pseudo-rigid body approach the imposed linear deformation (4.1) neither verifies the differential equations of linear elasticity nor the appropriate boundary conditions. At the end, we obtain an elastic rigidity of the body, with respect to the centrifugal force, that is not so different from that obtained by [52] (see Section 4.4). We remark that Love obtains a nonlinear deformation as a solution of a linear equation and we propose a linear deformation as an approximation to the real solution.

Motivated by the ideas in the previous paragraph, we propose that under the small ellipticity hypothesis the equations for the motion of a body that is isolated, incompressible, and spherically symmetric at rest, are given by:

$$\begin{aligned}\ddot{B}+\nu\dot{B}+\gamma B &= -\Omega^2+\frac{1}{3}\,\mathrm{Tr}\,(\Omega^2)\,\mathrm{Id}\,, & (4.3)\\ \dot{\Omega}+B\dot{\Omega}+\dot{\Omega}B &= -\left(\Omega\dot{B}+\dot{B}\Omega+[\Omega^2,B]\right), & (4.4)\end{aligned}$$

where

ν is an effective viscosity constant $(1/sec)$;

γ is an effective rigidity constant $(1/sec^2)$;

Ω is the average angular velocity matrix of the body $(1/sec)$

$$\Omega=\begin{pmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{pmatrix}\qquad\text{with}\qquad \|\Omega\|^2=2(\omega_1^2+\omega_2^2+\omega_3^2); \tag{4.5}$$

$B(t)$ is the deformation matrix ($dimensionless$) that is proportional to the "quadrupole moment tensor" $Q(t)$:

$$B_{ij}=\frac{1}{3\mathrm{I}_\circ}Q_{ij},\quad\text{where}\qquad Q_{ij}=\int(3x_ix_j-|x|^2\delta_{ij})\rho(x,t)d^3x\qquad\text{and}$$

$\mathrm{I}_\circ$ is the angular momentum $(kg\cdot m^2)$ of the body at rest around an arbitrary axis passing through its center of mass.

For this model, Lagrangian, dissipation, and energy functions, and angular momentum with respect to an inertial frame, are given, respectively, by:

$$\begin{aligned}\mathscr{L}(B,\dot{B},\Omega) &= \frac{\mathrm{I}_\circ}{4}\left(\|\dot{B}\|^2+\|\Omega\|^2+2\,\mathrm{Tr}\,(\Omega\Omega^TB)\right)-\frac{\mathrm{I}_\circ}{4}\gamma\|B\|^2, & (4.6)\\ \mathscr{D}(\dot{B}) &= \nu\frac{\mathrm{I}_\circ}{4}\|\dot{B}\|^2, & (4.7)\\ E(B,\dot{B},\Omega) &= \frac{\mathrm{I}_\circ}{4}\left(\|\dot{B}\|^2+\|\Omega\|^2+2\,\mathrm{Tr}\,(\Omega\Omega^TB)\right)+\frac{\mathrm{I}_\circ}{4}\gamma\|B\|^2 & (4.8)\\ L(Y,\Omega,B) &= \mathrm{I}_\circ[Y(\Omega+\Omega B+B\Omega)Y^T]. & (4.9)\end{aligned}$$

The angular momentum is conserved under time evolution. The energy function is nonnegative if $\|B\| < 1/2$ (Lemma 16), which is implied by our underlying hypothesis $\|B\| \ll 1$. Moreover, a simple computation gives $\dot{E} = -2\mathscr{D} \leq 0$ with equality being reached if and only if $\dot{B} = 0$. These facts imply (Theorem 4) that any solution to the equations of motion that is initially in the set

$$\left\{(B,\dot{B},\Omega) : 0 \leq E < \mathrm{I}_\circ\gamma/20, \|B\| < 1/2\right\}$$

is attracted to a relative equilibrium solution where $\dot{B} = 0$ and $\dot{\Omega} = 0$. The eigenvalues of the linearized problem are easily computed from equations (4.3) and (4.4) as it is done in Section 4.4. In this section we also use center-manifold arguments to show that the dynamics of the simplified equations (4.3) and (4.4) is qualitatively the same, and quantitatively almost the same, as the dynamics of the pseudo-rigid body equations given in Section 4.3 provided that initially $\|B(0)\|, \|\dot{B}(0)\|, \|\Omega(0)\|$ are small.

Sections 4.2, 4.3, and 4.4 are based on the assumption of an idealized homogeneous elastic body. For this idealized body it is possible to compute from first principles all the constants $I_\circ$, γ, and ν, and to establish the relation between B and Q. For the idealized body, let: M be the mass, R be the radius of the undeformed spherical body, $g = MG/R^2$ be the acceleration of gravity at the body surface, ρ be the density, μ be the elastic (shear) modulus of rigidity (kg/ms^2) [49], and η be the viscosity (shear) coefficient (kg/ms) [49] [50]. For the idealized body:

$$\begin{aligned}
\mathrm{I}_\circ &= MR^2 2/5 \quad \text{is the moment of inertia of the solid ball,} \\
\gamma &= \frac{4}{5}\frac{GM}{R^3}\left(1+\frac{25}{2}\frac{\mu}{g\rho R}\right), \\
\nu &= 40\pi\eta R/3M.
\end{aligned} \tag{4.10}$$

These μ and η are "molecular constants" that in principle can be measured by means of simple laboratory experiments. Nevertheless, it is well-known that these molecular constants are inappropriate for use in most geophysical and astronomical models (see, for instance, [12], for a discussion about η). So, even for an approximately homogeneous body "effective" constants γ and ν must replace μ and η.

Planets and particularly stars are not homogeneous bodies. Their density is almost radially symmetric with an increasing value towards the center. It is not possible to use the idealized homogeneous body hypothesis in a naive way to study real celestial bodies. For instance, the real moment of inertia of the Sun is $\mathrm{I}_\circ = 0.059MR^2$ while the moment of inertia of the idealized homogeneous Sun is $0.4MR^2$. So, in order to obtain the correct expression $(\mathrm{I}_\circ/2)\sum \omega_i^2$ for the rotational kinetic energy we replaced the idealized moment of inertia $0.4MR^2$ in the Lagrangian function (4.41) in Section 4.4 by $\mathrm{I}_\circ$ in the Lagrangian function (4.6). This is equivalent to change the visual radius R of the body for an effective inertial radius (or "radius of gyration") R_g such that $0.4MR_g^2 = \mathrm{I}_\circ$. Let $\omega = (\omega_1, \omega_2, \omega_3)$ be the angular velocity vector associated to the angular velocity matrix Ω and, similarly, let $\boldsymbol{L}$ be the angular momentum vector associated to the angular momentum matrix L. Then the relation $\mathbf{L} = \mathrm{I}\omega$ (valid for inhomogeneous bodies) and equation (4.9) imply that the inertia matrix $\mathrm{I}(t)$ of the body at time t must satisfy

$$\mathrm{I}_{ij} = \mathrm{I}_\circ(\delta_{ij} - B_{ij})$$

This expression and the following relation between the moment of inertia tensor and the moment of quadrupole tensor (valid for non-homogeneous bodies),

$$Q_{ij} = -3\mathrm{I}_{ij} + (\mathrm{Tr}\,\mathrm{I})\delta_{ij}$$

imply that $B_{ij} = Q_{ij}/(3\mathrm{I}_\circ)$ as stated above. Finally, the angular velocity matrix Ω in the Lagrangian function (4.6) must be interpreted as the average angular velocity of the body in the sense that instantaneously $\boldsymbol{L} = \mathrm{I}\omega$. Through this procedure Ω is well-defined even for the Sun where the gas angular velocity is known to vary considerably with the latitude and with the distance to the center (see [65] and [1]). In this way we are able to extrapolate the results for an idealized homogeneous body given in Section 4.4 to the results for a inhomogeneous body given above.

For the Sun and for some planets of the solar system the value of γ can be obtained in the following way. Let Ω and Q be the steady angular velocity and moment of quadrupole tensor of the body given by

$$\Omega = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad Q = 3\mathrm{I}_\circ B = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}.$$

Defining the dynamic form factor $J_2 = \lambda/(MR^2)$, we get from equation (4.3) that

$$\gamma = \frac{\mathrm{I}_\circ}{MR^2}\frac{\omega^2}{J_2}. \tag{4.11}$$

Table 4.1: Data from bodies of the Solar System

Body	$\omega\,(10^{-5}s^{-1})$	$I_\circ/MR^2$	$J_2\,(10^{-6})$	$\gamma\,(10^{-6}s^{-2})$	$\tilde{\gamma}$
Sun	0.338	0.059	0.218	3.092	0.5562
Mercury	0.1240	0.35	50.3	0.0107	0.0072
Venus	0.0299	0.33	4.458	0.0066	0.0042
Earth	7.2921	0.3308	1082.63	1.625	0.9909
Moon	0.26617	0.394	202.7	0.01377	0.0180
Mars	7.0882	0.366	1960.45	0.9380	0.9331
Jupiter	17.5852	0.254	14736.	0.5330	0.9093
Saturn	16.3788	0.21	16298.	0.3457	0.8556
Uranus	10.1237	0.225	3343.43	0.6897	1.024
Neptune	10.8338	0.2555	3411.	0.8792	1.225

The constant γ has the dimension $1/sec^2$. A hypothetical homogeneous body with radius R_g and with elastic modulus of rigidity $\mu = 0$ has $\gamma = (4/5)GM/R_g^3$ according to equation (4.10). This value can be used to define a "dimensionless γ" as:

$$\tilde{\gamma} = \frac{\gamma}{\frac{4}{5}\frac{GM}{R_g^3}} \qquad \text{where} \qquad R_g = \sqrt{\frac{5}{2}\frac{I_\circ}{M}} \tag{4.12}$$

In Table 4.1 the values of γ and $\tilde{\gamma}$ are given for several bodies of the Solar system. For the planets we used the data provided in (`http://nssdc.gsfc.nasa.gov/planetary/factsheet/`). For the Sun the value of $I_\circ$ was taken from (`http://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfact.html`). Since the angular velocity of the gas in the Sun varies considerably with the position, the average angular velocity ω of the Sun was obtained from the formula $\omega = \|L\|/I_\circ = 3.38\times 10^{-6}s^{-1}$ where the Sun angular momentum $\|L\| = 1.92\times 10^{41}\,kg\,m^2\,s^{-1}$ was taken from [45]. The $J_2 = 2.18\times 10^{-7}$ of the Sun was taken from [1], see also [65]. The moment of inertia of Neptune ($I_\circ/(MR^2) = 0.2555$) was taken from [58].

As expected $\tilde{\gamma}$ is close to one for most of the celestial bodies in Table 4.1. The differences $\tilde{\gamma}-1$ may be mostly explained by the lack of radial homogeneity of the bodies, especially in the case of the Sun. The low values of $\tilde{\gamma}$ found for Mercury, Venus, and Moon cannot be explained by the lack of radial homogeneity. In this case a possible explanation is that these bodies do not have a spherical equilibrium shape at rest, which violates one of our hypotheses. Notice that a small residual plastic deformation, and its consequent residual J_{2res}, combined with a small value of angular velocity ω yield a low value for $\omega^2/(J_{2res}+\Delta J_2)$, where ΔJ_2 is the part of J_2 that is caused by the rotation of the body. For large values of ω, and so of ΔJ_2, the residual value J_{2res} lacks importance in this ratio.

In order to estimate an effective value for the viscosity η using equations (4.3) and (4.4) it is necessary to have measurements of non steady solutions to these equations. This is difficult. The value of η can be more easily estimated using tide measurements due to the gravitational interaction of the body with a second one. Such an analysis have already been performed in Chapters 2 and 3. In the present chapter, the extension to the two-body problem is presented in Section 4.5.

4.2 Some aspects of Euler-Lagrange equations with dissipation function

Since we are interested in restricting the degrees of freedom of the system, in this section we give some details on how the Euler Lagrange equations with dissipation, discussed in Section 2.4.1, can be established on a manifold.

Consider $L,D:\mathbb{R}^{2n}\longrightarrow\mathbb{R}$ two smooth functions. Avoiding mistakes, denote by $(Q,\dot{Q})$ the coordinates of $\mathbb{R}^{2n}$. Now suppose that a smooth curve $\gamma(t)\in\mathbb{R}^n$ solves the equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{Q}}(\gamma(t),\gamma'(t))\right)-\frac{\partial L}{\partial Q}(\gamma(t),\gamma'(t))+\frac{\partial D}{\partial\dot{Q}}(\gamma(t),\gamma'(t))=0,\quad\forall t,\tag{4.13}$$

where the symbol $\partial/\partial Q$ denotes the gradient, in the corresponding variable, with respect to the Euclidean metric. Hence consider a diffeomorphism $\varphi:U\longrightarrow V$, where $U,V\subset\mathbb{R}^n$ are open sets and $\gamma(0)\in U$. If we define $\mathscr{L}(q,\dot{q}):=L(\varphi(q),d\varphi_q(\dot{q}))$, $\mathscr{D}(q,\dot{q}):=D(\varphi(q),d\varphi_q(\dot{q}))$ and denote by $\partial_q f(q,\dot{q})$ the differential of a function in the respective variable, we see that along the same curve

$$\partial_q\mathscr{L}=\partial_Q L\circ d\varphi_q+\partial_{\dot{Q}}L\circ d^2\varphi_q(\dot{q},\cdot),\ \ \partial_{\dot{q}}\mathscr{L}=\partial_{\dot{Q}}L\circ d\varphi_q,\ \ \partial_{\dot{q}}\mathscr{D}_1=\partial_{\dot{Q}}D\circ d\varphi_q,$$

$$\Longrightarrow\frac{d}{dt}\partial_{\dot{q}}\mathscr{L}=\frac{d}{dt}\left(\partial_{\dot{Q}}L\right)\circ d\varphi_q+\partial_{\dot{Q}}L\circ d^2\varphi_q(\dot{q},\cdot),$$

where $d^2\varphi_q$ is the Hessian of φ at q. Hence,

$$\frac{d}{dt}\left(\partial_{\dot{q}}\mathscr{L}\right) - \partial_q\mathscr{L} + \partial_{\dot{q}}\mathscr{D} = \left[\frac{d}{dt}\left(\partial_{\dot{Q}}L\right) - \partial_Q L + \partial_{\dot{Q}}D\right] \circ d\varphi_q, \tag{4.14}$$

and since the right hand side is vanishing, for all t, we conclude that (4.13) also holds for $\mathscr{L}$ and $\mathscr{D}$ along the curve $\varphi^{-1}(\gamma(t))$. In other words, these equations do not depend on coordinate systems.

So given a submanifold $M \subset \mathbb{R}^n$ and two smooth functions $L, D : TM \longrightarrow \mathbb{R}$, we say that a smooth curve $\gamma : I \subset \mathbb{R} \longrightarrow M$, with $0 \in I$, $\gamma(0) = x$, solves the Euler-Lagrange equations with dissipation on x (relative to L and D) if there exists a chart $\psi : U \subset M \longrightarrow \tilde{U} \subset \mathbb{R}^k$, $x \in U$, such that

$$\frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{Q}}(d\psi(\gamma(t),\gamma'(t)))\right) - \frac{\partial\mathscr{L}}{\partial Q}(d\psi(\gamma(t),\gamma'(t))) \\ + \frac{\partial\mathscr{D}}{\partial\dot{Q}}(d\psi(\gamma(t),\gamma'(t))) = 0, \quad \forall t, \tag{4.15}$$

where the functions $\mathscr{L}, \mathscr{D} : \tilde{U} \times \mathbb{R}^k \longrightarrow \mathbb{R}$ are given by $\mathscr{L}(Q,\dot{Q}) = L \circ d\psi^{-1}(Q,\dot{Q})$, $\mathscr{D}(Q,\dot{Q}) = D \circ d\psi^{-1}(Q,\dot{Q})$, the expressions for L, D in such a chart. We stress that this approach can be performed intrinsically, through the use of fiber derivatives (see [9]), and the use of the metric is superfluous.

In this setting, we state the following theorem which is a reformulation of the Lagrange's Multipliers Theorem to the dissipative case. This version is enunciated in [2], but we present a proof for completeness.

Theorem 1. *Let $\mathscr{L}, \mathscr{D} : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ be smooth functions and $M \subset \mathbb{R}^n$ a submanifold. Suppose that $\gamma : I \subset \mathbb{R} \longrightarrow M$ solves the Euler-Lagrange equations with dissipation, relative to $\mathscr{L}|_{TM}$ and $\mathscr{D}|_{TM}$. Then γ also satisfies*

$$\pi_{\gamma(t)}\left[\frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{Q}}(\gamma(t),\gamma'(t))\right) - \frac{\partial\mathscr{L}}{\partial Q}(\gamma(t),\gamma'(t)) + \frac{\partial\mathscr{D}}{\partial\dot{Q}}(\gamma(t),\gamma'(t))\right] = 0, \tag{4.16}$$

where $\pi_x : \mathbb{R}^n \longrightarrow T_xM$ is the orthogonal projection relative to the Euclidean metric.

Proof. Given $x \in M$, consider a diffeomorphism $\tilde{\varphi} : V_1 \times V_2 \subset \mathbb{R}^k \times \mathbb{R}^{n-k} \longrightarrow U \subset \mathbb{R}^n$, $x \in U$, such that $\tilde{\varphi}(Q_1, 0) = \varphi(Q_1) \in M$, for all $Q_1 \in V_1$, where $\varphi : V_1 \longrightarrow M$ is a chart of M. So, let $L = \mathscr{L} \circ d\tilde{\varphi}, D = \mathscr{D} \circ d\tilde{\varphi}$ and $\tilde{\gamma} =$

$\tilde{\varphi}^{-1} \circ \gamma$ be the expressions of these functions in this chart with coordinates $(Q_1, Q_2, \dot{Q}_1, \dot{Q}_2) \in (V_1 \times V_2) \times \mathbb{R}^n$. The hypotheses ensure that the functional

$$\frac{d}{dt}\left(\partial_{\dot{Q}} L\right) - \partial_Q L + \partial_{\dot{Q}} D$$

vanishes on the vectors $(v_1, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ along the curve $\tilde{\gamma}$. The same computation that provided (4.14) shows that

$$\frac{d}{dt}\left(\partial_{\dot{q}}\mathscr{L}\right) - \partial_q \mathscr{L} + \partial_{\dot{q}}\mathscr{D} = \left[\frac{d}{dt}\left(\partial_{\dot{Q}} L\right) - \partial_Q L + \partial_{\dot{Q}} D\right] \circ (d\tilde{\varphi}_Q)^{-1},$$

hence, along the curve γ, the left hand side vanishes on the vectors $d\tilde{\varphi}_{\tilde{\gamma}(t)}(v_1, 0) = d\varphi_{\tilde{\gamma}(t)} v_1$, $\forall v_1 \in \mathbb{R}^k$. Since this is the tangent space $T_{\gamma(t)}M$, we see that this statement is equivalent to (4.16). □

Therefore, we stress that the essential advantages of this formalism rest on the independence on coordinate systems and the behaviour about restrictions to submanifolds (constraints).

In the modeling, it is imposed linearity of dissipative forces on the velocities. Usually this leads to a two-degree homogeneous dissipation function (on the velocities, of course), i.e. $\forall \lambda \in \mathbb{R}$, $\mathscr{D}(Q, \lambda \dot{Q}) = \lambda^2 \mathscr{D}(Q, \dot{Q})$. In this case, defining the "energy function" $E : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$,

$$E(Q, \dot{Q}) := \left\langle \frac{\partial \mathscr{L}}{\partial \dot{Q}}(Q, \dot{Q}), \dot{Q} \right\rangle - \mathscr{L}(Q, \dot{Q}) \tag{4.17}$$

a straightforward calculation shows that along a solution γ,

$$\frac{d}{dt} E(\gamma(t), \gamma'(t)) = -\left\langle \frac{\partial \mathscr{D}}{\partial \dot{Q}}(\gamma(t), \gamma'(t)), \gamma'(t) \right\rangle = -2\mathscr{D}(\gamma(t), \gamma'(t)), \tag{4.18}$$

where $<,>$ means the Euclidean inner product as stated in Theorem 1. In the last equality we used the Euler's theorem. We remark that although the equations of motion are only satisfied on the submanifold, they can be freely used in the above computation since we take the inner product of all the terms with the vector $\gamma'(t)$.

For natural Lagrangians $\mathscr{L}(Q, \dot{Q}) = T(Q, \dot{Q}) - V(Q)$, with T two-degree homogeneous on velocities, we note that the energy is

$$E(Q, \dot{Q}) = \left.\frac{d}{d\lambda}\right|_{\lambda=1} T(Q, \lambda \dot{Q}) - T(Q, \dot{Q}) + V(Q) = T(Q, \dot{Q}) + V(Q).$$

Remark that this homogeneity condition is also coordinate-free.

4.2.1 Dissipative Noether's theorem

We need to adapt Noether's theorem of Lagrangian mechanics to these modified systems.

Let $M \subset \mathbb{R}^n$ be a submanifold, G a Lie group and $\Phi : G \times M \longrightarrow M$ a smooth action. We say a dissipative system ($\mathbb{R}^n$, $\mathscr{L}$, $\mathscr{D}$) *admits a* $1-$*parameter symmetry* $\gamma : I \longrightarrow G$, on M, if:

$$\mathscr{L}\left(d\Phi_{\gamma(s)}(x,\dot{x})\right) = \mathscr{L}(x,\dot{x}), \qquad \partial_{\dot{x}}\mathscr{D}(x,\dot{x})\left(\frac{\partial}{\partial s}\bigg|_{s=0}\Phi_{\gamma(s)}(x)\right) = 0, \tag{4.19}$$

$\forall s \in I, (x,\dot{x}) \in TM$, with the notation $\Phi_g(x) := \Phi(g,x)$ and $d\Phi_g(x,\dot{x}) = (\Phi_g(x), D\Phi_g(x)[\dot{x}])$.

Theorem 2. *If a dissipative system* $(\mathbb{R}^n, \mathscr{L}, \mathscr{D})$ *admits a symmetry* $\gamma(s)$ *(*$\gamma(0) = e$*) on* M*, then it has the first integral:*

$$h(x,\dot{x}) := \partial_{\dot{x}}\mathscr{L}(x,\dot{x})\left(\frac{\partial}{\partial s}\bigg|_{s=0}\Phi_{\gamma(s)}(x)\right). \tag{4.20}$$

Proof. Using the chain rule and the equations of motion we see,

$$\frac{d}{dt}h(x,\dot{x}) = \frac{d}{dt}\frac{\partial}{\partial s}\bigg|_{s=0}\mathscr{L}\left(d\Phi_{\gamma(s)}(x,\dot{x})\right) - \partial_{\dot{x}}\mathscr{D}(x,\dot{x})\left(\frac{\partial}{\partial s}\bigg|_{s=0}\Phi_{\gamma(s)}(x)\right) \equiv 0.$$

□

Here, as before, the partial derivative denotes a functional.

4.3 Pseudo-rigid bodies

As mentioned in Chapter 2, a crucial point in the kinematics of a moving body with fixed center of mass is the factorization of the motion $\phi(t,x)$ into a rotation $Y(t)$ and a deformation $u(t,x)$, $\phi(t,x) = Y(t)u(t,x)$. If the body is rigid, $u(t,x) = x$, then an orthonormal reference frame can be fixed to the body. The motion of the body $\phi(t,x) = Y(t)x$ is determined by the "moving frame". There is no way to fix an orthonormal reference frame to a deformable body. The choice of a moving frame that captures the motion of a deformable body may be a difficult task. Though not unique, this choice is easier for the motion $\phi(t,x) = G(t)x$ of a pseudo-rigid body. In this case, a standard method (see [10], [17], [31], [47]) is to use the *singular value decomposition*

$$G = R\tilde{A}S^T, \qquad R,S \in \mathrm{SO}(3), \ \tilde{A} = \mathrm{diag}(a_1,a_2,a_3) > 0, \tag{4.21}$$

where $\mathrm{SO}(3)$ is the group of 3×3 orthogonal matrices. The matrix R represents the rotation of the body (shape) and S the circulation of the matter in its interior. This decomposition is not unique and there exists a smooth curve $t \to G(t)$ that does not admit any smooth singular decomposition (if $t \to G(t)$ is analytic, there is always an analytic singular decomposition, see [31]). For pseudo-rigid body motions another choice of moving frame is provided by the polar decomposition. Let $\mathrm{SL}(3)$ be the group of 3×3 matrices with determinant one and $\mathrm{SSym}_+(3)$ be the subset of $\mathrm{SL}(3)$ of symmetric and positive matrices. Then given $G \in \mathrm{SL}(3)$, there are unique matrices $Y \in \mathrm{SO}(3)$ and $A \in \mathrm{SSym}_+(3)$ such that $G = YA$. Therefore, the mapping

$$\Phi : \mathrm{SSym}_+(3) \times \mathrm{SO}(3) \longrightarrow \mathrm{SL}(3),$$

$\Phi(A,Y) := YA$, is bijective. Moreover, we claim the following well known lemma, whose proof is presented for completeness.

Lemma 1. *Let* $\mathrm{Sym}(3)$ *be the set of symmetric* 3×3 *matrices, which is diffeomorphic to* $\mathbb{R}^6$*. We have that* $\mathrm{SSym}_+(3) \subset \mathrm{Sym}(3)$ *is a smooth 5-dimensional submanifold and the map* Φ *is a diffeomorphism.*

Proof. We claim that $\mathrm{SSym}_+(3)$ is a $5-$dimensional submanifold of $\mathrm{Sym}(3)$. Indeed, $\mathrm{SSym}_+(3)$ is the preimage of one by the function $\det : \mathrm{Sym}(3) \longrightarrow \mathbb{R}$. Moreover, the derivative of det is non-singular on $\mathrm{SSym}_+(3)$ because if $M \in \mathrm{SSym}_+(3)$

$$D\det_M(M) = \det(M)\,\mathrm{Tr}\,(\,\mathrm{Id}\,) = 3.$$

Therefore one is a regular value of $\det : \mathrm{Sym}(3) \longrightarrow \mathbb{R}$, which ensures that $\mathrm{SSym}_+(3)$ is a submanifold of $\mathrm{Sym}(3)$. Now recall that Φ is bijective and its domain and image have the same dimensions. So its enough to show that the kernel of $D\Phi_{(A,Y)}$ is trivial, for all A,Y. Given $A_0 \in \mathrm{SSym}_+(3)$, define the smooth function $f_{A_0} : \mathrm{SSym}_+(3) \longrightarrow \mathrm{SSym}_+(3)$, $f_{A_0}(A) :=$

$\sqrt{A_0}A\sqrt{A_0}$. Note f_{A_0} is a diffeomorphim, with $f_{A_0}^{-1} = f_{A_0^{-1}}$. So, we conclude

$$T_{A_0}\,\mathrm{SSym}_+(3) \equiv Df_{A_0}(T_{\mathrm{Id}}\ \mathrm{SSym}_+(3)) = \{\sqrt{A_0}B\sqrt{A_0} : B = B^T,\ \mathrm{Tr}\,B = 0\}$$
$$= \{H \in \mathrm{Sym}(3)\ :\ \mathrm{Tr}\,(A_0^{-1}H) = 0.\} \tag{4.22}$$

Now, take an arbitrary path $(A(t),Y(t)) \in \mathrm{SSym}_+(3)\times \mathrm{SO}(3)$, with $(A(0),Y(0)) = (A_0,Y_0)$. Supposing $D\Phi_{(A_0,Y_0)}(\dot A(0),\dot Y(0)) = 0$,

$$0 = \left.\frac{d}{dt}\right|_{t=0} Y(t)A(t) = Y_0\xi A_0 + Y_0\sqrt{A_0}B\sqrt{A_0},$$

where $\xi = Y_0^T\dot Y(0) \in \mathrm{skew}(3)$.

$$\Rightarrow -\sqrt{A_0}B\sqrt{A_0}^{-1} = \xi = (-\xi)^T = \sqrt{A_0}^{-1}B\sqrt{A_0} \Rightarrow \det B = -\det B.$$

So, the symmetric $B \in T_{\mathrm{Id}}\ \mathrm{SSym}_+(3)$ have determinant and trace vanishing. Then, for some $Y \in \mathrm{SO}(3)$, $Y^TBY = \mathrm{diag}(\lambda,-\lambda,0)$. From the equality $BA_0 = -A_0B$, we get

$$\left\langle e_1, Y^TBYY^TA_0Ye_1\right\rangle = -\left\langle e_1, Y^TA_0YY^TBYe_1\right\rangle$$
$$\Rightarrow \left\langle Y^TBYe_1, Y^TA_0Ye_1\right\rangle = -\lambda\left\langle e_1, Y^TA_0Ye_1\right\rangle$$
$$\Rightarrow \lambda\left\langle Ye_1, A_0Ye_1\right\rangle = -\lambda\left\langle Ye_1, A_0Ye_1\right\rangle \Rightarrow \lambda = 0,$$

where we have used $Ye_1 \neq 0$ and that A_0 is positive. Therefore $B = 0, \xi = 0 \Rightarrow (\dot A(0),\dot Y(0)) = (0,0)$. □

Therefore to each smooth motion $G(t) \in \mathrm{SL}(3)$ corresponds a unique smooth motion $(A(t),Y(t)) \in \mathrm{SSym}_+(3)\times\mathrm{SO}(3)$ (see [25] for an algorithm to find $(A(t),Y(t))$). So, in the following we use the polar decomposition to investigate the dynamics of an incompressible homogeneous body ($\rho(x) = \rho =$ constant, $\mu(x) = \mu =$ constant, $\eta(x) = \eta =$ constant, $\cdots$) under the pseudo-rigid body hypothesis (4.1).

Let $\phi(t,x) = Y(t)A(t)x$, $x \in \mathscr{B}$, be the function that describes the motion of a body that at rest has the shape of a ball $\mathscr{B} = \{x \in \mathbb{R}^3 : \|x\| \le R\}$. Using that the total mass of the body is $M = 4\pi R^3\rho/3$ and that for every 3×3 matrix

$$\int_{\mathscr{B}} \langle x, Cx\rangle\, dx = \frac{4\pi R^5}{15}\,\mathrm{Tr}\,(C), \tag{4.23}$$

we obtain that the kinetic energy (2.53) of the body is given by

$$T(Y,A,\dot{Y},\dot{A}) = \frac{MR^2}{10}\left(\operatorname{Tr}(\dot{A}^2) - \operatorname{Tr}(\Omega^2 A^2) + 2\operatorname{Tr}(\dot{A}\Omega A)\right). \tag{4.24}$$

Using that the pseudo-rigid body deformation is $u(t,x) = A(t)x$, the strain tensors (2.45) become:

$$\mathscr{E} = \frac{1}{2}(A + A^T) - \mathrm{Id} = A - \mathrm{Id}, \qquad \dot{\mathscr{E}} = \dot{A}.$$

So, the elastic potential (2.47) and dissipation function (2.52) become

$$V_{el}(A) = \frac{4\pi R^3}{3}\left(\mu\left(\operatorname{Tr}(A^2) - \frac{1}{3}\operatorname{Tr}(A)^2\right) + \frac{K}{2}\left(\operatorname{Tr}(A) - 3\right)^2\right), \tag{4.25}$$

$$\mathscr{D}(\dot{A}) = \frac{4\pi R^3}{3}\left(\eta\left(\operatorname{Tr}(\dot{A}^2) - \frac{1}{3}\operatorname{Tr}(\dot{A})^2\right) + \frac{\zeta}{2}\operatorname{Tr}(\dot{A})^2\right), \tag{4.26}$$

and the self-gravitational potential (2.49) becomes

$$V_g(A) = -\frac{\rho^2 G}{2}\int_{\mathscr{B}}\int_{\mathscr{B}} \frac{1}{\|A(x-y)\|}dxdy. \tag{4.27}$$

It is easy to check (since $A > 0$) that $V_g(A)$ is differentiable. The function $V_g(A)$ can also be written as

$$V_g(A) = -\frac{3M^2 G}{10R}\int_0^\infty \frac{1}{\sqrt{\det(A^2 + \lambda\,\mathrm{Id})}}d\lambda, \quad \forall A \in \mathrm{SSym}_+(3), \tag{4.28}$$

which is formula known to Dirichlet. For a proof, see [17] or [71] and for a recent discussion see [46]. The next three lemmas show that the potentials above have some natural properties. The first shows that the elastic stresses generated by (4.25) tend to restore the body to the relaxed shape, like a spring.

Lemma 2. *Take any* $\mu, K \geq 0$*, with at least one positive, and* $A \in \mathrm{SSym}_+(3)$. *Then we have*

(i) $\operatorname{Tr}(A)^2 \leq 3\operatorname{Tr}(A^2)$*;*

(ii) $\operatorname{Tr}(A) \geq 3$*;*

(iii) $V_{el}(A) \geq 0$.

And the equalities are reached if and only if $A = \mathrm{Id}$.

Proof. The Cauchy-Schwarz inequality for the inner product $\langle H_1, H_2\rangle := \mathrm{Tr}\,(H_1^T H_2)$ gives: $\mathrm{Tr}\,(A) = \langle A, \mathrm{Id}\,\rangle \leq \|A\| \|\,\mathrm{Id}\,\| = \sqrt{3}\sqrt{\mathrm{Tr}\,(A^2)}$. The equality holds if and only if $A = \lambda\,\mathrm{Id}$, i.e., $A = \mathrm{Id}$. Let $x, c > 0$ be such that x^{-1}, c^{-1}, xc are the eigenvalues of A. Defining $f_c(x) := cx + x^{-1} + c^{-1}$, $f_c'(x) = cx^{-2}(x^2 - c^{-1})$. So $f_c(x)$, for $x > 0$, has a global minimum at $x = c^{-\frac{1}{2}}$. Hence, $f_c(x) \geq f_c(c^{-\frac{1}{2}}) = h(c) := 2c^{\frac{1}{2}} + c^{-1}$ $\forall x, c > 0$. But, $h'(c) = c^{-2}(c^{\frac{3}{2}} - 1)$, then $h(c) \geq h(1) = 3$. Thus, $\mathrm{Tr}\,(A) = f_c(x) \geq h(c) \geq 3$, and since the minima are strict, the equality holds only for $A = \mathrm{Id}$. The last assertion follows from (i), (ii) and (4.25). ☐

The next lemma shows that the self-gravitational potential (4.28) has a global minimum exactly when the body has the shape of a ball.

Lemma 3. *For all* $A \in \mathrm{SSym}_+(3)$,

$$-\frac{3M^2G}{5R} \leq V_g(A) < 0.$$

The equality holds if and only if $A = \mathrm{Id}$.

Proof. Let $a_1, a_2, a_3 > 0$ be the eigenvalues of A. Thus, $\det(A^2 + \lambda\,\mathrm{Id}\,) = \lambda^3 + (a_1^2 + a_2^2 + a_3^2)\lambda^2 + (a_1^2a_2^2 + a_1^2a_3^2 + a_2^2a_3^2)\lambda + (a_1a_2a_3)^2 = \lambda^3 + \mathrm{Tr}\,(A^2)\lambda^2 + \mathrm{Tr}\,(\tilde{A})\lambda + 1$, where $\tilde{A} = \mathrm{diag}(a_1^2a_2^2, a_1^2a_3^2, a_2^2a_3^2) \in \mathrm{SSym}_+(3)$. Hence, from (ii) of Lemma 2, follows $\det(A^2 + \lambda\,\mathrm{Id}\,) \geq \lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3$. So,

$$V_g(A) \geq -\frac{3M^2G}{10R}\int_0^\infty (\lambda + 1)^{-\frac{3}{2}} d\lambda = -\frac{3M^2G}{5R}.$$

☐

The next lemma shows that energy dissipation ceases along with the internal motion.

Lemma 4. *Take any* $\eta > 0$, $\zeta \geq 0$ *and* $\dot{A} \in T_A\,\mathrm{SSym}_+(3)$. *Therefore* $\mathscr{D}(\dot{A}) \geq 0$, *and equality holds if and only if* $\dot{A} = 0$.

Proof. Note that the arguments used in Lemma 2, item (i), can be applied here, ensuring that $\mathscr{D} \geq 0$. Thus, if $\mathscr{D}(\dot{A}) = 0$, then $\dot{A} = \lambda\,\mathrm{Id}$. But, $0 = \mathrm{Tr}\,(A^{-1}\dot{A}) = \lambda\,\mathrm{Tr}\,(A^{-1})$. So $\dot{A} = 0$. ☐

In order to obtain the equations of motion for the deformation A, we use Theorem 1. Recalling the expression for $T_A\,\mathrm{SSym}_+(3)$ from (4.22), the orthogonal projection in this case is given by the following.

Lemma 5. *The orthogonal projection* $P_A : M(3) \longrightarrow T_A\,\mathrm{SSym}_+(3)$ *is*

$$P_A(H) = \frac{1}{2}(H + H^T) - \frac{\mathrm{Tr}\,(A^{-1}H)}{\mathrm{Tr}\,(A^{-2})}A^{-1}, \quad \forall H \in M(3).$$

Proof. Note that $P_A(H) \in \mathrm{Sym}(3)$, $\forall H \in M(3)$, $\mathrm{Tr}\,(A^{-1}P_A(H)) = 0$ and is a linear map. So, from (4.22), P_A is well-defined. If $H \in T_A\,\mathrm{SSym}_+(3)$, $P_A(H) = H$. Then P_A is surjective and $P_A^2 = P_A$. So, we only need to verify that its kernel is orthogonal to its image. This follows, since $\forall H_1, H_2$:

$$\begin{aligned}\langle P_A(H_1), H_2\rangle = \frac{1}{2}(\,\mathrm{Tr}\,(H_1H_2) + \mathrm{Tr}\,(H_1^TH_2))\\ - \frac{1}{\mathrm{Tr}\,(A^{-2})}\mathrm{Tr}\,(A^{-1}H_1)\,\mathrm{Tr}\,(A^{-1}H_2) = \langle H_1, P_A(H_2)\rangle\,.\end{aligned}$$

□

Finally defining $\mathscr{L} = T - V_{el} - V_g$, the equations of motion are given by

$$P_A\left[\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{A}}\right) - \frac{\partial \mathscr{L}}{\partial A} + \frac{\partial \mathscr{D}}{\partial \dot{A}}\right] = 0,$$

so

$$\begin{aligned}&\ddot{A} + \frac{1}{2}[\dot{\Omega}, A] + [\Omega, \dot{A}] + \frac{1}{2}(\Omega^2 A + A\Omega^2) + \\ &\frac{20\pi R}{3M}\left[2\mu A + \left(K - \frac{2\mu}{3}\right)\mathrm{Tr}\,(A)\,\mathrm{Id} - 3K\,\mathrm{Id} + 2\eta\dot{A} + \left(\zeta - \frac{2\eta}{3}\right)\mathrm{Tr}\,(\dot{A})\,\mathrm{Id}\right] \\ &\qquad + \frac{3MG}{2R^3}\int_0^\infty \frac{A(A^2 + \lambda\,\mathrm{Id})^{-1}}{\sqrt{\det(A^2 + \lambda\,\mathrm{Id})}}d\lambda + \chi A^{-1} = 0 \qquad (4.29)\end{aligned}$$

where,

$$\chi = \frac{1}{\mathrm{Tr}\,(A^{-2})}\Bigg\{ 2\,\mathrm{Tr}\,(A^{-1}\dot{A}\Omega) - \mathrm{Tr}\,(A^{-1}\dot{A}A^{-1}\dot{A}) - \mathrm{Tr}\,(\Omega^2)$$
$$- \frac{20\pi R}{3M}\left[6\mu + \left(K - \frac{2\mu}{3}\right)\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(A^{-1}) - 3K\,\mathrm{Tr}\,(A^{-1})\right.$$
$$\left. + \left(\zeta - \frac{2\eta}{3}\right)\mathrm{Tr}\,(\dot{A})\,\mathrm{Tr}\,(A^{-1})\right] - \frac{3MG}{2R^3}\int_0^\infty \frac{\mathrm{Tr}\,((A^2+\lambda\,\mathrm{Id}\,)^{-1})}{\sqrt{\det(A^2+\lambda\,\mathrm{Id}\,)}}d\lambda \Bigg\}.$$

Notice that the initial condition $(A_0, \dot{A}_0)$ must satisfy the constraints: $A_0 \in \mathrm{SSym}_+(3)$, $\dot{A}_0$ symmetric, and $\mathrm{Tr}\,(A_0^{-1}\dot{A}_0) = 0$. Since $\det A(t) = 1$ the positiveness of $A(t)$ is ensured for all t.

The equations of motion for the rotation Y are obtained in the same way as those for A. The set of matrices Y is considered as a subset of the vector-space of 3×3 matrices that satisfy $Y^TY = \mathrm{Id}$. We have the auxiliary lemmas.

Lemma 6. *Taking $Y \in \mathrm{SO}(3)$, let $P_Y : M(3) \longrightarrow T_Y\,\mathrm{SO}(3)$ be the orthogonal projection. For every $H \in M(3)$,*

$$P_Y H = Y(Y^TH)_a, \tag{4.30}$$

where $B_a := 1/2(B - B^T)$ is the skew-symmetric part of B.

Proof. Remark that $f_Y : M(3) \longrightarrow M(3)$, $f_Y(H) := YH$, is an isometry for all $Y \in \mathrm{SO}(3)$. So we have the orthogonal decomposition $M(3) = Y\,\mathrm{skew}(3) \oplus Y\,\mathrm{Sym}(3) = T_Y\,\mathrm{SO}(3) \oplus Y\,\mathrm{Sym}(3)$. Since $H = Y((Y^TH)_a + (Y^TH)_s)$, $\forall H \in M(3)$, the lemma follows. □

Lemma 7. *Suppose that for a smooth function $T : M(3)\times M(3) \longrightarrow \mathbb{R}$ exists a smooth function $K :\ \mathrm{skew}(3) \longrightarrow \mathbb{R}$ such that $T(Y,\dot{Y}) = K(Y^T\dot{Y})$, $\forall (Y,\dot{Y}) \in T\,\mathrm{SO}(3)$. So, along every smooth path $Y(t) \in \mathrm{SO}(3)$,*

$$P_Y\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Y}}\right) - \frac{\partial T}{\partial Y}\right) = Y\left(\frac{dW}{dt}(t) + [\Omega, W(t)]\right), \tag{4.31}$$

where $\Omega = Y^T\dot{Y} \in \mathrm{skew}(3)$ and $W(t) = \left(\frac{\partial K}{\partial \Omega}\right)_a$.

Proof. For every $H \in M(3)$,

$$\left\langle \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Y}}\right) - \frac{\partial T}{\partial Y}, H \right\rangle = \frac{d}{dt}\left(\frac{d}{d\lambda}\bigg|_{\lambda=0} K(Y^T(\dot{Y}+\lambda H)) \right)$$

$$- \frac{d}{d\lambda}\bigg|_{\lambda=0} K((\dot{Y}+\lambda H)^T \dot{Y}) = \left\langle Y \frac{d}{dt}\left(\frac{\partial K}{\partial \Omega}\right) + \dot{Y}\left(\frac{\partial K}{\partial \Omega} - \frac{\partial K}{\partial \Omega}^T\right), H \right\rangle,$$

hence

$$Y\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Y}}\right) - \frac{\partial T}{\partial Y}\right) = \frac{d}{dt}\left(\frac{\partial K}{\partial \Omega}\right) + \Omega\left(\frac{\partial K}{\partial \Omega} - \frac{\partial K}{\partial \Omega}^T\right),$$

and the result follows by Lemma 6. □

The function $\mathscr{L}$ fulfils the hypotheses of Lemma 7. So, we have

$$W = \frac{MR^2}{10}(A^2\Omega + \Omega A^2 + \dot{A}A - A\dot{A}),$$

$$\frac{dW}{dt}(t) + [\Omega, W(t)] = 0$$

$$\Rightarrow A^2\dot{\Omega} + \dot{\Omega}A^2 + 2\Omega\dot{A}A + 2A\dot{A}\Omega + [\Omega^2, A^2] = [A, \ddot{A}]. \tag{4.32}$$

Finally we use (4.29) to eliminate $\ddot{A}$ from equation (4.32) to obtain

$$\Psi_A(\dot{\Omega}) + 2\Omega\dot{A}A + 2A\dot{A}\Omega + 2A\Omega\dot{A} + 2\dot{A}\Omega A + [\Omega^2, A^2] = 0, \tag{4.33}$$

where $\Psi_A : \mathrm{skew}(3) \longrightarrow \mathrm{skew}(3)$ is the linear operator $\Psi_A(H) := A^2H + HA^2 + 2AHA$.

Lemma 8. *For every $A \in \mathrm{SSym}_+(3)$, the operator Ψ_A is invertible.*

Proof. Take $H \in \mathrm{skew}(3)$ such that $\Psi_A(H) = 0$. In a basis such that $A = \mathrm{diag}(a_1, a_2, a_3)$,

$$H = \begin{pmatrix} 0 & -h_3 & h_2 \\ h_3 & 0 & -h_1 \\ -h_2 & h_1 & 0 \end{pmatrix}$$

we see that H must satisfy

$$\begin{pmatrix} 0 & -(a_1+a_2)^2 h_3 & (a_1+a_3)^2 h_2 \\ (a_1+a_2)^2 h_3 & 0 & -(a_2+a_3)^2 h_1 \\ -(a_1+a_3)^2 h_2 & (a_2+a_3)^2 h_1 & 0 \end{pmatrix} = 0.$$

So, $H = 0$. □

Therefore, we have a well-posed problem, since we can write the system of equations (4.29), (4.33) and $\dot{Y} = Y\Omega$ in an explicit form.

Now, we describe the qualitative aspects of the motion.

Lemma 9. *The angular momentum on an inertial frame*

$$L(\Omega, A, \dot{A}) := \frac{1}{2} Y(A^2\Omega + \Omega A^2 + [\dot{A}, A])Y^T = \frac{5}{MR^2} YWY^T \qquad (4.34)$$

is conserved by the motion.

Proof. Consider the action $\Phi: \mathrm{SO}(3) \times (\mathrm{SO}(3) \times \mathrm{SSym}_+(3)) \longrightarrow \mathrm{SO}(3) \times \mathrm{SSym}_+(3)$, $\Phi(U,Y,A) := (UY, A)$. For every $\xi \in \mathrm{skew}(3)$, we have $\mathscr{L}(d\Phi_{e^{s\xi}}((Y,A),(\dot{Y},\dot{A}))) = \mathscr{L}((e^{s\xi}Y, A),(e^{s\xi}\dot{Y}, \dot{A})) = \mathscr{L}((Y,A),(\dot{Y},\dot{A}))$. Note that $\mathscr{D}$ is independent of $Y, \dot{Y}$, then the second condition of (4.19) is fulfilled. Hence, by Theorem 2,

$$\mathrm{Tr}\left((YA^2\Omega Y^T + Y\Omega A^2 Y^T - 2YA\dot{A}Y^T)\xi^T\right)$$

is a first integral, for all $\xi \in \mathrm{skew}(3)$. Then, its skew-symmetric part is conserved. □

Now, consider the action $\Phi(U,Y,A) := (YU^T, UAU^T)$. Note that

$$\begin{aligned} \mathscr{L}(d\Phi_{e^{s\xi}}((Y,A),(\dot{Y},\dot{A}))) &= \mathscr{L}((Ye^{-s\xi}, e^{s\xi}Ae^{-s\xi}),(\dot{Y}e^{-s\xi}, e^{s\xi}\dot{A}e^{-s\xi})) \\ &= \mathscr{L}((Y,A),(\dot{Y},\dot{A})). \end{aligned}$$

However,

$$\partial_{\dot{x}}\mathscr{D}\left(\frac{\partial}{\partial s}\bigg|_{s=0} \Phi_{e^{s\xi}}(x)\right) = \frac{16\pi R^3 \eta}{3} \mathrm{Tr}(\dot{A}\xi A) \qquad (4.35)$$

does not vanish, except for $\eta = 0$. In the case $\eta = 0$, given any $\xi \in \mathrm{skew}(3)$ we have the additional first integral

$$h = \partial_{\dot{x}}\mathscr{L}(-Y\xi, \xi A - A\xi) = \mathrm{Tr}(\xi^T(A\dot{A} + A\Omega A)).$$

This implies that, for $\eta = 0$, $\Sigma := [A, \dot{A}] + 2A\Omega A$ is conserved. This Σ is called *vorticity* in [10] (equation (29)) and *circulation* in [17].

Equilibria

One easily checks that both $\mathscr{L}(Y,A,\dot{Y},\dot{A})$, $\mathscr{D}(Y,A,\dot{Y},\dot{A})$ are two-degree homogeneous functions in the velocities. So, from (4.18) we see that the energy $E := T + V_g + V_{el} + 3M^2G/(5R)$ is such that

$$\dot{E} = -2\mathscr{D} \leq 0,$$

where the last inequality comes from Lemma 4. In other words, $E : T(\,\mathrm{SO}(3) \times \mathrm{SSym}_+(3)\,) \longrightarrow \mathbb{R}$ is a Liapunov function. Note that the kinetic energy (4.24) can be rewritten as

$$T(Y,A,\dot{Y},\dot{A}) = \frac{MR^2}{10}\|\dot{A} + \Omega A\|^2 \geq 0.$$

This and Lemmas 2 and 3 imply that $E \geq 0$.

Lemma 10. *Suppose that $\mu > 0$. Then given $E_0 \geq 0$, the set $E^{-1}([0,E_0]) \subset T\,\mathrm{SSym}_+(3) \times \mathrm{skew}(3)$ is compact where $\mathrm{skew}(3)$ denote the set of 3×3 skew-symmetric matrices.*

Proof. Notice that $E^{-1}([0,E_0])$ is closed in $T\,\mathrm{SSym}_+(3) \times \mathrm{skew}(3)$ that is closed in the set in $W = M \times M \times \mathrm{skew}(3)$ where M is the vector space of 3×3 matrices. We will show that there is no sequence $(A_n, \dot{A}_n, \Omega_n) \in E^{-1}([0,E_0])$ that is unbounded in W. The definition of E, Lemma 3, and the positivity of T imply:

$$\frac{4\pi\mu R^3}{3}\left(\mathrm{Tr}\,(A_n^2) - \frac{1}{3}\mathrm{Tr}\,(A_n)^2\right) \leq E \leq E_0, \quad \forall n, \tag{4.36}$$

$$\frac{MR^2}{10}\left(\|\dot{A}_n + \Omega_n A_n\|^2\right) \leq E \leq E_0. \quad \forall n. \tag{4.37}$$

From (4.36) we see that the norms of the vectors

$$A_n - \frac{< A_n, \mathrm{Id} >}{\|\,\mathrm{Id}\,\|^2}\,\mathrm{Id}$$

are bounded. So, the projection of the sequence A_n on any vector orthogonal to Id is bounded. Then, writing $A_n = s_n\,\mathrm{Id} + \alpha_n$, where $<\alpha_n, \mathrm{Id}>= 0$, we get that the $\|\alpha_n\|$ is bounded and the sequence $\|A_n\|$ is unbounded if, and only if, the sequence $|s_n|$ is unbounded. But if $|s_n|$ is sufficiently large

$s_n^{-1}\alpha_n$ is close to zero and $\det(A_n) = s_n^3 \det(\mathrm{Id} + s_n^{-1}\alpha_n) > 1$ that is impossible because $\det(A_n) = 1$. So A_n is bounded. Denote $(A_n)_{ij} = a^n_{ij}$ and $(\dot{A}_n)_{ij} = b^n_{ij}$. Equation (4.37) shows that $\dot{A}_n + \Omega_n A_n$ is bounded, as well as its skew-symmetric part $(A_n\Omega_n + \Omega_n A_n)/2$. Taking A_n diagonal again, we see that $S(A_n\Omega_n + \Omega_n A_n) = ((a^n_{22} + a^n_{33})\omega^n_1, (a^n_{11} + a^n_{33})\omega^n_2, (a^n_{11} + a^n_{22})\omega^n_3)$ (S defined in 2.58). Since none of the coefficients $(a^n_{ii} + a^n_{jj})$ can accumulate at zero, because $a^n_{11}a^n_{22}a^n_{33} = 1$, we conclude that the sequence Ω_n is bounded. Hence $\dot{A}_n$ must also be bounded. □

Therefore, we are able to apply the LaSalle invariance principle in the same spirit that [4].

Lemma 11 (LaSalle's Invariance Principle)**.** *Let $\gamma^+(x_0)$ be the positive orbit of the initial condition x_0 in the phase space M. If $\gamma^+(x_0)$ is bounded and V is a Liapunov function on M, then the $\omega-$limit set of this solution is contained in the largest invariant subset (by the flow) of $\{x \in M : \dot{V}(x) = 0\}$.*

For a proof, consult [41]. We denote the largest invariant subset under the flow contained in $\{X = (Y, \Omega, A, \dot{A}) : \dot{E}(X) = 0\}$ by $\mathscr{A}$. So, by Lemma 10, the $\omega-$limit set of *every* initial condition is contained in $\mathscr{A}$. Taking an initial condition on $\mathscr{A}$, since it is invariant, its flow is such that $\dot{E}(X(t)) = -2\mathscr{D}(X(t)) \equiv 0$. Then, by Lemma 4, $A(t) = A$, $\forall t \geq 0$. The next lemma shows that any point in the attracting set $\mathscr{A}$ is a relative equilibrium.

Lemma 12. $\dot{\Omega} = 0$ *on the set* $\mathscr{A}$.

Proof. Since for all $R_0 \in \mathrm{SO}(3)$, $(R_0 Y(t) R_0^T, R_0\Omega(t)R_0^T, R_0 A R_0^T, 0)$ is also a solution, without loss of generality, we may assume $A = \mathrm{diag}(a_1, a_2, a_3)$. Since $\dot{A} = 0$ on $\mathscr{A}$, equations (4.29) and (4.33) imply

$$A^2\dot{\Omega} + \dot{\Omega}A^2 + 2A\dot{\Omega}A + [\Omega^2, A^2] = 0,$$

$$\frac{1}{2}[\dot{\Omega}, A] + \frac{1}{2}(\Omega^2 A + A\Omega^2) + D = 0,$$

where D is a diagonal matrix. The first of these equations imply:

$$\begin{cases} (a_1^2 + a_2^2)\dot{\omega}_3 + 2a_1a_2\dot{\omega}_3 - (a_2^2 - a_1^2)\omega_1\omega_2 = 0 \\ (a_1^2 + a_3^2)\dot{\omega}_2 + 2a_1a_3\dot{\omega}_2 + (a_3^2 - a_1^2)\omega_1\omega_3 = 0 \\ (a_2^2 + a_3^2)\dot{\omega}_1 + 2a_2a_3\dot{\omega}_1 - (a_3^2 - a_2^2)\omega_2\omega_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (a_1+a_2)\dot{\omega}_3+(a_1-a_2)\omega_1\omega_2=0 \\ (a_1+a_3)\dot{\omega}_2+(a_3-a_1)\omega_1\omega_3=0 \\ (a_2+a_3)\dot{\omega}_1+(a_2-a_3)\omega_2\omega_3=0 \end{cases}$$

The off diagonal terms of the second of those equations imply:

$$\begin{cases} (a_2-a_1)\dot{\omega}_3+(a_1+a_2)\omega_1\omega_2=0 \\ (a_1-a_3)\dot{\omega}_2+(a_1+a_3)\omega_1\omega_3=0 \\ (a_3-a_2)\dot{\omega}_1+(a_2+a_3)\omega_2\omega_3=0 \end{cases}$$

so,

$$\begin{cases} ((a_1+a_2)^2+(a_2-a_1)^2)\dot{\omega}_3=0 \\ ((a_1+a_3)^2+(a_1-a_3)^2)\dot{\omega}_2=0 \\ ((a_2+a_3)^2+(a_3-a_2)^2)\dot{\omega}_1=0 \end{cases}$$

and the conclusion follows. □

We use again that equations (4.29) and (4.33) are invariant under rotations to see that the system can be initially rotated such that the angular momentum vector, equation (4.34), has the form $(0,0,\ell)$. We aim to understand qualitatively the set of all relative equilibria of equations (4.29) and (4.33) for a fixed value of ℓ. Since $\dot{\Omega}=0$ and $\dot{Y}(t)=Y(t)\Omega$ we conclude that Ω commutes with $Y(t)$. Differentiating with respect to t the angular momentum equation $2Y^TLY=A^2\Omega+\Omega A^2$ we conclude that L commutes with Ω and therefore the vector Ω is also of the form $(0,0,\omega_3)$. Equation (4.33) with $\dot{A}=0$ and $\dot{\Omega}=0$ gives

$$0=[\Omega^2,A^2]=A[\Omega^2,A]+[\Omega^2,A]A$$

that implies $[\Omega^2,A]=0$ because A is positive definite. This equation implies that A has the form:

$$A=\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}=\tilde{A}+\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \tag{4.38}$$

The relative equilibria can be of three types depending on the equilibrium shape of the body being: a sphere, an ellipsoid of revolution or a triaxial ellipsoid. The spherical shape, $A=\mathrm{Id}$, can only occur for $\ell=0$ and it is the only relative equilibrium in this case. If the equilibrium shape is an ellipsoid then one of the semi-major axis is in the vertical direction. For a given L, if there exists one relative equilibrium with the shape of a triaxial ellipsoid, then there are infinitely many others: one for each angle of rotation ($\mod \pi$) around the vertical axis. For one of these relative equilibria $a_{12} = a_{21} = 0$ and $a_{11} > a_{22}$. Notice that the invariance of ellipsoids of revolution under rotations around the vertical axis imply that there may exist a unique relative equilibrium of this kind for a fixed value of ℓ (as it happens to the spherical shape for $\ell = 0$). Our quantitative study of the relative equilibria is restricted to the case of small angular momentum.

Lemma 13. *There exists $|\ell_0| > 0$ such that for each angular momentum L with $0 < \|L\| = |\ell| < |\ell_0|$ there exists a unique relative equilibrium with the shape of an ellipsoid of revolution and with angular momentum L. For this equilibrium, $A = \exp(B)$ where B is a symmetric traceless matrix given approximately by*

$$B = -\frac{1}{\gamma}\left(\Omega^2 - \frac{\mathrm{Tr}\,(\Omega^2)}{3}\,\mathrm{Id}\right) + \mathscr{O}(\|\Omega\|^{3/2}) = \frac{\ell^2}{12\gamma}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \mathscr{O}(|\ell|^{3/2})$$

Proof. The angular momentum definition (4.34) and the above discussion imply that $\ell = \omega_3\,\mathrm{Tr}\,(\tilde{A}^2)$, where $\tilde{A}$ is the matrix defined in equation (4.38). We use this equation to replace ω_3 for ℓ in equation (4.29) after imposing $\dot{A} = 0$, $\dot{\Omega} = 0$. As a result we obtain a set of three scalar equations that can be represented as $F(\ell^2, A) = 0$, where A has the form in equation (4.38). For a given ℓ^2, the three scalar equations $F(\ell^2, A) = 0$ must be solved for the four unknowns in A under the additional constraint $\det(A) - 1 = 0$. Notice that $F(0, \mathrm{Id}) = 0$. So, to finish the proof it is enough to show that the matrix $\partial_A F(0, Id)$ is nonsingular when restricted to the tangent space to $\det(A) - 1$ at $A = \mathrm{Id}$. The lemma follows from the implicit function theorem. The best way to compute $\partial_A F(0, Id)$ is to write $A = \mathrm{Id} + \varepsilon B$ and to expand $F(0, \mathrm{Id} + \varepsilon B) = \varepsilon \partial_A F(0, Id) + \mathscr{O}(\varepsilon^2)$. The constraint $0 = \det(\mathrm{Id} + \varepsilon B) - 1 = \varepsilon\,\mathrm{Tr}\,(B) + \mathscr{O}(\varepsilon^2)$ implies that $\mathrm{Tr}\,(B) = 0$. So substituting $A = \mathrm{Id} + \varepsilon B$ into equation (4.29) with $\dot{\Omega} = 0$, $\dot{A} = 0$, and expanding up to order ε we obtain:

$$0 = \left(\Omega^2 - \frac{\mathrm{Tr}\,(\Omega^2)}{3}\,\mathrm{Id}\right) + \varepsilon\left(\gamma B + \frac{\Omega^2 B + B\Omega^2}{2} + \frac{\mathrm{Tr}\,(\Omega^2)}{3} B\right) + \mathscr{O}(\varepsilon^2) \tag{4.39}$$

where $\gamma > 0$ is given in equation (4.10). Using that $\ell = 0 \Leftrightarrow \Omega = 0$ we get $\partial_A F(0, Id)B = \gamma\mu B$, so the lemma is proved. The stated form of B for ℓ small follows directly from equation (4.39) and $\ell = 2\omega_3 + \mathscr{O}(|\omega_3|^{3/2})$. $\square$

Furthermore, it can be shown that there exists only one relative equilibrium with the shape of a ellipsoid of revolution for any $|\ell| \geq 0$, but the appearance of triaxial ellipsoids can take place.

For $\mu > 0$, we know that the ω-limit set of any solution to the equations of motion (4.29) and (4.33) is contained in the set $\mathscr{A}$, which is the set of relative equilibria. Lemma 13 states that there is exactly one point in $\mathscr{A}$ with a given angular momentum L if $\|L\| < \ell_0$. Therefore we get the following.

Theorem 3. *For $\mu > 0$, every solution to equations (4.29) and (4.33) with small angular momentum L, $\|L\| < \ell_0$, is attracted to the unique relative equilibrium given in Lemma 13. The asymptotic shape is an oblate ellipsoid of revolution.*

We remark that a theorem similar to Theorem 3 holds in the case $\mu = 0$. In this case some additional bound on the energy of the initial condition must be imposed. The gravitational force is not strong enough to restrain the growth of very energetic initial conditions.

4.4 Small deformations regime

Our goal in this section is to simplify the pseudo-rigid body equations of motion (4.29) and (4.33) under the small ellipticity hypothesis $A \approx \mathrm{Id}$. Recall from (4.22) that $\mathrm{ssym}(3) := T_{\mathrm{Id}}\,\mathrm{SSym}_+(3) = \{B \in \mathrm{Sym}(3) : \mathrm{Tr}\,(B) = 0\}$. In this case, we claim that the map

$$\exp : \mathrm{ssym}(3) \longrightarrow \mathrm{SSym}_+(3) \tag{4.40}$$

is a global parametrization of $\mathrm{SSym}_+(3)$, i.e. for every $A \in \mathrm{SSym}_+(3)$, $A = \exp(B)$ and it has a smooth inverse.

Lemma 14. *The map (4.40) is a global diffeomorphism.*

Proof. Recognize that $\exp(\mathrm{ssym}(3)) \subset \mathrm{SSym}_+(3)$. A standard calculation shows that there is a ball of radius ε centered at 0, $V_\varepsilon \subset \mathrm{ssym}(3)$

(euclidean metric), where exp is a diffeomorphism. Suppose that there are $B, C \in \text{ssym}(3)$ such that $\exp(B) = \exp(C)$. Taking $n \in \mathbb{N}$ such that $n^{-1}B, n^{-1}C \in V_\varepsilon$, we conclude $\exp(n^{-1}B)^n = \exp(B) = \exp(C) = \exp(n^{-1}C)^n$. Since these matrices are symmetric, positive definite, we get $\exp(n^{-1}B) = \exp(n^{-1}C)$. Then $B = C$, i.e., exp is injective. Given $A \in \text{SSym}_+(3)$, take $Y \in \text{SO}(3)$ such that $YAY^T = \text{diag}(a_1, a_2, a_3)$. Defining $B = Y^T \text{diag}(\ln(a_1), \ln(a_2), \ln(a_3))Y$, we have $\exp(B) = Y^T \text{diag}(a_1, a_2, a_3)Y = A$. Hence exp is onto.
Given $B \in V_\varepsilon$ and $H \in \text{ssym}(3)$, $H \neq 0$, we know that $d\exp_B(H) \neq 0$. Thus, suppose that

$$\left.\frac{d}{dt}\right|_{t=0} \exp(2B + 2tH) = d\exp_B(H)\exp(B) + \exp(B)\,d\exp_B(H) = 0.$$

We may take $\exp(B)$ diagonal. So, recovering that $\exp(B)$ is positive and $d\exp_B(H)$ is symmetric, we verify that the above equality can not hold. Hence, $d\exp_{2B}(H) \neq 0$, and by induction we can prove that $d\exp_{2^n B}(H) \neq 0$, $\forall n \in \mathbb{N}$.
Thus, given $B_0 \in \text{ssym}(3)$, take $n \in \mathbb{N}$ such that $2^{-n}B_0 \in V_\varepsilon$. So, for every nonzero $H \in \text{ssym}(3)$, $d\exp_{B_0}(H) = d\exp_{2^n(2^{-n}B_0)}(H) \neq 0$. Therefore, exp is a global diffeomorphism. □

We wish to insert a small scaling parameter $\varepsilon > 0$ in order to express the small amplitude of the internal vibrations of $A = \exp(\varepsilon B)$. This requires also an scaling for Ω, since small deformations can only exist for small angular velocities. In order to balance the scalings for B and Ω we use the relative equilibrium expression given in Lemma 13. Replacing B by εB in this expression we obtain that the correct scaling for Ω is $\sqrt{\varepsilon}\Omega$. So, we introduce the modified coordinates $A = \exp(\varepsilon B)$ and $\sqrt{\varepsilon}\Omega$ into the Lagrangian and dissipation functions presented in the last section. Performing their Taylor expansions up to order $\varepsilon^{\frac{5}{2}}$, we obtain:

$$\mathscr{L}(B, \dot{B}, \Omega) = \frac{MR^2}{10}\left(\varepsilon^2 \operatorname{Tr}(\dot{B}^2) - \varepsilon \operatorname{Tr}(\Omega^2) - 2\varepsilon^2 \operatorname{Tr}(\Omega^2 B)\right) - \varepsilon^2 \frac{MR^2}{10}\gamma \operatorname{Tr}(B^2) + \mathscr{O}(\varepsilon^{\frac{5}{2}}), \quad (4.41)$$

where γ is given in equation (4.10) in the Introduction. See equations (A.1) and (A.2) from Appendix A for further details. We also have,

$$\mathscr{D}(\dot{B}) = \varepsilon^2 \frac{4\pi\eta R^3}{3} \operatorname{Tr}(\dot{B}^2) + \mathscr{O}(\varepsilon^3).$$

The parameter ε was introduced by means of a change of variables, only to understand the relative scale between B and Ω. So, we neglect all terms in $\mathscr{O}(\varepsilon^{\frac{5}{2}})$ from the above Lagrangian function and reverse the change of variables, or equivalently take $\varepsilon = 1$. The result is the Lagrangian function (4.6) given in the Introduction. In the same way we obtain the dissipation function (4.7) given in the Introduction.

Recover from Lemma 5 that the orthogonal projection $P : M(3) \longrightarrow \mathrm{ssym}(3)$ is

$$P(H) = \frac{1}{2}(H + H^T) - \frac{\operatorname{Tr}(H)}{3} \operatorname{Id}, \quad \forall H \in M(3). \tag{4.42}$$

Now we apply again Theorem 1 and Lemma 7. Since the Lagrangian (4.41) fulfills the hypothesis of the later, we obtain the system of differential equations

$$\ddot{B} + \nu\dot{B} + \gamma B = -\Omega^2 + \frac{1}{3}\operatorname{Tr}(\Omega^2)\operatorname{Id}, \tag{4.43}$$

$$\Phi_{\varepsilon B}(\dot{\Omega}) = \varepsilon\left(\Omega\dot{B} + \dot{B}\Omega + [\Omega^2, B]\right) \tag{4.44}$$

where $\nu := 40\pi\eta R/(3M)$ and $\Phi_{\varepsilon B} : \mathrm{skew}(3) \longrightarrow \mathrm{skew}(3)$ is the linear operator:

$$\Phi_{\varepsilon B}(H) := H + \varepsilon BH + \varepsilon HB, \qquad \forall H \in \mathrm{skew}(3), \tag{4.45}$$

We remark that the deformations behave like a damped harmonic oscillator, externally forced by the non-inertial effects of the rotation. Notice that all the variables are coupled.

The existence of solutions is the first problem we must deal with. Likewise Lemma 8, we state the following.

Lemma 15. *For every $B \in \mathrm{ssym}(3)$, $\|B\| < 1/2$, the operator Φ_B is symmetric, positive definite and, therefore, invertible.*

Proof. Taking arbitrary $H_1, H_2 \in \mathrm{skew}(3)$,

$$\langle \Phi_B(H_1), H_2 \rangle = -\operatorname{Tr}(H_1 H_2) - \operatorname{Tr}(H_1 B H_2) - \operatorname{Tr}(B H_1 H_2) = \langle H_1, \Phi_B(H_2) \rangle.$$

Now, take $H \in \ker(\Phi_B)$. So, $(\,\mathrm{Id} + 2B)H + H(\,\mathrm{Id} + 2B) = 0$. Let $v \in \mathbb{R}^3$ be an unitary eigenvector of $(\,\mathrm{Id} + 2B)$. Then, $\exists \lambda \in \mathbb{R}$:

$$v + 2Bv = \lambda v \Rightarrow Bv = \frac{\lambda - 1}{2} v \Rightarrow \frac{|\lambda - 1|}{2} < \frac{1}{2}.$$

Hence, $\lambda > 0$, i.e, $A = (\,\mathrm{Id} + 2B)$ is symmetric, positive definite. By taking $A = \mathrm{diag}(a_1, a_2, a_3)$, and $S(H) = (h_1, h_2, h_3)$, the previous condition implies:

$$\begin{pmatrix} 0 & -(a_1 + a_2)h_3 & (a_1 + a_3)h_2 \\ (a_1 + a_2)h_3 & 0 & -(a_2 + a_3)h_1 \\ -(a_1 + a_3)h_2 & (a_2 + a_3)h_1 & 0 \end{pmatrix} = 0.$$

So, $H = 0$.
Let β and H be eigenvalue and eigenvector of Φ_B. So, we have

$$(\beta - 1)H = (BH + HB) \Rightarrow |\beta - 1| \leq 2\|B\| < 1.$$

Therefore, $\beta \in (0, 2)$. □

This lemma shows that the equation for $\dot{\Omega}$ can be written in explicit form for $\|B\| < 1/2$ and therefore in this region the standard existence and uniqueness theorems for ordinary differential equations hold.

The functions $\mathscr{L}(Y, B, \dot{Y}, \dot{B})$, $\mathscr{D}(Y, B, \dot{Y}, \dot{B})$ are two-degree homogeneous functions in the velocities. The energy function associated to $\mathscr{L}$ is given by

$$E(B, \dot{B}, \Omega) = \frac{MR^2}{10} \left(\|\dot{B}\|^2 + \|\Omega\|^2 + 2\,\mathrm{Tr}\,(\Omega\Omega^T B) \right) + \frac{MR^2}{10} \gamma \|B\|^2, \quad (4.46)$$

analogously to (4.8). So, from equation (4.18) we get

$$\dot{E} = -2\mathscr{D} \leq 0,$$

where the equality is reached if and only if $\dot{B} = 0$. The next lemma establishes a region where the solutions to the equations of motion are defined for all time.

Lemma 16. *Let $\beta = MR^2\gamma/50$ and $\mathscr{V}$ be the connected component of*

$$\left\{ (B, U, \Omega) \in E^{-1}([0, \beta)) : \|B\| < 1/2 \right\} \quad (4.47)$$

that contains the origin. The set $\mathscr{V}$ is bounded and any solution to the equations (4.43) and (4.44) initially in $\mathscr{V}$ remains in $\mathscr{V}$ for all $t > 0$.

Proof. Recall that the energy is given by:

$$E(B,\dot{B},\Omega) = \frac{MR^2}{10}\left(\mathrm{Tr}\,(\dot{B}^2) - \mathrm{Tr}\,((\,\mathrm{Id} + 2B)\Omega^2) + \gamma \mathrm{Tr}\,(B^2)\right).$$

Inside $\mathscr{V}$, $A := \mathrm{Id} + 2B > 0$, so $-\mathrm{Tr}\,((\,\mathrm{Id}+2B)\Omega^2) = \mathrm{Tr}\,((\sqrt{A}\Omega)^T(\sqrt{A}\Omega)) \geq 0$. Lemma 15 ensures the existence and uniqueness. Taking one such solution, we know that $E(t) < \beta$. Suppose that exists a sequence $\{t_n\}$ such that $\|B(t_n)\| < 1/2$, $\forall n$, and $\lim_{n\to+\infty} \|B(t_n)\| = 1/2$. But, in this case,

$$E(0) \geq E(t_n) \geq \frac{MR^2\gamma}{10}\|B(t_n)\|^2 \Rightarrow E(0) \geq \frac{MR^2\gamma}{40},$$

which is a contradiction. So, $\mathscr{V}$ is invariant. The set is bounded because, $\forall (B,\dot{B},\Omega) \in \mathscr{V}$, $\|B\| < 1/2$, $\|\dot{B}\|^2 < 10\beta/MR^2$, $\|\sqrt{A}\Omega\|^2 < 10\beta/MR^2$. □

Again, we can apply the LaSalle's invariance principle 11. Let $\mathscr{A}$ be the largest invariant subset under the flow contained in the set $\{(Y,\Omega,B,\dot{B}) : \dot{B} = 0\} \cap \mathscr{V}$. The $\omega-$limit set of any solution in $\mathscr{V}$ is contained in $\mathscr{A}$. Since $\mathscr{A}$ is invariant, for an initial condition in $\mathscr{A}$, $\dot{B}(t) = 0$ for all t. Then, by (4.43),

$$B = -\gamma^{-1}(\Omega^2 - \frac{1}{3}\mathrm{Tr}\,(\Omega^2)\,\mathrm{Id}\,) \Rightarrow [B,\Omega^2] = 0$$

that implies $\dot{\Omega} \equiv 0$. So, every point in the attracting set $\mathscr{A}$ is a relative equilibrium.

As in the previous section the equations of motion are invariant under rotations. Therefore we may assume that the angular velocity vector of a relative equilibrium has the form $(0,0,\omega_3)$. The relative equilibrium obtained from equations (4.43) and (4.44) is that given by the first term in the expression in Lemma 13. Let ε be the ellipticity as defined in equation (4.2). Using that $A = \exp(B)$, we obtain that the ellipticity associated to this relative equilibrium is:

$$\varepsilon = e^{\frac{\omega^2}{3\gamma}} - e^{-\frac{2\omega^2}{3\gamma}} \approx \frac{\omega^2}{\gamma} = \frac{1}{2}\frac{\omega^2 R^3}{GM}h_2, \quad h_2 := \frac{5}{2}\frac{1}{\left(1 + \frac{25}{2}\frac{\mu}{\rho g R}\right)},$$

where h_2 is analogous to the *second Love number* that is a long-standing known value, appearing in the works of [71] and [52], given by

$$h_2 = \frac{5}{2}\frac{1}{\left(1 + \frac{19}{2}\frac{\mu}{\rho g R}\right)}.$$

The factor $25/2$ appearing in our formula is different from the $19/2$ appearing in the second Love number because the pseudo-rigid body assumption (4.1) implies an overestimation of the stresses in the body. Since in both models, Love's and ours, the effective constant μ must be estimated using the model itself this difference is irrelevant. We remark that as the radius R becomes larger, we may neglect the term corresponding to the elasticity. So, we obtain the standard flattening:

$$\varepsilon_g \approx \frac{5}{4}\frac{\omega^2 R^3}{GM}. \tag{4.48}$$

By defining the same action as in Lemma 9, a calculation similar to that in Lemma 9 gives that the angular momentum given in equation (4.9) is conserved by this new system. We remark that the angular momentum in equation (4.9) is an approximation to the original angular momentum L given in equation (4.34) (for small values of $\|L\|$).

Finally the same arguments used to prove Theorem 3 can be used to prove the following.

Theorem 4. *Every solution to equations (4.43) and (4.44) initially in the set $\mathscr{V}$, given in Lemma 16, is attracted to the unique relative equilibrium that has the same angular momentum as the solution. The asymptotic shape is an oblate ellipsoid of revolution.*

So, for small angular momentum the asymptotic behaviour of the system studied in this section and that from the previous section are essentially the same.

4.4.1 Quantitative analysis

We recall from [15] some basic results which we apply to this specific model. In the general case, take $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and let $A_1 \in M(n)$, $A_2 \in M(m)$ be square matrices, $f : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$ be smooth functions such that $f(0,0) = 0$, $Df(0,0) = 0$, $g(0,0) = 0$ and $Dg(0,0) = 0$. Construct the system

$$\begin{cases} \dot{x} = A_1 x + f(x,y) \\ \dot{y} = A_2 y + g(x,y) \end{cases}, \tag{4.49}$$

supposing that all the eigenvalues of A_1 have zero real parts and all the eigenvalues of A_2 have negative real parts.

Let $h : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, with $h(0) = 0$ and $Dh(0) = 0$, be a smooth function. If its graph $y = h(x)$ is an invariant manifold for the flow of (4.49), it is called a *center manifold* for (4.49). In this case the flow on this manifold is given by $(u(t), h(u(t)))$, where $u(t) \in \mathbb{R}^n$ is solution of

$$\dot{u} = A_1 u + f(u, h(u)). \tag{4.50}$$

The following theorem is proved in [15].

Theorem 5. *Suppose that the zero solution of (4.50) is stable. Then*

(i) The zero solution of (4.49) is stable.

(ii) Let $(x(t), y(t))$ be solution of (4.49), with $(x(0), y(0))$ sufficiently small, and $\sigma := \min\{|Re(\lambda)| : \lambda \in Spectrum(A_2)\}$. So, exist constants $C_1, C_2 > 0$ and a solution $u(t)$ of (4.50) such that

$$\|x(t) - u(t)\| \le C_1 e^{-\sigma t}, \quad \|y(t) - h(u(t))\| \le C_2 e^{-\sigma t} \tag{4.51}$$

Now, we apply the rescaling $B \to \varepsilon B$ and $\Omega \to \sqrt{\varepsilon}\Omega$ directly at the equations of motion (4.29), (4.33) and (4.43), (4.44). The first system becomes

$$\begin{aligned} \ddot{B} + \nu\dot{B} + \gamma B &= -\Omega^2 + \frac{1}{3}\,\mathrm{Tr}\,(\Omega^2)\,\mathrm{Id} + \mathcal{O}(\varepsilon^{\frac{1}{2}}) \\ \dot{\Omega} &= -\varepsilon\Omega\dot{B} - \varepsilon\dot{B}\Omega - \varepsilon^{\frac{3}{2}}\frac{1}{2}[\Omega^2, B] + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{4.52}$$

and the second

$$\begin{aligned} \ddot{B} + \nu\dot{B} + \gamma B &= -\Omega^2 + \frac{1}{3}\,\mathrm{Tr}\,(\Omega^2)\,\mathrm{Id} \\ \dot{\Omega} &= -\varepsilon\Omega\dot{B} - \varepsilon\dot{B}\Omega - \varepsilon^{\frac{3}{2}}[\Omega^2, B] + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{4.53}$$

So, we see that the systems coincide when $\varepsilon = 0$ and have the form

$$\begin{aligned} \dot{\Omega} &= 0 \\ \begin{pmatrix} \dot{B} \\ \dot{U} \end{pmatrix} &= A_2 \begin{pmatrix} B \\ U \end{pmatrix} + \begin{pmatrix} 0 \\ -\Omega^2 + 1/3\,\mathrm{Tr}\,(\Omega^2)\,\mathrm{Id} \end{pmatrix}, \end{aligned} \tag{4.54}$$

where $A_2(B,U) := (U, -\nu U - \gamma B)$. Note that this system satisfies the hypotheses of Theorem 5, with $x = \Omega$, $y = (B,U)$ and $A_1(\Omega) = 0$. If $A_2(B,U) = \lambda(B,U)$, then $\lambda^2 + \nu\lambda + \gamma = 0$, so

$$\lambda = -\frac{\nu}{2} \pm \frac{1}{2}\sqrt{\nu^2 - 4\gamma}$$

whose real part is always negative.

We remark that the graph of the function $(B,U) = h(\Omega) = (\gamma^{-1}(-\Omega^2 + 1/3\,\mathrm{Tr}\,(\Omega^2)\,\mathrm{Id}\,), 0)$ defines a global center manifold. Indeed, each point on this graph is a stable relative equilibrium. So, the whole graph is invariant under the flow. We can see explicitly from (4.54) that for each $(\Omega(0), B(0), U(0))$ in this graph there exist constants $C_1, C_2 > 0$ such that

$$\|\Omega(t) - \Omega(0)\| \le C_1 e^{-\sigma t}, \|\gamma B(t) - \left(-\Omega(0)^2 + \frac{1}{3}\,\mathrm{Tr}\,(\Omega(0)^2)\,\mathrm{Id}\right)\| \le C_2 e^{-\sigma t}, \tag{4.55}$$

where $\sigma = Re\left(-\nu/2 + \sqrt{\nu^2 - 4\gamma}/2\right)$. Quoting the Section 9 from [34], we see that for a wide range of examples in the Solar System, such eigenvalues are real, i.e., the corresponding harmonic oscillator is overdamped. Remark that the constant γ considered in [34] is equal to $25\gamma/(4\nu)$.

Therefore, we see that this graph is a normally hyperbolic invariant manifold and using the main theorems from [32] we see that for each $\varepsilon > 0$ sufficiently small, there is an invariant manifold, diffeomorphic to this one, for each one of the systems [(4.43), (4.44)] and [(4.29), (4.33)]. Such manifolds are also attractive in the sense of (4.51). For both systems [(4.43), (4.44)] and [(4.29), (4.33)] these are manifolds of equilibria.

4.5 Two-body problem

This section is devoted to an extension of the previous approach to the two-body problem. Due to its Lagrangian formulation this extension can be immediately performed without any further assumptions.

Now we need to consider the translation of the pseudo-rigid body. The motion of an arbitrary point $\mathbf{x} \in \mathscr{B}$ is given by

$$\phi(t, \mathbf{x}) := G(t)\mathbf{x} + \mathbf{y}(t), \tag{4.56}$$

where $G = YA$ is the same from (4.1), where we consider the same smooth decomposition, and $\mathbf{y} \in \mathbb{R}^3$ describes the position of its center of mass.

We remind the hypothesis of small deformations of the pseudo-rigid body, where we introduced the small parameter $0 < \varepsilon << 1$ with $A = \exp(\varepsilon B)$. Such a hypothesis also induced an assumption of small angular velocities of the pseudo-rigid body. This angular velocity $\Omega = Y^T \dot{Y}$ was multiplied by $\sqrt{\varepsilon}$ in order to keep its adequate scale relative to B.

Our first step is to couple the motion of this pseudo-rigid body with the motion of a point mass whose spatial coordinates are denoted by $\mathbf{q} \in \mathbb{R}^3$. The validity regime of this model is defined by the conditions:

$$\frac{R_1}{\|\mathbf{q}\|} << 1, \quad \frac{R_2}{\|\mathbf{q}\|} << 1,$$

where R_1 is the radius of the pseudo-rigid body and R_2 is the radius of the external body. The former ensures the small deformation of the pseudo-rigid body and the later allows us to neglect the dimensions of the external body.

In the second part, we deal with the problem of two pseudo-rigid bodies under influence of mutual gravitation.

In order to be consistent with the modeling of the isolated body, we need to scale the coordinate $\mathbf{q}$ with a suitable power of the factor ε. We know that the force acting upon the particle is only the gradient of the interaction potential

$$V_{int} = -Gm\rho \int_{\mathscr{B}} \frac{1}{\|\mathbf{q} - \mathbf{y} - Ye^{\varepsilon B}\mathbf{x}\|} d\mathbf{x} = -Gm\rho \int_{\mathscr{B}} \frac{1}{\|Y^T(\mathbf{r}) - e^{\varepsilon B}\mathbf{x}\|} d\mathbf{x}, \tag{4.57}$$

where $\mathbf{r} = \mathbf{q} - \mathbf{y}$ is the relative position between the bodies. From Appendix B, its Taylor expansion up to order ε is

$$V_{int}(Y, B, \mathbf{r}) = -\frac{GM_1M_2}{r} - \frac{3GM_1M_2R_1^2}{5r^5} \varepsilon \operatorname{Tr}\left((Y^T\mathbf{r})(Y^T\mathbf{r})^T B\right) + \mathscr{O}(\varepsilon^2), \tag{4.58}$$

where $r = \|\mathbf{r}\|$.

So, the equation of motion for $\mathbf{r}$ is

$$\tilde{m}\ddot{\mathbf{r}} = -\nabla_{\mathbf{r}} V_{int}, \tag{4.59}$$

where $\tilde{m} = M_1M_2/(M_1 + M_2)$ is the reduced mass. Adopting a reference frame, let us assume that M_1 describes a steady rotation around the axis z, with constant angular velocity $\sqrt{\varepsilon}\omega$. Let us also assume that the system has a synchronous circular orbital motion around the same axis, $\mathbf{r}(t) = r(\cos(\sqrt{\varepsilon}\omega t), \sin(\sqrt{\varepsilon}\omega t), 0)$. Therefore, (4.59) imposes that

$$-\omega^2\varepsilon = -GMr^{-3} + \alpha\varepsilon r^{-5}.$$

From this equation we see that the adequate scale is $r^{-3} = \mathscr{O}(\varepsilon)$. Hence, in the computation of the energies we replace $\mathbf{r}$ by $\varepsilon^{-\frac{1}{3}}\mathbf{r}$.

4.5.1 Kinetic and potential energies and dissipation function

Suppose that the pseudo-rigid body is homogeneous, so extending (2.53) its kinetic energy is

$$T_{ps}(Y,\dot{Y},A,\dot{A},\mathbf{y},\dot{\mathbf{y}}) = \frac{\rho}{2}\int_{\mathscr{B}} \left\|\Omega A\mathbf{x} + \dot{A}\mathbf{x} + \dot{\mathbf{y}}\right\|^2 d\mathbf{x} = \frac{M_1}{2}\dot{\mathbf{y}}^2 + \frac{\rho}{2}\int_{\mathscr{B}} \|\Omega A\mathbf{x}\|^2 + \|\dot{A}\mathbf{x}\|^2 + 2\left\langle \mathbf{x}, \dot{A}\Omega A\mathbf{x}\right\rangle + \left\langle (\Omega A + \dot{A})\mathbf{x}, Y^T\dot{\mathbf{y}}\right\rangle d\mathbf{x},$$

where ρ is the density of the body. We remark that the last term in the integral is an odd function relative to $\mathbf{x}$, so its integral over the ball vanishes. The other terms were obtained in Section 4.3,

$$T_{ps} = \frac{M_1}{2}\dot{\mathbf{y}}^2 + \frac{M_1R_1^2}{10}(\operatorname{Tr}(\dot{A}^2) - \operatorname{Tr}(\Omega^2A^2) + 2\operatorname{Tr}(\dot{A}\Omega A)),$$

hence the total kinetic energy is given by

$$T = \frac{M_1R_1^2}{10}(\operatorname{Tr}(\dot{A}^2) - \operatorname{Tr}(\Omega^2A^2) + 2\operatorname{Tr}(\dot{A}\Omega A)) + \frac{\tilde{m}}{2}\dot{\mathbf{r}}^2 + \frac{M_1+M_2}{2}\dot{\mathbf{Q}}_{CM}^2 \tag{4.60}$$

where $\mathbf{Q}_{CM}$ is the position of the system's center of mass.

The total potential energy is given by $V = V_{sg} + V_{el} + V_{int}$ following (4.28), (4.25) and (4.57). The dissipation is still given by (4.26). The Lagrangian of the system is given by $\mathscr{L}(Y,A,\mathbf{r},\mathbf{Q}_{CM},\dot{Y},\dot{A},\dot{\mathbf{r}},\dot{\mathbf{Q}}_{CM}) = T - V$. Remark that $\mathbf{Q}_{CM}$ is an ignorable coordinate, so we neglect its occurrence in $\mathscr{L}$. As announced in the beginning of this section, we take the rescaling $A = e^B \to e^{\varepsilon B}$, $\Omega \to \varepsilon^{\frac{1}{2}}\Omega$, $\mathbf{r} \to \varepsilon^{-\frac{1}{3}}\mathbf{r}$. So, define new Lagrangian and dissipation functions by

$$\tilde{\mathscr{L}}(Y,B,\mathbf{r},\dot{Y},\dot{B},\dot{r}) = \varepsilon^{\frac{2}{3}}\mathscr{L}(Y,\varepsilon B,\varepsilon^{-\frac{1}{3}}\mathbf{r},\varepsilon^{\frac{1}{2}}\dot{Y},\varepsilon\dot{B},\varepsilon^{-\frac{1}{3}}\dot{\mathbf{r}}) \quad (4.61)$$

$$\tilde{\mathscr{D}}(B,\dot{B}) = \varepsilon^{\frac{2}{3}}\mathscr{D}(\varepsilon B,\varepsilon\dot{B}) \quad (4.62)$$

and neglect all its terms with order greater than $\varepsilon^{\frac{8}{3}}$. So, by the same computations from Section 4.4 and by (4.58), we get

$$\begin{aligned}\tilde{\mathscr{L}} &= \frac{M_1R_1^2}{10}\varepsilon^{\frac{5}{3}}\left(\varepsilon\,\mathrm{Tr}\,(\dot{B}^2) - \mathrm{Tr}\,(\Omega^2) - 2\varepsilon\,\mathrm{Tr}\,(\Omega^2B) - \gamma\varepsilon\,\mathrm{Tr}\,(B^2)\right)\\ &+ \frac{\tilde{m}}{2}\dot{\mathbf{r}}^2 + \frac{GM_1M_2}{r}\varepsilon + \frac{3GM_1M_2R_1^2}{5r^5}\varepsilon^{\frac{8}{3}}\,\mathrm{Tr}\,((Y^T\mathbf{r})(Y^T\mathbf{r})^TB), \quad (4.63)\\ \tilde{\mathscr{D}} &= \varepsilon^{\frac{8}{3}}\frac{4\pi\eta R_1^3}{3}\,\mathrm{Tr}\,(\dot{B}^2). \quad (4.64)\end{aligned}$$

We recall that the equations of motion for B are computed by

$$P\left(\frac{d}{dt}\left(\frac{\partial\tilde{\mathscr{L}}}{\partial\dot{B}}\right) - \frac{\partial\tilde{\mathscr{L}}}{\partial B} + \frac{\partial\tilde{\mathscr{D}}}{\partial\dot{B}}\right) = 0 \quad (4.65)$$

where P is the orthogonal projection given by (4.42). The equations are

$$\ddot{B} + \nu\dot{B} + \gamma B = -\Omega^2 + \frac{1}{3}\,\mathrm{Tr}\,(\Omega^2)\,\mathrm{Id} + \frac{3GM_2}{r^5}\left((Y^T\mathbf{r})(Y^T\mathbf{r})^T - \frac{1}{3}r^2\,\mathrm{Id}\right). \quad (4.66)$$

where

$$\nu = 40\pi\eta R_1/3M_1. \quad (4.67)$$

For the variable Y, we use again Lemma 7 to get

$$\frac{dW}{dt} + [\Omega,W] = Y^TP_Y\left(\frac{\partial V}{\partial Y}\right) \quad (4.68)$$

where $W = (M_1R_1\varepsilon^{\frac{5}{3}}/5)(\Omega + \varepsilon(B\Omega + \Omega B))$. Explicitly, we have

$$\Phi_{\varepsilon B}\left(\dot{\Omega}\right) = -\varepsilon\left(\Omega\dot{B} + \dot{B}\Omega\right) + \varepsilon[B,\Omega^2] + \frac{3Gm}{r^5}\varepsilon[(Y^T\mathbf{r})(Y^T\mathbf{r})^T,B], \quad (4.69)$$

where the inertia tensor $\Phi_{\varepsilon B}:\ \mathrm{skew}(3) \longrightarrow \mathrm{skew}(3)$ is given by (4.45).

Since the coordinate $\mathbf{r}$ has no constraint, its equations of motion are obtained simply by equation (4.59). So, a straightforward computation shows

$$\ddot{\mathbf{r}} = -\frac{G\tilde{M}\varepsilon}{r^3}\mathbf{r} - \frac{3G\tilde{M}\varepsilon^{\frac{8}{3}}R_1^2}{r^7}\left\langle YBY^T\mathbf{r},\mathbf{r}\right\rangle \mathbf{r} + \frac{6G\tilde{M}\varepsilon^{\frac{8}{3}}R_1^2}{5r^5}YBY^T\mathbf{r}. \qquad (4.70)$$

where $\tilde{M} = M_1 + M_2$ is the total mass. Remark that the two additional terms correspond to the gravitational attraction of the raising tide.

4.5.2 Angular momentum and mechanical energy

From (4.63) and (4.64) we see that the kinetic energy and the dissipation function are two-degree homogeneous in the velocities $\dot{B}, \dot{Y}, \dot{\mathbf{r}}$. So, from Section 4.2 we see that the function

$$E = \frac{M_1R_1^2}{10}\varepsilon^{\frac{5}{3}}\left(\varepsilon\,\mathrm{Tr}\,(\dot{B}^2) - \mathrm{Tr}\,(\Omega^2) - 2\varepsilon\,\mathrm{Tr}\,(\Omega^2 B) + \gamma\varepsilon\,\mathrm{Tr}\,(B^2)\right)$$
$$+\frac{\tilde{m}}{2}\dot{\mathbf{r}}^2 - \frac{GM_1M_2}{r}\varepsilon - \frac{3GM_1M_2R_1^2}{5r^5}\varepsilon^{\frac{8}{3}}\,\mathrm{Tr}\,((Y^T\mathbf{r})(Y^T\mathbf{r})^TB) \qquad (4.71)$$

is such that $\dot{E} = -\left(\varepsilon^{\frac{8}{3}}8\pi\eta R_1^3/3\right)\mathrm{Tr}\,(\dot{B}^2) \leq 0$ along any solution of the system [(4.66),(4.69),(4.70)]. Remark that $\dot{E} = 0$ if and only if $\dot{B} = 0$.

Moreover, consider the action $\Psi:\ \mathrm{SO}(3) \times (\,\mathrm{SO}(3) \times \mathrm{ssym}(3) \times \mathbb{R}^3)) \longrightarrow \mathrm{SO}(3) \times \mathrm{ssym}(3) \times \mathbb{R}^3$, $\Psi(U,Y,B,\mathbf{r}) := (UY,B,U\mathbf{r})$. Remark that for every $U \in \mathrm{SO}(3)$, $\tilde{\mathscr{L}}(d\Psi_U((Y,B,\mathbf{r}),(\dot{Y},\dot{B},\dot{\mathbf{r}}))) = \tilde{\mathscr{L}}((UY,B,U\mathbf{r}),(U\dot{Y},\dot{B},U\dot{\mathbf{r}})) = \tilde{\mathscr{L}}((Y,B,\mathbf{r}),(\dot{Y},\dot{B},\dot{\mathbf{r}}))$. Now, for each $\xi \in \mathrm{skew}(3)$, consider $\rho(s) = e^{s\xi} \in \mathrm{SO}(3)$. Since $\tilde{\mathscr{D}}$ depends only on $\dot{B}$, the second condition of (4.19) also holds. By Theorem 2, the system has the first integral

$$h = \mathrm{Tr}\left(\left(\frac{M_1R_1^2\varepsilon^{\frac{5}{3}}}{5}Y\Phi_{\varepsilon B}(\Omega)Y^T + \frac{\tilde{m}}{2}(\dot{\mathbf{r}}\mathbf{r}^T - \mathbf{r}\dot{\mathbf{r}}^T)\right)\xi\right), \quad \forall\xi \in \mathrm{skew}(3)\,.$$

Hence the total angular momentum

$$L = \frac{M_1R_1^2\varepsilon^{\frac{5}{3}}}{5}Y\Phi_{\varepsilon B}(\Omega)Y^T + \frac{\tilde{m}}{2}(\dot{\mathbf{r}}\mathbf{r}^T - \mathbf{r}\dot{\mathbf{r}}^T), \qquad (4.72)$$

is conserved by the system [(4.66),(4.69),(4.70)].

4.5.3 Relative equilibria

Since the function E from (4.71) is also a Liapunov function we may use again the LaSalle's Invariance Principle 11. However, in this problem, we have serious difficulties to determine the set of initial conditions that generate complete orbits. In Section 4.4, about the isolated body problem, it could be shown that this set contains an open region. Here, the singularity of the field, $\mathbf{r} = 0$, can allow bounded maximal solutions defined in a bounded interval (collision in finite time). We know that in the Kepler problem ($B \equiv 0, \Omega \equiv 0$) such orbits only take place when $L = 0$, but for this general system such a extension is not at all clear.

So, let us suppose that for an initial condition $(Y_0, B_0, \mathbf{r}_0, \dot{Y}_0, \dot{B}_0, \dot{\mathbf{r}}_0)$ the solution is defined for all $t \geq 0$, and also $\|B(t)\| < 1/2\varepsilon$. Hence, by Lemma 11, the ω-limit set of this solution is nonempty and is contained in the subset $\{\dot{E} = 0\} = \{\dot{B} = 0\}$. Now, take an initial condition in this ω-limit. Since this set is invariant and compact, this new solution is defined for all $t \geq 0$, and the deformation $B(t) \equiv B_1$ is a constant function. Therefore by (4.66) we have, for all $t \geq 0$

$$B_1 = \gamma^{-1}\left(-\Omega^2 + \frac{1}{3}\operatorname{Tr}(\Omega^2)\operatorname{Id} + \frac{3GM_2}{r^5}\left((Y^T\mathbf{r})(Y^T\mathbf{r})^T - \frac{1}{3}r^2\operatorname{Id}\right)\right). \tag{4.73}$$

Substituting (4.73) in (4.69), all the right hand side vanishes, so we get $\Phi_{\varepsilon B_1}(\dot{\Omega}) = 0, \forall t \geq 0$. Our hypothesis ensures that $\Phi_{\varepsilon B_1}$ is invertible, hence $\Omega(t) \equiv \Omega_1$, a constant angular velocity, for all $t \geq 0$. So, we can state the following lemma.

Lemma 17. *Assume, without loss of generality, that the pseudo-rigid body rotates around the z axis, $S(\Omega_1) = (0, 0, \omega_1)$. Then the relative position vector $\mathbf{r}$ describes an uniform circular motion around the z axis with frequency ω_1.*

Proof. We can take

$$Y(t) = \begin{pmatrix} \cos(\omega_1 t) & -\sin(\omega_1 t) & 0 \\ \sin(\omega_1 t) & \cos(\omega_1 t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.74}$$

Furthermore, since B and Ω are constant, (4.73) implies that

$$Z = \frac{1}{r^5}Y^T\left(\mathbf{r}\mathbf{r}^T - \frac{1}{3}r^2\operatorname{Id}\right)Y \tag{4.75}$$

is constant. Remark that $\|Z\|^2 = 2/(3r^6)$, so $r(t) = \|\mathbf{r}(t)\|$ also is constant. Consider the coordinates $\mathbf{r} = (r_1, r_2, r_3)$ and note that the last component of Z is $r^{-5}(r_3^2 - r^2/3)$. So, $r_3(t)$ also is constant as well as $r_1^2(t) + r_2^2(t)$. Therefore, we can state that $\mathbf{r}(t) = (a\cos(\alpha(t)), a\sin(\alpha(t)), c)$, where $\alpha(t)$ is a smooth function and $a, c \in \mathbb{R}$ constants. Remark that the constant matrix Z has the form

$$\frac{1}{3r^5}\begin{pmatrix} 3a^2\cos^2(\varphi(t)) - a^2 - c^2 & 3a^2\cos(\varphi(t))\sin(\varphi(t)) & 3ca\cos(\varphi(t)) \\ 3a^2\cos(\varphi(t))\sin(\varphi(t)) & 3a^2\sin^2(\varphi(t)) - a^2 - c^2 & 3ca\sin(\varphi(t)) \\ 3ac\cos(\varphi(t)) & 3ac\sin(\varphi(t)) & 2c^2 - a^2 \end{pmatrix},$$

where $\varphi(t) = \alpha(t) - \omega_1 t$. So, $\alpha(t) = \omega_1 t + \varphi_0$, and we see that in this model the only admissible conservative motions are the *synchronous* ones. Now, define $\tilde{\mathbf{r}}(t) = a(\cos(\omega_1 t + \varphi_0), \sin(\omega_1 t + \varphi_0), 0)$ and remark that (4.70) reduces to

$$-\omega_1^2\tilde{\mathbf{r}} = \psi(\mathbf{r})\mathbf{r} + \frac{6G\tilde{M}\varepsilon^{\frac{8}{3}}\omega_1^2 R_1^2}{5r^5}\tilde{\mathbf{r}}, \tag{4.76}$$

where $\psi(\mathbf{r})$ is a scalar function. Note that, since $\tilde{\mathbf{r}} \neq \mathbf{0}$, (4.76) implies that $c = 0$, i.e. the plane of translation of the bodies is orthogonal to the z axis. □

We claim that the semi-major axis of the equilibrium ellipsoid is aligned with the relative position vector. In fact, define $\mathbf{e}_1(t) = (\cos(\omega_1 t + \varphi_0), \sin(\omega_1 t + \varphi_0), 0)$, $\mathbf{e}_2(t) = (-\sin(\omega_1 t + \varphi_0), \cos(\omega_1 t + \varphi_0), 0)$ and $\mathbf{e}_3(t) = (0, 0, 1)$, and note that

$$Y(\mathrm{Id} + \varepsilon B_1)Y^T\mathbf{e}_1 = \left(1 + \frac{2}{3\gamma}\varepsilon\omega_1^2 + \frac{2GM_2\varepsilon}{\gamma r^3}\right)\mathbf{e}_1,$$

$$Y(\mathrm{Id} + \varepsilon B_1)Y^T\mathbf{e}_3 = \left(1 - \frac{GM_2\varepsilon}{\gamma r^3}\right)\mathbf{e}_3,$$

$$Y(\mathrm{Id} + \varepsilon B_1)Y^T\mathbf{e}_2 = \left(1 + \frac{2}{3\gamma}\varepsilon\omega_1^2 - \frac{GM_2\varepsilon}{\gamma r^3}\right)\mathbf{e}_2,$$

so their directions correspond to the principal axis of this (triaxial) ellipsoid and the highest eigenvalue is associated to $\mathbf{e}_1 = r^{-1}\mathbf{r}$. We stress that there is no phase lag in this final state.

Now, recognize that (4.70) becomes equivalent to

$$18G^2\tilde{M}M_2R_1^2\gamma^{-1}\varepsilon^{\frac{5}{3}}(r^{-1})^8+6G\tilde{M}R_1^2\gamma^{-1}\varepsilon^{\frac{5}{3}}\omega_1^2(r^{-1})^5+G\tilde{M}(r^{-1})^3=\omega_1^2, \tag{4.77}$$

and remark that on the left hand side we have a polynomial in r^{-1} whose coefficients are positive, and hence monotone for $r>0$. So, we see that for each ω_1^2 (4.77) has an unique root $r(\omega_1^2)$. Note that if $r\le\beta:=\varepsilon^{\frac{1}{3}}\sqrt[5]{6G\tilde{M}\gamma^{-1}R_1^2}$, the equation (4.77) can not be solved, hence the radius of all the circular orbits of this model must be greater than this bound. Furthermore, (4.77) can also be solved by a smooth function $\omega_1^2(r)$, $\forall r>\beta$, the inverse of the previous one, which is decreasing and such that $\lim_{r\to\beta+}\omega_1^2=+\infty$, $\lim_{r\to+\infty}\omega_1^2=0$ and $\lim_{r\to+\infty}r^3\omega_1^2=G\tilde{M}$. So, for each $r>\beta$, the norm of the angular momentum (4.72) becomes

$$\|L\|(r)=\left(\frac{M_1R_1^2\varepsilon^{\frac{5}{3}}}{5}\left(1+\frac{2\omega_1^2(r)\varepsilon}{3\gamma}+\frac{GM_2\varepsilon}{\gamma r^3}\right)+\frac{\tilde{m}r^2}{2}\right)\omega_1(r). \tag{4.78}$$

Remark that $\lim_{r\to\beta+}\|L\|=\lim_{r\to+\infty}\|L\|=+\infty$. Hence we see that $\|L\|$ attains a global minimum at some $r_0>\beta$ (at least), moreover each higher level admits at least two equilibrium radii. Note that this minimum angular momentum is nonvanishing. We stress that for r close to β the angular velocity becomes unbounded, what violates the condition $\|B_1\|<(2\varepsilon)^{-1}$, see (4.73). On such a level of angular momentum, it is plausible to consider exclusively the remaining external equilibrium radius.

We stress the surprising fact that no initial condition, whose norm of its angular momentum is less than the minimum of (4.78), generates a bounded solution that meets the requirement $\|B(t)\|<1/(2\varepsilon)$, for all $t\ge0$. Furthermore, we recall that the existence of a minimal radius for the equilibrium solutions is not at all surprising, whereas in the literature it is known as the Roche radius or Roche limit, see chapter 8 of [17] for instance. However notice that the methods to determine such minimal radii are distinct, we use the conservation of angular momentum and the orbital equation for $\mathbf{r}$ while [17] uses an equation that relates r with the eccentricity e of the (section of the) equilibrium ellipsoid, estimating its minimum value.

Substituting the value that γ assumes for a homogeneous body, formula (4.10), and reversing the scaling of the coordinates, we get the expression for the minimum radius

$$r_{min} = \left(\frac{15(M_1+M_2)}{2M_1} \frac{1}{\left(1+\frac{25}{2}\frac{\mu}{g\rho R_1}\right)} \right)^{\frac{1}{5}} R_1. \tag{4.79}$$

Neglecting the ratios $\mu/(g\rho R_1)$ and M_2/M_1, we get $r_{min} \approx 1.2394 \cdot R_1$.

We recover [60] where the authors proposed a similar model, including the polar decomposition for the matrix of deformation. They also observed a minimum distance of approximation between the bodies and discussed the differences relative to the results of [64] and [17]. They also found steady states where the centers of mass of the bodies do not perform a planar motion, the type II steady motions. We recall that our model does not allow such a motion asymptotically.

4.6 Singular perturbation, averaged equations and the Mignard's torque

We recall from [54] the model of interaction between an orbiting celestial body and a deformable body together with its induced tide. It is assumed explicitly that the angular velocity of the deformable body is fixed, the deformable body is viscous and the tide generated by the gravitational force is not due to the actual position of the orbiting body, but to its position in a delayed time. Such a hypothesis can be based on the analogy with the damped harmonic oscillator forced by a periodic external force, see the discussions in Chapter 2 and in [29]. Further supposing that the time lag Δt is small (and constant) and taking the expansion of the potential energy in this parameter, [54] deduces an equation for the orbiting body

$$\ddot{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r} - \frac{3k_2 GmR^5}{r^8}\left(\mathbf{r} - \Delta t\left(\dot{\mathbf{r}} + \mathbf{r}\times\tilde{\boldsymbol{\omega}} + 2\frac{<\mathbf{r},\dot{\mathbf{r}}>}{r^2}\mathbf{r}\right)\right), \tag{4.80}$$

where k_2 is the second Love number of the deformable body and $\tilde{\boldsymbol{\omega}}$ is its angular velocity. We remark that the second central term was neglected in [54], as evinced in equation (20) from [28]. In this section, we show that this model can be viewed as an asymptotic approximation of [(4.66),(4.69),(4.70)], together with some simplifying hypotheses.

4.6.1 Singular Perturbation Theory

We recall some main results of the Singular Perturbation Theory following basically the Section 2.1 from [8].

We call a *slow-fast system* a system of ODE's with two (vector) variables $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and a small parameter $\varepsilon > 0$. If $f : U \subset \mathbb{R}^{n+m} \times \mathbb{R} \longrightarrow \mathbb{R}^n$, and $g : U \subset \mathbb{R}^{n+m} \times \mathbb{R} \longrightarrow \mathbb{R}^m$ are smooth functions, such a system must have the form

$$\begin{aligned} \dot{x} &= f(x,y,\varepsilon) \\ \dot{y} &= \varepsilon g(x,y,\varepsilon). \end{aligned} \tag{4.81}$$

Performing a time rescaling, $\tau = t/\varepsilon$, the system (4.81) becomes

$$\begin{aligned} \varepsilon x' &= f(x,y,\varepsilon) \\ y' &= g(x,y,\varepsilon), \end{aligned} \tag{4.82}$$

where x', y' denote the derivatives with respect to τ. We call (4.81) the *fast system* and (4.82) the associated *slow system*, because as $\varepsilon \approx 0$, τ varies abruptly with small variations of t. The variables x and y are called the *fast* and *slow* variables, respectively.

The *reduced system* is obtained by taking $\varepsilon = 0$ only on the left hand side of (4.82). So we call *slow manifold* the set $\mathscr{M} := \{(x,y) : x = x^\star(y), y \in V\}$, where $x = x^\star(y)$ is solution of the equation

$$f(x,y,\varepsilon) = 0. \tag{4.83}$$

The implicit function theorem states that we can solve (4.83) locally if we show that the matrix

$$A^\star(y) := \frac{\partial f}{\partial x}(x^\star(y), y, \varepsilon) \tag{4.84}$$

is invertible in a given point. If, for all y, the real part of the eigenvalues of $A^\star(y)$ is negative and uniformly bounded away from zero, we say that $\mathscr{M}$ is uniformly asymptotically stable. The solution of the reduced system is the curve $(x^\star(y(\tau)), y(\tau))$, where $y(\tau)$ is solution of

$$y' = g(x^\star(y), y, \varepsilon). \tag{4.85}$$

Under these hypotheses, for each sufficiently small $\varepsilon > 0$, there exists a manifold $\mathscr{M}_\varepsilon = \{(x,y) : x = \bar{x}(y,\varepsilon), y \in V\}$, diffeomorphic to $\mathscr{M}$ and invariant by the flow of (4.82). Moreover, $\bar{x}(y,\varepsilon) = x^\star(y) + \mathscr{O}(\varepsilon)$. The set $\mathscr{M}_\varepsilon$ is called adiabatic manifold. Solutions of (4.82) on this manifold can be obtained by

$$y' = g(\bar{x}(y,\varepsilon), y, \varepsilon). \tag{4.86}$$

From center manifold theory, for instance in [15], the following theorem can be stated.

Theorem 6. *For each sufficiently small $\varepsilon > 0$, there exists a neighbourhood of $\mathscr{M}_\varepsilon$, $\mathscr{N}_\varepsilon \subset \mathbb{R}^{n+m}$, such that for each initial condition $(x_0, y_0) \in \mathscr{N}_\varepsilon$, with solution $(x(\tau), y(\tau))$ of (4.82), there are constants $K, \alpha > 0$ and a solution of the reduced system (4.86), $\hat{y}(\tau)$, such that*

$$\|(x(\tau), y(\tau)) - (\bar{x}(\hat{y}(\tau),\varepsilon), \hat{y}(\tau))\| \leq K\|(x_0, y_0) - (\hat{x}_0, \hat{y}_0)\| e^{-\alpha\tau/\varepsilon}, \tag{4.87}$$

for all $\tau > 0$ such that the system (4.86) is defined.

So, the solution of the reduced system is an approximation of solutions of the complete system. Furthermore, we can present successive approximations of the adiabatic manifold. The first correction is

$$\bar{x}(y,\varepsilon) = x^\star(y) - \varepsilon A^\star(y)^{-2}\partial_y f(x^\star(y), y) g(x^\star(y), y) + \mathscr{O}(\varepsilon^2). \tag{4.88}$$

4.6.2 Averaging the equations of motion

Recall that the system [(4.66),(4.69),(4.70)] can be written as

$$\begin{aligned}
\dot{B} &= U \\
\dot{U} &= -\nu U - \tilde{\mu} B + F(Y, \Omega, \mathbf{r}) \\
\dot{\Omega} &= \varepsilon G(Y, \Omega, B, U, \mathbf{r}, \varepsilon) \\
\dot{\mathbf{r}} &= \sqrt{\varepsilon}\mathbf{v} \\
\dot{\mathbf{v}} &= \sqrt{\varepsilon} h(Y, B, \mathbf{r}) \\
\dot{Y} &= \sqrt{\varepsilon} Y \Omega
\end{aligned} \tag{4.89}$$

which is a slow-fast system in scale $\sqrt{\varepsilon}$, where the fast and slow variables are $x = (B, U)$ and $y = (\mathbf{r}, \mathbf{v}, Y, \Omega)$, respectively. Remark that the slow manifold is given by

$$0 = U, \quad 0 = -\nu U - \gamma B + F(Y, \Omega, \mathbf{r}),$$

therefore, it can be solved for $x=(B,U)=x^\star(y)=(\gamma^{-1}F(Y,\Omega,\mathbf{r}),0)$, $\mathbf{r}\neq\mathbf{0}$. Note that $[F,\Omega^2]+3Gmr^{-5}[(Y^T\mathbf{r})(Y^T\mathbf{r})^T,F]=0$ and

$$YFY^T\mathbf{r}=\gamma^{-1}\left(\frac{2Gm}{r^3}\mathbf{r}-\langle\tilde{\boldsymbol{\omega}},\mathbf{r}\rangle\tilde{\boldsymbol{\omega}}+\frac{2}{3}\tilde{\omega}^2\mathbf{r}\right),$$

where $\tilde{\boldsymbol{\omega}}=S(Y\Omega Y^T)$ is the angular velocity in an inertial frame, as in (4.80). So, the reduced system (4.85) is given by

$$\Omega'+\frac{\varepsilon}{\gamma}\big(F(Y,\Omega,\mathbf{r})\Omega'+\Omega'F(Y,\Omega,\mathbf{r})\big)=0 \qquad (4.90)$$

$$\mathbf{r}''=-\frac{G\tilde{M}}{r^3}\mathbf{r}-\frac{6G\tilde{M}R^2\varepsilon^{\frac{5}{3}}}{5\gamma r^7}\left(\left(\frac{3GM_2}{r}+\frac{r^2\tilde{\omega}^2}{2}-\frac{5}{2}\langle\mathbf{r},\tilde{\boldsymbol{\omega}}\rangle^2\right)\mathbf{r}+r^2\langle\tilde{\boldsymbol{\omega}},\mathbf{r}\rangle\tilde{\boldsymbol{\omega}}\right). \qquad (4.91)$$

Remark that on the region $\|F(Y,\Omega,\mathbf{r})\|<\gamma/2\varepsilon$, (4.90) implies that $\Omega(\tau)=\Omega_0$, constant. We stress that it is a conservative system, with Hamiltonian

$$H(\mathbf{r},\mathbf{r}')=\frac{\tilde{m}}{2}\mathbf{r}'^2-\frac{GM_1M_2}{r}-\frac{GM_1M_2R^2\varepsilon^{\frac{5}{3}}}{5\gamma r^3}\left(\tilde{\omega}_0^2+\frac{3GM_2}{r^3}-\frac{3\langle\tilde{\omega}_0,\mathbf{r}\rangle^2}{r^2}\right). \qquad (4.92)$$

The next approximation of the adiabatic manifold (4.88) requires the computation of $A^\star$. For every y and $H_1,H_2\in\mathrm{ssym}(3)$,

$$A^\star(y)(H_1,H_2)=\frac{\partial f}{\partial x}(x^\star(y),y,\varepsilon)(H_1,H_2)=(H_2,-\nu H_2-\gamma H_1), \qquad (4.93)$$

hence,

$$A^\star(y)^{-2}(H_1,H_2)=\gamma^{-2}(\nu H_2+(\nu^2-\gamma)H_1,-\gamma(H_2+\nu H_1)).$$

Remark that, for all y, the eigenvalues of $A^\star(y)$ are the roots of $\lambda^2+\nu\lambda+\gamma=0$, so the system is uniformly asymptotically stable, fulfilling the hypotheses of Theorem 6. A straightforward computation shows that

$$\bar{x}(y,\varepsilon)=\left(\gamma^{-1}F-\frac{\nu\sqrt{\varepsilon}}{\gamma^2}(\partial_Y F(Y\Omega)+\partial_{\mathbf{r}}F(\mathbf{v})),\frac{\sqrt{\varepsilon}}{\gamma}(\partial_Y F(Y\Omega)+\partial_{\mathbf{r}}F(\mathbf{v}))\right)+\mathcal{O}(\sqrt{\varepsilon}^2) \qquad (4.94)$$

Performing the above derivatives, we see that to the equation (4.91) of the reduced system is added the term

$$\mathbf{F}_{Mig} = \frac{18G^2\tilde{M}R_1^2\varepsilon^{\frac{13}{6}}\nu}{5\gamma^2 r^8}\left(\mathbf{r}' + \mathbf{r}\times\tilde{\boldsymbol{\omega}} + 2\frac{\langle \mathbf{r},\mathbf{r}'\rangle}{r^2}\mathbf{r}\right) + \mathscr{O}(\varepsilon^{\frac{8}{3}}). \tag{4.95}$$

Recalling Lemmas 11 and 17 we see that for all solutions of [(4.66),(4.69) ,(4.70)], which are bounded and not close to collision, the inner product $\langle \mathbf{r}, \tilde{\boldsymbol{\omega}}\rangle \to 0$ as $t \to +\infty$. Hence, adding the hypothesis $\langle \mathbf{r}, \tilde{\boldsymbol{\omega}}\rangle = 0$ to the reduced system, we get the orbital equation

$$\begin{aligned}\mathbf{r}'' &= -\frac{G\tilde{M}}{r^3}\mathbf{r} - \frac{3G\tilde{M}R_1^2\varepsilon^{\frac{5}{3}}}{5\gamma r^5}\tilde{\omega}^2\mathbf{r}\\ &- \frac{3GM_2R_1^5\varepsilon^{\frac{13}{6}}}{r^8}\left(\frac{6G\tilde{M}}{5\gamma R_1^3}\right)\left(\mathbf{r} - \frac{\nu}{\gamma}\left(\mathbf{r}' + \mathbf{r}\times\tilde{\boldsymbol{\omega}} + 2\frac{\langle \mathbf{r},\mathbf{r}'\rangle}{r^2}\mathbf{r}\right)\right) + \mathscr{O}(\varepsilon^{\frac{8}{3}}).\end{aligned} \tag{4.96}$$

Note that only the second component is absent in (4.80). It corresponds to the gravitational attraction of the tide generated by the spin of the deformable body. Equation (4.96) imposes the interpretations of the time lag and Love number

$$\Delta t = \frac{\nu}{\gamma} = \frac{\eta}{\mu + \frac{2}{25}g\rho R}, \qquad k_2 = \frac{6GM}{5\gamma R^3} = \frac{3}{2}\frac{1}{\left(1 + \frac{25}{2}\frac{\mu}{\rho g R}\right)}. \tag{4.97}$$

A discussion on the deviation from the Love Number deduced in the traditional literature can be found in [63].

4.7 Time delay in a forced regime

An usual approach in tide theory is to study the reaction of the deformable body under an external periodic force. To do so, we impose the steady motion of the relative position vector $\mathbf{r}(t) = r_0(\cos(\tilde{\sigma}t), \sin(\tilde{\sigma}t), 0)$, where $\tilde{\sigma} > 0$ is constant. Suppose further that the angular velocity of the extended body is fixed and orthogonal to the orbit $S(\Omega) = \omega_1(0,0,1)$. So, $Y(t)$ have the same expression as in (4.74).

In this case, we can explicitly solve (4.66). Denoting by $\lambda_{1,2}$ the roots of the characteristic polynomial, we neglect all the transients terms $e^{\lambda_i t}$ of this solution, and the result is

$$YBY^T = \frac{1}{\lambda_1\lambda_2} Diag\left(\frac{\omega_1^2}{3}+\frac{GM_2}{2r_0^3}, \frac{\omega_1^2}{3}+\frac{GM_2}{2r_0^3}, -\frac{2\omega_1^2}{3}-\frac{GM_2}{r_0^3}\right)$$
$$+\frac{3GM_2}{2(\lambda_1-\lambda_2)r_0^3}\begin{pmatrix} h_1 & h_2 & 0 \\ h_2 & -h_1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$h_1 = \frac{-\lambda_1\cos(2\tilde{\sigma}t)+2\sigma\sin(2\tilde{\sigma}t)}{4\sigma^2+\lambda_1^2}+\frac{\lambda_2\cos(2\tilde{\sigma}t)-2\sigma\sin(2\tilde{\sigma}t)}{4\sigma^2+\lambda_2^2},$$

$$h_2 = -\frac{\lambda_1\sin(2\tilde{\sigma}t)+2\sigma\cos(2\tilde{\sigma}t)}{4\sigma^2+\lambda_1^2}+\frac{\lambda_2\sin(2\tilde{\sigma}t)+2\sigma\cos(2\tilde{\sigma}t)}{4\sigma^2+\lambda_2^2},$$

and $\sigma := \tilde{\sigma}-\omega_1$ is the frequency of the orbiting body relative to the surface of the extended body.

So, the matrix that gives the motion of the matter becomes

$$Y(\mathrm{Id}+\varepsilon B) = \gamma^{-1}\begin{pmatrix} a_1\begin{pmatrix}\cos(\omega_1 t) & -\sin(\omega_1 t)\\ \sin(\omega_1 t) & \cos(\omega_1 t)\end{pmatrix} & \begin{matrix}0\\0\end{matrix} \\ \begin{matrix}0 & \quad 0\end{matrix} & a_2\end{pmatrix}+\frac{3GM_2k}{2r_0^3}\varepsilon\cdot$$
$$\begin{pmatrix}\cos(\tilde{\sigma}t-\delta) & -\sin(\tilde{\sigma}t-\delta) & 0\\ \sin(\tilde{\sigma}t-\delta) & \cos(\tilde{\sigma}t-\delta) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos(\sigma t) & \sin(\sigma t) & 0\\ \sin(\sigma t) & -\cos(\sigma t) & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{4.98}$$

where

$$a_1 = \gamma+\varepsilon\left(\frac{\omega_1^2}{3}+\frac{GM_2}{2r_0^3}\right), \quad a_2 = \gamma-\varepsilon\left(\frac{2\omega_1^2}{3}+\frac{GM_2}{r_0^3}\right),$$

and

$$\delta = \tan^{-1}\left(\frac{2\sigma\nu}{\gamma-4\sigma^2}\right), \quad k = \frac{1}{\sqrt{4\sigma^2\nu^2+(\gamma-4\sigma^2)^2}}. \tag{4.99}$$

In this case, a straightforward analysis shows that $\lambda = \gamma^{-1}a_1 + 3GM_2k\varepsilon/(2r_0^3)$ is the highest eigenvalue of $\mathrm{Id} + \varepsilon B$, with eigenvector $\mathbf{v} = (\cos(\tilde{\sigma}t - \delta/2), \sin(\tilde{\sigma}t - \delta/2), 0)$. So, in the stationary state the bulge generated by the interaction follows the position of the orbiting body with phase lag

$$\Delta\varphi = \frac{1}{2}\tan^{-1}\left(\frac{2\sigma\nu}{\gamma - 4\sigma^2}\right). \tag{4.100}$$

We emphasize that the phase lag presented in (2.41) is similar to the just obtained. The approximation becomes better if we neglect σ^2, what is equivalent to neglect the inertial term $\ddot{B}$ in (4.66). The qualitative situation is the same illustrated in Figure 2.6. Moreover, the stationary motion of the complete system [(4.66), (4.69),(4.70)] corresponds to case $\sigma = 0$ and hence $\Delta\varphi = 0$.

The first matrix from (4.98) shows that the deformable body acquires a permanent flattening due to its rotation. In fact, most of the body's bulk rotates with angular velocity ω_1. The second matrix represents the motion of the bulge, which is consistently a small perturbation of the first. Remark that in the synchronous case, $\tilde{\sigma} = \omega_1$, the phase lag is not present.

Now we remark that substituting YBY^T, in the stationary state, into equation (4.70) we get

$$\begin{aligned}\ddot{\mathbf{r}} = -\frac{G\tilde{M}}{r^3}\mathbf{r} - \frac{3G\tilde{M}R_1^2}{r_0^7}\left(\frac{3}{5\gamma}\left(\frac{\omega_1^2 r_0^2}{3} + \frac{GM_2}{2r_0^3}\right)\right)\mathbf{r} \\ - \frac{3GM_2R_1^5}{r_0^7}\left(\frac{3G\tilde{M}k}{R_1^3}\right)\left(\frac{3}{5}\frac{\mathbf{r}}{r_0} + \frac{\langle \mathbf{r}, \mathbf{f}\rangle}{r_0^3}\mathbf{r} - \frac{2}{5}\frac{\mathbf{f}}{r_0}\right),\end{aligned} \tag{4.101}$$

where $\mathbf{f}(t) = \mathbf{r}_{del}(t) - \mathbf{r}(t)$, and $\mathbf{r}_{del}(t) = r_0(\cos(\sigma t - \delta), \sin(\sigma t - \delta))$ is the position of the orbiting body at the instant $t - \Delta t$. Notice that the last term of (4.101) is essentially the same present in equation (20) of [28]. So, our model imposes the following (frequency dependent) Love number

$$k_2(\sigma) = \frac{3G\tilde{M}}{5R_1^3}\frac{1}{\sqrt{4\sigma^2\nu^2 + (\gamma - 4\sigma^2)^2}}, \tag{4.102}$$

which is incompatible with the Love number obtained by [24], page 18,

$$k_2(\sigma) = k_f\sqrt{\frac{1 + \sigma^2\eta^2/\mu^2}{1 + (\sigma^2\eta^2/\mu^2)(1 + 19\mu/2g\rho R_1)^2}}, \tag{4.103}$$

where k_f is the fluid Love number, see also [19]. The main reason for this difference is that Darwin considered the body with the Maxwell rheology, modifying substantially the equation of motion.

4.8 Extension to the two pseudo-rigid body problem

In this section, we generalize the problem considering that the second body with mass M_2 is also pseudo-rigid, with spherical rest shape. We keep the same relative scales discussed in Section 4.5. We remark that the kinetic, potential elastic and potential of self-gravitation energies and the dissipation function are addictive, hence their expressions are easily adapted from those already deduced. Only the potential of interaction requires some discussion.

So, consider two homogeneous pseudo-rigid bodies whose center of mass is defined by the vectors $\mathbf{Q}_1, \mathbf{Q}_2 \in \mathbb{R}^3$. Defining again $\mathbf{r} = \mathbf{Q}_2 - \mathbf{Q}_1$, we have the potential of interaction

$$V_{int}(\varepsilon) = -G\rho_1\rho_2 \int_{\mathscr{B}_1}\int_{\mathscr{B}_2} \frac{1}{\|Y_2 e^{\varepsilon B_2}\mathbf{y} - Y_1 e^{\varepsilon B_1}\mathbf{x} + \mathbf{r}\|} d\mathbf{y}d\mathbf{x}.$$

Since the domains of the integrals are balls, we can assert that

$$V_{int}(0) = -G\rho_1\rho_2 \int_{\mathscr{B}_1}\int_{\mathscr{B}_2} \frac{1}{\|\mathbf{y} - \mathbf{x} + \mathbf{r}\|} d\mathbf{y}d\mathbf{x} = -\frac{GM_1M_2}{\|\mathbf{r}\|},$$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} V_{int}(\varepsilon) = G\rho_1\rho_2 \int_{\mathscr{B}_1}\int_{\mathscr{B}_2} \frac{\left\langle \mathbf{y} - \mathbf{x} + \mathbf{r}, Y_2B_2Y_2^T\mathbf{y} - Y_1B_1Y_1^T\mathbf{x}\right\rangle}{\|\mathbf{y} - \mathbf{x} + \mathbf{r}\|^3} d\mathbf{y}d\mathbf{x}.$$

The expression above is symmetrical with respect to the indices 1,2, hence it is enough to compute

$$\begin{aligned} &\int_{\mathscr{B}_1}\int_{\mathscr{B}_2} \frac{\left\langle \mathbf{y} - \mathbf{x} + \mathbf{r}, Y_1B_1Y_1^T\mathbf{x}\right\rangle}{\|\mathbf{y} - \mathbf{x} + \mathbf{r}\|^3} d\mathbf{y}d\mathbf{x} \\ &\qquad = \int_{\mathscr{B}_1} \left\langle Y_1B_1Y_1\mathbf{x}, \int_{\mathscr{B}_2} \frac{\mathbf{x} - \mathbf{r} - \mathbf{y}}{\|\mathbf{x} - \mathbf{r} - \mathbf{y}\|^3} d\mathbf{y} \right\rangle d\mathbf{x}. \qquad (4.104) \end{aligned}$$

From the well known formula

$$\int_{\mathscr{B}_2} \frac{\mathbf{x}-\mathbf{r}-\mathbf{y}}{\|\mathbf{x}-\mathbf{r}-\mathbf{y}\|^3} d\mathbf{y} = \frac{4\pi R_2^3}{3} \frac{\mathbf{x}-\mathbf{r}}{\|\mathbf{x}-\mathbf{r}\|^3},$$

we see that (4.104) becomes

$$\frac{4\pi R_2^3}{3} \int_{\mathscr{B}_1} \frac{\langle Y_1 B_1 Y_1 \mathbf{x}, \mathbf{x}-\mathbf{r}\rangle}{\|\mathbf{x}-\mathbf{r}\|^3} d\mathbf{x} = \frac{4\pi R_2^3}{3} \frac{4\pi R_1^5}{5\|\mathbf{r}\|^5} \operatorname{Tr}\left((Y_1^T \mathbf{r})(Y_1^T \mathbf{r})^T B_1\right),$$

where we have used again the equation (B.1). So,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} V_{int}(\varepsilon) = -\frac{3GM_1M_2}{\|\mathbf{r}\|^5} \operatorname{Tr}\left(\mathbf{r}\mathbf{r}^T (R_1^2 Y_1 B_1 Y_1^T + R_2^2 Y_2 B_2 Y_2^T\right).$$

Now we also impose the rescale $\mathbf{r} \to \varepsilon^{-\frac{1}{3}}\mathbf{r}$, hence

$$V_{int} = -\frac{GM_1M_2\varepsilon^{\frac{1}{3}}}{\|\mathbf{r}\|} - \frac{3GM_1M_2\varepsilon^2}{\|\mathbf{r}\|^5} \operatorname{Tr}\left(\mathbf{r}\mathbf{r}^T \left(R_1^2 Y_1 B_1 Y_1^T + R_2^2 Y_2 B_2 Y_2^T\right)\right) + \mathscr{O}(\varepsilon^3) \tag{4.105}$$

Therefore, the Lagrangian and dissipation functions acquire the form

$$\mathscr{L} = \sum_{i=1}^{2} \frac{M_i R_i^2}{10} \varepsilon^{\frac{5}{3}} \left(\varepsilon \operatorname{Tr}(\dot{B_i}^2) - \operatorname{Tr}(\Omega_i^2) - 2\varepsilon \operatorname{Tr}(\Omega_i^2 B_i) - \gamma_i \varepsilon \operatorname{Tr}(B_i^2)\right)$$
$$+ \frac{\tilde{m}}{2}\dot{\mathbf{r}}^2 + \frac{GM_1M_2\varepsilon}{\|\mathbf{r}\|} + \frac{3GM_1M_2\varepsilon^{\frac{8}{3}}}{\|\mathbf{r}\|^5} \operatorname{Tr}\left(\mathbf{r}\mathbf{r}^T \left(R_1^2 Y_1 B_1 Y_1^T + R_2^2 Y_2 B_2 Y_2^T\right)\right), \tag{4.106}$$

$$\mathscr{D} = \frac{4\pi}{3} \varepsilon^{\frac{8}{3}} \sum_{i=1}^{2} \eta_i R_i^3 \operatorname{Tr}(\dot{B}_i^2). \tag{4.107}$$

Hence, for $(i,j) = (1,2)$ and $(i,j) = (2,1)$, we have the equations of motion

$$\ddot{B}_i + \nu_i \dot{B}_i + \gamma_i B_i = -\Omega_i^2 + \frac{1}{3}\,\mathrm{Tr}\,(\Omega_i^2)\,\mathrm{Id} + \frac{3GM_j}{r^5}\left((Y_i^T\mathbf{r})(Y_i^T\mathbf{r})^T - \frac{1}{3}r^2\,\mathrm{Id}\right) \tag{4.108}$$

$$\dot{\Omega}_i + \varepsilon\left(B_i\dot{\Omega}_i + \dot{\Omega}_i B_i + \Omega_i \dot{B}_i + \dot{B}_i \Omega_i\right) = \varepsilon[B_i, \Omega_i^2] + \frac{3GM_j}{r^5}\varepsilon[(Y_i^T\mathbf{r})(Y_i^T\mathbf{r})^T, B_i] \tag{4.109}$$

$$\ddot{\mathbf{r}} = -\frac{G\tilde{M}\varepsilon}{r^3}\mathbf{r} - \frac{3G\tilde{M}\varepsilon^{\frac{8}{3}}}{r^7}\left\langle (R_1^2 Y_1 B_1 Y_1^T + R_1^2 Y_2 B_2 Y_2^T)\mathbf{r}, \mathbf{r}\right\rangle \mathbf{r} + \frac{6G\tilde{M}\varepsilon^{\frac{8}{3}}}{5r^5}(R_1^2 Y_1 B_1 Y_1^T + R_2^2 Y_2 B_2 Y_2^T)\mathbf{r}. \tag{4.110}$$

We remark that the Noether's theorem (Theorem 2) can also be applied to this case, which provides the total angular momentum as first integral

$$L = \frac{M_1 R_1^2 \varepsilon^{\frac{5}{3}}}{5} Y_1 \Phi_{\varepsilon B_1}(\Omega_1) Y_1^T + \frac{M_2 R_2^2 \varepsilon^{\frac{5}{3}}}{5} Y_2 \Phi_{\varepsilon B_2}(\Omega_2) Y_2^T + \frac{\tilde{m}}{2}(\dot{\mathbf{r}}\mathbf{r}^T - \mathbf{r}\dot{\mathbf{r}}^T). \tag{4.111}$$

Likewise the previous case, the kinetic energy and dissipation function are quadratic in the velocities, so $E = T + V$ is such that $\dot{E} = -2\mathscr{D} \leq 0$. The equality $\dot{E} = 0$ is reached if, and only if $\dot{B}_1 = \dot{B}_2 = 0$. Consequently, by the LaSalle invariance principle (Lemma 11) we see that every bounded solution that stays far from collision approximates of a rigid motion. The determination of such asymptotic motions is a corollary of Lemma 17, which we state in the following.

Corollary 1. *Suppose that a solution of system [(4.108),(4.109),(4.110)] is such that $B_1(t), B_2(t)$ are constant, for all $t \geq 0$, and $\|B_i\| < (2\varepsilon)^{-1}$ for $i = 1, 2$. Then, the relative position vector* $\mathbf{r}$ *describes an uniform circular motion and the angular velocities of the two bodies are constant and orthogonal to the plane of translation. Moreover, the bodies prescribe a synchronous motion.*

Proof. We see from equations (4.108) that

$$\varepsilon[B_i, \Omega_i^2] + \frac{3GM_j}{r^5}\varepsilon[(Y_i^T\mathbf{r})(Y_i^T\mathbf{r})^T, B_i] = 0, \quad i, j = 1, 2.$$

Since the operators $\Phi_{\varepsilon B_i}$ are invertible, we conclude from (4.109) that $\dot{\Omega}_1(t) = \dot{\Omega}_2(t) = 0$, $\forall t \geq 0$, hence Ω_1, Ω_2 are constant. So, from (4.108) again we deduce that

$$Z_i = \frac{1}{r^5} Y_i^T \left(\mathbf{r}\mathbf{r}^T - \frac{1}{3} r^2 \,\mathrm{Id} \right) Y_i, \quad i = 1, 2,$$

are constant, too. This is the same expression as (4.75), hence by the proof of Lemma 17 we conclude that at some reference $\mathbf{r}(t) = r_0(\cos(\omega t), \sin(\omega t), c)$ and the angular velocity of both bodies are equal and parallel to the z axis. We also see that $\|\boldsymbol{\omega_1}\| = \|\boldsymbol{\omega_2}\| = \omega$. Likewise equation (4.76), the orbital equation (4.110) implies that

$$-\omega^2 \tilde{\mathbf{r}} = \psi(\mathbf{r})\mathbf{r} + \frac{6G\tilde{M}\varepsilon^{\frac{8}{3}} \omega^2 (R_1^2 + R_2^2)}{5r^5} \tilde{\mathbf{r}}, \tag{4.112}$$

and we conclude that $c = 0$, so the motion is in fact planar. □

We remark that the plane of the orbital motion is orthogonal to the angular momentum vector, which is the same along the time evolution. Moreover, the same considerations of Lemma 17 ensure that the bulges induced on both surfaces point towards the line connecting the centers of mass and no time delay is present. Now, the angular momentum (4.111) is such that

$$\|L\| = \left(\frac{\varepsilon^{\frac{5}{3}}}{5} \left((M_1 R_1^2 + M_2 R_2^2) + \frac{2\omega^2 \varepsilon}{3} \left(\frac{M_1 R_1^2}{\gamma_1} + \frac{M_2 R_2^2}{\gamma_2} \right) \right.\right.$$
$$\left.\left. + \frac{G M_1 M_2 \varepsilon}{r^3} \left(\frac{R_1^2}{\gamma_1} + \frac{R_2^2}{\gamma_2} \right) \right) + \frac{\tilde{m} r^2}{2} \right) \omega, \tag{4.113}$$

and the orbital equation (4.110) implies that

$$18G^2\tilde{M}(M_2 R_1^2 \gamma_1^{-1} + M_1 R_2^2 \gamma_2^{-1}) \varepsilon^{\frac{5}{3}} (r^{-1})^8$$
$$+ 6G\tilde{M}(R_1^2 \gamma_1^{-1} + R_2^2 \gamma_2^{-1}) \varepsilon^{\frac{5}{3}} \omega^2 (r^{-1})^5 + G\tilde{M}(r^{-1})^3 = \omega^2. \tag{4.114}$$

Since [(4.113),(4.114)] and [(4.77),(4.78)] have the same form, we can analogously conclude the existence of a minimum radius (Roche's limit) $r_{min} = \varepsilon^{\frac{1}{3}} \sqrt[5]{6G\tilde{M}(R_1^2 \gamma_1^{-1} + R_2^2 \gamma_2^{-1})}$ which is higher than that estimated in the previous problem.

Chapter 5

Nongravitational Forces

By far the most important force acting on large celestial bodies is gravity, except for bodies in gaseous environment where gas drag can be equally or more important. As the size of the celestial body decreases other forces may acquire importance: Lorentz force, radiation pressure, Poynting-Robertson drag, etc... In this chapter we briefly discuss each of these forces.

5.1 Electromagnetic forces in planetary magnetospheres

Most of the macroscopic celestial matter is either neutral or has an electric charge that is negligible when compared to its mass. This is a consequence of the neutrality of atoms and molecules. So, most of the electrically charged particles in space are either elementary particles, for instance electrons, or ions, that means molecules or atoms with an unbalanced number of protons and electrons. A fluid composed of charged particles, a gas of ions, is called plasma. The dynamics of a plasma is determined by the usual equation of fluid mechanics (Euler or Navier-Stokes) plus additional terms corresponding to electromagnetic forces.

Small micrometric particles, or simply dust, embedded in a plasma acquire net electric charge. Since plasma is abundant in space and the mechanism of electric charging of dust particles can be different depending on the plasma conditions we decided to focus on a particular but important situation: the motion of charged particles in planetary magnetospheres specially those that form planetary rings. Dust is plentiful in planetary rings like those of Jupiter and Saturn. Since dust density is low, collisions can be neglected and the dynamics of dust in the rings can be well described by single particle dynamics. Parts of the following introduction to the subject were taken literally from [39].

A charged particle in a neighborhood of a planet is under the action of Keplerian gravity and of a rotating magnetic dipole. While the gravity field of a spherical planet is not affected by its rotation, its electromagnetic field is. A hypothesis usually accepted (see [40]) is that the plasmasphere (the region of the magnetosphere which contains plasma) of a planet co-rotates with it. This implies that in a rotating reference frame, where the planet is at rest, the electromagnetic field inside the plamasphere is purely magnetic. Indeed, if there were an electric field inside the plamasphere then the plasma would move with respect to the co-rotating reference frame what would violate the assumed hypothesis. Therefore, in the co-rotating reference frame the observed electromagnetic field is just that of a magnetic dipole. For simplicity, assume that the magnetic dipole is aligned with the planet axis of rotation and is located at the center of the planet. The Lorentz force F (CGS units) acting upon a particle of electric charge q, with velocity v, and under a magnetic field B is

$$F = \frac{q}{c} v \times B,$$

where c is the light velocity. The equations of motion of a charged particle in that rotating reference frame under Newton gravitational force and Lorentz magnetic force can be easily written. When transformed to an inertial reference frame with origin at the center of the planet and z-axis aligned and co-oriented with the planet's spin axis, these equations of motion write in CGS units as:

$$\ddot{\mathbf{r}} = \frac{q\mathscr{M}}{cm}\left[\dot{\mathbf{r}} - \Omega(\mathrm{e}_z \times \mathbf{r})\right] \times \left[-\frac{\mathrm{e}_z}{|\mathbf{r}|^3} + 3\frac{(\mathrm{e}_z \cdot \mathbf{r})\mathbf{r}}{|\mathbf{r}|^5}\right] - MG\frac{\mathbf{r}}{|\mathbf{r}|^3} \tag{5.1}$$

where: m is the mass of the particle, $\Omega > 0$ is the spin angular velocity of the planet, $\mathscr{M}$ is the magnitude of the magnetic dipole moment of the planet ($\mathscr{M}$ is positive if the dipole is co-oriented with the angular velocity of the planet, otherwise it is negative), M is the mass of the planet, and G is the universal constant of gravitation. A detailed analysis of the dynamics generated by equation (5.1) for particles initially close to the equatorial plane is given in [39]. In this analysis an important role is played by the dimensionless parameter

$$\delta = \frac{q}{m}\frac{\mathscr{M}\Omega}{M}\frac{1}{cG} \tag{5.2}$$

Notice that for a given planet the constants $\mathscr{M}$, Ω, and M are fixed, c and G are universal constants, and so δ represents the ratio charge over mass in a dimensionless way.

A circumplanetary dust grain can acquire net electrical charge by two main mechanisms. The first is due to collisions with plasma ions and electrons of the circumplanetary plasma. These collisions tend to negatively charge dust grains, because at the same temperature, electrons are faster than positive ions and have more collisions with dust particles. The second mechanism is due to ultraviolet photo-emission (photo-electric effect) which remove electrons from a grain to turn it positively charged. These two competing effects essentially determine the net charge of a particle (besides [40], [14], see [44] and [66]). Both mechanisms strongly depend on the location of the particle inside the plasmasphere and its relative position with respect to the Sun. So, the net charge of a dust grain is not a property of the grain itself but it is a function of its position and time. In spite of this fact, it is usually assumed in the literature that the net charge of a dust particle is constant in time. Due to this strong hypothesis we must be cautious when trying to apply the results obtained from equation (5.1) to real questions of circumplanetary dust.

A discussion on the magnitude of the important parameter δ is given in section 4.2 of [39]. Here we just mention that the radius of a typical dust particle in either Jupiter or Saturn ring is $1\mu m$ and that the electromagnetic force on it is one hundred times weaker than the gravity force (see [14]). So, the dynamics of a typical dust particle in one of these rings is just the usual gravity-dynamics of celestial mechanics perturbed by the Lorentz force (other forces acting upon the particle are even weaker than the Lorentz force). These planetary rings also contain a large fraction of sub-micrometric particles for which electromagnetic forces become more important. For instance, for a particle with $0.1\mu m$ of radius the electromagnetic force is typically greater than the gravitational force. In the limit of very small particles (like electrons or ions) the gravitational force becomes negligible compared to the electromagnetic force. See more details in [39] and in the review articles [40] and [14].

5.2 Radiation forces: radiation pressure and Poynting-Robertson drag

Photons carry momentum that can be transferred to matter by means of collision, absorption, or emission. As a result, a radiation source gives rise to a force that tends to push particles away from the source. Since the intensity of radiation falls off quadratically with distance, the same happens with the radiation force. For instance, the Sun radiation pressure on a planar totally reflecting surface placed transversally to the Sun rays is equal to $9.08 \times 10^{-6} N/m^2$. The radiation force is usually significant for particles which have large surface-area-to-mass ratios (tenths to tens of micrometers for dust in planetary rings [14]), since inertial and gravitational effects decrease cubically with linear dimension.

The detailed computation of the radiation force is quite involved and depends on the geometry of the particle, on the fraction of radiation that is absorbed, which depends on the radiation frequency, etc. The radiation force can be decomposed into two parts: a radiation pressure term and a Poynting-Robertson drag (or mass-loading drag) term. The following presentation of the concepts of radiation pressure and Poynting-Robertson drag is due to Burns, Lamy, and Soter [13].

Let S $(ergs \cdot cm^{-2} \cdot sec^{-1})$ be the flux of energy of a radiation beam integrated over all the radiation spectrum. The energy per second absorbed by a stationary, perfectly absorbing particle of geometrical cross section A is SA. Suppose that the radiation source is, for instance, the Sun and consider a polar coordinate (r, θ) system centered at it with coordinate unit vectors $(\mathrm{e}_r, \mathrm{e}_\theta)$. If the particle is moving relative to the Sun with velocity v, then S must be replaced by $S' = S(1 - \dot{r}/c)$, where $\dot{r} = v \cdot \mathrm{e}_r$ and c is the speed of light. The factor in parentheses is due to the Doppler effect, which alters the incident energy flux by shifting the received wavelengths. The momentum per second absorbed by the particle from the incident beam is

$$\frac{S'A}{c}\mathrm{e}_r = \frac{SA}{c}\left(1 - \frac{\dot{r}}{c}\right)\mathrm{e}_r, \tag{5.3}$$

which is the part of the radiation force upon the particle due to the **radiation pressure**.

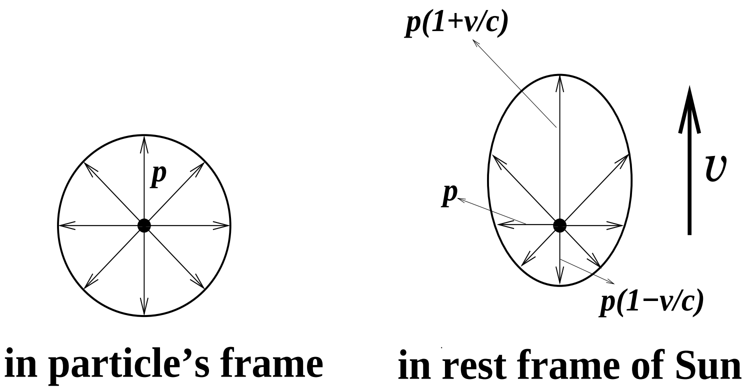

Figure 5.1: A schematic illustration of the balance of momentum reradiated by a spherical particle in isothermal equilibrium. The letter p denotes the momentum of the emitted photon.

At thermodynamic equilibrium, the same amount of energy per second $S'A$ absorbed by the particle must be reradiated. In a reference frame fixed on the particle, the reradiation is nearly isotropic (see Figure 5.1, adapted from Fig. 2 in [13]), because the temperature of small particles is nearly uniform. However, as seen from the solar frame of reference, supposed inertial, the reradiation preferentially emits $S'Av/c^2$ more momentum in the forward direction, because the frequencies (and momenta) of the quanta reradiated in the forward direction are increased over those in the backward direction. Conservation of momentum implies a reaction force due to reradiation that equals to

$$F_{PR} = -S'A\frac{v}{c^2} = -\frac{SA}{c}\frac{v}{c}\left(1-\frac{\dot{r}}{c}\right) \approx -\frac{SA}{c}\frac{v}{c}, \tag{5.4}$$

where terms of order $\|v/c\|^2$ were neglected. The part of the radiation force given in equation (5.4) is called **Poynting-Robertson drag**.

When the particle is not perfectly absorbing then the reflected radiation must also be taken into account in the balance of momentum. In this case, after averaging over the radiation spectrum and over the scattering angle α (that is measured from the incident beam direction; $\alpha = 0$ corresponding to forward scattering) a radiation pressure coefficient Q_{pr} is defined as

$$Q_{pr} = Q_{abs} + Q_{sca}(1 - \langle \cos\alpha \rangle) \tag{5.5}$$

where Q_{abs} and Q_{sca} are the absorption and scattering coefficients, respectively; and $\langle \cos\alpha \rangle$ is the anisotropy parameter that measures the preferential direction of scattering of radiation. In perfect forward-scattering, $Q_{pr} = Q_{abs}$; with isotropic scattering, $Q_{pr} = Q_{abs} + Q_{sca}$; while for perfect back-scattering, $Q_{pr} = Q_{abs} + 2Q_{sca}$. Finally, the radiation force (CGS units) upon a heliocentric particle with velocity $v = \dot{r}\mathrm{e}_r + r\dot{\theta}\mathrm{e}_\theta$ is

$$F_{rad} = \frac{SA}{c} Q_{pr} \left(1 - \frac{\dot{r}}{c}\right) \mathrm{e}_r - \frac{SA}{c} Q_{pr} \frac{v}{c}, \tag{5.6}$$

where the radial term comes from the radiation pressure, equation (5.3), and the other term comes from the Poynting-Robertson drag, equation (5.4). The radiation force can also be written as

$$F_{rad} = \frac{SA}{c} Q_{pr} \left[\left(1 - 2\frac{\dot{r}}{c}\right) \mathrm{e}_r - \frac{r\dot{\theta}}{c} \mathrm{e}_\theta \right]. \tag{5.7}$$

Many times the velocity dependent part of the radiation force, as given in equation (5.7), is called the Poynting-Robertson drag while the constant radial term the radiation pressure; sometimes the radial part of this force is called radiation pressure and the transverse term the Poynting-Robertson drag. So, there is no consensus in the literature about the use of the expressions "radiation pressure" and "Poynting-Robertson drag".

5.3 Radiation forces: Yarkovsky effect

In order to compute the Poynting-Robertson drag in equation (5.4) we assumed that the temperature within a dust particle is constant. This assumption fails to be true for sufficiently large bodies (more than $10cm$) because thermal resistivity becomes large and a temperature gradient establishes between the hot side of the body, which faces the radiation source, and the cold side. If the body is rotating around an axis that is not directed to the radiation source then the part of the body that is facing the radiation source changes continuously and due to thermic inertia the point of maximum temperature in the body is no longer that facing the radiation source. This is the cause of the so called diurnal Yarkovsky effect [11].

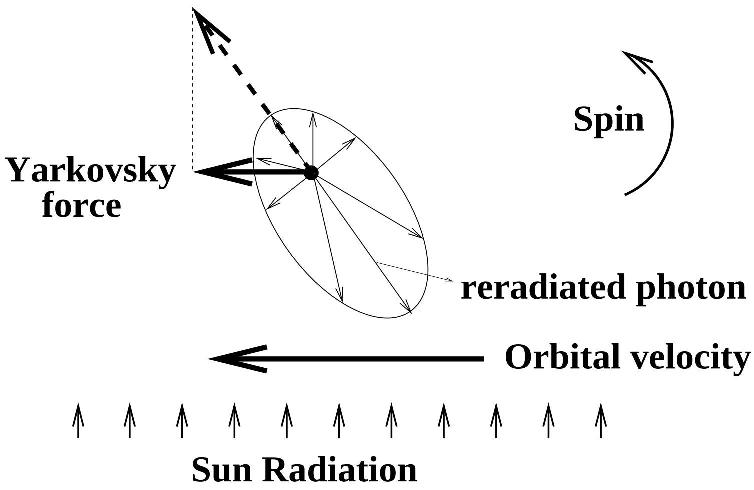

Figure 5.2: A schematic illustration of the balance of momentum reradiated by a prograde asteroid that generates a diurnal Yarkovsky force that points to the same direction of the orbital velocity.

For simplicity, we will explain the diurnal Yarkovsky effect on Earth. Due to the thermal inertia of the Earth the highest temperature during the day is not achieved at noon but in the afternoon (around 2PM [11]). Since the power radiated by a heated surface is proportional to the fourth power of the temperature (Kelvin degree), the angular distribution of the energy radiated by the Earth is not uniform (in contrast to what happens in the analysis of the Poynting-Robertson drag in the reference frame of the body, see Figure 5.1). As in the analysis of the Poynting-Robertson drag, we conclude that the angular distribution of the momentum carried by the radiated photons is not uniform but has a distribution illustrated in Figure 5.2. A balance of momentum results in a force that has a component in the direction of the orbital motion, this is the **Yarkovsky diurnal effect**.

Notice that the Yarkovsky force in Figure 5.2 points to the same direction as the orbital velocity. The opposite occurs when the body has a retrograde rotation, namely the spin rotation is opposite to the orbital rotation, see Figure 5.3.

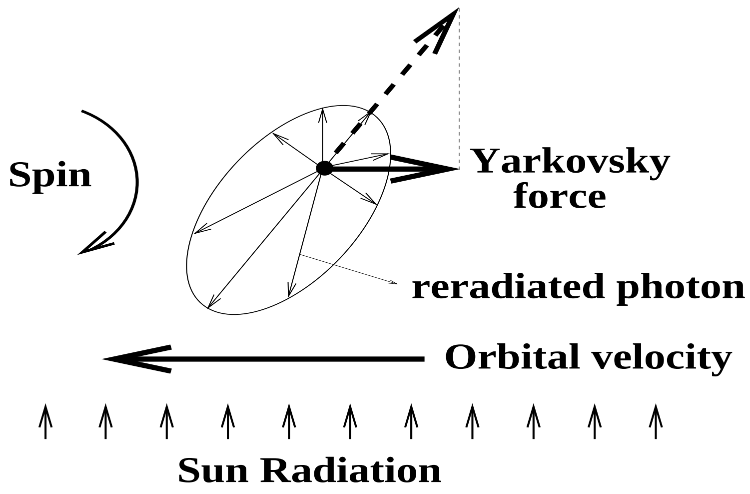

Figure 5.3: A schematic illustration of the balance of momentum reradiated by a retrograde asteroid that generates a diurnal Yarkovsky force that points to the opposite direction of the orbital velocity.

The diurnal Yarkovsky effect is mainly a consequence of the thermal inertia and the spin of a rotating body. There is a second force of the same nature that is a consequence of the thermal inertia, the orbital rotation of the body, and the inclination of the spin axis with respect to the orbital plane: the **seasonal Yarkovsky effect**. For simplicity, consider an asteroid with a circular orbit around the Sun and with spin axis lying in the orbital plane, see Figure 5.4. The northern hemisphere of the asteroid is heated when it is facing the Sun. Due to thermal inertia it remains hot when compared to the southern hemisphere even when the southern hemisphere begins to face the Sun. A balance of momentum of thermal photons reradiated implies on a radiation force upon the asteroid that has a component with the same direction of but opposite to the orbital velocity. Notice that this force is always opposite to the orbital velocity independently on on the sense of rotation of the asteroid.

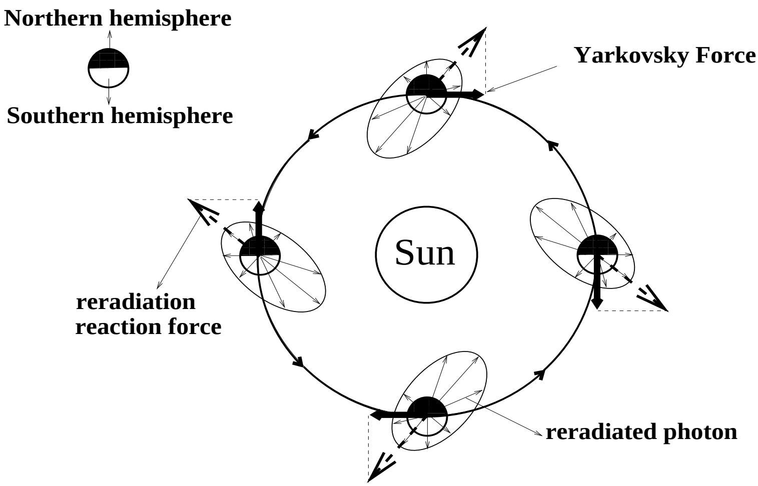

Figure 5.4: A schematic illustration of the balance of momentum reradiated by an asteroid that generates a seazonal Yarkovsky force.

The quantitative evaluation of the Yarkovsky effects depends on several properties of the body: albedo, geometry, thermal conductivity, etc... The theoretical estimates of the Yarkovsky force are not precise and the subject is currently under experimental investigation (see [11]).

Appendix A

Self-Gravitational Potential

We perform the Taylor expansion of (4.27) up to order ε^2. If $A = \exp(\varepsilon B) \in \mathrm{SSym}_+(3)$ then:

$$\phi(\varepsilon B) = -\frac{G\rho^2}{2}\int_{\mathscr{B}}\int_{\mathscr{B}} \frac{1}{\|\exp(\varepsilon B)(x-y)\|}dxdy + C$$
$$= -\gamma\int_0^\infty \frac{1}{\sqrt{\det(\exp(2\varepsilon B)+\lambda\,\mathrm{Id}\,)}}d\lambda + C.$$

Thus, (choosing $\phi(0) = 0$),

$$\phi(\varepsilon B) = \varepsilon D\phi(0)B + \frac{1}{2}\varepsilon^2 D^2\phi(0)B^2 + \mathscr{O}(\varepsilon^3).$$

Since the derivatives of the integrand are continuous and bounded, we may perform the following calculations

$$D\phi(\varepsilon B)B = \frac{d}{d\varepsilon}\phi(\varepsilon B) = \gamma\int_0^\infty \frac{\mathrm{Tr}\,\big((\exp(2\varepsilon B)+\lambda\,\mathrm{Id}\,)^{-1}\exp(2\varepsilon B)B\big)}{\sqrt{\det(\exp(2\varepsilon B)+\lambda\,\mathrm{Id}\,)}}d\lambda,$$

Hence,

$$D\phi(0)B = 0,$$

Extending this computation, we can see that

$$D^2\phi(0)B^2 = -4\gamma\int_0^\infty \frac{1}{2(1+\lambda)^{\frac{3}{2}}}\big((1+\lambda)^{-2} - (1+\lambda)^{-1}\big)\,d\lambda\,\mathrm{Tr}\,(B^2)$$

$$D^2\phi(0)B^2 = \frac{8}{15}\gamma \operatorname{Tr}(B^2).$$

Recalling that $\gamma = 3M^2G/(10R)$, we get

$$\phi(\varepsilon B) = \frac{2}{25}\frac{M^2G}{R}\varepsilon^2 \operatorname{Tr}(B^2) + \mathcal{O}(\varepsilon^3). \tag{A.1}$$

About the potential (4.25), a straightforward calculation shows that

$$\begin{aligned} V_{el}(\varepsilon B) &= \frac{4\pi R^3}{3}\left(\mu\left(\operatorname{Tr}(e^{2\varepsilon B}) - \frac{1}{3}\operatorname{Tr}(e^{\varepsilon B})^2\right) + \frac{K}{2}\left(\operatorname{Tr}(e^{\varepsilon B}) - 3\right)^2\right) \\ &= \frac{4\pi R^3\mu}{3}\varepsilon^2 \operatorname{Tr}(B^2) + \mathcal{O}(\varepsilon^3). \end{aligned} \tag{A.2}$$

Appendix B

Interaction Potential

Defining from (4.57)

$$V_{int}(Y,B,\mathbf{r}) = -GM_2\rho\phi(\varepsilon B),$$

and using the symmetries of the integral, we see that

$$\begin{aligned}\phi(\varepsilon B) &= \phi(0) + \varepsilon D\phi(0)B + \mathscr{O}(\varepsilon^2)\\ &= \int_{\mathscr{B}} \frac{1}{\|Y^T\mathbf{r}-\mathbf{x}\|}d\mathbf{x} + \varepsilon\int_{\mathscr{B}} \frac{\langle Y^T\mathbf{r}-\mathbf{x}, B\mathbf{x}\rangle}{\|Y^T\mathbf{r}-\mathbf{x}\|^3}d\mathbf{x} + \mathscr{O}(\varepsilon^2),\end{aligned}$$

with

$$\int_{\mathscr{B}} \frac{\langle \mathbf{r}-\mathbf{x}, YBY^T\mathbf{x}\rangle}{\|\mathbf{r}-\mathbf{x}\|^3}d\mathbf{x} = \operatorname{Tr}\left(YBY^T\int_{\mathscr{B}}\frac{\mathbf{x}(\mathbf{r}-\mathbf{x})^T}{\|\mathbf{r}-\mathbf{x}\|^3}d\mathbf{x}\right).$$

So, we only need to compute

$$\int_{\mathscr{B}} \frac{(\mathbf{r}-\mathbf{x})\mathbf{x}^T}{\|\mathbf{r}-\mathbf{x}\|^3}d\mathbf{x}.$$

Recalling the following well-known formula (for $\mathbf{u} \notin \mathscr{B}$),

$$\int_{\mathscr{B}} \|\mathbf{u}-\mathbf{x}\|d\mathbf{x} = \frac{4\pi}{3}\left(\frac{R_1^5}{5\|\mathbf{u}\|} + R_1^3\|\mathbf{u}\|\right),$$

and defining

$$G(\mathbf{r}) = \frac{4\pi}{3}\left(\frac{R_1^5}{5\|\mathbf{r}\|} + R_1^3\|\mathbf{r}\|\right),$$

we see that

$$\begin{aligned}\frac{4\pi}{3}R_1^3\frac{\mathbf{r}}{\|\mathbf{r}\|} - \int_{\mathscr{B}} \frac{\mathbf{x}}{\|\mathbf{r}-\mathbf{x}\|}d\mathbf{x} &= \int_{\mathscr{B}} \frac{\mathbf{r}-\mathbf{x}}{\|\mathbf{r}-\mathbf{x}\|}d\mathbf{x} \\ &= \nabla G(\mathbf{r}) = \frac{4\pi}{3}\left(-\frac{R_1^5}{5\|\mathbf{r}\|^3} + \frac{R_1^3}{\|\mathbf{r}\|}\right)\mathbf{r},\end{aligned}$$

so,

$$\int_{\mathscr{B}} \frac{\mathbf{x}}{\|\mathbf{r}-\mathbf{x}\|}d\mathbf{x} = \frac{4\pi}{15}\frac{R_1^5}{\|\mathbf{r}\|^3}\mathbf{r} =: (U_1(\mathbf{r}), U_2(\mathbf{r}), U_3(\mathbf{r})).$$

Remark that each function U_i satisfy

$$\int_{\mathscr{B}} \frac{x_i(\mathbf{r}-\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|^3}d\mathbf{x} = -\nabla U_i(\mathbf{r}) = -\frac{4\pi}{15}\frac{R_1^5}{\|\mathbf{r}\|^3}\mathbf{e}_i + \frac{4\pi}{5}\frac{R_1^5}{\|\mathbf{r}\|^5}r_i\mathbf{r}.$$

Therefore,

$$\int_{\mathscr{B}} \frac{(\mathbf{r}-\mathbf{x})\mathbf{x}^T}{\|\mathbf{r}-\mathbf{x}\|^3}d\mathbf{x} = -\frac{4\pi}{15}\frac{R_1^5}{\|\mathbf{r}\|^3}\,\mathrm{Id} + \frac{4\pi}{5}\frac{R_1^5}{\|\mathbf{r}\|^5}\mathbf{r}\mathbf{r}^T, \tag{B.1}$$

and hence we get the formula (4.58).

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