Boltzmann-type Equations and their Applications

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Chapter 1

Introduction

Modern Kinetic theory is full of applications not only for the understanding of complex phenomena but also for the development of accurate numerical schemes to resolve partial differential equations in many areas. Let us cite for instance applications in semiconductor modeling, radiative transfer, grain and polymer flows, biological systems, cellular mechanics, chain supply dynamics, quantitative finance, traffic models, wave propagation in random media, hydrodynamic and quantum models, understanding of boundary and interaction in multi-scale phenomena, and phase transitions. The main goal of this notes is precisely to present the reader an introduction to modern kinetic theory. We will cover some of the influential result in the area and give a baseline for research initiation in this topic leaving, of course, many important results out due to space and time. The list of reference is, by no means, exhaustive, yet, it is a good initial step for further reading and cross–reference.

These notes are divided in five chapters: Introduction, derivation of kinetic models from particle dynamics, classical Boltzmann equation, dissipative Boltzmann equation and radiative transfer equation. After this introduction, we start covering basic ideas that help to understand the kinetic modeling point of view. This translates mathematically in the rigorous derivation of kinetic models from systems of many particles. In some cases this process of going from particles to kinetics is known as mean field limit. The third chapter begins by covering elementary material about the Boltzmann equation such as physical interpretation, weak formulation, conservation laws and dissipation of entropy. It continues with a presentation of the classical theory of existence and uniqueness of weak solutions for the inhomogeneous Boltzmann equation given by Kaniel & Shinbrot and a short discussion of the celebrated theory introduced by DiPerna & Lions of renormalized solutions. After covering the basic material, the section moves to the homogeneous Boltzmann equation explaining the importance of the analysis of moments, propagation of integrability and regularity in the study of the equation. This section ends with a discussion on entropic methods and includes a short discussion of the celebrated result by Toscanni & Villanni on dissipation of entropy and its impact on the analysis for the long time asymptotic of the Boltzmann model. The fourth chapter will cover several relevant mathematical and physical aspects in the theory of viscoelastic materials modeled using the dissipative Boltzmann equation. Recent results on existence and uniqueness of solutions will be given by revisiting the Kaniel & Shinbrot method adding a short discussion on the discrepancies and difficulties with respect to the classical Boltzmann theory. Interesting phenomena present in dissipative dynamics such as self-similar profiles, overpopulated tails, intermediate asymptotic properties, propagation of regularity and Haff's law will be commented (all of them inexistent in the classical elastic theory!). Several examples of dissipative kinetic models will be given in this section mainly oriented to applications in biology and economics, such as the celebrated Cucker & Smale model, wealth distribution model and rod alignment model. The latter two fall directly in the theory of dissipative Boltzmann equation in one dimension. The notes ends with a chapter devoted to the study of the radiative transport equation. Classical theory on integrable scattering and recent results on the forward-peaked regime are presented. This equation will be used to motivate the theory of hypo-elliptic operators and fractional diffusions in mathematical physics.

Chapter 2

From particle systems to kinetic models

We start this notes with a generic example of a particle system that is widely used in physics, biomechanics, biology, economy, material sciences, traffic modeling and many other areas. The idea is simple and comes from elementary mechanics: in a system of large number of particles, particles essentially interact continuously by means of friction and elasticity. These interactions are of different nature, interaction by friction produces loss of mechanical energy while elasticity is related to storage of mechanic energy due to deformation. This is a generic model in material science, a typical example is the Kelvin–Voigt model for viscoelastic materials (viscosity and friction are equivalent terms here). Assume we have a system with N particles having position and velocity (x_i, v_i), the model that we briefly study in this section is given by the ODE system

$$\frac{dx_i}{dt} = v_i\,, \quad \frac{dv_i}{dt} = \frac{1}{N}\sum_{j\neq i} U_f\big(|x_j - x_i|\big)\big(v_j - v_i\big) + \frac{1}{N}\sum_{j\neq i} U_e'\big(|x_j - x_i|\big)\big(\widehat{x}_j - \widehat{x}_i\big)\,. \tag{2.1}$$

Here U_f and U_e are the frictional and elastic potentials respectively that we consider depending only on the distance between particles. A typical elastic potential is given by $U_e(s) = \frac{\kappa}{2}s^2$ which gives Hooke's law (linear law) in elasticity. A possible interpretation of model (2.1) is that a given particle experiments a weighted averaged frictional and elastic forces due to interaction with other particles, that is, each particle experiments a *mean field* interaction. These averages are presumably more influenced by close neighbors, thus, one expects such potentials to decay. Of course, the properties of the potentials, such as decay and smoothness, will completely determine the behavior of the system and the physics it models. Thus, it is natural to expect that the mathematical analysis will be highly dependent on the properties assumed for the potentials. For example, we refer to [7] for a numerical study of model (2.1) applied to cellular mechanics.

Before entering in mathematical details, let us formally discuss a particular case of (2.1), the celebrated model in animal behavior proposed by Cucker-Smale [36] which was first studied with mathematical rigor in [49]. The Cucker-Smale model is precisely the model (2.1) with zero elastic potential and frictional potential $U_f \geq 0$ enjoying certain properties.

2.1 Formal derivation of a mean field

The goal in this short discussion is to derive a kinetic model (mean filed model) for the particle model (2.1). Although this discussion is formal, it will help to introduce key ideas and concepts in kinetic theory that can be made rigorous in many instances. We start recalling that a Hamiltonian system is one that is completely described by a scalar function $\mathcal{H}(t, \mathbf{x}, \mathbf{v})$, the Hamiltonian. The evolution of the system is given by

$$\frac{d\mathbf{x}}{dt} = \partial_{\mathbf{v}}\mathcal{H} \quad \frac{d\mathbf{v}}{dt} = -\partial_{\mathbf{x}}\mathcal{H}\,,$$

where $\mathbf{x} = \big(x_1, \cdots, x_N\big)$ and $\mathbf{v} = \big(v_1, \cdots, v_N\big)$ are the vectors of positions and velocities of the particles respectively. The product space $(\mathbf{x}, \mathbf{v})$ is addressed as phase space. Many systems are Hamiltonian, including (2.1). There is a central result for Hamiltonian systems due to J. Gibbs: The distribution function of a Hamiltonian particle system is constant along any trajectory in phase space. Indeed, assume that the number of particles is large enough that it becomes meaningful to observe the N-particle density distribution

$$f^N(t,\mathbf{x},\mathbf{v}) = \frac{\text{number of particles at } (t,\mathbf{x},\mathbf{v})}{\text{volume in phase space}}\,.$$

If $\mathcal{J}$ is the flux of particles at any given point of phase space, one has for any measurable set A

$$\begin{aligned}\text{Variation number particles in } A &= \frac{d}{dt}\int_A f^N(t,\mathbf{x},\mathbf{v})\mathrm{d}V_{(\mathbf{x},\mathbf{v})}\\ &= \int_{\partial A}\mathcal{J}(t,\mathbf{x},\mathbf{v})\cdot\mathbf{n}\,\mathrm{d}S_{(\mathbf{x},\mathbf{v})} = \text{Net flux through } \partial A\,.\end{aligned}$$

Using the divergence theorem

$$\int_{\partial A}\mathcal{J}(t,\mathbf{x},\mathbf{v})\cdot\mathbf{n}\,\mathrm{d}S_{(\mathbf{x},\mathbf{v})} = -\int_A \nabla\cdot\mathcal{J}(t,\mathbf{x},\mathbf{v})\mathrm{d}V_{(\mathbf{x},\mathbf{v})}\,.$$

It readily follows that

$$\int_A\Big(\partial_t f^N(t,\mathbf{x},\mathbf{v}) + \nabla\cdot\mathcal{J}(t,\mathbf{x},\mathbf{v})\Big)\mathrm{d}V_{(\mathbf{x},\mathbf{v})} = 0\,.$$

Two points are made here: (1) The measurable set A is arbitrary, and (2) the flux is related to the density distribution by the formula

$$\mathcal{J}(t,\mathbf{x},\mathbf{v}) = f^N(t,\mathbf{x},\mathbf{v})\frac{d}{dt}(\mathbf{x},\mathbf{v})\,.$$

One concludes that

$$\begin{aligned}0 &= \partial_t f^N(t,\mathbf{x},\mathbf{v}) + \nabla\cdot\mathcal{J}(t,\mathbf{x},\mathbf{v})\\ &= \partial_t f^N(t,\mathbf{x},\mathbf{v}) +\\ &\qquad \frac{d}{dt}(\mathbf{x},\mathbf{v})\cdot\nabla f^N(t,\mathbf{x},\mathbf{v}) + \Big(\nabla\cdot\frac{d}{dt}(\mathbf{x},\mathbf{v})\Big)f^N(t,\mathbf{x},\mathbf{v})\,.\end{aligned}$$

Observe that for the latter term

$$\nabla\cdot\frac{d}{dt}(\mathbf{x},\mathbf{v}) = \nabla\cdot\big(\partial_{\mathbf{v}}\mathcal{H}, -\partial_{\mathbf{x}}\mathcal{H}\big) = \partial_{\mathbf{x}}\partial_{\mathbf{v}}\mathcal{H} - \partial_{\mathbf{v}}\partial_{\mathbf{x}}\mathcal{H} = 0\,.$$

The conclusion is an equation known as Liouville's equation

$$\partial_t f^N(t,\mathbf{x},\mathbf{v}) + \frac{d}{dt}\big(\mathbf{x},\mathbf{v}\big)\cdot\nabla f^N(t,\mathbf{x},\mathbf{v}) = 0\,, \qquad (2.2)$$

which is precisely Gibbs' statement. Now, Gibbs' statement is about the N-particle distribution function, what we really want is a closed equation for the *single* distribution function

$$f(t,x_1,v_1) := \int f^N(t,\mathbf{x},\mathbf{v})\mathrm{d}V^{N-1}_{(\mathbf{x},\mathbf{v})}\,,$$

where the superscript $N-1$ is added to the differential to denote an integration on the last $N-1$ coordinates (x_i,v_i) of the phase space. Since particles are indistinguishable, it is irrelevant which single density distribution we choose to describe. Of course, in a general situation finding a closed equation for the single particle distribution is an impossible task because particle trajectories are necessarily correlated (particles are interacting at all times), so any mathematical formalism will include dependence of all particles. However, it is possible to argue that in a situation of a large number of particles, one may find a good approximating model for the evolution of the single particle distribution. The argument goes like this for the Cucker-Smale model: Note that Liouville's equation in such case reduces to

$$\partial_t f^N + \sum_i v_i\cdot\nabla_{x_i} f^N + \frac{1}{N}\sum_i \nabla_{v_i}\cdot\Big(\sum_j U_f\big(|x_i-x_j|\big)\big(v_j-v_i\big)f^N\Big)\,. \qquad (2.3)$$

Integrate equation (2.3) in $(\mathbf{x},\mathbf{v})^{N-1}$ and observe that the divergence theorem leads to

$$\int\sum_i v_i\cdot\nabla_{x_i} f^N\,\mathrm{d}V^{N-1}_{(\mathbf{x},\mathbf{v})} = v_1\cdot\nabla_{x_1} f(t,x_1,v_1)\,.$$

Additionally,

$$
\begin{aligned}
&\frac{1}{N}\sum_i \int \nabla_{v_i}\cdot\Big(\sum_j U_f\big(|x_i-x_j|\big)\big(v_j-v_i\big)f^N\Big)\mathbf{d}V^{N-1}_{(\mathbf{x},\mathbf{v})}\\
&\qquad=\frac{1}{N}\int \nabla_{v_1}\cdot\Big(\sum_{j=2}^{N} U_f\big(|x_1-x_j|\big)\big(v_j-v_1\big)f^N\Big)\mathbf{d}V^{N-1}_{(\mathbf{x},\mathbf{v})}\\
&\qquad=\frac{N-1}{N}\int \nabla_{v_1}\cdot\Big(U_f\big(|x_1-x_2|\big)\big(v_2-v_1\big)f^N\Big)\mathbf{d}V^{N-1}_{(\mathbf{x},\mathbf{v})}\\
&\qquad=\frac{N-1}{N}\int \nabla_{v_1}\cdot\Big(U_f\big(|x_1-x_2|\big)\big(v_2-v_1\big)f^2\Big)\mathrm{d}x_2\mathrm{d}v_2\,.
\end{aligned}
$$

In the first equality we used divergence theorem which vanishes the last $N-1$ terms of the outer sum. In the second equality we used symmetry of f^N (particles are indistinguishable), thus, the interaction between particles $(1,j)$ equals $N-1$ times the interaction of particles $(1,2)$. And, for the last equality we used the obvious definition of the two-particle distribution function f^2. Now, a central issue rises here and it is known as molecular chaos. That is to say, for large number of particles

$$f^2(t,x_1,v_1,x_2,v_2)\approx f(t,x_1,v_1)\,f(t,x_2,v_2)\,. \tag{2.4}$$

This means that the specific position and velocity of one particular particle is almost uncorrelated, at any time, to the specific position and velocity of any other particle. Intuitively this should be the case for particles that follow the mean field of particles rather than a single one such us model (2.1). Molecular chaos should also holds in systems such as billiards (related to the Boltzmann equation) where two particles bear large number of interactions in between their particular interaction. That is, at the moment of their interaction such particles are essentially uncorrelated.

This argument leads to the approximated closed equation

$$
\begin{aligned}
&\partial_t f(t,x,v)+v\cdot\nabla_x f(t,x,v)\\
&+\tfrac{N-1}{N}\,\nabla_v f(t,x,v)\cdot\int U_f\big(|x_*-x|\big)\big(v_*-v\big)f(t,x_*,v_*)\,\mathrm{d}x_*\mathrm{d}v_*\approx 0\,,
\end{aligned}
$$

valid for large number of particles N. In the limit $N\to\infty$ the molecular chaos approximation (2.4) should become exact, thus, if the distribution sequence $f := f_N$ converges, say to f, one finds the kinetic description (mean field limit) of the Cucker-Smale particle model

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot Q(f,f) = 0\,, \tag{2.5}$$

where,

$$Q(f,f)(t,x,v) := f(t,x,v) \int U_f\big(|x_* - x|\big)\big(v_* - v\big) f(t,x_*,v_*)\, \mathrm{d}x_* \mathrm{d}v_*\,. \tag{2.6}$$

2.2 Rigorous derivation of a mean field

Let us give now a rigorous treatment of the mean field limit given in previous discussion for a particle model slightly more general that (2.1). We add some random fluctuations to the particles and perform the analysis following the program proposed in [70, 26]. Thus, consider a large system of N-interacting particles having positions $(x_i(t), v_i(t)) \in \mathbb{R}^{2d}$ and following the dynamics

$$\begin{aligned} \mathrm{d}x_i(t) = v_i(t)\mathrm{d}t\,, \quad \mathrm{d}v_i(t) &= \sqrt{2}\,\mathrm{d}B_i(t) \\ &- \frac{1}{N}\sum_{j\neq i} \mathcal{H}\big(x_i(t) - x_j(t), v_i(t) - v_j(t)\big)\mathrm{d}t\,, \end{aligned} \tag{2.7}$$

with independent initial data $(x_i(0), v_i(0))$ all having the *same* distribution law f_o. The processes $B_i(t)$ are independent standard brownian motions in $\mathbb{R}^d$. The interacting potential $\mathcal{H} : \mathbb{R}^{2d} \to \mathbb{R}^d$ is assumed to be Lipschitz continuous.

The central analytical result consists in proving that such process $(x_i(t), v_i(t))$ behave in the limit $N \to \infty$ like a process $(\bar{x}_i(t), \bar{v}_i(t))$ solving the McKean-Vlasov equation on $\mathbb{R}^{2d}$

$$\mathrm{d}\bar{x}_i(t) = \bar{v}_i(t)\mathrm{d}t\,, \quad \mathrm{d}\bar{v}_i(t) = \sqrt{2}\,\mathrm{d}B_i(t) - \big(\mathcal{H} * f\big)(t, \bar{x}_i(t), \bar{v}_i(t))\mathrm{d}t\,, \tag{2.8}$$

where the initial condition is given by $(\bar{x}_i(0), \bar{v}_i(0)) = (x_i(0), v_i(0))$ and $f(t,x,v)$ is the law of $(\bar{x}_i(t), \bar{v}_i(t))$. It is well know from the theory of Itô processes that the law of $(\bar{x}_i, \bar{v}_i(t))$ satisfies the Kolmogorov Forward equation, see for instance the tutorial [64]

$$\partial_t f + v \cdot \nabla_x f = \Delta_v f + \nabla \cdot \big(f\big(\mathcal{H} * f\big)\big)\,, \quad f(0) = f_o\,. \tag{2.9}$$

We refer to [26, Theorem 1.2] for a complete proof about existence and uniqueness of systems (2.7) and (2.8) under suitable conditions on the initial law f_o. In particular, the fact that equation (2.9) have unique solution implies that all processes are equally distributed which explain why we dropped the index in the law $f := f_i$. One fact that holds for solutions f of equation (2.9), which we will need below, is that spatial and velocity moments of order two are propagated, in order words

$$\int_{\mathbb{R}^{2d}} \big(|x|^2+|v|^2\big)\mathrm{d}f_o(x,v) \longrightarrow \sup_{t\in[0,T]}\int_{\mathbb{R}^{2d}} \big(|x|^2+|v|^2\big)\mathrm{d}f(t,x,v) \leq C_T\,. \tag{2.10}$$

Theorem 2.2.1. *Let f_o be a Borel probability measure having spatial and velocity moments of order two, and the initial state $(x_i(0), v_i(0))$ be independent random variables with common law f_o. Under the aforementioned conditions, there exists a constant C_T such that*

$$\sup_{t\in[0,T]}\max_{0\leq i\leq N}\Big(\mathbb{E}\big[|x_i(t)-\bar{x}_i(t)|^2\big]+\mathbb{E}\big[|v_i(t)-\bar{v}_i(t)|^2\big]\Big)$$
$$= \sup_{t\in[0,T]}\Big(\mathbb{E}\big[|x_1(t)-\bar{x}_1(t)|^2\big]+\mathbb{E}\big[|v_1(t)-\bar{v}_1(t)|^2\big]\Big) \leq \frac{C_T}{N}\,.$$

Proof. Define the fluctuations $x_i^e(t) := x_i(t)-\bar{x}_i(t)$ and $v_i^e(t) := v_i(t)-\bar{v}_i(t)$ for $i=1,\cdots,N$ and introduce the total error

$$e(t) = \max_{1\leq i\leq N}\big\{\mathbb{E}\big[|x_i^e(t)|^2+|v_i^e(t)|^2\big]\big\}\,.$$

Now, subtract the models (2.7) and (2.8). Thus, for the position fluctuations one has

$$\frac{1}{2}\frac{d}{dt}\mathbb{E}\big[|x_i^e(t)|^2\big] = \mathbb{E}\big[x_i^e(t)\cdot v_i^e(t)\big] \leq \frac{e(t)}{2}\,. \tag{2.11}$$

The velocity fluctuations require more work. One certainly has that the velocity fluctuations satisfy

$$\frac{1}{2}\frac{d}{dt}\mathbb{E}\big[|v_i^e(t)|^2\big] = -\frac{1}{N}\sum_{j\neq i}\mathbb{E}\Big[v_i^e(t)\cdot\Big(\mathcal{H}(x_i(t)-x_j(t), v_i(t)-v_j(t))$$
$$-\big(\mathcal{H}*f\big)(t,\bar{x}_i(t),\bar{v}_i(t))\Big)\Big] =: I_1+I_2\,, \tag{2.12}$$

where (dropping the t variable to ease notation)

$$I_1 = -\frac{1}{N}\sum_{j\neq i}\mathbb{E}\Big[v_i^e\cdot\Big(\mathcal{H}(x_i-x_j,v_i-v_j)-\mathcal{H}(\bar{x}_i-\bar{x}_j,\bar{v}_i-\bar{v}_j)\Big)\Big]$$

and,

$$\begin{aligned} I_2 &= -\frac{1}{N}\mathbb{E}\Big[v_i^e\cdot\sum_{j\neq i}\Big(\mathcal{H}(\bar{x}_i-\bar{x}_j,\bar{v}_i-\bar{v}_j)-\big(\mathcal{H}*f\big)(t,\bar{x}_i,\bar{v}_i)\Big)\Big]\\ &=: -\frac{1}{N}\mathbb{E}\Big[v_i^e\cdot\sum_{j\neq i}Y_{i,j}\Big]\,. \end{aligned}$$

The term I_1 is controlled using Lipschitz continuity of $\mathcal{H}$

$$\begin{aligned} \Big|\mathcal{H}(x_i-x_j,v_i-v_j)-&\mathcal{H}(\bar{x}_i-\bar{x}_j,\bar{v}_i-\bar{v}_j)\Big|\\ &\leq \|\mathcal{H}\|_{\mathrm{Lip}}\big(|x_i^e|+|v_i^e|+|x_j^e|+|v_j^e|\big)\,, \end{aligned}$$

as a consequence, a simple application of Young's inequality leads to

$$\big|I_1(t)\big| \leq \tfrac{5}{2}\|\mathcal{H}\|_{\mathrm{Lip}}\,e(t)\,. \tag{2.13}$$

For the term I_2 one has

$$\big|I_2\big| \leq \frac{1}{N}\sqrt{\mathbb{E}\big[|v_i^e|^2\big]}\sqrt{\mathbb{E}\Big[\Big|\sum_{j\neq i}Y_{i,j}\Big|^2\Big]}\,.$$

Furthermore, for any $j\neq k$

$$\begin{aligned} \mathbb{E}\big[Y^{i,j}\cdot Y^{i,k}\big] &= \mathbb{E}\Big[\mathbb{E}\big[Y^{i,j}\cdot Y^{i,k}\big|(\bar{x}_i,\bar{v}_i)\big]\Big]\\ &= \mathbb{E}\Big[\mathbb{E}\big[Y^{i,j}\big|(\bar{x}_i,\bar{v}_i)\big]\cdot\mathbb{E}\big[Y^{i,k}\big|(\bar{x}_i,\bar{v}_i)\big]\Big] \end{aligned}$$

since processes $(\bar{x}_j(t),\bar{v}_j(t))$ are uncorrelated. A direct computation shows then

$$\begin{aligned} &\mathbb{E}\big[Y^{i,j}\big|(\bar{x}_i,\bar{v}_i)\big]\\ &\qquad= \int_{\mathbb{R}^{2d}}\Big(\mathcal{H}(\bar{x}_i-x_*,\bar{v}_i-v_*)-\big(\mathcal{H}*f\big)(t,\bar{x}_i,\bar{v}_i)\Big)\mathrm{d}f(t,x_*,v_*)\\ &\qquad= \big(\mathcal{H}*f\big)(t,\bar{x}_i,\bar{v}_i)-\big(\mathcal{H}*f\big)(t,\bar{x}_i,\bar{v}_i)=0\,. \end{aligned}$$

Here we have use that the law of $(\bar{x}_j(t), \bar{v}_j(t))$ is precisely $f(t)$. Therefore,

$$\begin{aligned}\mathbb{E}\Big[\Big|\sum_{j\neq i} Y_{i,j}\Big|^2\Big] &= (N-1)\mathbb{E}\big[\big|Y_{1,2}\big|^2\big] \\ &\leq (N-1)\int_{\mathbb{R}^{4d}} \big|\mathcal{H}(x-x_*, v-v_*)\big|^2 \mathrm{d}f(t,x,v)\mathrm{d}f(t,x_*,v_*) \\ &\leq 4(N-1)\int_{\mathbb{R}^{4d}} \Big(\|\mathcal{H}\|^2_{\mathrm{Lip}}\big(\big|x-x_*\big|^2 + \big|v-v_*\big|^2\big) \\ &\qquad + \big|\mathcal{H}(0,0)\big|^2\Big)\mathrm{d}f(t,x,v)\mathrm{d}f(t,x_*,v_*) \leq C_T(N-1)\,,\end{aligned}$$

where in the last inequality we used (2.10). In summary,

$$\big|I_2(t)\big| \leq C_T \frac{\sqrt{e(t)}}{\sqrt{N}} \leq e(t) + \frac{\tilde{C}_T}{N}\,. \tag{2.14}$$

Gathering (2.11), (2.12), (2.13) and (2.14) one gets

$$\mathbb{E}\big[|x^e_i(t)|^2\big] + \mathbb{E}\big[|v^e_i(t)|^2\big] \leq c_o \int_0^t e(s)\mathrm{d}s + \frac{\tilde{C}_T}{N}\,, \quad 1 \leq i \leq N\,. \tag{2.15}$$

Here c_o is independent of $T > 0$. Since the right side of (2.15) is independent of the particle i, we can compute the max along the particles and use Gronwall's lemma to conclude

$$\sup_{t\in[0,T]} e(t) \leq \frac{\tilde{C}_T}{c_o\,N}\, e^{c_o T}\,.$$

The proof is concluded by noticing that all single marginals of the join probability of N–particles are equal because particles are indistinguishable. Thus,

$$e(t) = \mathbb{E}\big[|x_1(t) - \bar{x}_1(t)|^2 + |v_1(t) - \bar{v}_1(t)|^2\big]\,.$$

□

Convergence in mean square implies convergence in probability. Thus, Theorem 2.2.1 readily implies that $\lim_{N\to\infty} f^1_N(t) = f(t)$, where

$f_N^1(t)$ is the single marginal of the join probability of N-particles at time t. Furthermore, it also implies a precise quantitative version of molecular chaos. In order to see this, let us introduce the Wasserstein distance between Borel probability measures (μ, ν), see for instance [73], as

$$\mathrm{d}_2(\mu,\nu) = \inf_{(X,Y)} \sqrt{\mathbb{E}\big[|X-Y|^2\big]}\,, \tag{2.16}$$

where the infimum is taken over all couples of random variable (X, Y) with X having law μ and Y having law ν. Thus,

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathrm{d}_2(f_N^1(t), f(t))^2 \\
&\qquad \leq \mathbb{E}\Big[\big|(x_1(t),v_1(t)) - (\bar{x}_1(t),\bar{v}_1(t))\big|^2\Big] \\
&\qquad = \mathbb{E}\big[|x_1(t)-\bar{x}_1(t)|^2 + |v_1(t)-\bar{v}_1(t)|^2\big] \leq \frac{C_T}{N}\,.
\end{aligned}$$

Moreover, the k-marginal f_N^k converges towards the tensor $f^{\otimes k}$ as N increases since

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathrm{d}_2(f_N^k(t), f^{\otimes k}(t))^2 \\
&\leq \mathbb{E}\Big[\big|(x_1(t),v_1(t),\cdots,x_k(t),v_k(t)) - (\bar{x}_1(t),\bar{v}_1(t),\cdots,\bar{x}_k(t),\bar{v}_k(t))\big|^2\Big] \\
&= k\,\mathbb{E}\big[|x_1(t)-\bar{x}_1(t)|^2 + |v_1(t)-\bar{v}_1(t)|^2\big] \leq \frac{k\,C_T}{N}\,.
\end{aligned}$$

Chapter 3

Classical Boltzmann equation

We saw in the previous section that the mean field limit of particle systems interacting with smooth potentials is given by integro–differential equations. Solutions of such equations are interpreted as distributions of particles depending on space x (macroscopic variable), velocity v (microscopic or kinetic variable) and time t (which is, somehow, both a macro and a micro variable). The Boltzmann equation is also an integro–differential equation that represents the kinetic description of a many–particle system interacting through *collisions*. Such interaction is of different nature to that of friction or elasticity: a collision is a discontinuous process while interactions with smooth potentials is continuous. This seemingly banal difference proves to be crucial in the rigorous derivation of the Boltzmann model from particle dynamics. In fact, such derivation is still an open (and quite important) problem in statistical physics. Let us write down the model and try to explain it, at least, at the formal level

$$\partial_t f + v \cdot \nabla_x f = Q(f,f)\,, \qquad (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^{2d}\,, \tag{3.1}$$

complemented with an initial configuration $f(0) = f_o$. Here the operator $Q(f,f)$ will represent collision interactions between particles. More specifically, its bilinear form is defined, for any suitable functions f and g, as

$$Q(f,g)(v) := \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \Big(f('v)g('v_*) - f(v)g(v_*) \Big) B(v - v_*, \omega) \mathrm{d}\omega \mathrm{d}v_* \,. \tag{3.2}$$

We need to do some explaining with the introduced notation. The pair (v, v_*) represents velocities of two particles that just collide and had, before collision, velocities $'v$ and $'v_*$. In this way, the pair $('v, 'v_*)$ are *pre-collisional* velocities. Similar, it is common the notation (v', v'_*) to represent post-collisional velocities of a pair of particles having velocities v and v_* before collision. In the classical Boltzmann equation the collision law map $\mathfrak{C}_\omega : (v, v_*) \to (v', v'_*)$ is very special because must conserve microscopic momentum and energy, in other words, is such that

$$v' + v'_* = v + v_* \,, \qquad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2 \,. \tag{3.3}$$

One concludes that it *must* be the case that (see [34] or Lemma 3.0.2 below)

$$v' = v - (u \cdot \omega)\omega \,, \qquad v'_* = v_* + (u \cdot \omega)\omega \,, \tag{3.4}$$

where ω represents the unit vector perpendicular to the collision plane and $u := v - v_*$ is the relative velocity between particles. Now, the function $B \geq 0$ is commonly known as collision kernel and it describes the physics of the collision, we refer to [34] and [71] for extensive discussion. It is customary to assume the factorization in the mathematics community

$$B(u, \omega) = |u|^\gamma \, b(\widehat{u} \cdot \omega) \,, \quad \gamma \in (-d, 2] \,. \tag{3.5}$$

It is understood that $\widehat{u} = u/|u|$. The function b is known as scattering kernel and weights the probability of scattering at certain angle after a collision event. It is customary to assume the so-called cutoff hypothesis

$$\int_{\mathbb{S}^{d-1}} b(\widehat{u} \cdot \omega) \mathrm{d}\omega < \infty \,.$$

Although, cutoff is a realistic assumption, such hypothesis fails to be true in some relevant physical situations. The last section of these notes is brought precisely to give an introduction to the mathematical theory when such hypothesis is not met. The most typical example of (3.5) is the so called *hard-spheres* model which describes the dynamics of a 3-dimensional billiard and is given by $B(u,\omega) = |u \cdot \omega|$ which corresponds to $\gamma = 1$. Furthermore, in the mathematical literature, the cases $\gamma \in (-d, 0)$, $\gamma = 0$, and $\gamma \in (0, 1]$ are addressed as soft potentials, Maxwell molecules and hard potentials respectively. Properties of the collision law map $\mathfrak{C}_\omega$ are given in the following lemma,

Lemma 3.0.2. *For any $\omega \in \mathbb{S}^{d-1}$ it follows that: (1) $\mathfrak{C}_\omega \circ \mathfrak{C}_\omega = Id$, (2) $det\,\mathfrak{C}_\omega = -1$, (3) the only functions φ satisfying $\varphi + \varphi_* = \varphi' + \varphi'_*$ are given by*

$$\varphi(v) = a + b \cdot v + c\,|v|^2\,, \quad a,\, c \in \mathbb{R},\, b \in \mathbb{R}^d\,.$$

Such functions are called collision invariants (here $\varphi' = \varphi(v')$ and $\varphi'_* = \varphi(v'_*)$).

Proof. Let us denote the post-collisional relative velocity as $u' = v' - v'_*$. Item (1) is clear since $u' \cdot \omega = -u \cdot \omega$. A proof of (2) follows by introducing the map $\tilde{\mathfrak{C}}_\omega : (v, u) \to (v', u')$. Clearly, $\det\tilde{\mathfrak{C}}_\omega = \det\mathfrak{C}_\omega$, moreover, the matrix representation for $\tilde{\mathfrak{C}}_\omega$ is given by

$$[\tilde{\mathfrak{C}}_\omega] = \begin{bmatrix} \mathbf{1} & -\omega \otimes \omega \\ 0 & \mathbf{1} - 2\,\omega \otimes \omega \end{bmatrix}.$$

Thus, $\det \tilde{\mathfrak{C}}_\omega = \det\big(\mathbf{1} - 2\,\omega \otimes \omega\big) = \det\big(\text{diag}(-1, 1, \cdots, 1)\big) = -1$. For a proof of item (3) see for instance [65]. □

Particles are continuously colliding, thus, one may think that they are experiencing a birth-death process with respect to the velocity variable: at time t two particles occupying the same spatial point x will not longer have velocity (v, v_*) if they collide, that is, with approximate probability

$$\text{Prob. of death of a pair } (v, v_*) \approx f(t,x,v)\, f(t,x,v_*)\, B(u,\omega)\mathrm{d}\omega\mathrm{d}v_*\,.$$

Similarly, at time t two particles occupying the same spatial point x will create two particles with velocities (v, v_*) if they just collided having velocities $('v, 'v_*)$, that is, with approximate probability

$$\text{Prob. of birth of a pair } (v, v_*) \approx f(t,x,'v)\, f(t,x,'v_*)\, B(u,\omega)\mathrm{d}\omega\mathrm{d}v_*\,.$$

The collision operator is just the integration of this probabilities over all possible collision directions ω and velocities v_*. Note, that we have used propagation of chaos in computing these approximate probabilities, namely, the joint distribution of two particles is approximate the product of the single distributions. Intuitively, this should be very accurate since the velocity correlation between two particles is minimal in a large system of them sustaining numerous collisions. The proof of this fact is a notoriously difficult problem in the Boltzmann context. The reader can find a proof of the following proposition in [34, 65].

Proposition 3.0.3. *For a B satisfying* (3.5) *one has the following properties:*
(1) (Conservation) For all suitable functions f and φ

$$\int_{\mathbb{R}^d} Q(f,f)(v)\varphi(v)\,dv = \frac{1}{4}\int_{\mathbb{R}^d} Q(f,f)(v)\Big(\varphi' + \varphi'_* - \varphi - \varphi_*\Big)\,dv$$

(2) (Boltzmann's H-Theorem)

$$\int_{\mathbb{R}^d} Q(f,f)(v)\ln(f(v))\,dv \le 0\,.$$

(3) (Gaussian equilibria) And, for any $B > 0$ one have the equivalence

$$Q(F,F) = 0 \longleftrightarrow F(v) := \frac{\rho}{(2\pi T)^{d/2}}\,e^{-\frac{|v-v_o|^2}{2T}}\,.$$

for some ρ, $T \ge 0$ and $v_o \in \mathbb{R}^d$.

Note that, thanks to Proposition 3.0.3, solutions of the Boltzmann equation formally satisfy

$$\int_{\mathbb{R}^d} f(v)\begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix}\mathrm{d}v = 0\,. \tag{3.6}$$

Proposition 3.0.3 also leads to an important observation. Introduce the entropy and the dissipation of entropy as

$$\mathcal{H}(f) := \int_{\mathbb{R}^d} f \, \ln(f) \mathrm{d}v \mathrm{d}x \,,$$
$$0 \le \mathcal{D}(f) := \tag{3.7}$$
$$\frac{1}{4}\int_{\mathbb{R}^{3d}}\int_{\mathbb{S}^{d-1}} \big(f' f'_* - f f_*\big)\big(\ln(f' f'_*) - \ln(f f_*)\big) B \mathrm{d}\omega \mathrm{d}v_* \mathrm{d}v \mathrm{d}x \,.$$

Note the the dissipation of entropy is nonnegative because the logarithm is an increasing function. Then, it follows that a solution f of the Boltzmann equation formally satisfy

$$\partial_t \mathcal{H}(f) + \mathcal{D}(f) = 0 \,. \tag{3.8}$$

In other words, the entropy of our particle system does not increase, $\mathcal{H}(f) \le \mathcal{H}(f_o)$.

3.1 Well-posedness. Method of Kaniel & Shinbrot

The theory of well-posedness for the Cauchy problem of the Boltzmann equation for general data is incomplete despite the efforts of the mathematics community. However, in certain circumstances it is possible to give a complete proof of existence and uniqueness of non-negative solutions. One of the most celebrated methods, due to its simplicity and beauty, is the Kaniel & Shinbrot iterations, see [53], which we present here. This method can be used for short time existence with general initial data and for global well-posedness in some perturbative regimes. We sketch the latter by following the papers [51, 12]. First note that, under Grad cutoff assumption, the collision operator splits naturally in a gain and loss part (corresponding the the birth and death process respectively)

$$Q(f,f) = Q_+(f,f) - Q_-(f,f) \,.$$

Second, obseve that using characteristics $f^{\#}(t,x,v) := f(t, x+tv, v)$ it is possible to write the Boltzmann equation as

$$\frac{df^{\#}}{dt} + Q^{\#}_-(f,f) = Q^{\#}_+(f,f) \,. \tag{3.9}$$

Now, for simplicity assume the factorization of the scattering kernel (3.5). Then, it follows that the loss part of the collision operator reduces to

$$Q_-(f,f)(v) = f(v)\int_{\mathbb{R}^d} f(v_*)\big|v-v_*\big|^{\gamma}\mathrm{d}v_* =: f(v)\,R(f)(v).$$

Thus, integrating equation (3.9) follows that a solution of the Boltzmann equation satisfies the relation

$$\begin{aligned} f^{\#}(t,x,v) &= e^{-\int_0^t R^{\#}(f)(s,x,v)\mathrm{d}s} f_o(x,v) \\ &\quad + \int_0^t e^{-\int_s^t R^{\#}(f)(\tau,x,v)\mathrm{d}\tau} Q_+^{\#}(f,f)(s,x,v)\mathrm{d}s\,. \end{aligned} \tag{3.10}$$

Finally, introduce the Banach space $\mathcal{M}$ of functions with Gaussian (or Maxwellian) decay in space-velocity with norm

$$\|g\|_{\mathcal{M}} = \big\|g\; e^{|x|^2+|v|^2}\big\|_{\infty}\,.$$

With these notations and definitions, we are ready to proceed and give a well-posedness result for the Boltzmann equation in the so-called near vacuum regime, that is, when the initial data is sufficiently small in $\mathcal{M}$. The essence of the method consist in defining the following nested sequences of functions $\{l_n\}$ and $\{u_n\}$ as solutions of the linear problems

$$\begin{aligned} \frac{dl_n^{\#}}{dt} + Q_-^{\#}(l_n,u_{n-1}) &= Q_+^{\#}(l_{n-1},l_{n-1}) \quad \text{and} \\ \frac{du_n^{\#}}{dt} + Q_-^{\#}(u_n,l_{n-1}) &= Q_+^{\#}(u_{n-1},u_{n-1})\,, \end{aligned} \tag{3.11}$$

with the terms satisfying the initial condition $0 \le l_n(0) \le f_o \le u_n(0)$. The construction begins by choosing a pair (l_0,u_0) satisfying what Kaniel and Shinbrot called *the beginning condition*

$$0 \le l_0^{\#} \le l_1^{\#} \le u_1^{\#} \le u_0^{\#} \in \mathcal{M}\,. \tag{3.12}$$

It is precisely in the beginning condition where the methods fails for general initial data.

Theorem 3.1.1. *Assume Grad's cut off and factorization* (3.5) *hypotheses for the scattering kernel B. Assume also $-(d-1) < \gamma \leq 1$, and let $\{l_n\}$ and $\{u_n\}$ be the sequences defined by the mild solutions of the linear problems* (3.11)*. In addition, assume that the beginning condition* (3.12) *is satisfied. Then,*

(i) The sequences $\{l_n\}$ and $\{u_n\}$ are well defined for $n \geq 1$. In addition, $\{l_n\}$, $\{u_n\}$ are increasing and decreasing sequences respectively, and

$$l_n^{\#} \leq u_n^{\#} \quad a.e.$$

(ii) There exists $\varepsilon > 0$ such that if

$$\|u_0^{\#}\|_{\mathcal{M}} \leq \varepsilon \;\; and, \;\; 0 \leq l_n(0) = f_o = u_n(0) \;\; for \;\; n \geq 1\,,$$

then

$$\lim_{n\to\infty} l_n = \lim_{n\to\infty} u_n = f \quad a.e.$$

The nonnegative limit $f \in C(0,T;\mathcal{M})$, with $T > 0$, is the unique solution of the Boltzmann equation and fulfills

$$0 \leq l_0^{\#} \leq f^{\#} \leq u_0^{\#} \in \mathcal{M} \quad a.e.$$

Proof. Item *(i)* follows by induction where the beginning condition is exactly the first step of the induction. Assuming that $\{l_k\}$ and $\{u_k\}$ are increasing and decreasing respectively, and such that $l_k \leq u_k$ for $1 \leq k \leq n-1$, we can prove that same holds for $k = n$. Indeed, integration of the linear system (3.11) give us

$$\begin{aligned} l_n^{\#}(t) &= e^{-\int_0^t R^{\#}(u_{n-1})(s)\mathrm{d}s} l_n(0) \\ &\qquad + \int_0^t e^{-\int_s^t R^{\#}(u_{n-1})(\tau)\mathrm{d}\tau} Q_+^{\#}(l_{n-1}, l_{n-1})(s)\mathrm{d}s \\ &\leq e^{-\int_0^t R^{\#}(l_{n-1})(s)\mathrm{d}s} u_n(0) \\ &\qquad + \int_0^t e^{-\int_s^t R^{\#}(u_{n-1})(\tau)\mathrm{d}\tau} Q_+^{\#}(u_{n-1}, u_{n-1})(s)\mathrm{d}s \\ &= u_n^{\#}(t)\,. \end{aligned} \tag{3.13}$$

Same argument proves that $l_{n-1}^{\#} \leq l_n^{\#}$ and $u_n^{\#} \leq u_{n-1}^{\#}$. Let us present a lemma that will help us to prove item *(ii)*.

Lemma 3.1.2. *Assume $-(d-1) < \gamma \leq 1$. Then, for any $0 \leq s \leq t$ and functions $f^{\#}$, $g^{\#}$ that lie in $L^{\infty}\big([0,T);\mathcal{M}\big)$ the following inequality holds*

$$\int_s^t \Big|Q_+^{\#}(f,g)(\tau)\Big|\,d\tau \leq \\ C_{d,\gamma}\, e^{-|x|^2-|v|^2}\,\big\|f^{\#}\big\|_{L^\infty(0,T;\mathcal{M})}\,\big\|g^{\#}\big\|_{L^\infty(0,T;\mathcal{M})}\,, \tag{3.14}$$

where the constant $C_{d,\gamma}$ depends only on the dimension and γ. In other words,

$$\int_0^t \Big|Q_+^{\#}(f,g)(\tau)\Big|\,d\tau \in \mathcal{M}\,,\quad t\geq 0\,.$$

Proof. An explicit calculation yields the inequality,

$$\Big|Q_+^{\#}(f,g)(\tau,x,v)\Big| \leq e^{-|v|^2}\,\big\|f^{\#}\big\|_{L^\infty(0,T;\mathcal{M})}\,\big\|g^{\#}\big\|_{L^\infty(0,T;\mathcal{M})}\times \\ \int_{\mathbb{R}^d} e^{-|v_*|^2}\int_{\mathbb{S}^{d-1}} e^{-|x+\tau(v-{}'v)|^2-|x+\tau(v-{}'v_*)|^2} b(\widehat{u}\cdot\omega)\mathrm{d}\omega\,|u|^{\gamma}\mathrm{d}v_*. \tag{3.15}$$

Note that

$$|x+\tau(v-{}'v)|^2+|x+\tau(v-{}'v_*)|^2 = |x|^2+|x+\tau u|^2\,,$$

and,

$$\int_s^t e^{-|x+\tau u|^2}\mathrm{d}\tau \leq \int_{-\infty}^{\infty} e^{-|\tau u|^2}\mathrm{d}\tau \leq \sqrt{\pi}\,|u|^{-1}.$$

Therefore, integrating (3.15) in $[s,t]$

$$\int_s^t \Big|Q_+^{\#}(f,g)(\tau,x,v)\Big|\,\mathrm{d}\tau \leq \sqrt{\pi}\exp\big(-|x|^2-|v|^2\big) \\ \big\|f^{\#}\big\|_{L^\infty(0,T;\mathcal{M})}\,\big\|g^{\#}\big\|_{L^\infty(0,T;\mathcal{M})}\int_{\mathbb{R}^d} e^{-|v_*|^2}|u|^{\gamma-1}\mathrm{d}v_*.$$

Finally, the proof is completed by observing that,

$$\int_{\mathbb{R}^d} e^{-|v_*|^2}|u|^{\gamma-1}\mathrm{d}v_* \leq \int_{\{|v_*|<1\}}|u|^{\gamma-1}\mathrm{d}v_* + \int_{\{|v_*|\geq 1\}} e^{-|v_*|^2}\mathrm{d}v_* \\ \leq \frac{|\mathbb{S}^{d-1}|}{d+\gamma-1}+C_d\,.$$

□

Let us proceed to prove item *(ii)*. Define $\delta_n^{\#} = u_n^{\#} - l_n^{\#}$, thus, subtracting equations (3.11) follows

$$\frac{d\delta_n^{\#}}{dt} \le Q_+^{\#}(\delta_{n-1}, u_{n-1}) + Q_+^{\#}(l_{n-1}, \delta_{n-1}) + Q_-^{\#}(l_n, \delta_{n-1}) \,. \quad (3.16)$$

Integrating (3.16) in time, recalling that $\delta_n^{\#}(0) = \delta_n(0) = 0$ and using Lemma 3.1.2, it follows that

$$\begin{aligned} &\delta_n^{\#}(t) \le C_{d,\gamma}\, e^{-|x|^2-|v|^2} \times \\ &\quad \Big(\|u_{n-1}^{\#}\|_{L^\infty(\mathcal{M})} + \|l_{n-1}^{\#}\|_{L^\infty(\mathcal{M})} + \|l_n^{\#}\|_{L^\infty(\mathcal{M})}\Big)\|\delta_{n-1}^{\#}\|_{L^\infty(\mathcal{M})} \quad (3.17) \\ &\quad \le 3\, C_{d,\gamma}\, e^{-|x|^2-|v|^2}\|u_0^{\#}\|_{\mathcal{M}}\|\delta_{n-1}^{\#}\|_{L^\infty(\mathcal{M})}\,, \quad t \ge 0\,. \end{aligned}$$

The conclusion from (3.17) is that

$$\|\delta_n^{\#}\|_{L^\infty(\mathcal{M})} \le 3\, C_{d,\gamma}\|u_0^{\#}\|_{\mathcal{M}}\|\delta_{n-1}^{\#}\|_{L^\infty(\mathcal{M})}\,. \quad (3.18)$$

Taking $\varepsilon := 1/(4\, C_{d,\gamma})$ it follows directly from (3.18) that

$$\|\delta_n^{\#}\|_{L^\infty(\mathcal{M})} \le (3/4)^{n-1}\|\delta_0^{\#}\|_{L^\infty(\mathcal{M})} \le (3/4)^n\|u_0^{\#}\|_{\mathcal{M}}\,,$$

which proves *(ii)*. □

Theorem 3.1.3. *(Well-posedness near vacuum) Let B be a scattering kernel satisfying Grad's cut off and the factorization* (3.5) *with* $-(d-1) < \gamma \le 1$. *Then, there exists $\varepsilon_o > 0$ such that if $\|f_o\|_{\mathcal{M}} \le \varepsilon_o$, the Cauchy-Boltzmann problem has a unique global solution f satisfying the estimate*

$$\big\|f^{\#}\big\|_{L^\infty([0,T);\mathcal{M})} \le 2\,\varepsilon_o\,, \quad (3.19)$$

for any $0 \le T \le \infty$.

Proof. The key step to apply Theorem 3.1.1 is to find suitable functions that satisfy the *beginning condition* globally. The most natural (and simplest) choice for the first terms of the nested sequences $\{l_n\}$ and $\{u_n\}$ is

$$l_0^{\#} = 0 \quad \text{and} \quad u_0^{\#} = \varepsilon\, e^{-|x|^2-|v|^2}.$$

Here $\varepsilon > 0$ is the parameter given in Theorem 3.1.1, item *(ii)*. Now compute the following two terms

$$l_1^{\#}(t) = f_o\, e^{-\int_0^t R^{\#}(u_0)(\tau)\mathrm{d}\tau} \quad \text{and} \quad u_1^{\#}(t) = f_o + \int_0^t Q_+^{\#}(u_0, u_0)(\tau)\mathrm{d}\tau.$$

Clearly, $0 \le l_0^{\#} \le l_1^{\#} \le u_1^{\#}$. In addition, using Lemma 3.1.2 in the expression for $u_1^{\#}$ we conclude that, for all $t \ge 0$,

$$u_1^{\#}(t) \le \Big(\|f_o\|_{\mathcal{M}} + C_{d,\gamma}\|u_0^{\#}\|^2_{\mathcal{M}}\Big)\, e^{-|x|^2-|v|^2}\,.$$

Noting that $\|u_0^{\#}\|_{\mathcal{M}} = \varepsilon$, it suffices to satisfy the inequality

$$\|f_o\|_{\mathcal{M}} + C_{d,\gamma}\varepsilon^2 \le \varepsilon$$

in order to satisfy the beginning condition globally. This is actually possible as long as

$$\|f_o\|_{\mathcal{M}} \le \varepsilon_o := \frac{\varepsilon}{2} \le \frac{1}{4C_{d,\gamma}}\,.$$

□

3.2 The method of DiPerna & Lions

3.2.1 Velocity average

One of the most influential theories in the area of mathematical physics that has been in continuous development in the last couple of decades is the method of renormalized solutions introduced by DiPernal & Lions. This method was first used by the authors to prove existence of renormalized solutions for the inhomogeneous Boltzmann equation [38]. More recently, the method and its tools have been successfully implemented to tackle different relevant and challenging problems in kinetic theory, for instance, showing existence of solutions for kinetic equations and systems with rather general initial data, and proving the rigorous derivation of diffusion limits (such as Navier-Stokes equations) from kinetic models. A central result used in this theory is the so-called *average lemma* or *velocity averaging*.

The result is easily stated: assume $f(t,x,v)$ satisfies the transport equation

$$\partial_t f + v\cdot\nabla_x f = g\,,\qquad t\geq 0\,,\;\; x\,,v\in\mathbb{R}^d\,,$$

with $f(0,x,v)=f_o(x,v)$. Using the explicit expression of f in terms of f_o and g one gets convinced that the regularity of f is given by the lowest regularity between f_o and g. However, a velocity average

$$\bar f_\varphi(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\varphi(v)\mathrm{d}v\,,\quad \varphi\in C_c(\mathbb{R}^d)\,,$$

enjoys *higher* regularity. Velocity averages are central in kinetic theory because they represent what we can observe and measure in the macroscopic world (mass, momentum, temperature, pressure, etc). Thus, an average lemma is the mathematical expression of the intuitive idea that in the macroscopic world things should be smoother than at the kinetic level. The references in this area are extensive, here we mention some [41, 42, 25, 23, 40, 52, 48]. Our first result is the following classical result.

Proposition 3.2.1. *Fix $d\geq 2$ and let $f,\,g\in L^2_{x,v}$ satisfying the equation*

$$v\cdot\nabla_x f = g\,.$$

Then, the velocity average satisfies $\bar f_\varphi\in H^s_x$ for any $s\in\big(0,\tfrac12\big)$ with estimate

$$\|\bar f_\varphi\|_{H^s_x}\leq C_{d,s}(\varphi)\Big(\|f\|_{L^2_{x,x}}+\|g\|_{L^2_{x,v}}\Big)\,,$$

where the constant $C_{d,s}(\varphi)$ depends on $\varphi\in C_c(\mathbb{R}^d)$ through its supremum and support.

Proof. Applying Fourier transform in the spatial variable

$$\mathcal{F}\{g\}(\xi) = v\cdot\xi\,\mathcal{F}\{f\}(\xi) = |v|\,|\xi|\,\hat v\cdot\hat\xi\,\mathcal{F}\{f\}(\xi).$$

Then,

$$\begin{aligned}\big|\mathcal{F}\{\bar f_\varphi\}(\xi)\big|^2\,|\xi|^{2s} &= \Big|\int_{\mathbb{R}^d}\mathcal{F}\{f(\cdot,v)\}(\xi)\,\varphi(v)\,\mathrm{d}v\Big|^2|\xi|^{2s}\\ &= \Big|\int_{\mathbb{R}^d}\frac{\mathcal{F}\{g(\cdot,v)\}(\xi)}{|\xi|^{1-s}\,|v|\,\hat v\cdot\hat\xi}\,\varphi(v)\,\mathrm{d}v\Big|^2.\end{aligned}\tag{3.20}$$

But,

$$\left|\frac{\mathcal{F}\{g(\cdot,v)\}(\xi)}{|\xi|^{1-s}\,|v|\,\hat{v}\cdot\hat{\xi}}\right| = \frac{\big|\mathcal{F}\{f(\cdot,v)\}(\xi)\big|^{1-s}\big|\mathcal{F}\{g(\cdot,v)\}(\xi)\big|^{s}}{|v|^{s}|\hat{v}\cdot\hat{\xi}|^{s}}\,.$$

Then, putting the absolute value inside the integral in equation (3.20) and using Cauchy–Schwarz inequality one concludes

$$\begin{aligned}
&\big|\mathcal{F}\{\bar{f}_\varphi\}(\xi)\big|^2\,|\xi|^{2s}\\
&\leq \bigg(\int_{\mathbb{R}^d}\frac{\big|\mathcal{F}\{f(\cdot,v)\}(\xi)\big|^{1-s}\big|\mathcal{F}\{g(\cdot,v)\}(\xi)\big|^{s}}{|v|^{s}|\hat{v}\cdot\hat{\xi}|^{s}}\,\big|\varphi(v)\big|\,\mathrm{d}v\bigg)^2\\
&\leq \bigg(\int_{\mathbb{R}^d}\big|\mathcal{F}\{f(\cdot,v)\}(\xi)\big|^{2(1-s)}\big|\mathcal{F}\{g(\cdot,v)\}(\xi)\big|^{2s}\,\mathrm{d}v\bigg)\times\\
&\qquad\qquad\qquad\bigg(\int_{\mathbb{R}^d}\frac{|\varphi(v)|^2}{|v|^{2s}|\hat{v}\cdot\hat{\xi}|^{2s}}\,\mathrm{d}v\bigg).
\end{aligned}\tag{3.21}$$

Since $\varphi\in C_c(\mathbb{R}^d)$, it follows for any $d\geq 2$

$$\int_{\mathbb{R}^d}\frac{|\varphi(v)|^2}{|v|^{2s}|\hat{v}\cdot\hat{\xi}|^{2s}}\,\mathrm{d}v \leq C_{s,d}(\varphi)^2,\quad s\in\big(0,\tfrac{1}{2}\big)\,.\tag{3.22}$$

Additionally, using Young's inequality

$$\begin{aligned}
&\int_{\mathbb{R}^d}\big|\mathcal{F}\{f(\cdot,v)\}(\xi)\big|^{2(1-s)}\big|\mathcal{F}\{g(\cdot,v)\}(\xi)\big|^{2s}\,\mathrm{d}v\\
&\leq (1-s)\int_{\mathbb{R}^d}\big|\mathcal{F}\{f(\cdot,v)\}(\xi)\big|^2\mathrm{d}v + s\int_{\mathbb{R}^d}\big|\mathcal{F}\{g(\cdot,v)\}(\xi)\big|^2\mathrm{d}v\,.
\end{aligned}\tag{3.23}$$

Using (3.22) and (3.23) in (3.21) and integrating in ξ,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\big|\mathcal{F}\{\bar{f}_\varphi\}(\xi)\big|^2\,|\xi|^{2s}\mathrm{d}\xi\\
&\leq C_{s,d}(\varphi)^2\bigg(\int_{\mathbb{R}^{2d}}\big|\mathcal{F}\{f(\cdot,v)\}(\xi)\big|^2\mathrm{d}v\,\mathrm{d}\xi + \int_{\mathbb{R}^{2d}}\big|\mathcal{F}\{g(\cdot,v)\}(\xi)\big|^2\mathrm{d}v\,\mathrm{d}\xi\bigg).
\end{aligned}$$

The result follows applying Plancherel theorem in the ξ–variable. □

The L^1 space is a natural frame of work in kinetic equations since mass conservation is the basic property one expects for solutions of kinetic models. Thus, one may wonder if velocity averages also happen in this framework. The following result states that this is the case. Before entering in the details, note that the equation

$$\lambda f + v\cdot\nabla_x f = g\,,\qquad \lambda>0\,,\quad x,v\in\mathbb{R}^d\,, \tag{3.24}$$

has explicit solution

$$f(x,t) = \int_0^\infty e^{-\lambda s} g(x-vs,v)\mathrm{d}s\,. \tag{3.25}$$

Therefore,

$$\|\bar{f}_\varphi\|_{L^1_x} \le \|\varphi\|_{L^\infty}\|f\|_{L^1_{x,v}} \le \lambda^{-1}\|\varphi\|_{L^\infty}\|g\|_1\,. \tag{3.26}$$

Estimate (3.26) can also be obtained by direct integration in space-velocity of the equation (3.24).

Theorem 3.2.2. *Let $\{f^\epsilon\}$ a weakly compact family in $L^1_{x,v}$ such that $v\cdot\nabla_x f^\epsilon$ is a bounded family in $L^1_{x,v}$. Then, the velocity average $\bar{f^\epsilon}_\varphi$ is relatively compact in $L^1_{loc,x}$.*

Proof. We follow [48]. Let $g^\epsilon := v\cdot\nabla_x f^\epsilon$. Thus,

$$\lambda f^\epsilon + v\cdot\nabla_x f^\epsilon = \lambda f^\epsilon + g^\epsilon\,,\quad \lambda>0\,.$$

Now, write $f^\epsilon = f^\epsilon_{1,\alpha} + f^\epsilon_{2,\alpha}$ with $\alpha>0$, where

$$f^\epsilon_{1,\alpha} = \mathbf{1}_{\{|f^\epsilon|>\alpha\}}\, f^\epsilon\,,\qquad f^\epsilon_{2,\alpha} = \mathbf{1}_{\{|f^\epsilon|\le\alpha\}}\, f^\epsilon\,.$$

As a consequence, using linearity of (3.24) it follows that $f^\epsilon = c^\epsilon + b^\epsilon$ with

$$\lambda\, c^\epsilon + v\cdot\nabla_x c^\epsilon = \lambda f^\epsilon_{2,\alpha}\,,\qquad \lambda\, b^\epsilon + v\cdot\nabla_x b^\epsilon = \lambda f^\epsilon_{1,\alpha} + g^\epsilon\,.$$

Here c stands for compact and b for bounded. Let us estimate c^ϵ using Proposition (3.2.1). For any $s\in\big(0,\dfrac{1}{2}\big)$ one has

$$\begin{aligned}
\|\bar{c^\epsilon}_\varphi\|_{H^s_x} &\le C_{s,d}(\varphi)\Big(\lambda\|f^\epsilon_{2,\alpha}\|_{L^2_{x,v}} + \|c^\epsilon\|_{L^2_{x,v}}\Big)\\
&\le C_{s,d}(\varphi)(1+\lambda)\|f^\epsilon_{2,\alpha}\|_{L^2_{x,v}}\\
&\le C_{s,d}(\varphi)(1+\lambda)\sqrt{\alpha}\sqrt{\|f^\epsilon\|_{L^1_{x,v}}} \le C(\varphi)(1+\lambda)\sqrt{\alpha}\,,
\end{aligned} \tag{3.27}$$

where in the second inequality we used that $\|c^\epsilon\|_{L^2_{x,v}} \leq \|f^\epsilon_{2,\alpha}\|_{L^2_{x,v}}$. In the last inequality we used that $\{f^\epsilon\}$ is weakly compact, and thus, it is a bounded family. Using Rellich's compactness theorem, the family $\{\bar{c^\epsilon}_\varphi\}$ is relatively compact in $L^1_{loc,x}$. Furthermore, recalling (3.26)

$$\|\bar{b^\epsilon}_\varphi\|_{L^1_x} \leq \|\varphi\|_{L^\infty}\Big(\|f^\epsilon_{1,\alpha}\|_{L^1_{x,v}} + \lambda^{-1}\|g^\epsilon\|_{L^1_{x,v}}\Big)\,.$$

Since $\{f^\epsilon\}$ is weakly compact is equiintegrable. Thus, for any $\delta > 0$ there exists $\alpha > 0$ such that $\sup_\epsilon \|f^\epsilon_{1,\alpha}\|_{L^1_{x,v}} \leq \big(2\|\varphi\|_{L^\infty}\big)^{-1}\delta$. Additionally, we can choose $\lambda = 2\|\varphi\|_{L^\infty} \sup_\epsilon \|g^\epsilon\|_{L^1_{x,v}}\delta^{-1}$ to conclude that

$$\|\bar{b^\epsilon}_\varphi\|_{L^1_x} \leq \delta\,. \tag{3.28}$$

As a result of this discussion and the fact that $\bar{f^\epsilon}_\varphi = \bar{c^\epsilon}_\varphi + \bar{b^\epsilon}_\varphi$, we have proved that for any compact set K and $\delta > 0$, there exists a compact set $\mathcal{K}_\delta \in L^1_x(K)$ such that $\{\bar{f^\epsilon}_\varphi\} \subset \mathcal{K}_\delta + B(0,\delta)$. Consequently, the family $\{\bar{f^\epsilon}_\varphi\}$ is pre-compact, and since $L^1_x(K)$ is a Banach space, it is in fact compact. □

Corollary 3.2.3. *Assume the conditions of Theorem 3.2.2. Then, for every $\varphi \in \mathcal{C}^1_c(\mathbb{R}^{3d})$ the velocity average*

$$\int_{\mathbb{R}^d} f^\epsilon(x,v_*)\varphi(x,v,v_*)\,dv_*$$

belongs to a compact set of $L^1_{loc}(\mathbb{R}^{2d})$.

3.2.2 Renormalized solutions

The concept of renormalized solutions was introduced, at least in the Boltzmann equation setting, in [38]. An extensive discussion is found in the series [54, 55, 56] and, an example of the application of the theory to systems with bounded domains can be found in [58]. The idea goes like this: assume that $f \geq 0$ is a solution of the Boltzmann equation (3.1), thus, for any $\beta \in \mathcal{C}^1(\mathbb{R})$ one should have

$$\partial_t\beta(f) + v\cdot\nabla_x\beta(f) = \beta'(f)Q(f,f)\,, \qquad \beta(f(0)) = \beta(f_o)\,. \tag{3.29}$$

Thus, a renormalized solution for the Boltzmann equation is any non-negative function $f \in \mathcal{C}\big([0,\infty); L^1(\mathbb{R}^{2d})\big)$ satisfying (3.29), in the sense of distributions, for any β such that $\beta(0) = 0$ and $|\beta'(s)| \leq C(1+s)^{-1}$. More explicitly, for any $\varphi \in \mathcal{D}\big([0,T) \times \mathbb{R}^{2d}\big)$

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^{2d}} \Big(\beta(f)\big(\partial_t \varphi + v \cdot \nabla_x \varphi\big) + \beta'(f) Q(f,f)\varphi\Big)\mathrm{d}v\mathrm{d}x\mathrm{d}t \\ + \int_{\mathbb{R}^{2s}} \beta(f_o)\,\varphi\,\mathrm{d}v\mathrm{d}x = 0\,.\end{aligned} \tag{3.30}$$

Such suitable functions β are called renormalization functions. Renormalized solutions must also satisfy the natural *a priori* estimates coming from the conservation laws and entropy dissipation, see [38]

$$\begin{aligned}\sup_{t\in[0,T]} \int_{\mathbb{R}^{2d}} f\big(1 + |v|^2 + |x|^2 + |\log(f)|\big)\mathrm{d}v\mathrm{d}x \\ + \int_0^T \mathcal{D}(f)\mathrm{d}t \leq C(f_o, T) < \infty\,.\end{aligned} \tag{3.31}$$

In addition to time $T > 0$, the constant $C(f_o, T)$ depends on the mass, second moments and entropy of f_o. A central result in the theory of renormalized solutions for the Boltzmann equation is that they form a weakly stable set.

Theorem 3.2.4. *Fix any finite time $T > 0$ and, let $\{f^n\}$ be a sequence of renormalized solutions such that $\{f^n(0)\}$ satisfies* (3.31) *uniformly in $n \in \mathbb{Z}^+$ and converges weakly in $L^1(\mathbb{R}^{2d})$ to some f_o. Then, up to extraction of a subsequence, $\{f^n\}$ converges weakly in $L^1\big([0,T] \times \mathbb{R}^{2d}\big)$ to a renormalized solution f having initial value f_o.*

Proof. Let us present the argument of proof as discussed in [38, 56, 58] filling only the most relevant details. Since $\{f^n(0)\}$ satisfies (3.31) uniformly in $n \in \mathbb{Z}^+$, using Dunford-Pettis lemma one concludes that $\{f^n\}$ is weakly compact in $L^p(L^1_{x,v})$ for any $p \in [1,\infty)$. Thus, up to subsequence, one has the weak limit $f^n \rightharpoonup f \in L^p\big([0,T]; L^1(\mathbb{R}^{2d})\big)$. A convexity argument shows that f satisfies the estimate (3.31).

Now, for any fixed $\delta > 0$ define $\beta_\delta(s) := \frac{s}{1+\delta s}$. Then, $\beta'_\delta = (1+\delta s)^{-2}$ and β_δ is a valid renormalization function. It follows, in the sense of distributions, that

$$\partial_t \beta_\delta(f^n) + v\cdot\nabla_x \beta_\delta(f^n) = \beta'_\delta(f^n) Q(f^n, f^n)\,. \tag{3.32}$$

Since $\beta_\delta \le \delta^{-1}$, we may assume that

$$\beta_\delta(f^n) \rightharpoonup f_\delta\,,\ \text{ weakly-}\star \text{ in } L^\infty\big((0,T)\times\mathbb{R}^{2d}\big)\,. \tag{3.33}$$

The importance of the renormalization is that the sequences

$$\{\beta'_\delta(f^n) Q^{\pm}(f^n, f^n)\}$$

are weakly compact in $L^1_{t,x,v}$. This fact can be proved using only the natural estimate (3.31). Thus, we may also assume that

$$\beta'_\delta(f^n) Q(f^n, f^n) = \frac{Q(f^n, f^n)}{(1+\delta f^n)^2} \rightharpoonup Q_\delta\,,\ \text{ weakly in } L^1\big((0,T)\times\mathbb{R}^{2d}\big)\,.$$

We can pass to the limit in (3.32) and obtain the equation in the sense of distributions

$$\partial_t f_\delta + v\cdot\nabla_x f_\delta = Q_\delta\,, \tag{3.34}$$

complemented with initial condition $w_\delta := \lim_n \beta_\delta(f^n(0))$ (weak-$\star$ limit in $L^\infty_{x,v}$). The remainder of the proof consists in passing to the limit $\delta \to 0$ in equation (3.34). Note that for any $M > 0$

$$0 \le s - \beta_\delta(s) = \frac{\delta s^2}{1+\delta s} \le \delta M s + s\,\mathbf{1}_{\{s \ge M\}}\,,$$

hence, $0 \le f - f_\delta$. Thus, for any $\epsilon > 0$, there exists $n_o := n_o(\epsilon)$ such that

$$\begin{aligned}\|f - f_\delta\|_{L^1_{t,x,v}} &\le \|f^{n_o} - \beta_\delta(f^{n_o})\|_{L^1_{t,x,v}} + \epsilon \\ &\le \delta M \|f^{n_o}\|_{L^1_{t,x,v}} + \|f^{n_o}\mathbf{1}_{\{f^{n_o}\ge M\}}\|_{L^1_{t,x,v}} + \epsilon\,.\end{aligned}$$

Send $\delta \to 0$ and then $M \to \infty$ to conclude that

$$\limsup_{\delta} \|f - f_\delta\|_{L^1_{t,x,v}} \le \epsilon\,.$$

This proves the strong limit

$$\lim_{\delta} f_\delta = f\,, \text{ strongly in } L^1\big((0,T)\times\mathbb{R}^{2d}\big)\,. \tag{3.35}$$

Similarly, the initial condition in equation (3.34) satisfies the strong $L^1_{x,v}$ limit $w_\delta \to f_o$ as $\delta \to 0$. Therefore, we may pick an arbitrary renormalization function β and renormalize equation (3.34) to obtain

$$\partial_t\beta(f_\delta) + v\cdot\nabla_x\beta(f_\delta) = \beta'(f_\delta)Q_\delta\,. \tag{3.36}$$

Sending $\delta \to 0$ and using (3.35) one obtains the limit in the sense of distributions for the left-side in equation (3.36)

$$\partial_t\beta(f_\delta) + v\cdot\nabla_x\beta(f_\delta) \to \partial_t\beta(f) + v\cdot\nabla_x\beta(f)\,, \tag{3.37}$$

and also, for the initial condition

$$\beta(w_\delta) \to \beta(f_o)\,. \tag{3.38}$$

In order to finish the proof of Theorem 3.2.4 we need the following important result.

Lemma 3.2.5. *Let $B_R \subset \mathbb{R}^d$ be the ball with center at the origin and radius $R \in (0,\infty)$. Then, under previous setting*

$$\beta'(f_\delta)Q_\delta \to \beta'(f)Q(f,f)\,, \textit{ strongly in } L^1\big((0,T)\times B_R\times B_R\big)\,. \tag{3.39}$$

Assuming for the moment the validity of Lemma 3.2.5 and using (3.37-3.38) we can take the limit $\delta \to 0$ in (3.36) to obtain that

$$\partial_t\beta(f) + v\cdot\nabla_x\beta(f) = \beta'(f)Q(f,f)\,, \qquad \beta(f(0)) = \beta(f_o)\,. \tag{3.40}$$

We are allow to use the evaluation $f(0)$ because solutions, in the sense of distributions, of the transport equation with a $L^1_{t,x,v}$ right side (and $L^1_{x,v}$ initial data) are in fact $f \in \mathcal{C}\big([0,T);L^1(\mathbb{R}^{2d})\big)$. This proves that f is a renormalized solution. □

Proof of Lemma 3.2.5

We prove Lemma 3.2.5 assuming a Boltzmann collision operator having a smooth collision kernel $B(u,\omega) = \Phi(u)b(\widehat{u}\cdot\omega)$ with $\Phi \in \mathcal{C}^1_c(\mathbb{R}^d)$.

This assumption simplifies the technicalities and keeps the essential ideas of the argument intact. We consider separately $Q^{\pm}_{\delta}$ corresponding to the weak $L^1_{t,x,v}$ limits of $\{\beta'_\delta(f^n)Q^{\pm}(f^n,f^n)\}_n$ respectively starting with the loss part of the collision operator. Recall that

$$\beta'_\delta(f^n)Q^-(f^n,f^n) = \frac{f^n}{(1+\delta f^n)^2}\int_{\mathbb{R}^d} f^n(t,x,v_*)\Phi(v-v_*)\mathrm{d}v_*\,.$$

Using a version of Corollary 3.2.3 for the transport equation, one concludes that the velocity average is strongly convergent locally in $L^1_{t,x,v}$

$$\int_{\mathbb{R}^d} f^n(t,x,v_*)\Phi(v-v_*)\mathrm{d}v_* \longrightarrow \int_{\mathbb{R}^d} f(t,x,v_*)\Phi(v-v_*)\mathrm{d}v_*\,,\ \text{ strongly in } L^1((0,T)\times B_R\times B_R)\,.$$

In addition,

$$\frac{f^n}{(1+\delta f^n)^2}\longrightarrow \tilde{f}_\delta\,,\ \text{ weakly-}\star\ \text{ in } L^\infty\big((0,T)\times\mathbb{R}^{2d}\big)\,.$$

Therefore,

$$\beta'_\delta(f^n)Q^-(f^n,f^n)\longrightarrow \tilde{f}_\delta\int_{\mathbb{R}^d} f(t,x,v_*)\Phi(v-v_*)\mathrm{d}v_*\,,\ \text{ weakly in } L^1((0,T)\times B_R\times B_R)\,.$$

Clearly $\tilde{f}_\delta\le f_\delta$ and, as a consequence, for any renormalization function β

$$|\beta'(f_\delta)\tilde{f}_\delta| \le \frac{C_\beta \tilde{f}_\delta}{1+f_\delta}\le C_\beta\,. \tag{3.41}$$

The same argument given for f_δ also proves that $\tilde{f}_\delta$ converges to f strongly in $L^1_{t,x,v}$. Thus, these convergences are almost everywhere as well. The conclusion is $\lim_\delta \beta'(f_\delta)\tilde{f}_\delta = \beta'(f)f$ a.e. in $(0,T)\times B_R\times B_R$.

Using (3.41) and Lebesgue's dominated convergence theorem

$$\begin{aligned}\beta'(f_\delta)Q^-_\delta = \beta'(f_\delta)\tilde{f}_\delta \int_{\mathbb{R}^d} f(t,x,v_*)\Phi(v-v_*)\mathrm{d}v_* \longrightarrow \\ \beta'(f)f\int_{\mathbb{R}^d} f(t,x,v_*)\Phi(v-v_*)\mathrm{d}v_*\,, \\ \text{strongly in } L^1((0,T)\times B_R\times B_R)\,.\end{aligned} \tag{3.42}$$

This proves the result for the negative collision operator. Now, determining the limit for $\{\beta'(f_\delta)Q^+_\delta\}$ is a bit more involved. It relies in the following averaging result which is a direct consequence of the average lemmas, refer to [38, pg. 341–343] for a proof.

Lemma 3.2.6. *Fix $T<\infty$ and $\varphi \in L^\infty\big((0,T)\times\mathbb{R}^d\times\mathbb{R}^d\big)$. Then, under the conditions of Theorem 3.2.4 it follows that*

$$\int_{\mathbb{R}^d} Q^{\pm}(f^n,f^n)\varphi dv \to \int_{\mathbb{R}^d} Q^{\pm}(f,f)\varphi dv\,,\ \textit{in measure on}\ (0,T)\times B_R\,.$$

Having at hand Lemma 3.2.6 we finish the argument as presented in [56]. The strategy to prove strong convergence in L^1 for the sequence $\{\beta'(f_\delta)Q^+_\delta\}$ and identify its limit consists in showing that such sequence converges L^1–weakly and almost everywhere. Indeed, it is easily proved using Egorov's theorem and the equiintegrability characterization of weakly compact sets in L^1 that sequences enjoying weak and a.e. limits, in fact, converge L^1–strong. Of course, such limits agree. A first step is to use Arkeryd's inequality

$$Q^+(f^n,f^n) \le K\,Q^-(f^n,f^n) + \frac{e(f^n)}{\log(K)}\,, \quad K>1\,, \tag{3.43}$$

where

$$0 \le e(f) := \tfrac{1}{4}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \big(f'f'_* - ff_*\big)\log\Big(\frac{f'f'_*}{ff_*}\Big)B\,\mathrm{d}\omega\mathrm{d}v_*\,,$$

is the entropy dissipation rate at (t,x,v). Indeed, note that using estimate (3.31)

$$\int_0^T\int_{\mathbb{R}^{2d}} e(f) = \int_0^T \mathcal{D}(f) \le C(f_o,T) < \infty\,,$$

and therefore, we may assume that $e(f^n)$ is converging in the sense of measures to some nonnegative and bounded measure e in $[0,T)\times\mathbb{R}^{2d}$. As a consequence, multiplying (3.43) by $\beta'_\delta(f^n)$ and taking the limit, it is concluded that

$$Q^+_\delta \le K Q^-_\delta + \frac{e_o}{\log(K)}\,. \tag{3.44}$$

Here, e_o is the regular part of the measure e. This is allowed because we know that Q^+_δ lies in $L^1_{t,x,v}$, thus, it is a regular measure. Thus, that the sequence $\{\beta'(f_\delta)Q^+_\delta\}$ is weakly compact follows from the weak compactness of $\{\beta'(f_\delta)Q^-_\delta\}$. Now, note the easy inequality

$$Q^+(f^n,f^n) \ge \beta'_\delta(f^n)Q^+(f^n,f^n)\,. \tag{3.45}$$

Multiplying inequality (3.45) by a nonnegative $\varphi\in\mathcal{C}^1_o(\mathbb{R}^d)$, integrating in velocity and sending to the limit, it follow from Lemma 3.2.6 that

$$\int_{\mathbb{R}^d} Q^+(f,f)\varphi\,\mathrm{d}v \ge \int_{\mathbb{R}^d} Q^+_\delta\varphi\,\mathrm{d}v\,,\quad (0,T)\times B_R\,,$$

which readily implies that $Q^+(f,f)\ge Q^+_\delta$ a.e on $(0,T)\times\mathbb{R}^{2d}$. Thus,

$$Q^+(f,f) \ge \limsup_{\delta} Q^+_\delta\,,\quad \text{a.e on } (0,T)\times\mathbb{R}^{2d}\,. \tag{3.46}$$

For the opposite inequality set

$$L(f) := \int_{\mathbb{R}^d} f(v_*)\Phi(v-v_*)\mathrm{d}v_*\,,$$

and observe that (3.43) leads to

$$\begin{aligned}(1+\delta R)^{-2}&\frac{Q^+(f^n,f^n)}{1+\nu L(f^n)}\\ &\le \beta'_\delta(f^n)Q^+(f^n,f^n) + \frac{Q^+(f^n,f^n)\mathbf{1}_{\{f^n\ge R\}}}{1+\nu L(f^n)}\\ \le \beta'_\delta(f^n)&Q^+(f^n,f^n) + \frac{K}{\nu}f^n\mathbf{1}_{\{f^n\ge R\}} + \frac{e(f^n)}{\log(K)}\,.\end{aligned} \tag{3.47}$$

Furthermore, a slight variation of Lemma 3.2.6 implies that

$$\int_{\mathbb{R}^d} \frac{Q^+(f^n, f^n)}{1+\nu L(f^n)}\,\varphi\,\mathrm{d}v \longrightarrow \int_{\mathbb{R}^d} \frac{Q^+(f, f)}{1+\nu L(f)}\,\varphi\,\mathrm{d}v, \quad \text{in measure on } (0,T)\times B_R\,.$$

The key observation to prove this limit is the fact that average lemmas imply that $L(f^n)$ is converging strongly in $L^1_{t,x,v}$. Thus, multiplying inequality (3.47) by $\varphi \geq 0$, integrating in velocity, and sending to the limit

$$(1+\delta R)^{-2}\int_{\mathbb{R}^d} \frac{Q^+(f, f)}{1+\nu L(f)}\,\varphi\,\mathrm{d}v \leq \int_{\mathbb{R}^d} Q^+_\delta\,\varphi\,\mathrm{d}v + \frac{K}{\nu}\int_{\mathbb{R}^d} f_R\,\varphi\,\mathrm{d}v + \int_{\mathbb{R}^d} \frac{e}{\log(K)}\,\varphi\,\mathrm{d}v\,,$$

where f_R is the weak limit of $f^n\mathbf{1}_{\{f^n\geq R\}}$. And thus,

$$(1+\delta R)^{-2}\frac{Q^+(f, f)}{1+\nu L(f)} \leq Q^+_\delta + \frac{K}{\nu} f_R + \frac{e_o}{\log(K)}\,, \quad \text{a.e on } (0,T)\times\mathbb{R}^{2d}\,.$$

Take, in this order, the limits $\delta \to 0$, $R \to \infty$, $K \to \infty$ and $\nu \to 0$ to conclude that

$$Q^+(f,f) \leq \liminf_{\delta} Q^+_\delta\,, \quad \text{a.e on } (0,T)\times\mathbb{R}^{2d}\,. \tag{3.48}$$

Using estimates (3.46) and (3.48) one concludes that

$$\lim_{\delta} \beta'(f_\delta)Q^+_\delta = \beta'(f)Q^+(f,f)\,, \quad \text{a.e on } (0,T)\times\mathbb{R}^{2d}\,.$$

This concludes the proof. □

Theorem 3.2.4 is the essential tool to prove existence of renormalized solutions for the Boltzmann equation with initial data having finite second moments and entropy [38]. The main idea of the argument is to approximate the collision operator by a simpler operator involving some type of truncation and for which all conservation laws hold. The approximating problem is simple enough to find, using standard fixed point theory, existence of classical solutions. These classical solutions are, of course, renormalized solutions to the approximating problem. This method provides of a sequence of renormalized solutions that, by Theorem 3.2.4, should converge to a renormalized solution of the original problem. This is in fact the case provided the approximating operator converges sufficiently strong to the original collision operator.

3.3 Theory of moments

One of the most important quantities to be studied for a solution f of the Boltzmann equation, as for any probability distribution, are its moments

$$m_k(f)(x,t) = \int_{\mathbb{R}^d} f(t,x,v)|v|^k \mathrm{d}v\,. \tag{3.49}$$

Moments are associated to macroscopical quantities or observables. For example, the zero moment ($k = 0$) is the spatial density and the second moment ($k = 2$) is associated to the spatial temperature of the system. Moments are the basic quantities to study when one wants to pass from the kinetic scale to the macroscopical scale, in fact, they are the central quantities when deriving fluid equations (for instance, Navier-Stokes equations) from Boltzmann equation. In a general setting, the study of moments is a very difficult task due to the ample physical situations that may be modeled with the Boltzmann equation. However, in some particular regimes such as spatial homogeneous or quasi-homogeneous systems, the theory that has been developed in recent years is quite complete, see the seminal papers [18, 22]. Let us present here an introduction to this theory in the homogeneous case, that is, when spatial variations are completely neglected in the model

$$\partial_t f = Q(f,f)\,, \qquad (t,v) \in \mathbb{R}^+ \times \mathbb{R}^d\,. \tag{3.50}$$

A central tool of the moment analysis is the so called σ–representation which consists in performing the change of variables in the sphere

$$\sigma(\omega) = \widehat{u} - 2(\widehat{u}\cdot\omega)\omega \in \mathbb{S}^{d-1}\,,$$

where $\widehat{u}$ is an arbitrary, but fixed, unitary vector. One can compute the Jacobian of this transformation as [1]

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\omega} = 2^{d-1}|\widehat{u}\cdot\omega|^{d-2}\,. \tag{3.51}$$

The computations can be found in the appendix Lemma 6.0.1. Thus, it readily follows the identity

$$\int_{\mathbb{S}^{d-1}} b(\widehat{u}\cdot\omega)\,\varphi\big((u\cdot\omega)\omega\big)\mathrm{d}\omega \\ = \frac{1}{2^{d-1}}\int_{\mathbb{S}^{d-1}}\Big(\sqrt{\tfrac{1-\widehat{u}\cdot\sigma}{2}}\Big)^{2-d} b\Big(\sqrt{\tfrac{1-\widehat{u}\cdot\sigma}{2}}\Big)\varphi\Big(\frac{u-|u|\sigma}{2}\Big)\mathrm{d}\sigma\,.$$

As a conclusion, the following *weak representation* holds

$$\int_{\mathbb{R}^d} Q(f,g)(v)\varphi(v)\mathrm{d}v = \int_{\mathbb{R}^{2d}} f(v)g(v_*)\mathcal{S}(\varphi)(v,v_*)|u|^{\gamma}\mathrm{d}v_*\mathrm{d}v\,, \tag{3.52}$$

where

$$\mathcal{S}(\varphi)(v,v_*) := \int_{\mathbb{S}^{d-1}} b_o(\widehat{u}\cdot\sigma)\big(\varphi(v')+\varphi(v'_*)-\varphi(v)-\varphi(v_*)\big)\mathrm{d}\sigma\,. \tag{3.53}$$

Here, the new scattering kernel b_o is related to the original through the formula

$$b_o(\widehat{u}\cdot\sigma) = \tfrac{1}{2^d}\Big(\sqrt{\tfrac{1-\widehat{u}\cdot\sigma}{2}}\Big)^{2-d} b\Big(\sqrt{\tfrac{1-\widehat{u}\cdot\sigma}{2}}\Big)\,,$$

and, the collision laws in the σ-coordinates are given by the expressions

$$v' = v - \frac{u-|u|\sigma}{2} \qquad \text{and} \qquad v'_* = v_* + \frac{u-|u|\sigma}{2}\,. \tag{3.54}$$

For simplicity we consider only *Grad's cutoff* angular kernels normalized to unity, that is, $\int_{\mathbb{S}^{d-1}} b_o(\widehat{u}\cdot\sigma)\mathrm{d}\sigma = 1$. Observe carefully that Grad's cutoff assumption is stronger than the cutoff assumption

[1] Such change of variables was introduced in [17] for the relevant case of 3-dimensions.

introduced at the beginning of Section 3. Also, note that the collision kernel in the σ–representation for the important case of hard spheres simply reads

$$B(u,\sigma) = c_d\,|u|\,,$$

with c_d a constant depending only on the dimension. One application of the σ–representation is the following result known as Povzner lemma. It is specifically designed to study both propagation and generation of moments for solutions of the Boltzmann equation.

Lemma 3.3.1. *Fix $q \geq 1$ and let the angular scattering kernel satisfy $b_o \in L^q(\mathbb{S}^{d-1})$. Then, for any real $k \geq 1$, there exists an explicit constant $c_k > 0$ such that*

$$\begin{aligned}\mathcal{S}\big(|\cdot|^{2k}\big)(v,v_*) \leq -(1-c_k)\,\big(|v|^{2k} + |v_*|^{2k}\big)\\ + c_k\Big(\big(|v|^2 + |v_*|^2\big)^k - |v|^{2k} - |v_*|^{2k}\Big)\,.\end{aligned} \tag{3.55}$$

The map $k \to c_k$ has the following properties:

(1) c_k is strictly decreasing with $c_1 = 1$. In particular, $c_k < 1$ for $k \in (1,\infty)$.

(2) When $q > 1$, one has $c_k = O\big(k^{-1/q'}\big)$ for large k. Here $1/q + 1/q' = 1$. For the case $q = 1$, it still follows that $\lim_{k\to\infty} c_k = 0$.

Proof. Set $\varphi(v) = |v|^{2k} =: \psi_k(|v|^2)$, with $k \geq 1$. With obvious definitions for $\mathcal{S}^{\pm}$ we can write

$$\mathcal{S}(\varphi)(v,v_*) = \mathcal{S}^{+}(\varphi)(v,v_*) - \mathcal{S}^{-}(\varphi)(v,v_*)\,.$$

Let us focus in the term $\mathcal{S}^{+}$. Introduce the velocity of the center of mass $U = \dfrac{v+v_*}{2}$ to write the collision laws as

$$v' = U + \frac{|u|}{2}\sigma\,, \quad \text{and} \quad v'_* = U - \frac{|u|}{2}\sigma\,.$$

Expanding the squares,

$$\begin{aligned}&\psi_k\big(|v'|^2\big) + \psi_k\big(|v'_*|^2\big)\\ &\quad = \psi_k\Big(|U|^2 + \frac{|u|^2}{4} + |u||U|\widehat{U}\cdot\sigma\Big) + \psi_k\Big(|U|^2 + \frac{|u|^2}{4} - |u||U|\widehat{U}\cdot\sigma\Big)\\ &\quad = \psi_k\Big(E\,\frac{1+\xi\,\widehat{U}\cdot\sigma}{2}\Big) + \psi_k\Big(E\,\frac{1-\xi\,\widehat{U}\cdot\sigma}{2}\Big)\,,\end{aligned}$$

where we have set

$$E := |v|^2 + |v_*|^2 = 2|U|^2 + \frac{|u|^2}{2}\,, \quad \text{and} \quad \xi := 2\frac{|U|\,|u|}{E}\,.$$

Since $\psi_k(\cdot)$ is convex for $k \geq 1$, the mapping $\tilde{\psi}_k(s) := \psi_k(x+sy) + \psi_k(x-sy)$ is even and nondecreasing for $s \geq 0$ and $x,\, y \in \mathbb{R}$, see [22]. Therefore, using that $\xi \leq 1$ it follows that

$$\begin{aligned}\psi_k\big(|v'|^2\big) + \psi_k\big(|v'_*|^2\big) &\leq \psi_k\Big(E\,\frac{1+\widehat{U}\cdot\sigma}{2}\Big) + \psi_k\Big(E\,\frac{1-\widehat{U}\cdot\sigma}{2}\Big) \\ &= E^k\left(\psi_k\Big(\frac{1+\widehat{U}\cdot\sigma}{2}\Big) + \psi_k\Big(\frac{1-\widehat{U}\cdot\sigma}{2}\Big)\right)\,.\end{aligned}$$

Hence,

$$\begin{aligned}&\mathcal{S}^+\big(\varphi\big)(v,v_*) \\ &\leq E^k \int_{\mathbb{S}^{d-1}} b_o\big(\widehat{u}\cdot\sigma\big)\left(\psi_k\Big(\frac{1+\widehat{U}\cdot\sigma}{2}\Big) + \psi_k\Big(\frac{1-\widehat{U}\cdot\sigma}{2}\Big)\right)\mathrm{d}\sigma \quad (3.56) \\ &=: E^k\, c_k\big(\widehat{U},\,\widehat{u}\big)\,.\end{aligned}$$

Define $c_k := \sup_{\widehat{U},\,\widehat{u}} c_k\big(\widehat{U},\,\widehat{u}\big)$. Note that substituting $k=1$ in (3.56) readily implies that

$$c_1\big(\widehat{U},\,\widehat{u}\big) = \int_{\mathbb{S}^{d-1}} b_o\big(\widehat{u}\cdot\sigma\big)\mathrm{d}\sigma = 1\,.$$

Furthermore, using Hölder inequality in (3.56) and then computing explicitly

$$\begin{aligned}c_k &\leq \|b_0\|_{L^q}\left(|\mathbb{S}^{d-2}|\int_{-1}^{1}\left(\psi_k\Big(\frac{1+s}{2}\Big) + \psi_k\Big(\frac{1-s}{2}\Big)\right)^{q'}\mathrm{d}s\right)^{1/q'} \\ &\leq C_{d,q}\,k^{-1/q'}\,.\end{aligned}$$

The fact that c_k is strictly decreasing follows observing that the integrand in (3.56) strictly decreases as k increases. For the case $q=1$, the fact that $\lim_{k\to\infty} c_k = 0$ follows by dominated convergence theorem. Estimate (3.55) follows directly from the definition of c_k and (3.56). □

Let us explain how Povzner lemma help us to study the *generation of moments* in the Boltzmann model, namely, solutions of the Boltzmann equation have all moment finite, for any positive time, regardless the initial configuration. There is a caveat though, such property is exclusive for *hard potentials* with initial configurations having finite mass and energy

$$m_0(f_o) = \int_{\mathbb{R}^d} f_o(v)\mathrm{d}v = 1\,,\;\; m_2(f_o) = \int_{\mathbb{R}^d} f_o(v)|v|^2\mathrm{d}v < \infty\,, \quad (3.57)$$

where the mass is normalized to one for simplicity. In contrast, this property does not hold for soft potentials and Maxwell molecules. Let us consider $k \in (1,2)$ for simplicity, thus, we can write $k = 2\xi$ for some $\xi < 1$. It follows that

$$\big(|v|^2 + |v_*|^2\big)^k \le \big(|v|^{2\xi} + |v_*|^{2\xi}\big)^2 = |v|^{2k} + |v_*|^{2k} + 2|v|^k|v_*|^k\,.$$

Therefore, using (3.52) and (3.55) one concludes that

$$\begin{aligned}\int_{\mathbb{R}^d} & Q(f,f)(v)|v|^{2k}\mathrm{d}v \\ &= 2c_k \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(v)|v|^k f(v_*)|v_*|^k|u|^\gamma \mathrm{d}v_*\mathrm{d}v \qquad (3.58)\\ &- 2(1-c_k)\int_{\mathbb{R}^d} f(v)|v|^{2k}\Big(\int_{\mathbb{R}^d} f(v_*)|u|^\gamma\mathrm{d}v_*\Big)\mathrm{d}v\,.\end{aligned}$$

Note how Povzner lemma helped canceling the higher order moments contributing positively. Using the inequality $|u|^\gamma \ge |v|^\gamma - |v_*|^\gamma$ valid for $\gamma \in (0,1]$, one concludes that

$$\int_{\mathbb{R}^d} f(v_*)|u|^\gamma\mathrm{d}v_* \ge m_0(f)|v|^\gamma - m_\gamma(f)\,.$$

Additionally, using the inequality $|u|^\gamma \le |v|^\gamma + |v_*|^\gamma$ in the first term of (3.58)

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(v)|v|^k f(v_*)|v_*|^k|u|^\gamma\mathrm{d}v_*\mathrm{d}v \le 2m_{k+\gamma}(f)m_k(f)\,.$$

Using the last two estimates in (3.58) gives the estimate

$$\begin{aligned}\int_{\mathbb{R}^d} Q(f,f)(v)|v|^{2k}\mathrm{d}v \leq 4c_k m_{k+\gamma}(f)m_k(f)+ \\ + 2(1-c_k)m_{2k}(f)m_\gamma(f) - 2(1-c_k)m_0(f)m_{2k+\gamma}(f)\,.\end{aligned} \tag{3.59}$$

Now it is a matter of massaging inequality (3.59), mainly by using Lebesgue's interpolation, to obtain a suitable estimate. The key ingredient is that the Boltzmann model conserves mass and energy. Therefore,

$$\begin{aligned}m_0(f) = m_0(f_o) = 1\,,\\ m_\gamma(f) \leq m_0(f)^{\frac{2-\gamma}{2}} m_2(f)^{\frac{\gamma}{2}} = m_0(f_o)^{\frac{2-\gamma}{2}} m_2(f_o)^{\frac{\gamma}{2}}\,,\\ m_k(f) \leq m_0(f_o)^{1-\frac{k}{2}} m_2(f_o)^{\frac{k}{2}}\,.\end{aligned}$$

As a consequence, estimate (3.59) turns into

$$\begin{aligned}\int_{\mathbb{R}^d} Q(f,f)(v)|v|^{2k}\mathrm{d}v \leq \\ C(f_o)\big(c_k m_{k+\gamma}(f) + m_{2k}(f)\big) - 2(1-c_k)m_{2k+\gamma}(f)\,,\end{aligned} \tag{3.60}$$

where the constant $C(f_o)$ depends only on mass and energy of the initial configuration (we continue with such notation in the sequel). *A priori* the moments $k+\gamma$ and $2k$ are not controlled by the mass and the energy, however, they can be absorbed using the moment $2k+\gamma$ and Young's inequality

$$\begin{aligned}m_{k+\gamma}(f) \leq \varepsilon^{-\frac{2k+\gamma}{k}} m_0(f_o) + \varepsilon^{\frac{2k+\gamma}{k+\gamma}} m_{2k+\gamma}(f)\,,\\ m_{2k}(f) \leq \varepsilon^{-\frac{2k+\gamma}{\gamma}} m_0(f_o) + \varepsilon^{\frac{2k+\gamma}{2k}} m_{2k+\gamma}(f)\,,\end{aligned} \tag{3.61}$$

valid for any $\varepsilon > 0$. Choosing the parameter ε sufficiently small, it follows that (3.60) reduces to

$$\begin{aligned}\int_{\mathbb{R}^d} Q(f,f)(v)|v|^{2k}\mathrm{d}v &\leq C_k(f_o) - (1-c_k)m_{2k+\gamma}(f)\\ &\leq C_k(f_o) - (1-c_k)m_{2k}(f)^{\frac{2k+\gamma}{2k}}\,.\end{aligned} \tag{3.62}$$

This estimate will allow us to conclude the generation of moments result up to moment $2k < 4$. Indeed, assume f being a solution of the homogeneous Boltzmann equation (3.50) with initial condition f_o satisfying (3.57). We assume such solution conserves mass and energy, then, multiplying equation (3.50) by $|v|^{2k}$, integrating in velocity, and using estimate (3.62)

$$m_{2k}(f)'(t) + (1 - c_k) m_{2k}(f)^{\frac{2k+\gamma}{2k}}(t) \leq C_k(f_o)\,. \tag{3.63}$$

Invoking a classical comparison result in ODE's, estimate (3.63) implies that

$$m_{2k}(f)(t) \leq C(f_o)\Big(1 + \frac{1}{t^{1+\frac{2k}{\gamma}}}\Big), \quad 1 < k < 2\,.$$

It is important when invoking such comparison result for ODE's that the exponent $\frac{2k+\gamma}{2k} > 1$. Of course, this only happens for hard potentials. The result for higher moments follows using the same ideas and a little bit more work. Many of the ideas exposed here can be found in [22, 74, 10].

Theorem 3.3.2. *Let $f_o \geq 0$ an initial datum with finite mass and energy. Then, any solution f of the Boltzmann equation (for hard potentials) conserving mass and energy has all moments bounded for any positive time*

$$m_k(f)(t) \leq C(f_o)\Big(1 + \frac{1}{t^{1+\frac{k}{\gamma}}}\Big), \qquad k > 2\,.$$

Furthermore, if $m_k(f_o) < \infty$, then

$$\sup_{t\geq 0} m_k(f)(t) \leq \max\big\{m_k(f_o),\, C_k(f_o)\big\}\,, \qquad k > 2\,.$$

The latter result is known as propagation of moments.

3.4 Propagation of regularity

We study in this section the propagation of L^p-integrability and Sobolev regularity for the homogeneous Boltzmann equation. Although, moments are truly the physically meaningful quantities to study, propagation of regularity is an important mathematical issue that has ripple effects in the physics. For instance, it will be central in the study of long time behavior of the equation which is presented later on. In fact, rates of convergence in the long run are quite dependent of the smoothness of the initial data.

The following presentation will borrow ideas that can be found in the references [39, 11, 63] and started in [47]. Additional treatment about the integrability properties of the collision operator including discussion of sharp constants can be found in [6]. Essentially, there are 3 steps in the study of propagation of integrability: (1) proving an estimate for the operator Q^+ which is closely related to Young's inequality for convolutions, (2) proving a sharp lower bound for the operator Q^- which helps as absorption term (similar to the case of moment analysis) and (3) proving the so-called gain of integrability for the operator Q^+ which is related to a compactness property due to the angular averaging in $\mathbb{S}^{d-1}$. Recall that the angular averaging was essential in the Povzner's lemma. Furthemore, for the propagation of Sobolev regularity we will need an additional, and quite technical, result that essentially remarks that higher derivatives of the operator Q^+ can be controlled with lower derivatives. This was a result first observed for the collision operator by Lions [54, 55], but, simpler proofs can be found in [75, 63]. In fact, we will use an even simpler approach given in [24] suited to our purpose.

3.4.1 Step 1. L^p-bounds for Q^+

In this exposition the framework will be the weighted Lebesgue's space

$$L^p_k(\mathbb{R}^d) = \left\{ f \,:\, f\langle v\rangle^k \in L^p(\mathbb{R}^d)\right\},$$

where $\langle v\rangle := \sqrt{1+|v|^2}$. To ease the notation we may write $\|\cdot\|_{L^p_k} = \|\cdot\|_{p,k}$. We continue working with hard potential kernels of the form

$$B(u,\sigma) = |u|^\gamma b_o(\widehat{u}\cdot\sigma)\,, \qquad \gamma\in(0,1]\,.$$

Finally, let us observe an important issue in the Boltzmann equation that will help us with the proofs: it is a fact that the angular kernel b_o can be assumed to be supported in $\mathbb{S}^{d-1}_+$ due to symmetry of collisions.

Indeed, for any fixed vector u

$$\int_{\mathbb{S}^{d-1}} f('v)f('v_*)b_o(\widehat{u}\cdot\sigma)\mathrm{d}\sigma = \sum_{\pm}\int_{\{\pm\widehat{u}\cdot\sigma\geq 0\}} f('v)f('v_*)b_o(\widehat{u}\cdot\sigma)\mathrm{d}\sigma$$
$$= \int_{\{\widehat{u}\cdot\sigma\geq 0\}} f('v)f('v_*)\Big(b_o(\widehat{u}\cdot\sigma)+b_o(-\widehat{u}\cdot\sigma)\Big)\mathrm{d}\sigma\,,$$

where for the last identity we used the change of variables $\sigma\to-\sigma$ in the integral performed in the set $\{\widehat{u}\cdot\sigma\leq 0\}$. Note that this change of variables implies that $('v,'v_*)\to('v_*,'v)$. Thus, this accounts to have an equivalent scattering kernel defined as

$$\tilde{b}_o(z) := \big(b_o(z)+b_o(-z)\big)\mathbf{1}_{\{\widehat{u}\cdot\sigma\geq 0\}}\,.$$

We drop the tilde to ease the notation and continue considering an angular scattering kernel with normalized Grad's cut off assumption.

Lemma 3.4.1. *The positive scattering operator*

$$\mathcal{S}^+(\varphi)(v,v_*) = \int_{\mathbb{S}^{d-1}} \varphi(v')b_o(\widehat{u}\cdot\sigma)\,d\sigma$$

is a bounded operator $\mathcal{S}^+ : L^p(\mathbb{R}^d)\to L^\infty\big(\mathbb{R}^d_{v_*},L^p(\mathbb{R}^d_v)\big)$ *with norm estimated as*

$$\|\mathcal{S}^+\|_p \leq \|\mathcal{S}^+\|_1^{1/p} \leq 2^{d/p}\|b_o\|^{1/p}_{L^1(\mathbb{S}^{d-1})}\,,$$

where $\|\mathcal{S}^+\|_1$ *is defined in* (3.64)

Proof. Let us prove the result for L^1 and L^∞, and then, conclude using Marcinkiewicz interpolation theorem. We address the L^1 estimate since the L^∞ estimate is clear. Observe that,

$$\big\|\mathcal{S}^+(\varphi)\big\|_{L^1(\mathbb{R}^d_v)} = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \varphi(v')b_o(\widehat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v$$
$$= \int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d} \varphi(v')b_o(\widehat{u}\cdot\sigma)\mathrm{d}u\mathrm{d}\sigma\,.$$

Use the change of variables $\xi = \frac{1}{2}\big(u+|u|\sigma\big)$, for any fixed unitary vector σ, in the integral performed in the u variable noticing that the collision law can be written as

$$v' = v - \frac{u-|u|\sigma}{2} = v_* + \xi\,.$$

The Jacobian of the transformation $\xi(u)$ is easily computed as, see appendix Lemma 6.0.2

$$\frac{\mathrm{d}\xi}{\mathrm{d}u} = \frac{1+\widehat{u}\cdot\sigma}{2^d},$$

and therefore,

$$\begin{aligned}\left\|\mathcal{S}^+(\varphi)\right\|_{L^1(\mathbb{R}^d_u)} &= 2^d\int_{\mathbb{R}^d}\varphi(v_*+\xi)\int_{\mathbb{S}^{d-1}}\frac{b_o(\widehat{u}\cdot\sigma)}{1+\widehat{u}\cdot\sigma}\mathrm{d}\sigma\mathrm{d}\xi \\ &= 2^d\left|\mathbb{S}^{d-2}\right|\int_0^{\pi/2} b_o(\cos(\theta))\frac{\sin(\theta)^{d-2}}{1+\cos(\theta)}\mathrm{d}\theta\,\|\varphi\|_1 =: \|\mathcal{S}^+\|_1\|\varphi\|_1\,.\end{aligned} \tag{3.64}$$

Note that the integration in angle is performed in the interval $\theta\in[0,\pi/2]$ due to the support of b_o. This makes the norm bounded with estimate $\|\mathcal{S}^+\|_1\le 2^d\|b_o\|_{L^1(\mathbb{S}^{d-1})}$. □

Proposition 3.4.2. *For every $p\in[1,\infty]$, $\gamma\ge 0$ and $k\ge-\gamma$,*

$$\left\|Q^+(f,g)\right\|_{p,k}\le 2^{\gamma/2}\|\mathcal{S}^+\|_1^{1/p'}\|f\|_{p,k+\gamma}\|g\|_{1,k+2\gamma}\,.$$

Proof. Let us deal first with the case $k=0$. Fix nonnegative functions f and g and use duality to conclude that

$$\begin{aligned}\left\|Q^+(f,g)\right\|_p &= \sup_{\|\varphi\|_{p'}\le 1}\int_{\mathbb{R}^d}Q^+(f,g)(v)\varphi(v)\mathrm{d}v \\ &= \sup_{\|\varphi\|_{p'}\le 1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}f(v)g(v_*)|u|^\gamma\mathcal{S}^+(\varphi)(v,v_*)\mathrm{d}v_*\mathrm{d}v\,.\end{aligned}$$

Using the inequality $|u|\le\langle v\rangle\langle v_*\rangle$ and Lemma 3.4.1 one finds that the latter integral is bounded by

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}f(v)g(v_*)|u|^\gamma\mathcal{S}^+(\varphi)(v,v_*)\mathrm{d}v_*\mathrm{d}v \\ &\qquad\le\int_{\mathbb{R}^d}g(v_*)\langle v_*\rangle^\gamma\left(\int_{\mathbb{R}^d}f(v)\langle v\rangle^\gamma\mathcal{S}^+(\varphi)(v,v_*)\mathrm{d}v\right)\mathrm{d}v_* \\ &\qquad\qquad\le\|\mathcal{S}^+\|_1^{1/p'}\|g\|_{1,\gamma}\|f\|_{p,\gamma}\|\varphi\|_{p'}\,.\end{aligned}$$

This proves the case $k=0$. To incorporate weights fix $k \geq -\gamma$, introduce $u^+ := \dfrac{u+|u|\sigma}{2}$, and note the manipulation for the potential

$$|u|^\gamma = |u^+|^\gamma \left(\frac{|u|}{|u^+|}\right)^\gamma = |u^+|^\gamma \Big(\frac{2}{1+\widehat{u}\cdot\sigma}\Big)^{\gamma/2} \leq 2^{\gamma/2}|u^+|^\gamma\,,$$

where last inequality is valid in the support of b_o, namely, $\{\widehat{u}\cdot\sigma \geq 0\}$. Now, $u^+ = v - {}'v_*$, then

$$\begin{aligned}\langle v\rangle^k |u|^\gamma &\leq 2^{\gamma/2}\langle v\rangle^k |v - {}'v_*|^\gamma\\ &\leq 2^{\gamma/2}\langle v\rangle^{k+\gamma}\langle {}'v_*\rangle^\gamma \leq 2^{\gamma/2}\langle {}'v\rangle^{k+\gamma}\langle {}'v_*\rangle^{k+2\gamma}\,,\end{aligned}$$

where we used $\langle v\rangle \leq \langle {}'v\rangle\langle {}'v_*\rangle$ by conservation of energy, and the fact that $k+\gamma \geq 0$. Therefore,

$$Q^+_\gamma(f,g)(v)\langle v\rangle^k \leq 2^{\gamma/2} Q^+_o\big(f\langle\cdot\rangle^{k+\gamma}, g\langle\cdot\rangle^{k+2\gamma}\big)(v)\,.$$

The subscript in the operators indicate the potential order. The proof follows using the case $k=0$. □

3.4.2 Step 2. Sharp lower bound for Q^-

Lemma 3.4.3. *Let f be a solution of the homogeneous Boltzmann equation having initial datum f_o with zero momentum and moment $2+\mu$ finite*

$$m_{2+\mu}(f_o) < \infty \ \ \textit{for some}\ \ \mu > 0\,.$$

Then,

$$\big(f * |\cdot|^\gamma\big)(v) \geq C(f_o)\langle v\rangle^\gamma, \tag{3.65}$$

with $C(f_o) > 0$ depending only on the mass and the moment $2+\mu$ of f_o.

Proof. Notice that in the ball $B(0,r)$ one has for any $R > 0$ and

$\mu > 0$,

$$\begin{aligned}\int_{|v-w|\le R} & f(t,w)|v-w|^2\mathrm{d}w \\ &= \int_{\mathbb{R}^d} f(t,w)|v-w|^2\mathrm{d}w - \int_{|v-w|\ge R} f(t,w)|v-w|^2\mathrm{d}w \\ &= \int_{\mathbb{R}^d} f_o(w)|v-w|^2\mathrm{d}w - \int_{|v-w|\ge R} f(t,w)|v-w|^2\mathrm{d}w \\ &\ge C_0(f_o)\,\langle v\rangle^2 - \frac{1}{R^\mu}\int_{|v-w|\ge R} f(t,w)|v-w|^{2+\mu}\mathrm{d}w.\end{aligned}$$

For the last inequality we expanded the square in the first integral of the right side and used momentum zero. We now use in the second integral the inequality $|v-w| \le \langle v\rangle\langle w\rangle$ and the uniform propagation of the moment $2+\mu$ to conclude

$$\begin{aligned}\int_{|v-w|\le R} f(t,w)|v-w|^2\mathrm{d}w &\ge C_0(f_o)\langle v\rangle^2 - \frac{C_1}{R^\mu}\langle v\rangle^{2+\mu} \\ &\ge \frac{C_0(f_o)}{2}, \quad \forall\, v \in B(0,r)\,.\end{aligned}$$

We have taken $R := R(C_1, r)$ sufficiently large recalling that C_1 is a constant depending only on the mass and $m_{2+\mu}(f_o)$ thanks to Theorem 3.3.2. Therefore,

$$\int_{\mathbb{R}^d} f(t,w)|v-w|^\gamma\mathrm{d}w \ge \frac{1}{R^{2-\gamma}}\int_{|v-w|\le R} f(t,w)||v-w|^2\mathrm{d}w \ge \frac{C_0(f_o)}{2R^{2-\gamma}}\,,$$

valid for any $v \in B(0,r)$. Moreover,

$$\int_{\mathbb{R}^d} f(t,w)|v-w|^\gamma\mathrm{d}w \ge m_0(f_o)|v|^\gamma - C_2(f_o)\,,$$

as a consequence,

$$\begin{aligned}\int_{\mathbb{R}^d} & f(t,w)|v-w|^\gamma\mathrm{d}w \\ &\ge \frac{C_0(f_o)}{2R^{2-\gamma}}\,\mathbf{1}_{B(0,r)} + \big(m_0(f_o)|v|^\gamma - C_2(f_o)\big)\,\mathbf{1}_{B(0,r)^c}\,.\end{aligned} \tag{3.66}$$

Inequality (3.65) follows from (3.66) choosing r sufficiently large and then $R := R(C_1, r)$. □

3.4.3 Step 3. Gain of integrability of Q^+

The gain of integrability of the positive collision operator is referred to the property in which its L^p-norm is controlled by lower norms of its entries, that is, by L^q-norms with $q \in [1,p)$. In fact more is true, under certain conditions on the collision kernel, higher Sobolev norms of Q^+ are controlled by lower ones of its entries. This compactness fact is not easy to prove and we will postpone it for later. Instead, we follow a simple approach developed in [11] which avoids many technicalities.

Let f and g be suitable nonnegative functions. First, we introduce the Carleman's representation for $Q^+(f,g)$

$$Q^+(f,g)(v) = \\ 2^{n-1}\int_{\mathbb{R}^d}\frac{g(x)}{|v-x|}\int_{\{(v-x)\cdot z=0\}}\frac{\tau_{-x}f(z+(v-x))}{|z+(v-x)|^{n-2}}\,\tilde{B}\,(z,v-x)\,\mathrm{d}\pi_z\,\mathrm{d}x\,, \\ \text{where } \tilde{B}\,(z,v-x) := B\left(|z+(v-x)|, 1-2\tfrac{|z|^2}{|z+(v-x)|^2}\right). \quad (3.67)$$

Here τ is the translation operator and $\mathrm{d}\pi_z$ is the Lebesgue measure in the plane. The Carleman's representation has been quite used to study the collision operator, it is some sort of stereographic projection which allow to perform computations easier. In the appendix Lemma 6.0.3 we give a proof for such representation. Note that if $g = \delta_o$, it follows that

$$Q^+(f,\delta_o)(v) = \\ \frac{2^{n-1}}{|v|}\int_{\{v\cdot z=0\}}\frac{f(z+v)}{|z+v|^{n-2}}\,B\left(|z+v|, 1-\frac{2|z|^2}{|z+v|^2}\right)\mathrm{d}\pi_z\,. \quad (3.68)$$

Therefore combining (3.67) and (3.68) we obtain the double mixing convolution formula for the collision operator

$$Q^+(f,g)(v) = \int_{\mathbb{R}^d} g(x)\tau_x Q^+\big(\tau_{-x}f,\delta_o\big)(v)\mathrm{d}x\,. \quad (3.69)$$

Differences and similarities between a regular convolution and the positive collision operator are observed through identity (3.69). Let us compute,

$$
\begin{aligned}
\big\|Q^+(f,g)\big\|_2 &= \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} g(x)\tau_x Q^+(\tau_{-x}f,\delta_o)(v)\mathrm{d}x\right)^2 \mathrm{d}v\right)^{1/2} \\
&\le \int_{\mathbb{R}^d} g(x)\left(\int_{\mathbb{R}^d}\big(\tau_x Q^+(\tau_{-x}f,\delta_o)(v)\big)^2\mathrm{d}v\right)^{1/2}\mathrm{d}x\,.
\end{aligned}
\tag{3.70}
$$

Thus, we need to compute the L^2-norm of the bilinear gain operator with the second entry a Dirac point mass. In the sequel we adopt the notation Q^+_γ, where we allow any $\gamma \ge 0$ in this notation, to make explicit the kinetic scattering potential $|u|^\gamma$ in the operator.

Proposition 3.4.4. *Fix $d \ge 2$ and let $f \in L^{\frac{2d}{2d-1}}_{d-1}(\mathbb{R}^d)$. Then,*

$$
\big\|Q^+_{d-1}(f,\delta_o)(v)\big\|_2 \le C_d\|b_o\|_\infty\|f\|_{\frac{2d}{2d-1},d-1}\,,
$$

with C_d some explicit constant depending only on the dimension.

Proof. Using the expression of the scattering kernel

$$
B(u,\sigma) = |u|^\gamma b_o(\widehat{u}\cdot\sigma)
$$

in (3.68), we write

$$
Q^+_\gamma(f,\delta_o)(v) = \frac{2^{d-1}}{|v|}\int_{\{z\cdot v=0\}}\frac{f(z+v)}{|z+v|^{d-2-\gamma}}\, b_o\left(1-\frac{2|z|^2}{|z+v|^2}\right)\mathrm{d}\pi_z\,.
$$

Since $b_o(s)$ is supported in $\{s\ge 0\}$, it follows that $|v|\ge|z|$. As a consequence,

$$
\frac{|v|^2}{|z+v|^2} = \frac{|v|^2}{|z|^2+|v|^2} \ge \frac{1}{2}\,.
$$

Then,

$$
Q^+_\gamma(f,\delta_o)(v) \le 2^{d-1/2}\|b_o\|_\infty\int_{\{z\cdot v=0\}}\frac{f(z+v)}{|z+v|^{d-1-\gamma}}\mathrm{d}\pi_z\,.
$$

Set $\gamma = d-1$ and use polar coordinates $v = r\sigma$. Thus, it is possible to estimate its L^2-norm in the following way

$$\big(2^{d-1/2}\|b_o\|_\infty\big)^{-2}\int_{\mathbb{R}^d}\big(Q^+_{d-1}(f,\delta_o)(v)\big)^2\,\mathrm{d}v \le$$
$$\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^+}\int_{\{z_1\cdot\sigma=0\}}\int_{\{z_2\cdot\sigma=0\}} f(z_1+r\sigma)f(z_2+r\sigma)\mathrm{d}\pi_{z_1}\mathrm{d}\pi_{z_2}r^{d-1}\mathrm{d}r\mathrm{d}\sigma$$
$$=: I.$$

Next, perform the change of variables for fixed σ, $x := z_1 + r\sigma$ and note that $r = x\cdot\sigma$. Therefore,

$$I \le \int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{\{z_2\cdot\sigma=0\}} f(x)f(z_2+(x\cdot\sigma)\sigma)\mathrm{d}\pi_{z_2}|x\cdot\sigma|^{d-1}\mathrm{d}x\mathrm{d}\sigma\,.$$

Writing

$$z_2 + (x\cdot\sigma)\sigma = x + (z_2 + (x\cdot\sigma)\sigma - x) := x + z_3\,,$$

it easy to check that $z_3 \in \big\{z : z\cdot\sigma = 0\big\}$. Hence,

$$I \le \int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{\{z_3\cdot\sigma=0\}} f(x)f(x+z_3)\mathrm{d}\pi_{z_3}|x\cdot\sigma|^{d-1}\mathrm{d}x\mathrm{d}\sigma\,.$$

Using the identity

$$\int_{\mathbb{R}^d}\delta_o(z\cdot y)\,\varphi(z)\,\mathrm{d}z = |y|^{-1}\int_{\{z\cdot y=0\}}\varphi(z)\,\mathrm{d}\pi_z$$

valid for any smooth φ, we transform the integration in the hyperplane $\{z_3\cdot\sigma = 0\}$ into an integration in $\mathbb{R}^d$,

$$\int_{\{z_3\cdot\sigma=0\}} f(x+z_3)\mathrm{d}\pi_{z_3} = \int_{\mathbb{R}^d}\delta_o(z\cdot\sigma)f(x+z)\mathrm{d}z$$
$$= \int_{\mathbb{R}^d}\delta_o(\widehat{z}\cdot\sigma)\frac{f(x+z)}{|z|}\mathrm{d}z\,.$$

Hence,

$$I \le C_d\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{f(x)|x|^{d-1}f(z)}{|z-x|}\mathrm{d}z\mathrm{d}x,$$

where the constant can be taken as

$$C_d = \int_{\mathbb{S}^{d-1}} \delta_o(\widehat{z}\cdot\sigma)\mathrm{d}\sigma = \left|\mathbb{S}^{d-2}\right| \int_{-1}^{1} \delta_o(s)(1-s^2)^{\frac{d-3}{2}}\mathrm{d}s = \left|\mathbb{S}^{d-2}\right|.$$

Finally, recalling the Hardy-Littlewood-Sobolev inequality we have for any $r\in(1,\infty)$,

$$I \le C_d \big\| f\,|\cdot|^{d-1}\big\|_{r'} \big\| f * |\cdot|^{-1}\big\|_{r} \le C_{d,r}\big\| f\,|\cdot|^{d-1}\big\|_{r'}\|f\|_p,$$

where $1/p+1/d = 1+1/r$. Choosing $p=r'$, that is $p=\frac{2d}{2d-1}$, yields

$$I \le C_{d,r}\big\| f\,|\cdot|^{d-1}\big\|_{\frac{2d}{2d-1}}\big\|f\big\|_{\frac{2d}{2d-1}} \le C_{d,r}\,\|f\|^{2}_{\frac{2d}{2d-1},d-1}\,.$$

Since $r := r(d)$ the result follows. □

Let us come back to the estimation of the L^2-norm of Q^+. Fix a dimension $d\ge 2$ and recall that the physical accepted values of γ satisfy $0<\gamma\le 1\le d-1$. Thus,

$$|u|^{\gamma} \le \frac{\varepsilon^{s'}}{s'} + \frac{1}{s\varepsilon^{s}}|u|^{\gamma s} \qquad s=\tfrac{d-1}{\gamma}\,,\ \ \varepsilon>0\,. \tag{3.71}$$

Additionally, we write the scattering kernel $b_o = b_o^1 + b_o^\infty$ with

(1) $b_o^1\ge 0$ having small mass, say $\|b_o^1\|_{L^1(\mathbb{S}^{d-1})} = \delta > 0$.

(2) And, $b_o^\infty \ge 0$ essentially bounded and such that $b_o^\infty\le b_o^1$. Of course, $\|b_o^\infty\|_\infty$ will depend in general on δ.

Therefore, for any $k\ge -\gamma$ it follows that

$$\begin{aligned}\big\|Q^+_\gamma(f,g)\big\|_{2,k} &= \big\|Q^+_{\gamma,b_o^1}(f,g) + Q^+_{\gamma,b_o^\infty}(f,g)\big\|_{2,k}\\ &\le \big\|Q^+_{\gamma,b_o^1}(f,g)\big\|_{2,k} + \big\|Q^+_{\gamma,b_o^\infty}(f,g)\big\|_{2,k}\\ &\le \big\|Q^+_{\gamma,b_o^1}(f,g)\big\|_{2,k}\\ &\quad + \tfrac{\varepsilon^{s'}}{s'}\big\|Q^+_{o,b_o^\infty}(f,g)\big\|_{2,k} + \tfrac{1}{s\varepsilon^{s}}\big\|Q^+_{d-1,b_o^\infty}(f,g)\big\|_{2,k}\,.\end{aligned} \tag{3.72}$$

For the last inequality we have used (3.71). Using Proposition 3.4.2 it follows that

$$\begin{aligned}\big\|Q^+_{\gamma,b^1_o}(f,g)\big\|_{2,k} &\leq 2^{\frac{d+\gamma}{2}}\|b^1_o\|_1^{1/2}\|f\|_{2,k+\gamma}\|g\|_{1,k+2\gamma}\\ &= 2^{\frac{d+\gamma}{2}}\sqrt{\delta}\,\|f\|_{2,k+\gamma}\|g\|_{1,k+2\gamma}\,,\\ \big\|Q^+_{o,b^\infty_o}(f,g)\big\|_{2,k} &\leq 2^{\frac{d+\gamma}{2}}\|b^\infty_o\|_1^{1/2}\|f\|_{2,k^+}\|g\|_{1,k^+}\\ &\leq 2^{\frac{d+\gamma}{2}}\|f\|_{2,k^+}\|g\|_{1,k^+}\,,\end{aligned}$$

where, in the last inequality, we introduced $k^+ := \max\{0,k\}$. For the latter term in (3.72) use (3.70) and Proposition 3.4.4. Thus, when $k=0$

$$\begin{aligned}\big\|Q^+_{d-1,b^\infty_o}(f,g)\big\|_2 &\leq \int_{\mathbb{R}^d}\big\|\tau_x Q^+_{d-1,b^\infty_o}(\tau_{-x}f,\delta_o)\big\|_2\, g(x)\mathrm{d}x\\ &\leq C_d\|b^\infty_o\|_\infty\int_{\mathbb{R}^d}\big\|\tau_{-x}f\big\|_{\frac{2d}{2d-1},d-1}\, g(x)\mathrm{d}x\\ &\leq C_{d,\delta}\|f\|_{\frac{2d}{2d-1},d-1}\|g\|_{1,d-1}\,.\end{aligned}$$

Therefore, when a weight $k\geq-\gamma$ is used

$$\begin{aligned}\big\|Q^+_{d-1,b^\infty_o}(f,g)\big\|_{2,k} &\leq \big\|Q^+_{d-1,b^\infty_o}(f\langle\cdot\rangle^{k^+},g\langle\cdot\rangle^{k^+})\big\|_2\\ &\leq C_{d,\delta}\|f\|_{\frac{2d}{2d-1},k^++d-1}\|g\|_{1,k^++d-1}\,.\end{aligned}$$

This proves the following proposition.

Proposition 3.4.5. *Fix $d\geq2$ and $\gamma\in(0,1]$. The collision operator satisfies the estimate for any $\varepsilon>0$, $\delta>0$ and $k\geq-\gamma$*

$$\|Q^+_\gamma(f,g)\|_{2,k} \leq 2^{\frac{d+\gamma}{2}}\sqrt{\delta}\,\|f\|_{2,k+\gamma}\|g\|_{1,k+2\gamma} + \\ 2^{\frac{d+\gamma}{2}}\frac{\varepsilon^{s'}}{s'}\|f\|_{2,k^+}\|g\|_{1,k^+} + \frac{C_{d,\delta}}{s\varepsilon^s}\|f\|_{\frac{2d}{2d-1},k^++d-1}\|g\|_{1,k^++d-1}\,,$$

where $s=\frac{d-1}{\gamma}$, $k^+=\max\{0,k\}$, and $C_{d,\delta}$ depends only on d and δ. In the case $s=1$, the second term in the right side vanishes and $\varepsilon=1$.

We have all the ingredients to prove the main result of this section, propagation of L^2 integrability for the Boltzmann homogeneous equations.

Theorem 3.4.6. *Fix a dimension $d \geq 2$ and potential $\gamma \in (0,1]$. Assume Grad's cutoff in the angular kernel (normalized to one). Additionally, fix $k \geq 0$ and assume an initial estate $f_o \in L^1_{k_o} \cap L^2_k$. Then, the solution f to the homogeneous Boltzmann equation propagates L^2_k integrability*

$$\sup_{t\geq 0} \big\| f(t) \big\|_{2,k} \leq C(f_o)\,.$$

The constant depends on the mass, $m_{k_o}(f_o)$ and $\|f_o\|_{2,k}$, where

$$k_o = \max\Big\{ k + \tfrac{3\gamma}{2},\, k + \gamma + \tfrac{d}{d-1}\big(d-1-\gamma\big) \Big\}\,.$$

Proof. Fix $k \geq 0$ and multiply the Boltzmann equation (3.50) by $f\langle\cdot\rangle^k$ and integrate in velocity. This process leads to

$$\begin{aligned}\frac{d}{dt}\|f(t)\|^2_{2,\frac{k}{2}} = &\int_{\mathbb{R}^d} Q^+(f,f)(v) f(v) \langle v\rangle^k \mathrm{d}v \\ &- \int_{\mathbb{R}^d} Q^-(f,f)(v) f(v)\langle v\rangle^k \mathrm{d}v\,.\end{aligned} \tag{3.73}$$

Using Lemma 3.4.3 one obtains

$$\begin{aligned}&\int_{\mathbb{R}^d} Q^-(f,f)(v) f(v)\langle v\rangle^k \mathrm{d}v \\ &\qquad = \int_{\mathbb{R}^d} f(v)^2 \langle v\rangle^k \big(f * |\cdot|^\gamma\big)(v)\mathrm{d}v \geq C(f_o)\big\|f\big\|^2_{2,\frac{k+\gamma}{2}}\,.\end{aligned} \tag{3.74}$$

For the first term in (3.73) use Proposition 3.4.5, with $f = g$, to conclude

$$\begin{aligned}&\int_{\mathbb{R}^d} \Big(Q^+(f,f)(v)\langle v\rangle^{\frac{k-\gamma}{2}}\Big) f(v)\langle v\rangle^{\frac{k+\gamma}{2}}\mathrm{d}v \leq \\ &2^{\frac{d+\gamma}{2}}\Big(\sqrt{\delta}\,\|f\|_{2,\frac{k+\gamma}{2}}\|f\|_{1,\frac{k+3\gamma}{2}} + \frac{\varepsilon^{s'}}{s'}\|f\|_{2,\left(\frac{k-\gamma}{2}\right)^+}\|f\|_{1,\left(\frac{k-\gamma}{2}\right)^+} \\ &\quad + \frac{C_{d,\delta}}{s\varepsilon^s}\|f\|_{\frac{2d}{2d-1},\left(\frac{k-\gamma}{2}\right)^+ + d-1}\|f\|_{1,\left(\frac{k-\gamma}{2}\right)^+ + d-1}\Big)\big\|f\big\|_{2,\frac{k+\gamma}{2}}\,.\end{aligned} \tag{3.75}$$

Since $\frac{2d}{2d-1} < 2$, it is possible to use Lebesgue's interpolation

$$\|f\|_{\frac{2d}{2d-1},\left(\frac{k-\gamma}{2}\right)^+ + d-1} \leq \|f\|^{\frac{1}{d}}_{2,\frac{k+\gamma}{2}}\|f\|^{\frac{d-1}{d}}_{1,\frac{k}{2}+\gamma+\frac{d}{d-1}(d-1-\gamma)}\,. \tag{3.76}$$

Therefore, using (3.76) and propagation of moments Theorem 3.3.2 in (3.75), it follows that there exists constants $C^1(f_o)$ and $C^2_\delta(f_o)$ depending on the initial datum only through the mass and $m_{k_o}(f_o)$, with

$$k_o = \max\left\{\tfrac{k+3\gamma}{2},\, \tfrac{k}{2}+\gamma+\tfrac{d}{d-1}\big(d-1-\gamma\big)\right\},$$

and such that

$$\begin{aligned}\int_{\mathbb{R}^d} & Q^+(f,f)(v)f(v)\langle v\rangle^k \mathrm{d}v \\ &\le C^1(f_o)\big(\sqrt{\delta}+\varepsilon^{s'}\big)\|f\|^2_{2,\frac{k+\gamma}{2}} + \frac{C^2_\delta(f_o)}{\varepsilon^s}\|f\|^{1+\frac{1}{d}}_{2,\frac{k+\gamma}{2}}\,.\end{aligned} \tag{3.77}$$

Choosing δ and ε such that $C^1(f_o)\big(\sqrt{\delta}+\varepsilon^{s'}\big) = C(f_o)/2$ we can absorb every right side term in (3.77) with (3.74). For the latter right side term of (3.77) one can use Young's inequality because $1+\frac{1}{d}<2$. The conclusion is that there exists a constant $C^3(f_o)$ such that

$$\frac{d}{dt}\|f(t)\|^2_{2,\frac{k}{2}} \le C^3(f_o) - \frac{C(f_o)}{4}\|f\|^2_{2,\frac{k+\gamma}{2}} \le C^3(f_o) - \frac{C(f_o)}{4}\|f\|^2_{2,\frac{k}{2}}\,.$$

Simple integration of this differential inequality leads to the result. □

In the important physical cases $d = 2, 3$ this theorem gives a satisfactory result in terms of the initial datum moment order k_o needed for L^2-propagation. In higher dimension such order can sensibly be lowered by refining the analysis, see [63]. The result for L^p-propagation follows by interpolation, see for instance [63, 11]. Let us just state the general result here.

Theorem 3.4.7. *Fix a dimension $d \ge 2$, $p \in (1,\infty)$, and potential $\gamma \in (0,1]$. Assume Grad's cutoff in the angular kernel (normalized to one). Additionally, fix $k \ge 0$ and assume an initial estate $f_o \in L^1_{k_o} \cap L^p_k$ for some $k_o := k_o(k)$ sufficiently large. Then, the solution f to the homogeneous Boltzmann equation propagates L^p_k integrability*

$$\sup_{t\ge 0}\big\|f(t)\big\|_{p,k} \le C(f_o)\,.$$

The constant depends on the mass, $m_{k_o}(f_o)$ and $\|f_o\|_{p,k}$.

3.4.4 Step 4. Propagation of Sobolev regularity

Before entering in details about propagation of regularity for the homogeneous Boltzmann equation, we will need a result very much related to Proposition 3.4.4, yet, of slightly different nature. It is about the fact that under certain integrability condition on the angular kernel, higher Sobolev norms can be controlled by lower ones. There are several versions of this fact, see for instance [54, 55, 75, 63]. For our purpose, it will suffice to follow a neat result that was brought in [24].

Proposition 3.4.8. *Let $b_o \in L^2(\mathbb{S}^{d-1})$. Then*

$$\left\|Q^+_\gamma(f,g)\right\|_{\dot H^{\frac{d-1}{2}}(\mathbb{R}^d)} \le C_d \|b_o\|_2\, \|f\|_{2,\gamma+1}\|g\|_{2,\gamma+1}\,, \quad \gamma \ge 0\,.$$

Proof. The proof is based on the Fourier transform of the gain operator Q^+ using the weak formulation

$$\begin{aligned}\mathcal{F}\big\{Q^+(f,g)\big\}(\xi) &= \int_{\mathbb{R}^d} Q^+(f,g)(v) e^{-iv\cdot\xi}\mathrm{d}v\\ &= \int_{\mathbb{R}^{2d}} f(v)g(v_*)|u|^\gamma\Big(\int_{\mathbb{S}^{d-1}} e^{-iv'\cdot\xi} b_o(\widehat{u}\cdot\sigma)\mathrm{d}\sigma\Big)\mathrm{d}v_*\mathrm{d}v\,.\end{aligned}$$

For the average in the sphere, it follows that

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} e^{-iv'\cdot\xi} b(\widehat{u}\cdot\sigma)\mathrm{d}\sigma &= e^{-i(v-\frac{u}{2})\cdot\xi}\int_{\mathbb{S}^{d-1}} e^{-i\frac{|u||\xi|}{2}\widehat{\xi}\cdot\sigma} b_o(\widehat{u}\cdot\sigma)\mathrm{d}\sigma\\ &= e^{-i(v-\frac{u}{2})\cdot\xi}\int_{\mathbb{S}^{d-1}} e^{-i\frac{|u||\xi|}{2}\widehat{u}\cdot\sigma} b_o(\widehat{\xi}\cdot\sigma)\mathrm{d}\sigma\,.\end{aligned}$$

The last equality results after interchanging $\widehat{u} \leftrightarrow \widehat{\xi}$ which is allowed since the integral on the sphere is a function of the inner product $\widehat{u}\cdot\widehat{\xi}$ only (and the norms $|u|$ and $|\xi|$). Thus,

$$\begin{aligned}&\mathcal{F}\big\{Q^+(f,g)\big\}(\xi) =\\ &\int_{\mathbb{S}^{d-1}}\Big(\int_{\mathbb{R}^{2d}} f(v)g(v_*)|u|^\gamma e^{-i(\frac{\xi+|\xi|\sigma}{2})\cdot v} e^{-i(\frac{\xi-|\xi|\sigma}{2})\cdot v_*}\mathrm{d}v_*\mathrm{d}v\Big) b_o(\widehat{\xi}\cdot\sigma)\mathrm{d}\sigma\\ &\qquad = \int_{\mathbb{S}^{d-1}} \mathcal{F}\{F\}\Big(\tfrac{\xi+|\xi|\sigma}{2}, \tfrac{\xi-|\xi|\sigma}{2}\Big)\, b_o(\widehat{\xi}\cdot\sigma)\mathrm{d}\sigma\,. \qquad (3.78)\end{aligned}$$

where $F(v,v_*) := f(v)g(v_*)|u|^\gamma$. The use of the Fourier transform in the Boltzmann equation was proposed in [17] for Maxwell molecules and later refined to other potentials. Thus, using Cauchy-Schwarz's inequality

$$\big|\mathcal{F}\{Q^+(f,g)\}(\xi)\big|^2 \le \|b_o\|_2^2 \int_{\mathbb{S}^{d-1}} \Big|\mathcal{F}\{F\}\Big(\tfrac{\xi+|\xi|\sigma}{2}, \tfrac{\xi-|\xi|\sigma}{2}\Big)\Big|^2 \mathrm{d}\sigma\,. \tag{3.79}$$

Now, compute

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} \Big|\mathcal{F}\{F\}\Big(\tfrac{\xi+|\xi|\sigma}{2}, \tfrac{\xi-|\xi|\sigma}{2}\Big)\Big|^2 \mathrm{d}\sigma \\
&\quad= \int_{\mathbb{S}^{d-1}} \int_{|\xi|}^{\infty} -\partial_r \Big|\mathcal{F}\{f(v)g(v_*)|u|^\gamma\}\Big(\tfrac{\xi+r\sigma}{2}, \tfrac{\xi-r\sigma}{2}\Big)\Big|^2 \mathrm{d}r\mathrm{d}\sigma \\
&\quad= \int_{\mathbb{S}^{d-1}} \int_{|\xi|}^{\infty} \mathcal{F}\{F\}\Big(\nabla_2\mathcal{F}\{F\} - \nabla_1\mathcal{F}\{F\}\Big)\Big(\tfrac{\xi+r\sigma}{2}, \tfrac{\xi-r\sigma}{2}\Big)\cdot\sigma\mathrm{d}r\mathrm{d}\sigma \\
&\quad= \int_{\{|\eta|\ge|\xi|\}} \mathcal{F}\{F\}\,\mathcal{F}\{-i(v_*-v)\,F\}\Big(\tfrac{\xi+\eta}{2}, \tfrac{\xi-\eta}{2}\Big)\cdot\widehat{\eta}\,\frac{\mathrm{d}\eta}{|\eta|^{d-1}}\,.
\end{aligned} \tag{3.80}$$

As a consequence from (3.79) and (3.80)

$$\begin{aligned}
\|b_o\|_2^{-2}\big\|Q^+(f,g)\big\|^2_{\dot{H}^{\frac{d-1}{2}}} &= \int_{\mathbb{R}^d} \Big|\mathcal{F}\{Q^+(f,g)\}(\xi)|\xi|^{\frac{d-1}{2}}\Big|^2 \mathrm{d}\xi \\
&\le \int_{\mathbb{R}^{2d}} \Big|\mathcal{F}\{F\}\,\mathcal{F}\{-i(v_*-v)\,F\}\Big(\tfrac{\xi+\eta}{2}, \tfrac{\xi-\eta}{2}\Big)\Big|\mathrm{d}\eta\mathrm{d}\xi \\
&= 2^d\big\|\mathcal{F}\{F\}\mathcal{F}\{(v_*-v)F\}\big\|_{L^1(\mathbb{R}^{2d})} \\
&\le 2^d\big\|\mathcal{F}\{F\}\big\|_{L^2(\mathbb{R}^{2d})}\big\|\mathcal{F}\{(v_*-v)F\}\|_{L^2(\mathbb{R}^{2d})}\,.
\end{aligned}$$

Fourier isometry give us

$$\begin{aligned}
\big\|\mathcal{F}\{F\}\big\|_{L^2(\mathbb{R}^{2d})} &= (2\pi)^{2d}\big\|F\big\|_{L^2(\mathbb{R}^{2d})} \le (2\pi)^{2d}\big\|f\big\|_{2,\gamma}\big\|g\big\|_{2,\gamma}\,, \\
\big\|\mathcal{F}\{(v_*-v)F\}\big\|_{L^2(\mathbb{R}^{2d})} &= (2\pi)^{2d}\big\|(v_*-v)F\big\|_{L^2(\mathbb{R}^{2d})} \\
&\le (2\pi)^{2d}\big\|f\big\|_{2,\gamma+1}\big\|g\big\|_{2,\gamma+1}\,.
\end{aligned}$$

□

Now, the way one uses Proposition 3.4.8 to prove propagation of Sobolev regularity is the following. Set the dimension $d = 3$ for simplicity and consider an initial datum $f_o \in L^1_{k(\gamma)} \cap L^2_{1+\gamma}$, with $k(\gamma)$ sufficiently large, belonging to $H^1(\mathbb{R}^d)$. Then, differentiating the Boltzmann equation by v_i

$$\partial_t f_i = \partial_{v_i} Q^+(f,f) - f_i\big(f * |\cdot|^\gamma\big) - f\big(f * \partial_{v_i}|\cdot|^\gamma\big)\,. \tag{3.81}$$

Here of course $f_i := \partial_{v_i} f$. Recall the lower bound on the Q^- given in Lemma 3.4.3, and also note the simple estimate

$$f * \partial_{v_i}|\cdot|^\gamma \le c_\gamma\big(\|f\|_2 + m(f_o)\big) \le C(f_o)\,, \quad \gamma \in (0,1]\,.$$

The last inequality follows from Theorem 3.4.6 since $f_o \in L^2(\mathbb{R}^d)$. As a consequence, multiplying equation (3.81) by f_i and integrating in velocity one concludes

$$\tfrac{1}{2}\frac{d}{dt}\|f_i\|_2^2 \le \|\partial_{v_i} Q^+(f,f)\, f_i\|_1 + C(f_o)\|f_i\, f\|_1 - c(f_o)\|f_i\|^2_{2,\gamma/2}\,. \tag{3.82}$$

Using Proposition 3.4.8 it follows that (here $\frac{d-1}{2} = 1$)

$$\begin{aligned}\|\partial_{v_i} Q^+(f,f)\, f_i\|_1 &\le \|\partial_{v_i} Q^+(f,f)\|_2 \|f_i\|_2 \\ &\le C_d \|b_o\|_2 \|f\|^2_{2,\gamma+1}\|f_i\|_2 \le C(f_o)\|f_i\|_2\,.\end{aligned} \tag{3.83}$$

In summary, gathering (3.82) and (3.83)

$$\tfrac{1}{2}\frac{d}{dt}\|f_i\|_2^2 \le C(f_o)\|f_i\|_2 - c(f_o)\|f_i\|^2_{2,\gamma/2}\,. \tag{3.84}$$

This *a priori* estimate readily implies that

$$\sup_{t\ge 0}\|f\|_{\dot{H}^1} \le C(f_o)\,. \tag{3.85}$$

The dependence of the constant is written in terms of the $L^1_{k(\gamma)} \cap L^2_{1+\gamma}$ norms and the H^1-regularity of the initial data. Higher Sobolev regularity is implemented with induction using the differentiation Leibniz rule holding for the collision operator

$$\partial^\nu_v Q(f,g) = \sum_{\eta=0}^{\nu} \begin{pmatrix} \nu \\ \eta \end{pmatrix} Q\big(\partial^\eta_v f, \partial^{\nu-\eta}_v g\big)\,,$$

where ν and η are multi-indexes. The details are left to the reader or can be found in [63].

Theorem 3.4.9. *Let $\gamma \in (0,1]$ and $b_o \in L^2(\mathbb{S}^{d-1})$. Assume $f_o \in L^1_2 \cap H^k_{k(1+\gamma)}$. Then, the solution of the homogeneous Boltzmann equation satisfies*

$$\sup_{t\geq 0} \|f\|_{H^k} \leq C_k(f_o)\,, \quad k \in \mathbb{N}.$$

3.5 Entropy dissipation method and time asymptotic

The starting point of this section is an essential process in physics: the Ornstein–Uhlenbeck process. Such process is describing the velocity of a particle that experiences friction and Brownian forces (thermalization)

$$\mathrm{d}v(t) = -\nu\, v(t)\mathrm{d}t + \eta\, \mathrm{d}B(t)\,, \quad v(0) = v_o\,. \tag{3.86}$$

Here the non negative parameters ν and η are the friction and thermalization coefficients respectively. The Brownian motion is added to the equation to model the influence of a *smaller* scale physics influencing the particle, such as, small molecules in continuous collision with this macroscopic particle. As a consequence, the friction term decreases the kinetic energy of the particle while the thermalization increases it. The Ornstein–Uhlenbeck stochastic equation has Kolmogorov forward equation (commonly known as Fokker–Planck equation)

$$\partial_t f = \nu\, \nabla_v \cdot (vf) + \frac{\eta^2}{2}\Delta_v f\,, \quad f(0) = f_o\,. \tag{3.87}$$

Equation (3.87) does not conserves momentum or energy, due to collisions with the micro-scale, only conserves mass. However, it is easy to see that energy will remain uniformly bounded. Furthermore, given the physics of the problem one expects a non trivial relaxation of this process as time evolve. Let us prove this by introducing the relative entropy (compare with 3.7)

$$\mathcal{H}\big(f\big|M\big) = \int_{\mathbb{R}^d} f(v)\, \log\frac{f(v)}{M(v)}\mathrm{d}v\,, \quad M(v) = \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{\frac{d}{2}}}\,. \tag{3.88}$$

Thus, set $\nu = 1$ and $\eta = \sqrt{2}$ for simplicity and multiply the Fokker–Planck equation (3.87) by $\log\big(\frac{f(v)}{M(v)}\big)$, one finds[2]

$$\frac{d}{dt}\mathcal{H}\big(f\big|M\big) = -\mathcal{I}\big(f\big|M\big)\,, \tag{3.89}$$

where $\mathcal{I}$ is the relative Fisher information

$$\mathcal{I}\big(f\big|M\big) := \int_{\mathbb{R}^d} f(v)\Big|\nabla_v \log\frac{f(v)}{M(v)}\Big|^2 \mathrm{d}v\,. \tag{3.90}$$

Using the Stam–Gross logarithmic Sobolev inequality, see [69]

$$\mathcal{H}\big(f\big|M\big) \le \frac{1}{2}\mathcal{I}\big(f\big|M\big)\,, \tag{3.91}$$

valid for any $f \in L^1(\mathbb{R}^d)$ having unit mass since M is a normalized Gaussian. Therefore, using (3.91) in (3.89) one gets

$$\mathcal{H}\big(f\big|M\big) \le e^{-2t}\mathcal{H}\big(f_o\big|M\big)\,.$$

With this at hand it is possible to invoke Csiszár–Kullback–Pinsker inequality

$$\|f - M\|_1 \le \sqrt{2\,\mathcal{H}\big(f\big|M\big)}\,,$$

to deduce that the distribution of the Ornstein–Uhlenbeck process experiments an exponential relaxation in the L^1-metric, provided the initial datum f_o has finite second moment (energy) and entropy, towards a Gaussian distribution. We refer to [71] for an ample discussion on the topic and references. Let us conclude this short discussion by noticing that multiplying equation (3.87) by $\log(f)$, it is concluded after integration by parts that

$$\frac{d}{dt}\mathcal{H}(f) = -\mathcal{I}(f) + d = -\mathcal{I}(f) + \mathcal{I}(M)\,.$$

Here $\mathcal{I}$ is the Fisher information (given by (3.90) with $M \equiv 1$). Now, integrate in the time interval $[0,\infty)$, use that $f(t) \to M$ in L^1 and

[2] Simply observe that $v = -\dfrac{\nabla M(v)}{M(v)}$.

the fact that f has second moment uniformly bounded, to conclude that

$$\mathcal{H}(f_o) - \mathcal{H}(M) = \int_0^\infty \big(\mathcal{I}(f(s)) - \mathcal{I}(M)\big)\mathrm{d}s\,. \tag{3.92}$$

Representation formula (3.92) will be important in the next subsection and it is valid for any function $f_o \in L^1(\mathbb{R}^d)$ with unitary mass and finite energy.

3.5.1 Relaxation of the linear Boltzmann model

The Ornstein–Uhlenbeck process, as important as it is, does not present big analytic challenges. In fact, the machinery just introduced previously it is not needed to prove relaxation to a Gaussian state. The reason is that one can easily compute the Green function of the Fokker-Planck equation (3.87): Assume a initial particle distribution $f_o(v) = \delta_o(v-w)$, then, the probability distribution of the Ornstein–Uhlenbeck process evolves as

$$h_t(v,w) = \frac{e^{-\frac{1}{2}\frac{|v-e^{-t}w|^2}{1-e^{-2t}}}}{\big(2\pi(1-e^{-2t})\big)^{\frac{d}{2}}}\,.$$

Thus, the semigroup $\mathcal{S}_t$ of the Fokker-Planck equation has the explicit form

$$\begin{aligned} \mathcal{S}_t f(v) &= \int_{\mathbb{R}^d} h_t(v,w)\, f(w)\mathrm{d}w = e^{dt}\big(H_t * F_t\big)(v)\,, \\ \text{where}\ \ H_t(v) &:= \frac{e^{-\frac{1}{2}\frac{|v|^2}{1-e^{-2t}}}}{\big(2\pi(1-e^{-2t})\big)^{\frac{d}{2}}}\,, \quad F_t(v) := f(e^t\, v)\,. \end{aligned} \tag{3.93}$$

From here it is clear that $\mathcal{S}_t f \to M$ as $t \to \infty$ at exponential rate. This observation, however, does not mean that the Ornstein–Uhlenbeck process is just a simple example to introduce advanced tools, we will see next that it plays a central role in more complicated processes having distributions evolving to Gaussians.

Let us try to apply the entropy relaxation method to the linear Boltzmann model with *Maxwell* molecules interactions, that is, when the collision operator has potential $\gamma = 0$. This problem is quite interesting because the Boltzmann model represents a jump process in contrast with the Fokker–Planck equation. The linear Boltzmann equation reads

$$\partial_t f = Q_o(f, M) =: \mathcal{L}(f)\,, \quad f(0) = f_o\,. \tag{3.94}$$

The physics representing this equation is a cloud of particles colliding with an homogeneous background (of particles) having a Gaussian distribution. This model approximates a regime where the background particles are many in comparison with the particles distributed with f. In such a case, particle–particle interaction is a secondary phenomena compared to the particle–background interaction. Thus, thermodynamics tells us that it should be the case that $f \to M$ as time evolves. A first indication that this is in fact the case comes from the equations for the momentum and energy of f

$$\begin{aligned} \frac{d\bar{v}(f)}{dt} &= -\tfrac{1-\lambda}{2}\bar{v}(f)\,, \qquad \bar{v}(f)(t) := \int_{\mathbb{R}^d} f(t,v)\,v\,\mathrm{d}v\,, \\ \frac{dm_2(f)}{dt} &= -\tfrac{1-\lambda}{2}\big(m_2(f) - m_2(M)\big)\,, \end{aligned} \tag{3.95}$$

where the parameter λ is given by

$$\lambda := \int_{\mathbb{S}^{d-1}} (\widehat{u}\cdot\sigma)\, b_o(\widehat{u}\cdot\sigma)\mathrm{d}\sigma \in (0,1)\,. \tag{3.96}$$

Equations (3.95) follows after multiplying (3.94) with v and $|v|^2$ and integrating in velocity. Thus, $\bar{v}(f) \to \bar{v}(M)$ and $m_2(t) \to m_2(M)$ at exponential rate $\frac{1-\lambda}{2}$. We observe that closed equations for moments is a benefit only happening for Maxwellian interactions due to homogeneity of the potential.

The problem of exponential converge of the equation (3.94) towards the Gaussian background has been nicely treated in [16] for Maxwell molecules and other potentials as well. We follow here their presentation and argument only in the case of Maxwell molecules and refer to [16] for a complete discussion to more general potentials and links to logarithmic Sobolev inequalities. The idea is, as presented previously, to control the relative entropy with the entropy dissipation which is the analogous of the Fisher information in the case of

Boltzmann

$$\frac{d}{dt}\mathcal{H}\big(f(t)\big|M\big) = \int_{\mathbb{R}^d} \mathcal{L}(f(t))(v)\log\frac{f(t,v)}{M(v)}\mathrm{d}v = -\mathcal{D}\big(f(t)\big)\,, \quad (3.97)$$

where

$$\begin{aligned}\mathcal{D}(f) := \tfrac{1}{2}\int_{\mathbb{R}^{2d}} & M(v)M(v_*)\times \\ & \left(\int_{\mathbb{S}^{d-1}}\Phi\Big(\frac{f(v)}{M(v)},\frac{f(v')}{M(v')}\Big)b_o(\widehat{u}\cdot\sigma)\mathrm{d}\sigma\right)\mathrm{d}v_*\mathrm{d}v\,,\end{aligned} \quad (3.98)$$

and $\Phi(x,y) := (x-y)\log(x/y) \geq 0$.

Theorem 3.5.1. *The functional inequality*

$$\mathcal{D}(f) \geq \tfrac{1-\lambda}{2}\,\mathcal{H}\big(f\big|M\big)$$

holds for any probability distribution $f \in L^1(\mathbb{R}^d)$. The constant λ is given in (3.96).

There are several interesting steps in proving Theorem 3.5.1, each of them important in their own right. The first step consists in the following commutativity property: Let $\mathcal{S}_t$ be the semigroup generated by the Fokker-Planck equation (3.93), then

$$Q^+(\mathcal{S}_t f, \mathcal{S}_t g) = \mathcal{S}_t Q^+(f,g)\,, \quad t \geq 0\,, \quad (3.99)$$

valid for any $f \in L^1(\mathbb{R}^d)$ with finite energy. This commutativity property looks odd, as one would expect such property holding for the generator of the Ornstein–Uhlenbeck process (the adjoint of $\mathcal{S}_t$) related to the Kolmogorov backward equation rather than for $\mathcal{S}_t$ (related to the Kolmogorov forward equation). Indeed, propagating two particles backward in time using the Ornstein–Uhlenbeck process and, then, perform a collision looks the same as collide the particles and, then, propagate backwards in time. However, one recalls that for elastic interactions the collision law is time symmetric, pre/pos collision laws are interchangeable, which explains, at least intuitively, identity (3.99). The proof of (3.99) goes by direct computation using

Fourier transform in velocity. Indeed, recalling formula (3.78)

$$
\begin{aligned}
&\mathcal{F}\big\{Q^+\big(H_t * F_t, H_t * G_t\big)\big\}(\xi) \\
&\qquad = \int_{\mathbb{S}^{d-1}} \mathcal{F}\big\{H_t * F_t\big\}(\xi^+)\,\mathcal{F}\big\{H_t * G_t\big\}(\xi^-)\,b_o(\widehat{\xi}\cdot\sigma)\mathrm{d}\sigma \\
&= \int_{\mathbb{S}^{d-1}} \mathcal{F}\big\{H_t\big\}(\xi^+)\mathcal{F}\big\{H_t\big\}(\xi^-)\mathcal{F}\big\{F_t\big\}(\xi^+)\,\mathcal{F}\big\{G_t\big\}(\xi^-)\,b_o(\widehat{\xi}\cdot\sigma)\mathrm{d}\sigma\,.
\end{aligned}
$$

Here $\xi^{\pm} = \frac{1}{2}(\xi \pm |\xi|\sigma)$. Now, H_t is a normalized Gaussian, so its Fourier transform is a unitary Gaussian. In addition, note that $|\xi|^2 = |\xi^+|^2 + |\xi^-|^2$, therefore

$$
\mathcal{F}\big\{H_t\big\}(\xi^+)\mathcal{F}\big\{H_t\big\}(\xi^-) = \mathcal{F}\big\{H_t\big\}(\xi)\,.
$$

As a consequence,

$$
\mathcal{F}\big\{Q^+\big(H_t * F_t, H_t * G_t\big)\big\}(\xi) = \mathcal{F}\big\{H_t\big\}\mathcal{F}\big\{Q^+\big(F_t, G_t\big)\big\}(\xi)\,.
$$

And thus,

$$
\begin{aligned}
Q^+(\mathcal{S}_t f, \mathcal{S}_t g) &= H_t * Q^+\big(F_t, G_t\big) \\
&= e^{dt}\big(H_t * Q^+\big(f, g\big)_t\big) = \mathcal{S}_t Q^+\big(f, g\big)\,.
\end{aligned}
$$

The second step is further proof of the importance of the Ornstein–Uhlenbeck process, the complete proof can be found in [57].

Proposition 3.5.2. *Let $\mathcal{I}$ be the Fisher information and Q^+ be the Boltzmann collision operator with Maxwellian interactions. Then,*

$$
\mathcal{I}\big(Q^+(f,g)\big) \leq \tfrac{1+\lambda}{2}\,\mathcal{I}(f) + \tfrac{1-\lambda}{2}\,\mathcal{I}(g)\,,
$$

valid for any sufficiently smooth probability densities f, g. The constant λ is given in (3.96)*.*

Corollary 3.5.3. *Let Q^+ be the Boltzmann collision operator with Maxwellian interactions. Then,*

$$
\mathcal{H}\big(Q^+(f,g)\big) \leq \tfrac{1+\lambda}{2}\,\mathcal{H}(f) + \tfrac{1-\lambda}{2}\,\mathcal{H}(g)
$$

for any probability densities f, g.

Proof. Note that identity (3.92) can be written in terms of the Fokker-Planck semigroup $\mathcal{S}_t$

$$\mathcal{H}(f) - \mathcal{H}(M) = \int_0^\infty \big(\mathcal{I}(\mathcal{S}_t f) - \mathcal{I}(M)\big)\mathrm{d}t$$

for any probability density $f \in L^1(\mathbb{R}^d)$ with finite energy. Thus, taking $f \equiv Q^+(f,g)$ it follows that

$$\begin{aligned}\mathcal{H}(Q^+(f,g)) - \mathcal{H}(M) &= \int_0^\infty \big(\mathcal{I}(\mathcal{S}_t Q^+(f,g)) - \mathcal{I}(M)\big)\mathrm{d}t \\ &= \int_0^\infty \big(\mathcal{I}(Q^+(\mathcal{S}_t f, \mathcal{S}_t g)) - \mathcal{I}(M)\big)\mathrm{d}t \\ &\le \tfrac{1+\lambda}{2}\int_0^\infty \big(\mathcal{I}(\mathcal{S}_t f) - \mathcal{I}(M)\big)\mathrm{d}t + \tfrac{1-\lambda}{2}\int_0^\infty \big(\mathcal{I}(\mathcal{S}_t g) - \mathcal{I}(M)\big)\mathrm{d}t \\ &= \tfrac{1+\lambda}{2}\mathcal{H}(f) + \tfrac{1-\lambda}{2}\mathcal{H}(g) - \mathcal{H}(M)\,.\end{aligned}$$

where we used the commutativity property (3.99) and Proposition 3.5.2. □

Corollary 3.5.4. *Let $\mathcal{L}$ be the linear Boltzmann collision operator with Maxwellian interactions. Then,*

$$\mathcal{H}\big(\mathcal{L}^+(f)\big|M\big) \le \tfrac{1+\lambda}{2}\,\mathcal{H}\big(f\big|M\big)$$

for any probability density f. Here, of course, $\mathcal{L}(f) =: \mathcal{L}^+(f) - f$.

Proof. Let us use Corollary 3.5.3 with $g \equiv M$ to estimate

$$\begin{aligned}\mathcal{H}\big(\mathcal{L}^+(f)\big|M\big) &= \mathcal{H}\big(\mathcal{L}^+(f)\big) - \int_{\mathbb{R}^d} \mathcal{L}^+ f\,\log M\mathrm{d}v \\ &\le \tfrac{1+\lambda}{2}\mathcal{H}\big(f\big) + \tfrac{1-\lambda}{2}\mathcal{H}\big(M\big) - \int_{\mathbb{R}^d} \mathcal{L}^+ f\,\log M\mathrm{d}v \\ &= \tfrac{1+\lambda}{2}\mathcal{H}\big(f\big|M\big) + \tfrac{1-\lambda}{2}\int_{\mathbb{R}^d}\big(M - f\big)\log M\mathrm{d}v - \int_{\mathbb{R}^d}\mathcal{L}(f)\log M\mathrm{d}v \\ &= \tfrac{1+\lambda}{2}\mathcal{H}\big(f\big|M\big)\,.\end{aligned}$$

We used conservation of mass and the equation for the energy given in (3.95) to conclude that the last two terms cancel out. □

Proof of Theorem 3.5.1

Estimate the dissipation of entropy as follows

$$\begin{aligned}\mathcal{D}(f) &= -\int_{\mathbb{R}^d} \mathcal{L}(f) \log \frac{f}{M} \mathrm{d}v \\ &= \mathcal{H}\big(f\big|M\big) - \mathcal{H}\big(\mathcal{L}^+(f)\big|M\big) + \mathcal{H}\big(\mathcal{L}^+(f)\big|f\big) \\ &\geq \mathcal{H}\big(f\big|M\big) - \mathcal{H}\big(\mathcal{L}^+(f)\big|M\big) \geq \tfrac{1-\lambda}{2}\, \mathcal{H}\big(f\big|M\big)\,.\end{aligned}$$

In the first inequality we used that $\mathcal{H}\big(\mathcal{L}^+(f)\big|f\big) \geq 0$ because the mass of f and $\mathcal{L}^+ f$ are both one. The second inequality follows from Corollary 3.5.4. □

As a consequence of Theorem 3.5.1, the solution of (3.94) satisfies the exponential relaxation

$$\mathcal{H}\big(f\big|M\big) \leq \mathcal{H}\big(f_o\big|M\big) e^{-\frac{1-\lambda}{2} t}\,,$$

which seems to be the sharp rate of relaxation (by comparing with the relaxation rate of the momentum and energy). Furthermore, Csiszár–Kullback–Pinsker inequality implies the relaxation in the L^1-metric

$$\|f - M\|_1 \leq \sqrt{2\, \mathcal{H}\big(f\big|M\big)} \leq \sqrt{2\mathcal{H}\big(f_o\big|M\big)} e^{-\frac{1-\lambda}{4} t}\,.$$

Let us finishing this discussion by mentioning that the nonlinear Boltzmann equation relaxation has already been studied with entropy methods, we refer to [67, 71] and their related references, by solving the so-called Cercignani's conjecture. Needless to say, in the nonlinear cases an entropy functional becomes more valuable since it is unique to the process in question as opposed to the linear case where there may be plenty of entropy functionals. Interestingly, the program presented here for the linear case follows essentially the same steps of the program given in [67] for the nonlinear case (up to mathematical technicalities).

3.6 More on the Cauchy theory for Boltzmann

There is a vast literature for the Cauchy problem treating kinetic equations. In particular, the theory of existence and uniqueness of solutions for the Boltzmann equation in a Gaussian–perturbative regime has been pursued quite successfully by an number of authors. The methods include linearization of the collision operator and the development of a mathematical machinery used to control the non linear dynamic of the model using the linear dynamics. These ideas are far reaching and have many applications, we refer to [32, 33, 44, 45, 46] for an initiation on this theory.

Chapter 4

Dissipative Boltzmann equation

4.1 Cucker-Smale model and self organization

We start this discussion on dissipative systems with an example that we discussed at the beginning of this note (Section 1), the Cucker-Smale (CS) model. We recall that the CS model represents a system of N particles interacting by an average law of friction through a friction potential

$$\frac{dx_i}{dt} = v_i\,, \qquad \frac{dv_i}{dt} = \frac{1}{N}\sum_{j\neq i} U_f\big(|x_j - x_i|\big)(v_j - v_i)\,, \tag{4.1}$$

where (x_i, v_i) represents the position and velocity of the i-particle. In the applied literature, the CS model has been used to explain "flocking" of birds, school of fish, swarming of bacteria and other biological organisms. The word flocking represents a phenomenon in which self-propelled individuals organize into an ordered motion. In the following lines we present a simple analysis on the particle system (4.1) and its kinetic equation to represent, up to some extent, mathematically this phenomenon. The main assumption simply read:

U_f is Lipschitz continuous, decreasing and such that

$$\frac{c}{(1+|x|^2)^\beta} \leq U_f(|x|) \leq C\,, \quad \beta \geq 0\,. \tag{4.2}$$

The position and velocity of the center of mass of the particles is given by

$$x_c(t) := \frac{1}{N}\sum_{i=1}^{N} x_i(t)\,, \quad v_c(t) := \frac{1}{N}\sum_{i=1}^{N} v_i(t)\,. \tag{4.3}$$

Summing over all velocities in the equation for the velocity in (4.1) and using the symmetry of the interaction it follows that

$$v_c(t) = v_c(0)\,, \quad x_c(t) = x_c(0) + t\, v_c(0)\,. \tag{4.4}$$

That is, the system conserves momentum, and hence, the trajectory of the center of mass is just a straight line. Define the fluctuations of the particles around the center of mass as $\tilde{x}_i(t) := x_i(t) - x_c(t)$ and $\tilde{v}_i(t) := v_i(t) - v_c(t)$ and introduce the variance, or fluctuation, functions

$$X(t) := \sum_{i=1}^{N} |\tilde{x}_i(t)|^2\,, \quad V(t) := \sum_{i=1}^{N} |\tilde{v}_i(t)|^2\,. \tag{4.5}$$

Then, we can show the formation of a flock by showing that $V(t)$ decays towards zero and $X(t)$ remains uniformly bounded. In other words, in the long run particles will travel in one group and sharing the same velocity. Moreover, if the decay on $V(t)$ is sufficiently fast, this behavior will be stable under external perturbations, refer to [49] for ample discussion. First, let us investigate the spatial fluctuation with the following computation

$$\begin{aligned} \frac{dX}{dt} &= \frac{2}{N}\sum_{i=1}^{N}\big(x_i - x_c\big)\cdot\Big(\frac{dx_i}{dt} - \frac{dx_c}{dt}\Big) \\ &= \frac{2}{N}\sum_{i=1}^{N}\big(x_i - x_c\big)\cdot\big(v_i - v_c\big) \leq 2\sqrt{X}\sqrt{V}\,, \end{aligned} \tag{4.6}$$

where the last estimate follows from Cauchy-Schwarz inequality. Second, let us estimate the velocity fluctuation in the following way

$$\begin{aligned}\frac{dV}{dt} &= -\frac{1}{N^2}\sum_{1\le i,j\le N} U_f(|x_j-x_i|)|v_j-v_i|^2 \\ &\le -\frac{U_f\big(2\sqrt{X}\big)}{N^2}\sum_{1\le i,j\le N}|v_j-v_i|^2 \qquad (4.7)\\ = -2\,U_f\big(2\sqrt{X}\big)&\Big(\frac{1}{N}\sum_{i=1}^{N}|v_i|^2 - v_c^2\Big) = -2\,U_f\big(2\sqrt{X}\big)\,V \le 0\,.\end{aligned}$$

For the inequality we have used that $U_f(|x_j-x_i|) \ge U_f\big(2\sqrt{X}\big)$, for any i, j, since $U_f(\cdot)$ is decreasing. Thus, thanks to (4.7) and (4.6) it follows that

$$V(t) \le V_o\,, \qquad \sqrt{X(t)} \le \sqrt{X_o} + \sqrt{V_o}\,t\,. \tag{4.8}$$

Theorem 4.1.1. *(Flock formation) Let (x_i, v_i) be solutions to the CS model* (4.1) *with U_f satisfying* (4.2)*. Then, the fluctuation functions satisfy*

$$\sqrt{X(t)} \le \sqrt{X_o} + C\,, \quad V(t) \le V_o\,e^{-c_1 t}\,, \quad \beta \in [0, \tfrac{1}{2}]\,.$$

The constants C and c_1 are positive, finite and explicit in terms of the initial fluctuation (X_o, V_o), the model parameter β and the constant c defining the lower bound for U_f.

Proof. Integrating inequality (4.7) gives

$$V(t) \le V_o\,e^{-2\int_0^t U_f\left(2\sqrt{X(s)}\right)\mathrm{d}s}\,. \tag{4.9}$$

Assume first that $\beta \in [0, 1/2)$. Thus, using estimate (4.8) on the spatial fluctuation and assumption (4.1), it is not very difficult to estimate the time integral

$$2\int_0^t U_f\big(2\sqrt{X(s)}\big)\mathrm{d}s \ge \frac{c}{(1-2\beta)V_o^{\beta}}\,t^{1-2\beta}\,.$$

Plugging this estimate in (4.9), one upgrades the estimate on the velocity fluctuation to

$$V(t) \leq V_o\, e^{-\frac{c}{(1-2\beta)V_o^\beta}t^{1-2\beta}}\,,$$

and bootstrapping into estimate (4.6) gives us

$$\sqrt{X(t)} \leq \sqrt{X_o} + \int_0^t \sqrt{V(s)}\,\mathrm{d}s \leq \sqrt{X_o} + C(V_o,\beta,c)\,.$$

This in turn implies that $2U_f(2\sqrt{X(t)}) \geq c_1 := c_1(X_o,V_o,\beta,c) > 0$ which together with (4.9) gives

$$V(t) \leq V_o\, e^{-c_1 t}\,.$$

The case $\beta = \dfrac{1}{2}$ follows the same lines: First, using (4.9) we upgrade the estimate on the velocity fluctuation to

$$V(t) \leq V_o\big(1 + \tfrac{2\sqrt{V_o}}{1+2\sqrt{X_o}}t\big)^{-\frac{c}{2\sqrt{V_o}}}\,.$$

Second, we use (4.6) and (4.9) successively to improve the estimate on fluctutions

$$\sqrt{X(t)} \leq \sqrt{X_o} + C\,t^{1-\frac{c}{4\sqrt{V_o}}} \longrightarrow V(t) \leq V_o \exp\big(-c\,t^{\frac{c}{4\sqrt{V_o}}}\big)$$
$$\longrightarrow \sqrt{X(t)} \leq \sqrt{X_o} + C \longrightarrow V(t) \leq V_o e^{-c_1 t}\,.$$

This completes the proof for any $\beta \in [0, \frac{1}{2}]$. □

Now that we have shown that the CS model exhibit flocking dynamics, it would be interesting to look at the mean field limit searching for similar behavior. Recall that such limit is given by the equations (2.5–2.6). Here, we write it in non conservative form as

$$\begin{aligned}\partial_t f + v\cdot\nabla_x f + L(f)\cdot\nabla_v f &= -f\,\nabla\cdot L(f)\,,\\ L(f)(t,x,v) := \int_{\mathbb{R}^{2d}} U_f\big(|x_*-x|\big)\big(v_*-v\big)\,&f(t,x_*,v_*)\mathrm{d}x_*\mathrm{d}v_*\,.\end{aligned} \tag{4.10}$$

The characteristics or trajectories $(x(t),v(t))$ for equation (4.10) are defined by the ODE system

$$\frac{dx(t)}{dt} = v(t)\,,\quad \frac{dv(t)}{dt} = L(f)(t,x(t),v(t))\,, \tag{4.11}$$

and complemented with initial conditions (x_o, v_o). The characteristics have the natural interpretation of being the pair position-velocity of a particle at time t provided it started from position x_o and velocity v_o. Such particle evolves through the mean field force $L(f)$ which is given by the superposition of the interactions with every other particle. The characteristics (4.11) are well defined provided we have a sufficiently smooth solution f, say Lipschitz continuous. We refer to [49] for a well-posedness theory of equation (4.10) in $\mathcal{C}^1$ in time, space and velocity. Observe that we can estimate the velocity of a particle in the mean field limit similarly as we estimated velocity in the discrete model. Indeed, multiplying the velocity characteristic equation by $v(t)$ we obtain

$$\begin{aligned}
\tfrac{1}{2}\frac{d|v(t)|^2}{dt} &= v(t)\cdot L(f)(t,x(t),v(t)) \\
&= \int_{\mathbb{R}^{2d}} U_f\big(|x_* - x(t)|\big)\big(v(t)\cdot v_* - |v(t)|^2\big)\, f(t,x_*,v_*)\mathrm{d}x_*\mathrm{d}v_* \\
&\le \tfrac{1}{2}\int_{\mathbb{R}^{2d}} U_f\big(|x_* - x(t)|\big)\big(|v_*|^2 - |v(t)|^2\big)\, f(t,x_*,v_*)\mathrm{d}x_*\mathrm{d}v_*\,.
\end{aligned}$$

Observe something quite interesting here: If the distribution of particles f at some time $t>0$ is compactly supported in velocity (uniformly in space), we can define a finite maximum speed as

$$v_{max}(t) := \sup_{x\in\mathbb{R}^d}\{|v| : f(t,x,v) > 0\}\,.$$

Then, any particle velocity $v(t)$ such that $|v(t)| \ge v_{max}(t)$ will satisfy

$$\frac{d|v(t)|^2}{dt} \le 0\,,$$

because in the support of f, at time t, one has that $|v_*|^2 - |v(t)|^2 \le 0$. In other words, at every time t, the ball with radii $v_{max}(t)$, denoted as $B_v\big(v_{max}(t)\big)$, is an attractive set for the velocity trajectories $v(t)$. As a consequence, if the initial configuration f_o is such that $v_{max}(0) < \infty$, then, at any further time $v_{max}(t) \le v_{max}(0) < \infty$. Indeed, the initial particle velocity v_o belongs to the support of f_o, that is, it belongs to the stable set $B_v\big(v_{max}(0)\big)$, henceforth, the particle velocity $v(t)$ will remain in such set at all times $t>0$. The rigorous proof of this fact is left as exercise. This reasoning leads us to the estimate for the velocity trajectories

$$|v(t)| \leq v_{max}(0)\,, \quad t \geq 0\,. \tag{4.12}$$

The estimate on the spatial trajectories follows from (4.11)

$$\frac{1}{2}\frac{d|x(t)|^2}{dt} = x(t)\cdot v(t) \leq |x(t)||v(t)|\,,$$

thus,

$$|x(t)| \leq |x_o| + v_{max}(0)\,t\,, \quad t \geq 0\,. \tag{4.13}$$

Estimates (4.12–4.13) lead us to conclude that if f_o is compactly supported in both space and velocity, the velocity support remains bounded and the spatial support grows linearly with time

$$\sup_{v\in\mathbb{R}^d} \text{diameter}\big\{\text{spatial support of } f(t,v,x)\big\} \leq D_o + 2\,v_{max}(0)\,t\,, \tag{4.14}$$

where D_o is the supremum (in velocity) of the initial spatial support diameters. Let us proceed now presenting proof for flocking dynamics in the mean field model by introducing the fluctuations

$$\begin{aligned} X(f)(t) &:= \int_{\mathbb{R}^{2d}} |x - x_c(t)|^2 f(t,x,v)\mathrm{d}v\mathrm{d}x\,, \\ x_c(t) &:= \int_{\mathbb{R}^{2d}} x\, f(t,x,v)\,\mathrm{d}v\mathrm{d}x\,. \\ V(f)(t) &:= \int_{\mathbb{R}^{2d}} |v - v_c(t)|^2 f(t,x,v)\mathrm{d}v\mathrm{d}x\,, \\ v_c(t) &:= \int_{\mathbb{R}^{2d}} v\, f(t,x,v)\,\mathrm{d}v\mathrm{d}x\,. \end{aligned} \tag{4.15}$$

Here, $(x_c(t), v_c(t))$ are the position and velocity of the center of mass. Using the equation (4.10), it is clear that $v_c(t) = v_c(0)$ and $x_c(t) =$

$x_c(0) + v_c(0)\, t$. Furthermore,

$$\begin{aligned}\frac{dV(f)}{dt} &= \int_{\mathbb{R}^{2d}} |v - v_c|^2 \partial_t f \,\mathrm{d}v\mathrm{d}x \\ = -\int_{\mathbb{R}^{2d}} |v - v_c|^2 v \cdot \nabla_x f \,\mathrm{d}v\mathrm{d}x &- \int_{\mathbb{R}^{2d}} |v - v_c|^2 \nabla_v \cdot \big(f L(f)\big) \,\mathrm{d}v\mathrm{d}x \\ &= -\int_{\mathbb{R}^{4d}} U_f\big(|x_* - x|\big) |v_* - v|^2 f_* \, f \,\mathrm{d}v_*\mathrm{d}x_*\mathrm{d}v\mathrm{d}x\,.\end{aligned} \tag{4.16}$$

For the last identity, we have used the divergence theorem and the symmetry of the expressions. Thus, in light of (4.14) and (4.16) one concludes that

$$\begin{aligned}\frac{dV(f)}{dt} &\le -2\, U_f\big(D_o + 2\, v_{max}(0)\, t\big) V(f)(t) \\ &\le -\frac{2c}{(1 + D_o + v_{max}(0)\, t)^{2\beta}}\, V(f)(t)\,.\end{aligned} \tag{4.17}$$

In this inequality we have assumed implicitly that the particle distribution f has unitary mass, thus

$$2V(f)(t) = \int_{\mathbb{R}^{4d}} |v_* - v|^2 f_* \, f \,\mathrm{d}v_*\mathrm{d}x_*\mathrm{d}v\mathrm{d}x\,.$$

Theorem 4.1.2. *Let f be a classical solution of the kinetic model (4.10) with compactly supported initial datum $f_o \in \mathcal{C}^1$ having unitary mass. Then, the spatial fluctuation satisfies*

$$\sqrt{X(f)(t)} \le \sqrt{X(f_o)} + C\,, \quad \beta \in [0, \tfrac{1}{2})\,,$$

and the velocity fluctuation satisfies

$$V(f)(t) \le V(f_o) \times \begin{cases} C\, e^{-c_1 t^{1-2\beta}} & \text{if } \beta \in [0, \frac{1}{2}) \\ (1+t)^{-c_2} & \text{if } \beta = \frac{1}{2}\,. \end{cases}$$

The constants C, c_1 and c_2 are positive, finite and depending only on the support of the initial configuration f_o, the parameter β and the constant c defining a lower bound for U_f in (4.2).

Proof. The estimate for the velocity fluctuation follows after direct integration of the differential inequality (4.17). The estimate for the spatial fluctuation is concluded first noticing the identity

$$\begin{aligned}\frac{dX(f)}{dt} &= -\int_{\mathbb{R}^{2d}} \nabla_x \cdot (vf)|x-x_c|^2 \, \mathrm{d}v\mathrm{d}x \\ &= 2\int_{\mathbb{R}^{2d}} f(v-v_c)\cdot(x-x_c)\,\mathrm{d}v\mathrm{d}x\,.\end{aligned}$$

Therefore, using Cauchy–Schwarz inequality

$$\frac{d\sqrt{X(f)}}{dt} \leq \sqrt{V(f)} \longrightarrow \sqrt{X(f)(t)} \leq \sqrt{X(f_o)} + \int_0^t \sqrt{V(f)(s)}\mathrm{d}s\,.$$

The result follows using the estimate for the velocity fluctuation. □

4.2 Modeling a granular material using Boltzmann equation

We have introduced and studied the Boltzmann model in previous sections. Recall, that such model is used to describe a large number of particles interacting elastically. That is, the total energy of a pair of interacting particles is unchanged in the interaction process. A typical situation where this is *not* the case is when particles deform in the process of interaction. Such deformation transforms kinetic energy in other type of energy such as heat. This is the typical case in granular materials where particles interact by physical contact. The key parameter introduced by physicist to model this issue is the *restitution coefficient*, defined as the proportion of relative impact velocity lost in a collision. The formula is

$$e := -\frac{u'\cdot\omega}{u\cdot\omega}\,, \quad \text{with } e\in[0,1]\,. \tag{4.18}$$

General speaking, the restitution coefficient may have a complicate dependence on the state of the system and on the microscopic variables, in particular, it may depend on the temperature of the system and on the relative velocity of particles. The so-called *viscoelastic* particles model is a simplification of this fact. Here, the restitution coefficient depends *inversely* of the impact velocity $e = e(u \cdot \omega)$, see [28]. This particular dependence is intuitive, since the deformation will be greater for higher impact velocity. In the following discussion we mostly assume, for simplicity, that the restitution coefficient is constant [61, 62] and give a short and intuitive discussion for the more interesting viscoelastic case, [10, 9]. One can deduce from (4.18) and momentum conservation that the collision laws are given by the formulas

$$\begin{aligned} v' &= v - \frac{1+e}{2}\big(u\cdot\omega\big)\omega \\ v'_* &= v_* + \frac{1+e}{2}\big(u\cdot\omega\big)\omega\,. \end{aligned} \tag{4.19}$$

In particular, the rate of kinetic energy dissipation is

$$|v'|^2 + |v'_*|^2 - |v|^2 - |v'|^2 = -\frac{1-e^2}{4}\big(u\cdot\omega\big)^2 \leq 0\,. \tag{4.20}$$

Note then, that $e = 1$ reduces to the elastic Boltzmann equation and $e = 0$ corresponds to sticky particles, that is, after collision particles travel together. Contrary to the elastic case, the collision law (4.19) does not have unitary Jacobian. In fact, after elementary calculations, one can prove that

$$\mathrm{d}v_*\mathrm{d}v = e\,\mathrm{d}v'_*\mathrm{d}v'\,. \tag{4.21}$$

Formula (4.21) is valid only for a constant restitutions coefficient, we refer to [4] for a more general situation that includes viscoelastic particles. Thus, the collision operator writes

$$Q_{in}(f,g) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\Big(\frac{f('v)g('v_*)}{e^2} - f(v)g(v_*)\Big)|u\cdot\omega|\mathrm{d}\omega\mathrm{d}v_*\,. \tag{4.22}$$

We have chosen to work in the notes with the relevant case of hard spheres interactions (recall that in granular material interactions happen through physical contact of the particles) in 3-D. Also, the form (4.22) has to be the correct one by conservation of mass and momentum. Indeed, the weak formulation follows readily as

$$\int_{\mathbb{R}^3} Q_{in}(f,g)(v)\psi(v)\mathrm{d}v =$$
$$\tfrac{1}{2}\int_{\mathbb{R}^3} f(v)g(v_*)\Big(\int_{\mathbb{S}^2} |u\cdot\omega|\big(\varphi(v')+\varphi(v'_*)-\varphi(v)-\varphi(v_*)\big)\mathrm{d}\omega\Big)\mathrm{d}v_*\mathrm{d}v\,. \tag{4.23}$$

Although, the form of the weak formulations for the elastic and inelastic interactions looks the same, they are not due to the collision laws defining the processes. In particular taking $\varphi\equiv 1$, $\varphi(v)=v$ and $\varphi(v)=|v|^2$ successively in (4.23) and using (4.20), one obtains conservation of mass and momentum and dissipation of energy

$$\begin{aligned}\int_{\mathbb{R}^3} Q_{in}(f,g)(v)\mathrm{d}v &= \int_{\mathbb{R}^3} Q_{in}(f,g)(v)\, v\,\mathrm{d}v = 0\,,\\ \int_{\mathbb{R}^3} Q_{in}(f,g)(v)|v|^2\mathrm{d}v &= -c_o\frac{1-e^2}{8}\int_{\mathbb{R}^3} f(v)g(v_*)|u|^3\mathrm{d}v_*\mathrm{d}v\,,\end{aligned} \tag{4.24}$$

with $c_o := \int_{\mathbb{S}^2} |\mathrm{e}_1\cdot\omega|^3\mathrm{d}\omega > 0$. A central observation is made when one tries to find a dissipation of entropy for Q_{in} by choosing $\varphi(v) = \log\big(f(v)\big)$,

$$\begin{aligned}\int_{\mathbb{R}^3} Q_{in}(f,f)(v)\log\big(f(v)\big)\mathrm{d}v &= -\mathcal{D}_{in}(f)\\ &\quad + c_1\frac{1-e^2}{e^2}\int_{\mathbb{R}^3} f(v)f(v_*)|u|\mathrm{d}v_*\mathrm{d}v\,,\end{aligned} \tag{4.25}$$

where $c_1 := \int_{\mathbb{S}^2} |\mathrm{e}_1\cdot\omega|\mathrm{d}\omega > 0$ and

$$\mathcal{D}_{in}(f) := \int_{\mathbb{R}^3} f(v)g(v_*)\Big(\int_{\mathbb{S}^2} |u\cdot\omega|\Psi\Big(\frac{f(v')f(v'_*)}{f(v)f(v_*)}\Big)\mathrm{d}\omega\Big)\mathrm{d}v_*\mathrm{d}v\,, \tag{4.26}$$

with $\Psi(x) = x - \log(x) - 1 \geq 0$. In other words, as long as the restitution coefficient $e\in[0,1)$, the dissipation of entropy does not have a sign and therefore the Boltzmann equation for granular materials

$$\partial_t f + v\cdot\nabla_x f = Q_{in}(f,f)\,,\quad (t,x,v)\in\mathbb{R}^+\times\mathbb{R}^6\,, \tag{4.27}$$

does not have *a priori* bounded entropy. This is a major set back for the Cauchy problem as we recall that DiPerna & Lions theory was based on the compactness brought by the entropy estimate. Moreover, near equilibria theories which rely heavily in linearization are difficult to apply as Gaussian distributions are not stationary solutions of equation (4.27). There are at least two instances in which problem (4.27) can be solved globally, one is in the so-called weakly inelastic regime, see [68], which reduces to assume $e \approx 1$. This regime is quite important as many important systems fall in this category, in addition, the validity of model (4.27) should hold precisely in this regime. Thus, the weakly inelastic regime is a perturbation of the elastic Boltzmann model and many of the machinery created for it can be applied. A second example is the near vacuum case, see [4]. Interestingly, the near vacuum case can be solved by the classical iteration of Kaniel & Shinbrot for general restitution coefficients using a Gaussian upper barrier. The reader can easily verify that all steps given in section 3.1 are valid for the inelastic model (4.27), in particular, the important Lemma 3.1.2. Apart from this, the in-homogenous granular Boltzmann equation continues being a big mystery, refer to [72] for additional comments and references.

In the sequel, we consider only the homogenous granular Boltzmann model, that is, the spatial dependence will be neglected in the model. In contrast to elastic systems, inelastic systems can develop strong spatial inhomogeneity even departing from homogenous configurations. Still, the homogeneous problem is quite relevant from both physical and mathematical point of view. It is an accurate approximation of weakly inelastic systems and thermalized systems presenting diffusive effects. Furthermore, for a vico-elastic system, numerical simulations and theoretical results [28, 10] show that the system tends to homogenize as it cools down. As a consequence, the homogeneous equation is a valid model in the long time asymptotic. Rigorous results for the homogeneous granular Boltzmann equation can be found for instance in [19, 20, 22, 39, 61, 62, 59, 8, 9, 10]. In these references the reader will find a mathematical analysis of phenomena happening exclusively of dissipative Boltzmann such as overpopulated tails and optimal cooling rate as well as the use of contracting distances and entropy in the inelastic context.

4.3 Rescaled problem and self-similar profile

Let us start by fixing an initial datum f_o having unitary mass, zero momentum and finite energy. Since mass and momentum are conserved for granular Boltzmann, it follows that for all $t \geq 0$

$$\int_{\mathbb{R}^3} f(t,v)\mathrm{d}v = 1\,, \quad \int_{\mathbb{R}^3} f(t,v)\,v\,\mathrm{d}v = 0\,.$$

As a consequence of Jensen's inequality $\int_{\mathbb{R}^3} f(t,v_*)|u|^3\mathrm{d}v_* \geq |v|^3$. Plugging this estimate into the dissipation of energy given in (4.24) it follows that

$$\frac{d}{dt}m_2(f) \leq -c_o\frac{1-e^2}{8}m_2(f)^{\frac{3}{2}} =: -\tilde{c}_o\, m_2(f)^{\frac{3}{2}}\,.$$

Simple integration leads to

$$m_2(f)(t) \leq \frac{4\,m_2(f_o)}{\big(2+\tilde{c}_o\sqrt{m(f_o)}\,t\big)^2}\,, \quad t \geq 0\,. \tag{4.28}$$

This proves that $f(t) \to \delta_o$ as time increases. Note, that such cooling effect is at odds with the in-homogenous granular Boltzmann model for near vacuum where Kaniel & Shinbrot iteration shows that the solution remains controlled by a Gaussian (in space and velocity!). Now, we want to understand better this cooling down process. For this matter, it is introduced a *closely* related problem where the stationary state is not a singular measure but rather a regular function. The key idea is to stop the continuous loss of energy of the dissipative problem using scaling: the scaling method can be interpreted as a "zooming in" process in the velocity variable at a very precise rate, so that, the rescaled function has a stationary state which is a function (no concentration of mass at zero velocity). Indeed, define the function g with the scaling formula

$$f(t,v) =: (1+t)^3\, g\big(\ln(1+t), (1+t)v\big)\,, \tag{4.29}$$

then, it is not difficult to prove that $g(s,\xi)$ solves

$$\partial_s g + \nabla_\xi \cdot (\xi\, g) = Q_{in}(g,g)\,, \quad (s,\xi) \in \mathbb{R}^+ \times \mathbb{R}^3\,, \tag{4.30}$$

where $s = \ln(1 + t)$ and $\xi = (1 + t)v$. Note that effectively the variable ξ is a "zoom in" version of the physical variable v. Trivially, the scaled function g has unitary mass and zero momentum for all time $s \geq 0$. In fact, the energy of g is simply estimated by the differential inequality

$$\frac{d}{ds} m_2(g) \leq -c_o \frac{1-e^2}{8} m_2(g)^{\frac{3}{2}} + 2\, m_2(g) \,.$$

Thus,

$$\sup_{s\geq 0} m_2(g)(s) \leq \max \left\{ m_2(f_o), \Big(\frac{16}{c_o(1-e^2)} \Big)^2 \right\} . \tag{4.31}$$

Given that the second moment of the rescaled problem remains finite, the machinery developed for the elastic Boltzmann equation can be proven to work also for the inelastic interaction operator Q_{in}, see [62].

Theorem 4.3.1. *Let g be a solution to the rescaled problem* (4.30) *with nonnegative initial datum f_o having unitary mass, zero momentum and finite energy. Then,*

$$\sup_{s\geq 0} m_k(g)(s) \leq \max \big\{ m_k(f_o), C_k \big\} \,, \quad k \geq 2 \,,$$

where the constant C_k depends only on the mass and energy of f_o and e. Moreover, if in addition $f_o \in L^p(\mathbb{R}^3)$ it follows that

$$\sup_{s\geq 0} \|g(s)\|_p \leq \max \big\{ \|f_o\|_p, \tilde{C}_p \big\} \,, \quad p \geq 1 \,,$$

where C_p depends only on the mass and energy of f_o and e.

Proof. With the exception of one, all the results given for $Q(f, f)$ in section 3.3 and 3.4 can be easily extended to $Q_{in}(f, f)$ in the case e constant. For more general restitution coefficient is technically more challenging, see [8]. The exception is the lower bound of Lemma 3.4.3 which is based on propagation of energy. There is not conservation of energy for g, thus, the best we can say at the moment is (using Jensen's inequality)

$$\big(g * |\cdot|\big)(\xi) \geq |\xi| \,.$$

At this point is where the term $\nabla_\xi \cdot (\xi\, g)$ comes to the rescue. Recall that this term is present due to the scaling. Thus, after integration by parts, it follows for any $p \geq 1$

$$\int_{\mathbb{R}^3} \nabla_\xi \cdot (\xi\, g) g^{p-1} \mathrm{d}\xi = \frac{3}{p'} \|g\|_p^p\,, \tag{4.32}$$

which translates in the estimate

$$\int_{\mathbb{R}^3} \Big(Q^-(g,g) + \nabla \cdot (\xi\, g) \Big) g^{p-1} \mathrm{d}\xi \geq c_p \|g\|^p_{p,1/p}\,.$$

This observation leads to the propagation of the L^p-norm for g but not for f. Indeed, the physical solution f cannot propagate uniformly the L^p-norm for $p > 1$ since it is converging to δ_o as time increases. □

The uniform propagation of the L^p-norm for g obtained in Theorem 4.3.1 readily implies that g cannot converge (if converges) to a singular measure. In particular, it cannot converge to δ_o. Furthermore, it also implies that

$$\inf_{s\geq 0} m_2(g)(s) \geq c\,,$$

for some constant $c > 0$, depending only on $\sup_{s \geq 0} \|g(s)\|_p$, which together with estimate (4.31) give us

$$c \leq m_2(g)(s) \leq C\,.$$

This translates into the optimal rate of cooling for the physical solution f. This rate is known as Haff's law

$$\frac{c}{(1+t)^2} \leq m_2(f)(t) \leq \frac{C}{(1+t)^2}\,.$$

This discussion lead us to believe that g must be converging, in some sense, to a L^1-solution of the stationary problem

$$\nabla_\xi \cdot (\xi\, G) = Q_{in}(G,G)\,, \quad \xi \in \mathbb{R}^3\,. \tag{4.33}$$

Such a stationary solution is called self-similar profile because the fact that $g \approx G$ for large time, translates into

$$f(t,v) \approx (1+t)^3 G\big((1+t)v\big)$$

in the physical solution, that is, there exists a profile G attracting f. Unfortunately, it is quite hard to prove the convergence of the rescale function g to the self-similar profile G even without explicit rate of convergence. Furthermore, uniqueness of solutions of equation (4.33) is not always guarantee. There are some instances where this can be done relatively easy such as in the example that we discuss in the next section, see also [10].

4.4 1-D inelastic Boltzmann

A good introductory example that helps understanding the tools used in the analysis of converge of dissipative systems is dissipative Maxwell particles interacting in the real line. In such a case, the interaction law (4.19) reduces to

$$v' = \frac{1-e}{2}v + \frac{1+e}{2}v_*\,, \quad v'_* = \frac{1+e}{2}v + \frac{1-e}{2}v_*\,, \tag{4.34}$$

and thus, the Boltzmann model reduces to

$$\begin{aligned}\partial_t f(v) &= Q_{in}(f,f)(v)\\ &= \int_{\mathbb{R}} \big(f({}'v)f({}'v_*) - f(v)f(v_*)\big)\mathrm{d}v_*\,, \quad v \in \mathbb{R}\,.\end{aligned} \tag{4.35}$$

The weak formulation follows directly from here as

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} f(v)\varphi(v)\mathrm{d}v =\\ &\tfrac{1}{2}\int_{\mathbb{R}^2} f(v)f(v_*)\big(\varphi(v') + \varphi(v'_*) - \varphi(v) - \varphi(v_*)\big)\mathrm{d}v_*\mathrm{d}v\,,\end{aligned} \tag{4.36}$$

valid for any suitable φ. The weak formulation shows that two conservation laws hold: conservation of mass $m_o(f)(t) = m_o(f_o)$ and conservation of momentum

$$m(f)(t) := \int_{\mathbb{R}} f(t,v)v\mathrm{d}v = m(f_o)\,, \quad t \geq 0\,.$$

It also shows the explicit dynamics of the global energy of the particles

$$m_2(f)(t) = \int_{\mathbb{R}} f(t,v)|v|^2 \mathrm{d}v = m_2(f_o)e^{-\frac{1-e^2}{2}t}\,. \tag{4.37}$$

Thus, rescaling the velocity variable as

$$g(t,\xi) := \sqrt{m_2(f)(t)}\, f\big(t, \sqrt{m_2(f)(t)}\,\xi\big)\,, \tag{4.38}$$

one obtains the rescaled equation as

$$\partial_t g(\xi) + \frac{1-e^2}{4}\partial_\xi\big(\xi\, g\big)(\xi) = Q_{in}(g,g)(\xi)\,. \tag{4.39}$$

Note that the rescaling (4.38) is exactly the one used in (4.29). It is given now in terms of the dissipation rate of the energy. This is quite intuitive as the "zooming in" process should be related with the exact rate of the dissipation of energy occurring in the system. Furthermore, scaling (4.38) is unique in the sense that give us conservation of *energy* for $g(t,v)$, that is, equation (4.39) conserves energy in addition to mass and momentum. Thus, philosophically speaking one uses rescaling to gain a conservation law in the dynamics. Observe, however, that such rescaling is well defined if the dynamics of the global energy $m_2(f)$ is known *a priori*. More complicated models such as hard spheres do not render such explicit forms. This issue complicate matters in non Maxwellian interactions. Now, taking advantage that we are in the Maxwell case, it is possible to apply Fourier transform in the scaled velocity variable ξ in equation (4.39) and obtain, using the shorthand $\hat{g} := \mathcal{F}\{g\}$, that

$$\partial_t \hat{g}(k) - \frac{1-e^2}{4}\, k\, \partial_k \hat{g}(k) = \hat{g}\big(\tfrac{1-e}{2}k\big)\hat{g}\big(\tfrac{1+e}{2}k\big) - \hat{g}(k)\hat{g}(0)\,. \tag{4.40}$$

Thus, in the Maxwell case it is possible to find a close form for the equation solved by the Fourier transform of the physical solution and, in addition, such equation is simpler than the original. The associated steady state problem to (4.40) is given by

$$-\frac{1-e^2}{4}\, k\, \partial_k \hat{g}_\infty(k) = \hat{g}_\infty\big(\tfrac{1-e}{2}k\big)\hat{g}_\infty\big(\tfrac{1+e}{2}k\big) - \hat{g}_\infty(k)\hat{g}_\infty(0)\,. \tag{4.41}$$

Interestingly, this problem has explicit solution that can be found by Taylor expansion, we refer to [20] for details. Indeed, normalizing g_∞ to have unitary mass and zero momentum it follows that

$$\hat{g}_\infty(k) = \big(1+|k|\big)\, e^{-|k|} \longleftrightarrow g_\infty(\xi) = \frac{2/\pi}{(1+\xi^2)^2}\,. \tag{4.42}$$

Thus, the problem of existence of stationary solutions is a relatively simple matter for 1-D homogenous Maxwell particles. Let us address now the convergence of the time dependent problem towards the stationary problem by following the discussion given in [31]. One of the main tools helping us here will be a Fourier-based metric d_s defined on suitable subspaces of $L^1(\mathbb{R})$

$$d_s(f,g) := \sup_{k\in\mathbb{R}} \frac{\big|\hat{f}(k)-\hat{g}(k)\big|}{|k|^s}\,, \quad s\geq 0\,. \tag{4.43}$$

That the Fourier-based metric is well defined depends on the modulus of continuity at the origin of the Fourier transform of a given function. This is closely related to the moments such function has.

Definition 4.4.1. *Fix $N\in\mathbb{N}$. We say that two functions $f,g\in L^1_N(\mathbb{R})$ have equal moments up to N if*

$$\int_{\mathbb{R}} \big(f(\xi)-g(\xi)\big)\xi^n\, d\xi = 0\,, \quad 0\leq n\leq N\,.$$

Proposition 4.4.2. *[31, Proposition 2.6] Consider two functions $f,g\in L^1_s(\mathbb{R})$ with $s>0$ and equal moments up to N where*

$$N = \begin{cases} s-1 & \textit{if } s\in\mathbb{N}\,,\\ [s] & \textit{otherwise}\,. \end{cases}$$

Then, $d_s(f,g)<\infty$.

Proposition 4.4.3. *Given functions $f_n, g_n\in L^1_s(\mathbb{R})$, with $n=1,2$ and $s>0$, having equal moments up to N (as defined in Proposition 4.4.2). Then, d_s enjoys the following properties:*
(1) Scaling

$$d_s\big(\alpha f_1(\alpha\,\cdot), \alpha g_1(\alpha\,\cdot)\big) = \alpha^{-s} d_s(f_1,g_1)\,, \quad \alpha>0\,. \tag{4.44}$$

(2) Convexity

$$d_s\big(\alpha f_1 + (1-\alpha) f_2, \alpha g_1 + (1-\alpha) g_2\big) \\ \leq \alpha d_s(f_1, g_1) + (1-\alpha) d_s(f_2, g_2)\,, \quad \alpha \in [0,1]\,. \tag{4.45}$$

(3) Super additivity with respect to convolution

$$d_s\big(f_1 * f_2,\, g_1 * g_2\big) \leq \|f_2\|_1\, d_s(f_1, g_1) + \|g_1\|_1\, d_s(f_2, g_2)\,. \tag{4.46}$$

Proof. This is an easy exercise. For example for item (3)

$$\frac{\big|\mathcal{F}\{f_1 * f_2\} - \mathcal{F}\{g_1 * g_2\}\big|}{|k|^s} = \frac{\big|\hat{f}_1 \hat{f}_2 - \hat{g}_1 \hat{g}_2\big|}{|k|^s} \\ \leq \frac{\big|(\hat{f}_1 - \hat{g}_1)\hat{f}_2\big|}{|k|^s} + \frac{\big|\hat{g}_1(\hat{f}_2 - \hat{g}_2)\big|}{|k|^s}\,.$$

The result follows since $\|\hat{f}\|_\infty \leq \|f\|_1$. □

Let us use the Fourier-based metric to prove stability of the 1-D inelastic Boltzmann with Maxwell interactions by assuming that for any $T < \infty$ there exist solutions f_1 and f_2 in $\mathcal{C}\big([0,T); L^1(\mathbb{R})\big)$ of (4.35) associated to nonnegative initial data f_o^1 and f_o^2 respectively. The initial data is assumed with unitary mass and zero momentum for simplicity. Then, Fourier transform implies that

$$\partial_t \hat{f}_n(k) = \hat{f}_n\big(\tfrac{1-e}{2}k\big)\hat{f}_n\big(\tfrac{1+e}{2}k\big) - \hat{f}_n(k)\,, \quad n = 1, 2\,.$$

Subtracting these equations and integrating in time

$$\hat{f}_1(k) - \hat{f}_2(k) = \big(\hat{f}_o^1(k) - \hat{f}_o^2(k)\big)e^{-t} \\ + \int_0^t e^{-(t-t')} F(\hat{f}_1, \hat{f}_2)(t', k)\mathrm{d}t'\,, \tag{4.47}$$

where

$$F(\hat{f}_1, \hat{f}_2)(t,k) := \hat{f}_1\big(\tfrac{1-e}{2}k\big)\hat{f}_1\big(\tfrac{1+e}{2}k\big) - \hat{f}_2\big(\tfrac{1-e}{2}k\big)\hat{f}_2\big(\tfrac{1+e}{2}k\big) \\ = \Big(\hat{f}_1\big(\tfrac{1-e}{2}k\big) - \hat{f}_2\big(\tfrac{1-e}{2}k\big)\Big)\hat{f}_1\big(\tfrac{1+e}{2}k\big) \\ + \hat{f}_2\big(\tfrac{1-e}{2}k\big)\Big(\hat{f}_1\big(\tfrac{1+e}{2}k\big) - \hat{f}_2\big(\tfrac{1+e}{2}k\big)\Big)\,.$$

Therefore, for any $s > 0$

$$\left| \frac{F(\hat{f}_1, \hat{f}_2)(t,k)}{|k|^s} \right| \leq \Big(\big(\tfrac{1-e}{2}\big)^s + \big(\tfrac{1+e}{2}\big)^s \Big) d_s(f_1, f_2) =: \kappa_{e,s}\, d_s(f_1, f_2)\,,$$

where we used conservation of mass to estimate $\|\hat{f}_n\|_\infty \leq \|f_o\|_1 = 1$ for $n = 1, 2$. As a consequence, estimate (4.47) implies that

$$d_s(f_1, f_2)(t) \leq d_s(f_o^1, f_o^2)e^{-t} + \kappa_{e,s} \int_0^t e^{-(t-t')} d_s(f_1, f_2)(t')\mathrm{d}t'\,.$$

Hence, Gronwall's lemma readily gives that

$$d_s(f_1, f_2)(t) \leq d_s(f_o^1, f_o^2)e^{-(1-k_{e,s})t}\,. \tag{4.48}$$

Thus, the dynamics of the inelastic Boltzmann for Maxwell interactions is contractive in any Fourier based norm with $s > 1$ since clearly $\kappa_{e,s} < 1$ for any $e \in [0, 1)$. Such restriction in s is natural since solutions f_1 and f_2 at least should share same mass and momentum, that is, the conserved quantities. Observe that (4.48) can be proved by working directly in the velocity space. This is the case because the collision operator can be written as a convolution of rescaled functions

$$Q_{in}(f, f) = \Big(\tfrac{2}{1-e} f\big(\tfrac{2}{1-e}\ \cdot\big)\Big) * \Big(\tfrac{2}{1+e} f\big(\tfrac{2}{1+e}\ \cdot\big)\Big) - f\,.$$

Thus, the result follows using the properties of d_s given in Proposition (4.4.3). In fact, estimate (4.48) and the scaling property of d_s give for the rescaled solutions g_1 and g_2

$$d_s(g_1,\, g_2)(t) \leq d_s(f_o^1, f_o^2)e^{-(1-k_{e,s})t}\theta(t)^{-s/2} = d_s(f_o^1, f_o^2)e^{-C_{e,s}t}\,, \tag{4.49}$$

provided the initial data share the same initial energy $m_2(f_o^1) = m_2(f_o^2) = 1$, so that, $m_2(f_1)(t) = m_2(f_2)(t) =: \theta(t)$ for all times t (recall equation (4.37)). The rate of contraction is given by

$$C_{e,s} = p^s + q^s - 1 - \frac{s}{2}\big(p^2 + q^2 - 1\big)\,, \quad p := \tfrac{1-e}{2}\,, \ \ q := \tfrac{1+e}{2}\,, \tag{4.50}$$

which can be shown to be positive, for any $e \in [0, 1)$, in the range $s \in (2, 3)$. Thus, recalling that equation (4.39) also conserves energy, one naturally must have two solutions g_1 and g_2 sharing same mass, momentum and energy to obtain a contractive dynamics between them.

Theorem 4.4.4. *Let g be a solution of the rescaled problem* (4.39) *with initial data f_o having unitary mass and energy, and zero momentum. Then, for any $e \in [0,1)$*

$$d_s(g,\, g_\infty)(t) \leq d_s(f_o, g_\infty)e^{-C_{e,s}t}\,, \quad s \in (2,3)\,,$$

where g_∞ and $C_{e,s} > 0$ are given by (4.42) *and* (4.50) *respectively.*

Proof. Simply take $g_1 \equiv g$ and $g_2 \equiv g_\infty$ in (4.49). Note that this is allowed since g_∞ is solution of the stationary problem, thus, it is solution of the time dependent problem with initial datum g_∞ which has unitary mass and energy, and zero momentum. □

As a corollary, Theorem 4.4.4 implies uniqueness for the stationary problem (4.41). Of course, a uniqueness result for such problem must be stated up to functions having a given mass, momentum and energy.

Corollary 4.4.5. *The function g_∞ given in* (4.42) *is the unique solution of the stationary problem* (4.41) *in the class of functions in $L^1(\mathbb{R})$ having unitary mass and energy, and zero momentum.*

Proof. Given any other solution $\tilde{g}_\infty$ having the desired properties one can take $g \equiv \tilde{g}_\infty$ in Theorem 4.4.4. Thus, for any fixed $s \in (2,3)$

$$0 \leq d_s(g_\infty, \tilde{g}_\infty) \leq d_s(g_\infty, \tilde{g}_\infty)e^{-C_{e,s}t}\,, \quad t \geq 0\,.$$

Since $C_{e,s} > 0$, it must be the case that $d_s(g_\infty, \tilde{g}_\infty) = 0$. □

Chapter 5

Radiative transfer equation

In the last section section of this note we bring an example of highly diffusive phenomena happening in kinetic equations. Diffusion commonly happens in particle systems due to interactions and, there are specific instances when one can see the action of the diffusion phenomena playing a central role in a specific proof. For example, a classical result in the homogeneous Boltzmann equation is: given any initial state in $L^1_2 \cap L^2$, then, the solution of the homogeneous Boltzmann equation has an uniform lower Gaussian barrier depending only on the mass, energy, and the L^2-norm of the initial state for any positive time. In other words, there is an infinite speed of propagation for the homogeneous Boltzmann equation (in the velocity variable), we refer to [66] for a proof of this fact. Now, if the interactions are sufficiently strong, it is reasonable to expect that stronger diffusive properties, such as instantaneous regularization, will appear in the model. This is the case for example in radiative transport in the forward peaked regime, see [13] and references therein.

Radiative transfer is the physical phenomenon of energy transfer/propagation in the form of electromagnetic radiation. The propagation of radiation through an inhomogeneous medium is described by absorption, emission, and scattering processes. In the case that the medium is free of absorption and emission, the radiative transfer equation (RTE) in free space reduces to

$$\partial_t u + \theta \cdot \nabla_x u = \mathcal{I}(u)\,, \quad (t,x,\theta) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{S}^{d-1}\,. \tag{5.1}$$

The equation is complemented with an initial data $u_o \geq 0$. The non negative solution $u := u(t,x,\theta)$ to equation (5.1) is understood as the radiation distribution at time $t > 0$, spatial location $x \in \mathbb{R}^d$, and propagation direction $\theta \in \mathbb{S}^{d-1}$. As usual, the operator $\mathcal{I}$ is the interaction operator which in the radiative transfer literature is called scattering operator. More explicitly,

$$\mathcal{I}(u) := \mathcal{I}_{b_s}(u) = \int_{\mathbb{S}^{d-1}} \big(u(\theta') - u(\theta)\big) b_s(\theta \cdot \theta')\mathrm{d}\theta'. \tag{5.2}$$

The function b_s is called scattering kernel (or phase function) and has the purpose of describing the scattering pattern of the waves in a specific medium. It is commonly assumed that b_s satisfies the normalized integrability condition

$$1 = \int_{\mathbb{S}^{d-1}} b_s(\theta \cdot \theta')\mathrm{d}\theta'\,. \tag{5.3}$$

Assumption (5.3) is satisfactory in diverse physical situations and many of the mathematical theory was built around it, we refer to [37, Chapter XXI]. However, it is common to find a propagation regime where the scattering pattern is highly peaked in the direction of propagation, the so-called *highly forward-peaked regime.* Such regime can be found in neutron transport, atmospheric radiative transfer and optical imaging of animal tissue among others. This peak in the scattering pattern is modeled as a quasi-singularity in the scattering kernel b_s happening at $\theta \cdot \theta' = 1$ which is barely integrable. As a consequence, for practical purposes condition (5.3) is not correct. Indeed, in the scattering physics literature it is customary to use the Henyey-Greenstein angular scattering kernel, introduced in [50], which in $3D$ reads

$$b^g_{HG}(\theta \cdot \theta') = \frac{1 - g^2}{\big(1 + g^2 - 2\,g\,\theta \cdot \theta'\big)^{\frac{3}{2}}}\,.$$

The anisotropic factor $g \in (0,1)$ measures the strength of forward-peakedness of the scattering kernel. As an example, typical values for this factor in animal tissues are in the range $0.9 \leq g \leq 0.99$. At this point, the reader may argue that as long as $g < 1$ the Henyey-Greenstein angular scattering kernel is integrable in $\mathbb{S}^2$, thus, it falls in the category given by assumption (5.3). This is correct from the mathematical point of view, yet, unsatisfactory from the modeling point of view. The reason is that waves are usually traveling long distances (measured in wavelengths) through the media, as a consequence, the peaked scattering will have a sizable cumulative effect on the radiation profile. To see this, let u_{HG} be the solution of the RTE with Henyey-Greenstein kernel and introduce the rescaled radiation profile u^g as

$$u^g(t,x,\theta) := \tfrac{1}{(1-g)^d}\, u_{HG}\big(\tfrac{t}{1-g}, \tfrac{x}{1-g}, \theta\big)\,,$$

where the time-space variables (t,x) are order one quantities. This rescaling is natural in order to observe the large spatial-time radiation profile (of the original problem) so that the highly forward-peaked scattering has a visible effect. A simple computation shows that u^g solves the radiative transfer equation (5.1) with new scattering kernel given by

$$b^g_s(\theta\cdot\theta') := \frac{b^g_{HG}(\theta\cdot\theta')}{1-g} = \frac{1+g}{\big(1+g^2-2\,g\,\theta\cdot\theta'\big)^{\frac{3}{2}}}\,,$$

and new initial condition $u^g_o := u_{HG}(0)$. Since

$$\lim_{g\to 1} b^g_s(\theta\cdot\theta') = \frac{1}{\sqrt{2}(1-\theta\cdot\theta')^{\frac{3}{2}}} =: b_{HG}(\theta\cdot\theta')\,,$$

it is reasonable to expect that the profile u^g is converging, in the limit $g \to 1$, to a solution of the equation (5.1) having scattering kernel b_{HG} which, in particular, does not satisfies assumption (5.3).

In the remainder of the discussion we consider scattering kernels of the form

$$b_s(\theta\cdot\theta') = \frac{D}{\big(1-\theta\cdot\theta'\big)^{\frac{d-1}{2}+s}} + h(\theta\cdot\theta'), \quad s \in \big(0, \min\{1, \tfrac{d-1}{2}\}\big)\,, \tag{5.4}$$

where $D > 0$ will be the scattering diffusion coefficient and $h \in L^1(\mathbb{S}^{d-1})$. This decomposition is commonly used to separate the forward-peaked scattering from others such as the Rayleigh scattering (modeled by h). Observe that the operator $\mathcal{I}$ in (5.2) is not well defined unless the radiation distribution u has sufficient regularity. A sufficient condition is u being twice continuously differentiable in the variable θ. This motivate the introduction of the weak formulation for the scattering operator $\mathcal{I}$: for any sufficiently regular functions u and φ,

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} \mathcal{I}(u)(\theta)\,\varphi(\theta)\,\mathrm{d}\theta :=&\\ -\tfrac{1}{2}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}} \big(u(\theta')-u(\theta)\big)\big(\varphi(\theta')-\varphi(\theta)\big)\,b_s(\theta,\theta')\mathrm{d}\theta'\mathrm{d}\theta& \qquad (5.5)\\ = \lim_{\epsilon\to 0}\int_{\mathbb{S}^{d-1}} u(\theta)\int_{\{1-\theta\cdot\theta'\geq\epsilon\}} \big(\varphi(\theta')-\varphi(\theta)\big)\,b_s(\theta,\theta')\mathrm{d}\theta'\mathrm{d}\theta\,.&\end{aligned}$$

The presentation that follows is borrowed from [13] where a detailed exposition of the techniques and a good list references can be found.

5.1 Scattering operator as a fractional diffusion

Interestingly we start this discussion with the stereographic projection $\mathcal{S} : \mathbb{S}^{d-1} \to \mathbb{R}^{d-1}$. Using subscripts to denote the coordinates of a vector, we write the stereographic projection as

$$\mathcal{S}(\theta)_i = \frac{\theta_i}{1-\theta_d}\,, \qquad 1 \leq i \leq d-1\,.$$

The stereographic projection is surjective and smooth (except in the north pole) with its inverse $\mathcal{J} : \mathbb{R}^{d-1} \to \mathbb{S}^{d-1}$ given by

$$\mathcal{J}(v)_i = 2\frac{v_i}{\langle v\rangle^2}\,, \quad 1 \leq i \leq d-1\,, \quad \text{and} \quad \mathcal{J}(v)_d = \frac{|v|^2-1}{\langle v\rangle^2}\,,$$

where $\langle v\rangle := \sqrt{1+|v|^2}$. The Jacobian of such transformations can be computed respectively as

$$\mathrm{d}v = \frac{\mathrm{d}\theta}{(1-\theta_d)^{d-1}}\,, \qquad \text{and} \qquad \mathrm{d}\theta = \frac{2^{d-1}\,\mathrm{d}v}{\langle v\rangle^{2(d-1)}}\,.$$

Using the notation $\theta = \mathcal{J}(v)$ and $\theta' = \mathcal{J}(v')$, it follows that

$$1-\theta\cdot\theta' = 2\,\frac{|v-v'|^2}{\langle v\rangle^2\langle v'\rangle^2}\,. \tag{5.6}$$

These simple facts about the stereographic projection are enough to prove the following result about the scattering operator in the peaked regime and its diffusive nature in the *angular* variable.

Proposition 5.1.1. *Let b_s be a scattering kernel satisfying* (5.4) *and write the scattering operator as $\mathcal{I}_{b_s} = \mathcal{I}_D + \mathcal{I}_h$. Then, for any sufficiently regular function u in the sphere the stereographic projection of the operator $\mathcal{I}_D$ is given by*

$$\begin{aligned}\frac{\big[\mathcal{I}_D(u)\big]_{\mathcal{J}}}{\langle\cdot\rangle^{d-1}} &= D\,\frac{2^{\frac{d-1}{2}-s}}{c_{d-1,s}}\langle v\rangle^{2s}\Big(-(-\Delta_v)^s w_{\mathcal{J}} + u_{\mathcal{J}}\,(-\Delta_v)^s\frac{1}{\langle\cdot\rangle^{d-1-2s}}\Big)\\ &= D\,\frac{2^{\frac{d-1}{2}-s}}{c_{d-1,s}}\langle v\rangle^{2s}\Big(-(-\Delta_v)^s w_{\mathcal{J}} + c_{d,s}\,\frac{u_{\mathcal{J}}}{\langle v\rangle^{d-1+2s}}\Big)\,,\end{aligned} \tag{5.7}$$

where $u_{\mathcal{J}} = u\circ\mathcal{J}$ (the projected function) and $w_{\mathcal{J}} := \dfrac{u_{\mathcal{J}}}{\langle\cdot\rangle^{d-1-2s}}$. In particular, one has the formula

$$\begin{aligned}&\frac{1}{D}\int \mathcal{I}_D(u)(\theta)\,\overline{u(\theta)}\,d\theta = \\ &\qquad -\,c_{d,s}\,\big\|(-\Delta_v)^{s/2}w_{\mathcal{J}}\big\|^2_{L^2(\mathbb{R}^{d-1})} + C_{d,s}\,\|u\|^2_{L^2(\mathbb{S}^{d-1})}\,,\end{aligned} \tag{5.8}$$

for some explicit positive constants $c_{d,s}$ and $C_{d,s}$ depending on s and d. Furthermore, defining the differential operator $(-\Delta_\theta)^s$ acting on functions defined on the sphere by the formula

$$\big[(-\Delta_\theta)^s u\big]_{\mathcal{J}} := \langle\cdot\rangle^{d-1+2s}\,(-\Delta_v)^s w_{\mathcal{J}}\,, \tag{5.9}$$

the scattering operator simply writes as the sum of a singular and a L^2_θ-bounded parts

$$\mathcal{I}_{b_s} = -\, D_o\,(-\Delta_\theta)^s + c_{d,s}\,\mathbf{1} + \mathcal{I}_h\,, \tag{5.10}$$

where $D_o = \dfrac{2^{\frac{d-1}{2}-s}}{c_{d-1,s}} D$ *is the diffusion parameter.*

Before entering in the details of the proof recall the characterization of the fractional Laplacian $(-\Delta_v)^s$ using the Fourier transform

$$\mathcal{F}\big\{(-\Delta_v)^s\varphi\big\}(\xi) = |\xi|^{2s}\,\mathcal{F}\{\varphi\}(\xi)\,, \quad s\in(0,1)\,, \tag{5.11}$$

valid for any suitable function φ. This characterization is equivalent to the singular integral relation

$$(-\Delta_v)^s\varphi(v) = c_{d,s}\int_{\mathbb{R}^{d-1}}\frac{\varphi(v)-\varphi(v+z)}{|z|^{d-1+2s}}\,\mathrm{d}z\,, \tag{5.12}$$

where the constant is given by

$$\frac{1}{c_{d,s}} = \int_{\mathbb{R}^d}\frac{1-e^{-i\hat{\xi}\cdot z}}{|z|^{d+2s}}\,\mathrm{d}z > 0\,.$$

Thus, Proposition 5.1.1 essentially claims that the scattering operator in the forward-peaked regime acts as a fractional Laplacian in angle. The degree of the fractional diffusivity is completely determined by the non integrable singularity of the scattering kernel which in our case is measured by $s\in(0,1)$.

Proof. The decomposition $\mathcal{I} = \mathcal{I}_D + \mathcal{I}_h$ is assured by assumption (5.4) on b_s. Cauchy-Schwarz inequality shows that the operator $\mathcal{I}_h$ is a bounded operator in $L^2(\mathbb{S}^{d-1})$

$$\big\|\mathcal{I}_h(u)\big\|_{L^2(\mathbb{S}^{d-1})} \le 2\,\big\|h\big\|_{L^1(\mathbb{S}^{d-1})}\|u\|_{L^2(\mathbb{S}^{d-1})}\,. \tag{5.13}$$

Let us concentrate on the operator $\mathcal{I}_D$. Using the stereographic projection and (5.6)

$$\begin{aligned}\big[\mathcal{I}_D(u)\big]_{\mathcal{J}}(v) &= 2^{\frac{d-1}{2}-s}\,D\,\langle v\rangle^{d-1+2s}\int_{\mathbb{R}^{d-1}}\frac{u_{\mathcal{J}}(v')-u_{\mathcal{J}}(v)}{|v-v'|^{d-1+2s}}\frac{\mathrm{d}v'}{\langle v'\rangle^{d-1-2s}}\\
&= 2^{\frac{d-1}{2}-s}\,D\,\langle v\rangle^{d-1+2s}\Bigg(\int_{\mathbb{R}^{d-1}}\frac{w_{\mathcal{J}}(v')-w_{\mathcal{J}}(v)}{|v-v'|^{d-1+2s}}\mathrm{d}v'\\
&\qquad + u_{\mathcal{J}}(v)\int_{\mathbb{R}^{d-1}}\frac{\frac{1}{\langle v\rangle^{d-1-2s}}-\frac{1}{\langle v'\rangle^{d-1-2s}}}{|v-v'|^{d-1+2s}}\mathrm{d}v'\Bigg)\\
&= \frac{2^{\frac{d-1}{2}-s}\,D}{c_{d-1,s}}\,\langle v\rangle^{d-1+2s}\Big(-(-\Delta_v)^s w_{\mathcal{J}} + u_{\mathcal{J}}(-\Delta_v)^s\frac{1}{\langle\cdot\rangle^{d-1-2s}}\Big)\\
&= \frac{2^{\frac{d-1}{2}-s}\,D}{c_{d-1,s}}\langle v\rangle^{d-1+2s}\Big(-(-\Delta_v)^s w_{\mathcal{J}} + c_{d,s}\,\frac{u_{\mathcal{J}}}{\langle v\rangle^{d-1+2s}}\Big)\,.\end{aligned} \tag{5.14}$$

For the last inequality we have used the identity which can be proved through elementary Bessel potential theory

$$(-\Delta_v)^s\frac{1}{\langle\cdot\rangle^{d-1-2s}}(v) = \frac{c_{d,s}}{\langle v\rangle^{d-1+2s}}\,.$$

This proves (5.7) and as a direct consequence,

$$\begin{aligned}\int_{\mathbb{S}^{d-1}}\mathcal{I}_D(u)(\theta)\,\overline{u(\theta)}\,\mathrm{d}\theta &= 2^{d-1}\int_{\mathbb{S}^{d-1}}\Big[\mathcal{I}_D(u)\Big]_{\mathcal{J}}(v)\,\overline{u_{\mathcal{J}}(v)}\,\frac{\mathrm{d}v}{\langle v\rangle^{2(d-1)}}\\
&= 2^{\frac{3(d-1)}{2}-s}\,\frac{D}{c_{d-1,s}}\bigg[-\big\|(-\Delta_v)^{s/2}w_{\mathcal{J}}\big\|^2_{L^2(\mathbb{R}^{d-1})} + \frac{c_{d,s}}{2^{d-1}}\|u\|^2_{L^2(\mathbb{S}^{d-1})}\bigg],\end{aligned} \tag{5.15}$$

which completes the proof. □

Proposition 5.1.1 readily implies the following central *a priori* energy estimate

$$\begin{aligned}&\tfrac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\big|u(t,x,\theta)\big|^2\mathrm{d}\theta\mathrm{d}x + \tilde{D}_o\int_{t'}^{t}\int_{\mathbb{R}^d}\big\|(-\Delta_v)^{s/2}w_{\mathcal{J}}\big\|^2_{L^2(\mathbb{R}^{d-1})}\mathrm{d}x\mathrm{d}\tau\\
&\le \tfrac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\big|u(t',x,\theta)\big|^2\mathrm{d}\theta\mathrm{d}x + C\int_{t'}^{t}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\big|u(\tau,x,\theta)\big|^2\mathrm{d}\theta\mathrm{d}x\mathrm{d}\tau\,,\end{aligned} \tag{5.16}$$

valid for any $0 < t' \leq t < \infty$. Here the parameters $\tilde{D}_o$ and C depend on d, s and D with C additionally depending on $\|h\|_{L^1(\mathbb{S}^{d-1})}$.

5.2 Regularization mechanism of the RTE

Energy estimate (5.16) is essentially pointing out the diffusive nature of the radiative transport equation in the angular variable due to the cumulative effect of the forward-peaked scattering. Such estimate is enough to prove instantaneous fractional Sobolev regularization in the angular variable for solutions of the RTE having quite general initial data say in $L^1_{x,\theta}$. In fact, $L^1_{x,\theta}$ (as opposed to $L^2_{x,\theta}$) is a natural space for the initial configuration since mass is the only conserved quantity for the RTE model (5.1). Interestingly, estimate (5.16) says nothing about the spatial variable. In order to shown instantaneous regularization in space, one is forced to dig deeper into the equation, namely, to use the action of the transport operator $\theta \cdot \nabla_x$. We saw previously that the transport operator has some weak regularizing effects that can be manifested, for instance, in the form of an average lemma or some type of compactness result as in the theory of DiPerna & Lions. In our particular case, such weak regularization will be amplified because of the *a priori* regularity in the angular variable which is transported to the spatial variable.

Theorem 5.2.1. *Fix any dimension $d \geq 3$ and assume that $u \in \mathcal{C}\big([t_0, t_1), L^2(\mathbb{R}^d \times \mathbb{S}^{d-1})\big)$ solve the transport problem*

$$\partial_t u + \theta \cdot \nabla_x u = \mathcal{I}(u)\,, \qquad t \in [t_0, t_1)\,. \tag{5.17}$$

Then, for any $s \in (0,1)$ there exists a constant $C := C(d,s)$ independent of time such that

$$\begin{aligned} &\big\|(-\Delta_x)^{\frac{s_0}{2}} u\big\|_{L^2([t_0,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1})} \\ &\quad \leq C\Big(\big\|u(t_0)\big\|_{L^2(\mathbb{R}^d\times\mathbb{S}^{d-1})} + \big\|u\big\|_{L^2([t_0,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1})} \\ &\qquad + \big\|(-\Delta_v)^{s/2} w_{\mathcal{J}}\big\|_{L^2([t_0,t_1)\times\mathbb{R}^d\times\mathbb{R}^{d-1})}\Big)\,, \qquad s_0 := \tfrac{s/4}{2s+1}\,. \end{aligned} \tag{5.18}$$

Proof. A proof for this theorem can be found in [13] which follows a technique based heavily on average lemmas [23, 25]. □

Although the proof of Theorem 5.2.1 is a bit technical, it is based in the following simple observation (recall Proposition 3.2.1): Consider a function $f(x,\theta)\in L^2(\mathbb{R}^d\times\mathbb{S}^{d-1})$ with $d\geq 3$ and such that

$$\theta\cdot\nabla_x f\in L^2_{x,\theta}\,.$$

Consider now the average in angle of such function

$$\bar{f}_\varphi(x):=\int_{\mathbb{S}^{d-1}} f(x,\theta)\,\varphi(\theta)\,\mathrm{d}\theta\,,$$

with function φ bounded. Then,

$$\|\bar{f}_\varphi\|_{H^s_x}\leq C_{s,d}\|\varphi\|_\infty\Big(\|f\|_{L^2_{x,\theta}}+\|\theta\cdot\nabla_x f\|_{L^2_{x,\theta}}\Big),\quad s\in\big(0,\tfrac12\big)\,. \tag{5.19}$$

Indeed, define $g:=\theta\cdot\nabla_x f$ and compute its Fourier transform in the spatial variable

$$\mathcal{F}\{g(\cdot,\theta)\}(\xi)=i\theta\cdot\xi\,\mathcal{F}\{f(\cdot,\theta)\}(\xi)=i|\xi|\theta\cdot\hat{\xi}\,\mathcal{F}\{f(\cdot,\theta)\}(\xi)\,.$$

As a consequence,

$$\begin{aligned}\big|\mathcal{F}\{\bar{f}_\varphi\}(\xi)\big|^2\,|\xi|^{2s}&=\Big|\int_{\mathbb{S}^{d-1}}\mathcal{F}\{f(\cdot,\theta)\}(\xi)\,\varphi(\theta)\,\mathrm{d}\theta\Big|^2|\xi|^{2s}\\&=\Big|\int_{\mathbb{S}^{d-1}}\frac{\mathcal{F}\{g(\cdot,\theta)\}(\xi)}{|\xi|^{1-s}\theta\cdot\hat{\xi}}\,\varphi(\theta)\,\mathrm{d}\theta\Big|^2.\end{aligned} \tag{5.20}$$

But,

$$\left|\frac{\mathcal{F}\{g(\cdot,\theta)\}(\xi)}{|\xi|^{1-s}\,\theta\cdot\hat{\xi}}\right|=\frac{\big|\mathcal{F}\{f(\cdot,\theta)\}(\xi)\big|^{1-s}\big|\mathcal{F}\{g(\cdot,\theta)\}(\xi)\big|^s}{|\theta\cdot\hat{\xi}|^s}\,.$$

Then, permuting the absolute value and the integral in equation (5.20), and using Cauchy–Schwarz inequality

$$\begin{aligned}&\big|\mathcal{F}\{\bar{f}_\varphi\}(\xi)\big|^2\,|\xi|^{2s}\leq\\&\quad\Big(\int_{\mathbb{S}^{d-1}}\frac{\big|\mathcal{F}\{f(\cdot,\theta)\}(\xi)\big|^{1-s}\big|\mathcal{F}\{g(\cdot,\theta)\}(\xi)\big|^s}{|\theta\cdot\hat{\xi}|^s}\,\big|\varphi(\theta)\big|\,\mathrm{d}\theta\Big)^2\\&\leq\Big(\int_{\mathbb{S}^{d-1}}\big|\mathcal{F}\{f(\cdot,\theta)\}(\xi)\big|^{2(1-s)}\big|\mathcal{F}\{g(\cdot,\theta)\}(\xi)\big|^{2s}\,\mathrm{d}\theta\Big)\times\\&\qquad\qquad\Big(\int_{\mathbb{S}^{d-1}}\frac{|\varphi(\theta)|^2}{|\theta\cdot\hat{\xi}|^{2s}}\,\mathrm{d}\theta\Big).\end{aligned} \tag{5.21}$$

Each term in the right can be estimated as follows,

$$\int_{\mathbb{S}^{d-1}} \frac{|\varphi(\theta)|^2}{|\theta\cdot\hat{\xi}|^{2s}}\,\mathrm{d}\theta \le \|\varphi\|_\infty^2 \big|\mathbb{S}^{d-2}\big| \int_0^\pi \frac{\sin^{d-2}(z)}{|\cos(z)|^{2s}}\mathrm{d}z \le C_{d,s}^2\|\varphi\|_\infty^2\,, \qquad s\in\big(0,\tfrac{1}{2}\big)\,. \tag{5.22}$$

Meanwhile, Young's inequality implies

$$\int_{\mathbb{S}^{d-1}} \big|\mathcal{F}\{f(\cdot,\theta)\}(\xi)\big|^{2(1-s)}\big|\mathcal{F}\{g(\cdot,\theta)\}(\xi)\big|^{2s}\,\mathrm{d}\theta \le (1-s)\int_{\mathbb{S}^{d-1}}\big|\mathcal{F}\{f(\cdot,\theta)\}(\xi)\big|^2\mathrm{d}\theta + s\int_{\mathbb{S}^{d-1}}\big|\mathcal{F}\{g(\cdot,\theta)\}(\xi)\big|^2\mathrm{d}\theta\,. \tag{5.23}$$

Using (5.22) and (5.23) in (5.21), and integrating in $\xi\in\mathbb{R}^d$,

$$\int_{\mathbb{R}^d}\big|\mathcal{F}\{\bar{f}_\varphi\}(\xi)\big|^2|\xi|^{2s}\mathrm{d}\xi \le C_{d,s}^2\|\varphi\|_\infty^2\bigg(\int\big|\mathcal{F}\{f(\cdot,\theta)\}(\xi)\big|^2\mathrm{d}\theta\,\mathrm{d}\xi + \int\big|\mathcal{F}\{g(\cdot,\theta)\}(\xi)\big|^2\mathrm{d}\theta\,\mathrm{d}\xi\bigg).$$

As a consequence, estimate (5.19) follows after applying Plancherel theorem in the ξ–variable on previous inequality. Stronger results, such as Theorem 5.2.1, which state regularization on the function f can be proven from the regularization of the averages $\bar{f}_\varphi$ and the *a priori* angle regularity. In essence, the proof is obtained by choosing $\varphi := \{\varphi_\epsilon\}_{\epsilon>0}$ as an approximation to the identity, and then, finding uniform estimates in the mollification index $\epsilon>0$. Finding such uniform estimate for $\|\bar{f}_{\varphi_\epsilon}\|_{H^s_x}$ is a trade-off process, thus, a fraction of regularization s must be lost.

Using Theorem 5.2.1 and the energy estimate (5.16) we obtain a complete *a priori* energy inequality

$$\tfrac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\big|u(t,x,\theta)\big|^2\mathrm{d}\theta\mathrm{d}x + \mathfrak{D}\int_{t'}^t\int_{\mathbb{R}^d}\Big(\big\|(-\Delta_v)^{s/2}w_{\mathcal{J}}\big\|^2_{L^2(\mathbb{R}^{d-1})} + \big\|(-\Delta_x)^{s_0/2}u\big\|^2_{L^2(\mathbb{S}^{d-1})}\Big)\mathrm{d}x\mathrm{d}\tau \le C_o\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\big|u(t',x,\theta)\big|^2\mathrm{d}\theta\mathrm{d}x + C_1\int_{t'}^t\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\big|u(\tau,x,\theta)\big|^2\mathrm{d}\theta\mathrm{d}x\mathrm{d}\tau\,, \tag{5.24}$$

valid for any $0 < t' \leq t < \infty$. The parameters $\mathfrak{D}$, C_o and C_1 depend on d, s and D with C_1 additionally depending on $\|h\|_{L^1(\mathbb{S}^{d-1})}$. Having at hand estimate (5.24), classical arguments used for parabolic equations imply instantaneous fractional Sobolev regularization of solutions for the RTE in the forward-peaked regime, we refer to [13]. Let us present here only the initial step of a regularization proof which is quite natural, namely, solutions in $L^1_{x,\theta}$ are in fact in $L^2_{x,\theta}$. Before entering into the details observe that (5.24) implies the following more standard version of parabolic energy estimate

$$\sup_{t' \leq t \leq \frac{1}{4C_1}} \tfrac{1}{4}\|u(t)\|^2_{L^2_{x,\theta}} + \mathfrak{D}\int_{t'}^{t}\int_{\mathbb{R}^d}\Big(\big\|(-\Delta_v)^{s/2} w_{\mathcal{J}}\big\|^2_{L^2(\mathbb{R}^{d-1})} + \big\|(-\Delta_x)^{s_0/2} u\big\|^2_{L^2(\mathbb{S}^{d-1})}\Big)\mathrm{d}x\mathrm{d}\tau \leq 3\,C_o\,\|u(t')\|_{L^2_{x,\theta}}\,, \qquad t' \leq t \leq \frac{1}{4C_1}\,. \tag{5.25}$$

Proposition 5.2.2. *Suppose u is a solution to the transport equation (5.1) on $[0, \frac{1}{4C_1}]$. Then, there exists constants $\omega := \omega(s) > 1$ and $C := C(m_o, \mathfrak{D}, s)$ such that*

$$\|u(t)\|^2_{L^2_{x,\theta}} \leq \frac{C}{t^{\frac{1}{\omega-1}}}\,, \qquad 0 < t \leq \frac{1}{4C_1}\,,$$

where $m_o = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} u_o(x,\theta)dxd\theta$ is the mass of u and C_1 is the constant appearing in (5.24).

Proof. Using Sobolev embedding on has that

$$\int_{\mathbb{R}^d}\Big(\big\|(-\Delta_v)^{s/2} w_{\mathcal{J}}\big\|^2_{L^2(\mathbb{R}^{d-1})} + \big\|(-\Delta_x)^{s_0/2} u\big\|^2_{L^2(\mathbb{S}^{d-1})}\Big)\mathrm{d}x \geq c\,\|u(t)\|^2_{L^p_{x,\theta}}\,,$$

for some $p := p(s) > 2$. Also, note that Lebesgue's interpolation gives

$$\|u(t)\|_{L^2_{x,\theta}} \leq \|u(t)\|^{1-1/\omega}_{L^1_{x,\theta}}\|u(t)\|^{1/\omega}_{L^p_{x,\theta}} = m_o^{1-1/\omega}\|u(t)\|^{1/\omega}_{L^p_{x,\theta}}\,,$$

for some $1/\omega \in (0,1)$ depending only on p (so, ω depends only on s). Putting these two together in estimate (5.25) it follows that

$$\int_{t'}^{t} \|u(\tau)\|^{2\omega}_{L^2_{x,\theta}} \mathrm{d}\tau \leq C^{1/\omega} \|u(t')\|^2_{L^2_{x,\theta}}\,, \quad t' \leq t \leq \frac{1}{4C_1}\,, \tag{5.26}$$

for some constant $C > 0$ depending on the mass and $\mathfrak{D}$. Denote $X(t) := \|u(t)\|^{2\omega}_{L^2_{x,\theta}}$, then, estimate (5.26) becomes

$$\left(\int_{t'}^{t} X(\tau)\mathrm{d}\tau\right)^{\omega} \leq C\, X(t')\,, \quad t' \leq t \leq \frac{1}{4C_1}\,.$$

If we fix $t = \frac{1}{4C_1}$ and further denote $Y(t') := \displaystyle\int_{t'}^{\frac{1}{4C_1}} X(\tau)\mathrm{d}\tau$, it is concluded that

$$C\,\frac{dY(t')}{dt} + Y^{\omega}(t') \leq 0\,.$$

Such differential inequality leads to the estimate

$$\int_{t'}^{\frac{1}{4C_1}} \|u(\tau)\|^{2\omega}_{L^2_{x,\theta}} \mathrm{d}\tau = Y(t') \leq \frac{C}{t'^{\frac{1}{\omega-1}}}\,, \qquad 0 < t' \leq \frac{1}{4C_1}\,, \tag{5.27}$$

where the constant C may have changed from line to line. Now, it is quite easy to show that solutions of the RTE do not increase its $L^2_{x,\theta}$ norm, hence

$$\|u(t)\|^{2\omega}_{L^2_{x,\theta}} \leq \|u(\tau)\|^{2\omega}_{L^2_{x,\theta}}\,, \qquad 0 < t' \leq \tau \leq t \leq \frac{1}{4C_1}\,. \tag{5.28}$$

Taking the average in $\tau \in (t', t)$ in (5.28) and using (5.27) we have

$$\begin{aligned}\|u(t)\|^{2\omega}_{L^2_{x,\theta}} &\leq \frac{1}{t-t'}\int_{t'}^{t} \|u(\tau)\|^{2\omega}_{L^2_{x,\theta}}\mathrm{d}\tau \\ &\leq \frac{C}{t-t'}\frac{1}{t'^{\frac{1}{\omega-1}}}\,, \quad 0 < t' < t \leq \frac{1}{4C_1}\,.\end{aligned}$$

In particular, if we take $t' = t/2$ it follows that

$$\|u(t)\|^{2\omega}_{L^2_{x,\theta}} \leq 2^{\frac{w}{w-1}}\frac{C}{t^{\frac{w}{w-1}}}\,, \qquad 0 < t \leq \frac{1}{4C_1}\,,$$

which is the desired result. □

5.3 On the Cauchy problem for singular scattering

As a final comment let us just mention that the Cauchy problem for the inhomogeneous Boltzmann equation with singular scattering, usually regarded to as without angular cutoff, has been studied by some authors in the community, refer for example to [1, 2, 3, 43, 35]. As good as these references are, these works are a first step towards a more complete strategy leading to a theory of well-posedness of solutions, in the singular case, for the inhomogeneous Boltzmann equation with general initial data.

Chapter 6

Appendix

Lemma 6.0.1. *Fix a unitary vector* $\widehat{u}$*. The map* $\sigma : \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ *given by*

$$\sigma(\omega) = \widehat{u} - 2(\widehat{u} \cdot \omega)\omega$$

has Jacobian

$$\frac{d\sigma}{d\omega} = 2^{d-1}\big|\widehat{u} \cdot \omega\big|^{d-2}.$$

Proof. Let $\mathcal{O}_{\widehat{u}}$ be the orthogonal space to $\widehat{u}$, α be the angle between $\widehat{u}$ and ω, and β be the angle between $\widehat{u}$ and σ. In this way one may write

$$\omega = \cos(\alpha)\widehat{u} + \omega_o\,, \qquad \sigma = \cos(\beta)\widehat{u} + \sigma_o$$

where ω_o, $\sigma_o \in \mathcal{O}_{\widehat{u}}$. Using spherical coordinates with north pole given by $\widehat{u}$, the measures $\mathrm{d}\omega$ and $\mathrm{d}\sigma$ are given by

$$\mathrm{d}\omega = \sin(\alpha)^{d-2}\,\mathrm{d}\widehat{\omega_o}\,\mathrm{d}\alpha\,, \qquad \mathrm{d}\sigma = \sin(\beta)^{d-2}\,\mathrm{d}\widehat{\sigma_o}\,\mathrm{d}\beta\,.$$

Here the measures $\mathrm{d}\widehat{\omega_o}$ and $\mathrm{d}\widehat{\sigma_o}$ are the Lebesgue measure in $\mathbb{S}^{d-2}$ parameterized with the vectors ω_o and σ_o respectively. Directly from the expression of the map one has

$$\cos(\beta) = \widehat{u} \cdot \sigma = 1 - 2\big(\widehat{u} \cdot \omega\big)^2 = 1 - 2\cos(\alpha)^2\,.$$

Then, it follows by direct differentiation that

$$-\sin(\beta)\mathrm{d}\beta = 4\cos(\alpha)\sin(\alpha)\mathrm{d}\alpha\,.$$

Now, choose a orthonormal base $\{\xi_i\}_{i=1}^{d-2}$ for $\mathcal{O}_{\widehat{u}}$. Compute again using the explicit expression of the map

$$\sigma_o = \sum_i (\sigma \cdot \xi_i)\xi_i = -2\big(\widehat{u} \cdot \omega\big) \sum_i (\omega \cdot \xi_i)\xi_i$$
$$= -2\big(\widehat{u} \cdot \omega\big)\omega_o = -2\cos(\alpha)\omega_o \,.$$

Thus, $\widehat{w_o} = \widehat{\sigma_o}$, and as a consequence, $\mathrm{d}\widehat{w_o} = \mathrm{d}\widehat{\sigma_o}$. Gathering these relations all together and using basic trigonometry

$$\mathrm{d}\omega = \left(\frac{\sin(\alpha)}{\sin(\beta)}\right)^{d-3} \frac{\mathrm{d}\sigma}{4\big|\cos(\alpha)\big|} = \frac{\mathrm{d}\sigma}{2^{d-1}\big|\cos(\alpha)\big|^{d-2}} \,.$$

This completes the proof. □

Lemma 6.0.2. *Fix* $\sigma \in \mathbb{S}^{d-1}$. *The map* $z : \mathbb{R}^d \to \mathbb{R}^d$ *given by*

$$z(u) = \tfrac{1}{2}\big(u + |u|\sigma\big)$$

has Jacobian

$$\frac{dz}{du} = \frac{1 + \widehat{u} \cdot \sigma}{2^d} \,.$$

Proof. Choose an orthonormal base $\{\sigma, \xi_i\}$, with $2 \leq i \leq d$. Then, the coordinates of this change of variables are

$$z_1 = z \cdot \sigma = \tfrac{1}{2}\big(u \cdot \sigma + |u|\big) = \tfrac{1}{2}\big(u_1 + |u|\big) \,,$$
$$z_i = z \cdot \xi_i = \tfrac{1}{2}u_i \,, \quad i = 2, \cdots, d \,.$$

Thus,

$$\partial_{u_1} z_1 = \tfrac{1}{2}\big(1 + \widehat{u} \cdot \sigma\big) \,, \qquad \partial_{u_j} z_i = \tfrac{1}{2}\delta_{ij} \,, \quad i = 2, \cdots, d \,,$$

and, therefore

$$\frac{dz}{du} = \prod_i \big|\partial_{u_i} z_i\big| = \frac{1 + \widehat{u} \cdot \sigma}{2^d} \,.$$

□

Lemma 6.0.3. *(Carlemman representation) For any sufficiently smooth functions f and g the gain collision operator has the representation*

$$Q^+(f,g)(v) = 2^{d-1} \int_{\mathbb{R}^d} \frac{g(x)}{|v-x|} \int_{\{y\cdot(v-x)=0\}} \frac{\tau_{-x} f(y+(v-x))}{|y+(v-x)|^{d-2}} \times B\Big(-y+(v-x), \frac{y+(v-x)}{|y+(v-x)|}\Big)\, d\pi_y\, dx\,.$$

Proof. We start with the identity

$$\int_{\mathbb{S}^{d-1}} \varphi(\sigma)\mathrm{d}\sigma = \int_{\mathbb{R}^d} \varphi(k)\delta_o\Big(\frac{|k|^2-1}{2}\Big)\mathrm{d}k\,,$$

and write

$$Q^+(f,g)(v) = \int_{\mathbb{R}^{2d}} f('v)g('v_*)B(u,k)\delta_o\Big(\frac{|k|^2-1}{2}\Big)\mathrm{d}k\mathrm{d}u\,.$$

recalling that $'v = v'$ and $'v_* = v'_*$ (for elastic interactions) one has

$$'v = v - u + \frac{u+|u|k}{2}\,, \quad 'v_* = v - \frac{u+|u|k}{2}\,.$$

Using the change of variables $z = -\dfrac{u+|u|k}{2}$, for fixed u, it follows that

$$\begin{aligned}
&Q^+(f,g)(v)\\
&= \int_{\mathbb{R}^{2d}} g(v+z)f(v-u-z)B\Big(u, -\frac{2z+u}{|u|}\Big)\delta_o\Big(\frac{2z\cdot(z+u)}{|u|^2}\Big)\frac{2^d}{|u|^d}\mathrm{d}z\mathrm{d}u\\
&= 2^{d-1}\int_{\mathbb{R}^{2d}} g(v+z)\frac{f(v-u-z)}{|u|^{d-2}}B\Big(u,-\frac{2z+u}{|u|}\Big)\delta_o\big(z\cdot(z+u)\big)\mathrm{d}z\mathrm{d}u\\
&= 2^{d-1}\int_{\mathbb{R}^{2d}} g(v+z)\frac{f(v+y)}{|y+z|^{d-2}}B\Big(-y-z, \frac{y-z}{|y+z|}\Big)\delta_o\big(z\cdot y\big)\mathrm{d}y\mathrm{d}z
\end{aligned}$$

Using the identity

$$\int_{\mathbb{R}^d} \delta_o\big(z\cdot y\big)\varphi(y)\mathrm{d}y = \frac{1}{|z|}\int_{\{z\cdot y=0\}} \varphi(y)\mathrm{d}\pi_y\,,$$

one finds

$$
\begin{aligned}
&Q^+(f,g)(v)\\
&= 2^{d-1}\int_{\mathbb{R}^d}\frac{g(v+z)}{|z|}\int_{\{y\cdot z=0\}}\frac{f(v+y)}{|y+z|^{d-2}}B\Big(-y-z,\frac{y-z}{|y+z|}\Big)\mathrm{d}\pi_y\mathrm{d}z\\
&= 2^{d-1}\int_{\mathbb{R}^d}\frac{g(x)}{|v-x|}\int_{\{y\cdot(v-x)=0\}}\frac{\tau_{-x}f(y+(v-x))}{|y+(v-x)|^{d-2}}\times\\
&\qquad\qquad\qquad B\Big(-y+(v-x),\frac{y+(v-x)}{|y+(v-x)|}\Big)\mathrm{d}\pi_y\mathrm{d}x\,.
\end{aligned}
$$

In the last inequality we used the change of variables $x=z+v$ and the fact that $y\cdot z=0$ to write $|y+z|=|y-z|$. □

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