Expansive Measures

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Preface

It is customary to say that a given phenomenum is *chaotic* if it cannot be predicted. This is what currently occurs in many circuntances like in weather prediction, particle behavior in physic or financial marked. But what the meanning of predictiblity is? A simple manner to answer this question is to model the given phenomenum as the trajectories of a dynamical system and, then, reinterpret the predictibility as the knowledgement of where trajectories go. For instance, in weather prediction or particle behavior or financial marked it is known that nearby initial conditions can produce very different outputs thus characterizing a very high degree of unpredictibility. Such a situation is easily described in dynamics with the notion of *sensitivity to initial conditions* in which every face point can be approached by points for which the corresponding trayectories eventually separate in the future (or in the past for invertible systems). The worst scenario appears precisely when the trajectory of *every* nearby point separate from the initial one, and this is what is commonly denominated as *expansive system*. In these terms expansivity manifests the most chaotic scenario in which predictions may have no sense at all.

The first researcher who considered the expansivity in dynamics was Utz in his seminal paper [86]. Indeed, he defined the notion of unstable homeomorphisms (nowadays known as *expansive homeomorphisms* [39]) and studied their basic properties. Since then an extensive literature about these homeomorphisms has been developed.

For instance, [90] proved that the set of points doubly asymptotic to a given point for expansive homeomorphisms is at most countable. Moreover, a homeomorphism of a compact metric space is expansive if it does in the complement of finitely many orbits [91]. In 1972 Sears proved the denseness of expansive homeomorphisms with respect to the uniform topology in the space of homeomorphisms of a Cantor set [80]. An study of expansive homeomorphisms using generators is given in [20]. Goodman [38] proved that every expansive homeomorphism of a compact metric space has a (nonnecessarily unique) measure of maximal entropy whereas Bowen [11] added specification to obtain unique equilibrium states. In another direction, [76] studied expansive homeomorphisms with canonical coordinates and showed in the locally connected case that sinks or sources cannot exist. Two years later, Fathi characterized expansive homeomorphisms on compact metric spaces as those exhibiting adapted hyperbolic metrics [34] (see also [78] or [30] for more about adapted metrics). Using this he was able to obtain an upper bound of the Hausdorff dimension and upper capacity of the underlying space using the topological entropy. In [54] it is computed the large deviations of irregular periodic orbits for expansive homeomorphisms with the specification property. The C^0 perturbations of expansive homeomorphisms on compact metric spaces were considered in [24]. Besides, the multifractal analysis of expansive homeomorphisms with the specification property was carried out in [84]. We can also mention [23] in which it is studied a new measure-theoretic pressure for expansive homeomorphisms.

From the topological viewpoint we can mention [67] and [74] proving the existence of expansive homeomorphisms in the genus two closed surface, the n-torus and the open disk. Analogously for compact surfaces obtained by making holes on closed surfaces different from the sphere, projective plane and Klein bottle [51]. In [46] it was proved that there are no expansive homeomorphisms of the compact interval, the circle and the compact 2-disk. The same negative result was obtained independently by Hiraide and Lewowicz in the 2-sphere [42], [59]. Mañé proved in [62] that a compact metric space exhibiting expansive homeomorphisms must be finite dimensional and, further, every minimal set of such homeomorphisms is zero dimensional. Previously he proved that the C^1 interior of the set of expansive diffeomorphisms of a closed manifold is composed by pseudo-Anosov (and hence Axiom A) diffeomorphisms. In 1993 Vieitez [87] obtained results about expansive homeomorphisms on closed 3-manifolds. In particular, he proved that the denseness of the topologically hyperbolic periodic points does imply constant dimension of the stable and unstable sets. As a consequence a local product property is obtained for such homeomorphisms. He also obtained that expansive homeomorphisms on closed 3-manifolds with dense topologically hyperbolic periodic points are both supported on the 3-torus and topologically conjugated to linear Anosov isomorphisms [88].

In light of these results it was natural to consider another notions of expansiveness. For example, G-expansiveness, continuouswise and pointwise expansiveness were defined in [29], [50] and [75] respectivelly. We also have the entropy-expansiveness introduced by Bowen [10] to compute the metric and topological entropies in a large class of homeomorphisms.

In this monograph we will consider a notion of expansiveness, located in between sensitivity and expansivity, in which Borel probability measures μ will play fundamental role. Indeed, we say that μ is an *expansive measure* of a homeomorphism f if the probability of two orbits remain close each other up to a prefixed radius is zero. Analogously, for continuous maps, we define *positively expansive measure* by considering positive orbits instead. The corresponding concepts for certain topological spaces (e.g. uniform spaces) likewise flows or topological group actions have been considered elsewhere [22], [66].

These concepts are closely related (and sometimes equivalent to) the concepts of *pairwise sensitivity* [27] and *the μ-sensitivity* [44] in which the sensitivity properties of these systems are emphasized. Here we give emphasize not in the sensitivity but, rather, in the expansivity properties of these systems.

In Chapter 1 we will give the precise definition of expansive measures for homeomorphisms f as well as some basic properties closely related to the expansive systems. For instance, we characterize the expansive measures as those for which the diagonal is almost invariant for $f \times f$ with respect to the product measure μ^2. In addition, we prove that the set of heteroclinic points has measure zero with respect to any expansive measure. In particular, the set of periodic orbits for these homeomorphisms is also of measure zero for such measures. We also prove that there are expansive measures for homeomorphisms in any compact interval and, in the circle, we prove that they exists solely for the the Denjoy maps. As an application we obtain probabilistic proofs of some result of expansive systems.

In Chapter 2 we will analyze the n-expansive systems which represent a particular (an interesting) example of nonexpansive systems for which every non-atomic Borel measure is expansive.

In Chapter 3 we study the class of positively expansive measures and prove that every ergodic invariant measure with positive entropy of a continuous map on a compact metric space is positively expansive. We use this property to prove, for instance, that the stable classes have measure zero with respect to any ergodic invariant measure with positive entropy. Moreover, continuous maps which either have countably many stable classes or are Lyapunov stable on their recurrent sets have zero topological entropy. We also apply our results to the Li-Yorke chaos.

Finally, in Chapter 4, we will extend the notion of expansivity to include measurable maps on measure spaces. Indeed, we study countable partitions for measurable maps on measure spaces such that for all point x the set of points with the same itinerary of x is negligible. We prove that in non-atomic probability spaces every strong generator [69] satisfies this property but not conversely. In addition, measurable maps carrying partitions with this property are aperiodic and their corresponding spaces are non-atomic. From this we obtain a characterization of nonsingular countable to one mappings with these partitions on non-atomic Lebesgue probability spaces as those having strong generators. Furthermore, maps carrying these partitions include the ergodic measure-preserving ones with positive entropy on probability spaces (thus extending a result by Cadre and Jacob [27]). Applications of these results will be given. At the end of each chapter we include some exercices whose difficulty was not estimated. Some basics of dynamical systems, ergodic and measure theory will be recommendable for the comprension of this text.

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Chapter 1

Expansive measures

1.1 Definition and examples

In this section we introduce the definition of expansive measures for homeomorphisms and present some examples. To motivate let us recall the concept of expansive homeomorphism.

Definition 1.1. *A homeomorphism $f : X \to X$ of a metric space X is* expansive *if there is $\delta > 0$ such that for every pair of different points $x, y \in X$ there is $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) > \delta$.*

An important remark is given below.

Remark 1.2. *Equivalently, f is expansive if there is $\delta > 0$ such that $\Gamma_\delta(x) = \{x\}$ for all $x \in X$ where*

$$\Gamma_\delta(x) = \{y \in X : d(f^i(x), f^i(y)) \leq \delta, \forall i \in \mathbb{Z}\}.$$

(Notation $\Gamma^f_\delta(x)$ will indicate dependence on f.)

This definition suggests further notions of expansiveness involving a given property (P) of the closed sets in X. More precisely, we say that f is *expansive with respect to (P)* if there is $\delta > 0$ such that $\Gamma_\delta(x)$ satisfies (P) for all $x \in X$.

For example, a homeomorphism is expansive it is expansive with respect to the property of being a single point. Analogously, it is *h-expansive* (c.f. [10]) if it is expansive with respect to the property of being a zero entropy set. In this vein it is natural to consider the property of having zero measure with respect to a given Borel probability measure μ of X. By *Borel measure* we mean a non-negative σ-additive function μ defined in the Borel σ-algebra of X which is non-zero in the sense that $\mu(X) > 0$.

Definition 1.3. *A* expansive measure *of homeomorphism $f : X \to X$ of a metric space X is a Borel measure μ for which there is $\delta > 0$ such that $\mu(\Gamma_\delta(x)) = 0$ for all $x \in X$. The constant δ will be referred to as an* expansivity constant *of μ.*

Let us present some examples related to this definition. Recall that a Borel measure μ of a metric space X is a *probability* if $\mu(X) = 1$ and *non-atomic* if $\mu(\{x\}) = 0$ for all $x \in X$.

Example 1.4. *Every expansive measure is non-atomic. Therefore, every metric space carrying homeomorphisms with expansive (probability) measures also carries a non-atomic Borel (probability) measure.*

In the converse direction we have the following relation between expansive homeomorphisms and expansive measures for homeomorphisms.

Example 1.5. *If $f : X \to X$ is an expansive homeomorphism of a metric space X, then every non-atomic Borel measure of X (if it exists) is an expansive measure of f. Moreover, all such measures have a* common expansivity constant*.*

Example 1.5 motivates the question whether a homeomorphism is expansive if it satisfies that every non-atomic Borel measure (if it exists) is expansive with a common expansivity constant. We shall give a partial positive answer based on the following definition (closely related to that of expansive homeomorphism).

Definition 1.6. *A homeomorphism $f : X \to X$ of a metric spaces X is* countably-expansive *if there is $\delta > 0$ such that $\Gamma_\delta(x)$ is countable, $\forall x \in X$.*

Clearly, every expansive homeomorphism is countably-expansive but not conversely (as we shall see in Chapter 2). In addition, every countably-expansive homeomorphism satisfies that all non-atomic Borel probability measures (if they exist) are expansive with common expansivity constant. The following result proves the converse of this last assertion for *Polish metric spaces*, i.e., metric spaces which are both complete and separable.

Proposition 1.7. *The following properties are equivalent for every homeomorphism $f : X \to X$ of a Polish metric space X:*

1. *f is countably-expansive.*
2. *All non-atomic Borel probability measures of X (if they exit) are expansive with a common expansivity constant.*

Proof. By the previous discussion we only have to prove that (2) implies (1). Suppose by contradiction that all non-atomic Borel probability measures are expansive measures with a common expansivity constant (say δ) but f is not countably-expansive. Then, there is $x \in X$ such that $\Gamma_\delta(x)$ is uncountable. Since $\Gamma_\delta(x)$ is also a closed subset of X which is a Polish metric space, we have that $\Gamma_\delta(x)$ is a Polish metric space too. Then, we can apply a result in [73] (e.g. Theorem 8.1 p. 53 in [72]) to obtain a non-atomic Borel probability μ of X supported on $\Gamma_\delta(x)$. For such a measure we would obtain $\mu(\Gamma_\delta(x)) = 1$ a contradiction. □

In light of this proposition it is natural to ask what can happen if we still assume that all non-atomic Borel probability measure (if it exist) are expansive but without assuming that they have a common expansivity constant.

This question emphasizes the role of metric spaces for which there are non-atomic Borel probability measures. For the sake of convenience we call these spaces *non-atomic metric spaces*. The aforementioned result in [73] (stated in Theorem 8.1 p. 53 in [72]) implies that every uncountable Polish metric space is a non-atomic metric space. This includes the compact metric space containing perfect subsets [55]. Every non-atomic metric space is uncountable.

Another related definition is as follows.

Definition 1.8. *A homeomorphism $f : X \to X$ of a non-atomic metric space X is* measure-expansive *if every non-atomic Borel probability measure is expansive for f.*

It is clear that every expansive homeomorphism of a non-atomic metric space is measure-expansive. Moreover, as discussed in Example 1.5, *every countably-expansive homeomorphism of a non-atomic metric space is measure-expansive.* Although we obtain in Example 3.44 that there are measure-expansive homeomorphisms of compact non-atomic metric spaces which are not expansive, we don't know any example of one which is not countably-expansive (see Problem 1.46). Some dynamical consequences of measure-expansivity resembling expansivity will be given later on.

Further examples of homeomorphisms without expansive measures can be obtained as follows. Recall that an *isometry* of a metric space X is a map $f : X \to X$ satisfying $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

Example 1.9. *Every isometry of a separable metric space has no expansive measures. In particular, the identity map in these spaces (or the rotations in $\mathbb{R}^2$ or translations in $\mathbb{R}^n$) are not measure-expansive homeomorphisms.*

Proof. Suppose by contradiction that there is a an expansive measure μ for some isometry f of a separable metric space X. Since f is an isometry we have $\Gamma_\delta(x) = B[x, \delta]$, where $B[x, \delta]$ denotes the closed δ-ball around x. If δ is an expansivity constant, then $\mu(B[x, \delta]) = \mu(\Gamma_\delta(x)) = 0$ for all $x \in X$. Nevertheless, since X is separable (and so Lindelöf), we can select a countable covering $\{C_1, C_2, \cdots, C_n, \cdots\}$ of X by closed subsets such that for all n there is $x_n \in X$ such that $C_n \subset B[x_n, \delta]$. Thus, $\mu(X) \leq \sum_{n=1}^{\infty} \mu(C_n) \leq \sum_{n=1}^{\infty} \mu(B[x_n, \delta]) = 0$ which is a contradiction. This proves the result. □

Example 1.10. *Endow $\mathbb{R}^n$ with a metric space with the Euclidean metric and denote by Leb the Lebesgue measure in $\mathbb{R}^n$. Then, Leb is an expansive measure of a linear isomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ if and only if f has eigenvalues of modulus less than or bigger than* 1.

Proof. Since f is linear we have $\Gamma_\delta(x) = \Gamma_\delta(0) + x$ thus $Leb(\Gamma_\delta(x)) = Leb(\Gamma_\delta(0))$ for all $x \in \mathbb{R}^n$ and $\delta > 0$. If f has eigenvalues of modulus less than or bigger than 1, then $\Gamma_\delta(0)$ is contained in a proper subspace of $\mathbb{R}^n$ which implies $Leb(\Gamma_\delta(0)) = 0$ thus Leb is expansive. □

Example 1.11. *As we shall see later, a homeomorphism of a compact interval has no expansive measures. In the circle the sole homeomorphisms having such measures are the Denjoy ones.*

Recall that a subset $Y \subset X$ is *invariant* if $f^{-1}(Y) = Y$.

Example 1.12. *A homeomorphism f has an expansive measure if and only if there is an invariant borelian set Y of f such that the restriction f/Y has an expansive measure.*

Proof. We only have to prove the only if part. Assume that f/Y has an expansive measure ν. Fix $\delta > 0$. Since Y is invariant we have either $\Gamma^f_{\delta/2}(x) \cap Y = \emptyset$ or $\Gamma^f_{\delta/2}(x) \cap Y \subset \Gamma^{f/Y}_\delta(y)$ for some $y \in Y$. Therefore, either $\Gamma^f_{\delta/2}(x) \cap Y = \emptyset$ or $\mu(\Gamma^f_{\delta/2}(x)) \leq \mu(\Gamma^{f/Y}_\delta(y))$ for some $y \in Y$ where μ is the Borel probability of X defined by $\mu(A) = \nu(A \cap Y)$. From this we obtain that for all $x \in X$ there is $y \in Y$ such that $\mu(\Gamma^f_{\delta/2}(x)) \leq \nu(\Gamma^{f/Y}_\delta(y))$. Taking δ as an expansivity constant of f/Y we obtain $\mu(\Gamma^f_{\delta/2}(x)) = 0$ for all $x \in X$ thus μ is expansive with expansivity constant $\delta/2$. □

The next example implies that the property of having expansive measures is a conjugacy invariant. Given a Borel measure μ in X and a homeomorphism $\phi : X \to Y$ we denote by $\phi_*(\mu)$ the *pullback* of μ defined by $\phi_*(\mu)(A) = \mu(\phi^{-1}(A))$ for all borelian A.

Example 1.13. *Let μ be an expansive measure of a homeomorphism $f : X \to X$ of a compact metric space X. If $\phi : X \to Y$ is a homeomorphism of compact metric spaces, then $\phi_*(\mu)$ is an expansive measure of $\phi \circ f \circ \phi^{-1}$.*

Proof. Clearly ϕ is uniformly continuous so for all $\delta > 0$ there is $\epsilon > 0$ such that $\Gamma^{\phi\circ f\circ\phi}_\epsilon(y) \subset \phi(\Gamma^f_\delta(\phi^{-1}(y)))$ for all $y \in Y$. This implies

$$\phi_*(\mu)(\Gamma^{\phi\circ f\circ\phi}_\epsilon(y)) \leq \mu(\Gamma^f_\delta(\phi^{-1}(y))).$$

Taking δ as the expansivity constant of μ we obtain that ϵ is an expansivity constant of $\phi_*(\mu)$. □

For the next example recall that a *periodic point* of a homeomorphism (or map) $f : X \to X$ is a point $x \in X$ such that $f^n(x) = x$ for some $n \in \mathbb{N}^+$. The *nonwandering set* of f is the set $\Omega(f)$ of points $x \in X$ such that for every neighborhood U of x there is $n \in \mathbb{N}^+$ satisfying $f^n(U) \cap U \neq \emptyset$. Clearly a periodic point belongs to $\Omega(f)$ but not every point in $\Omega(f)$ is periodic. If $X = M$ is a *closed* (i.e. compact connected boundaryless Riemannian) manifold and f is a diffeomorphism we say that an invariant set H is *hyperbolic* if there are a continuous invariant tangent bundle decomposition $T_H M = E^s_H \oplus E^u_H$ and positive constants K, $\lambda > 1$ such that

$$\|Df^n(x)/E^s_x\| \leq K\lambda^{-n} \qquad \text{and} \qquad m(Df^n(x)/E^u_x) \geq K^{-1}\lambda^n,$$

for all $x \in H$ and $n \in I\!N$ (m denotes the co-norm operation in M). We say that f is *Axiom A* if $\Omega(f)$ is hyperbolic and the closure of the set of periodic points.

Example 1.14. *Every Axiom A diffeomorphism with infinite nonwandering set of a closed manifold has expansive measures.*

Proof. Consider an Axiom A diffeomorphism f of a closed manifold. It is well known that there is a spectral decomposition $\Omega(f) = H_1 \cup \cdots \cup H_k$ consisting of finitely many disjoint homoclinic classes $H_1, \cdots, H_k$ of f (see [40] for the corresponding definitions). Since $\Omega(f)$ is infinite we have that $H = H_i$ is infinite for some $1 \leq i \leq k$. As is well known f/H is expansive. On the other hand, H is compact without isolated points since it is a homoclinic class. It follows from Example 1.5 that f/H has an expansive measure, so, f also has by Example 1.12. □

We shall prove in the next section that every homeomorphism with expansive measures of a compact metric space has uncountable nonwandering set.

1.2 Expansive invariant measures

Let $f : X \to X$ be a continuous map of a metric space X. We say that a Borel measure μ of X is *invariant* if $f_*\mu = \mu$. In this section we investigate the existence of expansive invariant measures for homeomorphisms on compact metric spaces.

Indeed, every homeomorphism of a compact metric space carries invariant measures, but not necessarily expansive measures (e.g. the circle rotations). On the other hand, the homeomorphism $f(x) = 2x$ on the real line exhibits expansive probability measures (e.g. the Lebesgue measure supported on the unit interval) but not expansive *invariant* probability measures. The result of this section will show that the situation described in this example does not occur on compact metric spaces. More precisely, we will show that every homeomorphism exhibiting expansive probability measures of a compact metric space also exhibit expansive invariant probability measures.

We start with the following observation where f is assumed to be a bijective map, namely,

$$f(\Gamma_\delta(x)) = \Gamma_\delta(f(x)), \qquad \forall (x, \delta) \in X \times \mathbb{R}^+.$$

Using it we obtain the elementary lemma below.

Lemma 1.15. *Let $f : X \to X$ be a homeomorphism of a metric space X. If μ is an expansive measure with expansivity constant δ of f, then so does $f_*\mu$.*

Proof. Applying the previous observation to f^{-1} we obtain

$$f_*\mu(\Gamma_\delta(x)) = \mu(f^{-1}(\Gamma_\delta(x)) = \mu(\Gamma_\delta(f^{-1}(x))) = 0$$

for all $x \in X$. □

Another useful observation is as follows. Given a bijective map $f : X \to X$, $x \in X$, $\delta > 0$ and $n \in \mathbb{N}^+$ we define

$$V[x, n, \delta] = \{y \in X : d(f^i(x), f^i(y)) \leq \delta, \text{ for all } -n \leq i \leq n\},$$

i.e.,

$$V[x, n, \delta] = \bigcap_{i=-n}^{n} f^{-i}(B[f^i(x), \delta]).$$

(when necessary we write $V_f[x, n, \delta]$ to indicate dependence on f.) It is then clear that

$$\Gamma_\delta(x) = \bigcap_{n \in \mathbb{N}^+} V[x, n, \delta]$$

and that $V[x,n,\delta] \supset V[x,m,\delta]$ for $n \leq m$. Consequently,

$$\mu(\Gamma_\delta(x)) = \lim_{l\to\infty} \mu(V[x,k_l,\delta]) \tag{1.1}$$

for every $x \in X$, $\delta > 0$, every Borel probability measure μ of X, and every sequence $k_l \to \infty$. From this we have the following lemma.

Lemma 1.16. *Let $f : X \to X$ be a homeomorphism of a metric space X. A Borel probability measure μ is an expansive measure of f if and only if there is $\delta > 0$ such that*

$$\liminf_{n\to\infty} \mu(V[x,n,\delta]) = 0, \qquad \textit{for all } x \in X.$$

We shall use this information in the following lemma.

Lemma 1.17. *If $f : X \to X$ is a homeomorphism of a metric space X, then every invariant measure of f which is the limit (with respect to the weak-* topology) of a sequence of expansive probability measures with a common expansivity constant of f is expansive for f.*

Proof. Denote by $\partial A = \mathrm{Cl}(A) \setminus Int(A)$ the closure of a subset $A \subset X$. Let μ be an invariant probability measure of f. As in the proof of Lemma 8.5 p. 187 in [40] for all $x \in X$ we can find $\frac{\delta}{2} < \delta_x < \delta$ such that

$$\mu(\partial(B[x,\delta_x])) = 0.$$

This allows us to define

$$W[x,n] = \bigcap_{i=-n}^{n} f^{-i}(B[f^i(x), \delta_{f^i(x)}]), \qquad \forall (x,n) \in X \times \mathbb{N}.$$

Since $\frac{\delta}{2} < \delta_x < \delta$ we can easily verify that

$$V\left[x,n,\frac{\delta}{2}\right] \subset W[x,n] \subset V[x,n,\delta], \qquad \forall (x,n) \in X \times \mathbb{N}. \tag{1.2}$$

Moreover, as f (and so f^{-i}) are homeomorphisms one has

$$\partial(W[x,n]) = \partial\left(\bigcap_{i=-n}^{n} f^{-i}(B[f^i(x), \delta_{f^i(x)}])\right) \subset$$

$$\bigcup_{i=-n}^{n} \partial\left(f^{-i}(B[f^i(x), \delta_{f^i(x)}])\right) = \bigcup_{i=-n}^{n} f^{-i}\left(\partial(B[f^i(x), \delta_{f^i(x)}])\right),$$

and, since μ is invariant,

$$\mu(\partial(W[x,n])) \leq \sum_{i=-n}^{n} \mu(f^{-i}\left(\partial(B[f^i(x), \delta_{f^i(x)}])\right)) =$$

$$\sum_{i=-n}^{n} \mu(\partial(B[f^i(x), \delta_{f^i(x)}])) = 0,$$

proving

$$\mu(\partial(W[x,n])) = 0, \qquad \forall (x,n) \in X \times \mathbb{N}. \tag{1.3}$$

Now, suppose that μ is the weak-* limit of a sequence of expansive probability measures μ_n withoutcommon expansivity constant δ of f. Clearly, μ is also a probability measure. Fix $x \in X$. Since each μ_n is a probability we have $0 \leq \mu_m(W[x,n]) \leq 1$ for all $n, m \in \mathbb{N}$. Then, we can apply the Bolzano-Weierstrass Theorem to find sequences $k_l, r_s \to \infty$ for which the double limit

$$\lim_{l,s\to\infty} \mu_{r_s}(W[x,k_l])$$

exists.

On the one hand, for fixed l, using (1.3), $\mu_n \to \mu$ and well-known properties of the weak-* topology (e.g. Theorem 6.1-(e) p. 40 in [72]) one has that the limit

$$\lim_{s\to\infty} \mu_{r_s}(W[x,k_l]) = \mu(W[x,k_l])$$

exists.

On the other hand, the second inequality in (1.2) and (1.1) imply for fixed s that

$$\lim_{l\to\infty} \mu_{r_s}(W[x,k_l]) \leq \lim_{l\to\infty} \mu_{r_s}(V[x,k_l,\delta]) = \mu_{r_s}(\Gamma_\delta(x)) = 0.$$

Consequently, the limit

$$\lim_{l\to\infty} \mu_{r_s}(W[x,k_l]) = 0$$

also exists for fixed s.

From these assertions and well-known properties of double sequences one obtains

$$\lim_{l\to\infty}\lim_{s\to\infty}\mu_{r_s}(W[x,k_l])=\lim_{s\to\infty}\lim_{l\to\infty}\mu_{r_s}(W[x,k_l])=0.$$

But (1.2) implies

$$\liminf_{n\to\infty}\mu\left(V\left[x,n,\frac{\delta}{2}\right]\right)\le\lim_{l\to\infty}\mu\left(V\left[x,k_l,\frac{\delta}{2}\right]\right)\le\lim_{l\to\infty}\mu(W[x,k_l])$$

and $\mu_n\to\mu$ together with (1.3) yields

$$\lim_{l\to\infty}\mu(W[x,k_l])=\lim_{l\to\infty}\lim_{s\to\infty}\mu_{r_s}(W[x,k_l])$$

so

$$\liminf_{n\to\infty}\mu\left(V\left[x,n,\frac{\delta}{2}\right]\right)=0$$

and then μ is expansive by Lemma 1.16. □

Using these lemmas we obtain the following result.

Theorem 1.18. *A homeomorphisms of a compact metric space has an expansive probability measure if and only if it has an expansive* invariant *probability measure.*

Proof. Let μ be an expansive measure (with expansivity constant δ) of a homeomorphism $f:X\to X$ of a compact metric space X. By Lemma 1.15 we have that $f^i_*\mu$ is also an expansive probability measure with expansivity constant δ ($\forall i\in\mathbb{Z}$). Therefore,

$$\mu_n=\frac{1}{n}\sum_{i=0}^{n-1}f^i_*\mu,\qquad n\in\mathbb{N}^+$$

is a sequence of expansive probability measures of f with common expansivity constant δ. As X is compact there is a subsequence $n_k\to\infty$ such that μ_{n_k} converges to a Borel probability measure μ. Since μ is clearly invariant we can apply Lemma 1.17 to this sequence to obtain that μ is expansive. □

A direct consequence of Theorem 1.18 is as follows. First of all denote by supp(μ) the *support* of a Borel measure μ. Given a metric space X and a map $f : X \to X$ we define the *omega-limit set* of $x \in X$,

$$\omega(x) = \left\{ y \in X : y = \lim_{k \to \infty} f^{n_k}(x) \text{ for some sequence } n_k \to \infty \right\}.$$

The *recurrent set* of f is given by

$$R(f) = \{x \in X : x \in \omega(x)\}.$$

With these definitions we have the following corollary. Denote by supp(μ) the support of a Borel measure μ of a metric space X.

Corollary 1.19. *The recurrent (and hence the nonwandering) sets of every homeomorphism with expansive probability measures of a compact metric space is uncountable.*

Proof. Let $f : X \to X$ be a homeomorphism with an expansive probability measure μ of a compact metric space X. By Theorem 1.18 we can assume that μ is invariant, and so, $\text{supp}(\mu) \subset R(f)$ by the Poincaré Recurrent Theorem. If $R(f)$ were countable we would have $\mu(\text{supp}(\mu)) \subset \mu(R(f)) = 0$ which is absurd thus $R(f)$ is uncountable. □

1.3 Equivalences

In this section we present some equivalences for the expansivity of a given measure. Hereafter all metric spaces X under consideration will be compact unless otherwise stated. We also fix a Borel probability measure μ of X and a homeomorphism $f : X \to X$.

To start we observe an apparently weak definition of expansive measure saying that μ is an expansive measure of f if there is $\delta > 0$ such that $\mu(\Gamma_\delta(x)) = 0$ *for* μ*-a.e.* $x \in X$. However, this definition and the previous one are in fact equivalent by the following lemma.

Lemma 1.20. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. Then, a Borel probability measure μ of X is an expansive measure of f if and only if there is $\delta > 0$ such that $\mu(\Gamma_\delta(x)) = 0$ for μ-a.e. $x \in X$.*

Proof. We only need to prove the if part. Let $\delta > 0$ be such that $\mu(\Gamma_\delta(x)) = 0$ for μ-a.e. $x \in X$. We shall prove that $\delta/2$ is an expansiveness constant of μ. Suppose by contradiction that it is not so. Then, there is $x_0 \in X$ such that $\mu(\Gamma_{\delta/2}(x_0)) > 0$. Denote $A = \{x \in X : \mu(\Gamma_\delta(x)) = 0\}$ so $\mu(A) = 1$. Since μ is a probability measure we obtain $A \cap \Gamma_{\delta/2}(x_0) \neq \emptyset$ so there is $y_0 \in \Gamma_{\delta/2}(x_0)$ such that $\mu(\Gamma_\delta(y_0)) = 0$.

Now, since $y_0 \in \Gamma_{\delta/2}(x_0)$ we have $\Gamma_{\delta/2}(x_0) \subset \Gamma_\delta(y_0)$. Indeed $d(f^i(x), f^i(x_0)) \leq \delta/2$ ($\forall i \in \mathbb{N}$) implies

$$d(f^i(x), f^i(y_0)) \leq d(f^i(x), f^i(x_0)) + d(f^i(x_0), f^i(y_0)) \leq$$

$$\delta/2 + \delta/2 = \delta, \qquad \forall i \in \mathbb{N}$$

proving the assertion. It follows that $\mu(\Gamma_{\delta/2}(x_0)) \leq \mu(\Gamma_\delta(y_0)) = 0$ which is a contradiction. This proves the result. $\square$

In particular, we have the following corollary.

Corollary 1.21. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. Then, a Borel probability measure μ is an expansive measure of f if and only if there is $\delta > 0$ such that $\mu(\Gamma_\delta(x)) = 0$ for all $x \in supp(\mu)$.*

A direct application of Lemma 1.16 is the following version of a well-known property of the expansive homeomorphisms (see Corollary 5.22.1-(ii) of [89]).

Proposition 1.22. *Let $f : X \to X$ a homeomorphism and μ be a Borel probability measure of a compact metric space X. If $n \in \mathbb{Z}\setminus\{0\}$, then μ is an expansive measure of f if and only if it is an expansive measure of f^n.*

Proof. We can assume that $n > 0$. First notice that $V_f[x, n \cdot m, \delta] \subset V_{f^n}[x, m, \delta]$. If μ is an expansive measure of f^n is expansive then by Lemma 1.16 there is $\delta > 0$ such that for every $x \in X$ there is a sequence $m_j \to \infty$ such that $\mu(V_{f^n}[x, m_j, \delta]) \to 0$ as $j \to \infty$. Therefore $\mu(V_f[x, n \cdot m_j, \delta]) \to 0$ as $j \to \infty$ yielding $\liminf_{n\to\infty} \mu(V_f[x, n, \delta]) = 0$. Since x is arbitrary we conclude that μ is expansive with constant δ.

Conversely, suppose that μ is an expansive measure of f with constant δ. Since X is compact and n is fixed we can choose $0 < \epsilon < \delta$ such that if $d(x, y) \leq \epsilon$, then $d(f^i(x), f^i(y)) < \delta$ for all $-n \leq i \leq n$. With this property one has $\Gamma_\epsilon^{f^n}(x) \subset \Gamma_\delta^f(x)$ for all $x \in X$ thus μ is an expansive measure of f^n with constant ϵ. $\square$

One more equivalence is motivated by a well known condition for expansiveness. Given metric spaces X and Y we always consider the *product metric* in $X \times Y$ defined by

$$d((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2).$$

If μ and ν are measures in X and Y respectively we denote by $\mu \times \nu$ their product measure in $X \times Y$. If $f : X \to X$ and $g : Y \to Y$ we define their product $f \times g : X \times Y \to X \times Y$,

$$(f \times g)(x, y) = (f(x), g(y)).$$

Notice that $f \times g$ is a homeomorphism if f and g are. Denote by $\Delta = \{(x, x) : x \in X\}$ the diagonal of $X \times X$.

Given a map g of a metric space Y we call an invariant set I *isolated* if there is a compact neighborhood U of it such that

$$I = \{z \in U : g^n(z) \in U, \forall n \in \mathbb{Z}\}.$$

As is well known, a homeomorphism f of X is expansive if and only if the diagonal Δ is an isolated set of $f \times f$ (e.g. [4]). To express the corresponding version for expansive measures we introduce the following definition. Let ν be a Borel probability measure of Y. We call an invariant set I of g *ν-isolated* if there is a compact neighborhood U of I such that

$$\nu(\{z \in Y : g^n(z) \in U, \forall n \in \mathbb{Z}\}) = 0.$$

With this definition we have the following result in which we write $\mu^2 = \mu \times \mu$.

Theorem 1.23. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. Then, a Borel probability measure μ of X is an expansive measure of f if and only if the diagonal Δ is a μ^2-isolated set of $f \times f$.*

Proof. Fix $\delta > 0$ and a δ-neighborhood $U_\delta = \{z \in X \times X : d(z, \Delta) \leq \delta\}$ of Δ. For simplicity we set $g = f \times f$.

We claim that

$$\{z \in X \times X : g^n(z) \in U_\delta, \ \forall n \in \mathbb{Z}\} = \bigcup_{x \in X} (\{x\} \times \Gamma_\delta(x)). \qquad (1.4)$$

In fact, take $z = (x, y)$ in the left-hand side set. Then, for all $n \in \mathbb{Z}$ there is $p_n \in X$ such that $d(f^n(x), p_n) + d(f^n(y), p_n) \leq \delta$ so $d(f^n(x), f^n(y)) \leq \delta$ for all $n \in \mathbb{Z}$ which implies $y \in \Gamma_\delta(x)$. Therefore z belongs to the right-hand side set. Conversely, if $z = (x, y)$ is in the right-hand side set then $d(f^n(x), f^n(y)) \leq \delta$ for all $n \in \mathbb{Z}$ so $d(g^n(x, y), (f^n(x), f^n(x))) = d(f^n(x), f^n(y)) \leq \delta$ for all $n \in \mathbb{Z}$ which implies that z belongs to the left-hand side set. The claim is proved.

Let F be the characteristic map of the left-hand side set in (1.4). It follows that $F(x, y) = \chi_{\Gamma_\delta(x)}(y)$ for all $(x, y) \in X \times X$ where χ_A if the characteristic map of $A \subset X$. So,

$$\mu^2(\{z \in X \times X : g^n(z) \in U_\delta, \ \forall n \in \mathbb{Z}\}) =$$

$$\int_X \int_X \chi_{\Gamma_\delta(x)}(y) d\mu(y) d\mu(x). \qquad (1.5)$$

Now suppose that μ is an expansive measure of f with constant δ. It follows that

$$\int_X \chi_{\Gamma_\delta(x)}(y) d\mu(y) = 0, \qquad \forall x \in X$$

therefore $\mu^2(\{z \in X \times X : g^n(z) \in U_\delta, \ \forall n \in \mathbb{Z}\}) = 0$ by (1.5).

Conversely, if $\mu^2(\{z \in X \times X : g^n(z) \in U_\delta, \ \forall n \in \mathbb{Z}\}) = 0$ for some $\delta > 0$, then (1.5) implies that $\mu(\Gamma_\delta(x)) = 0$ for μ-almost every $x \in X$. Then, μ is expansive by Lemma 1.20. This ends the proof. $\square$

Our final equivalence is given by using the idea of generators (see [89]). Call a finite open covering $\mathcal{A}$ of X *μ-generator* of a homeomorphism f if for every bisequence $\{A_n : n \in \mathbb{Z}\} \subset \mathcal{A}$ one has

$$\mu\left(\bigcap_{n \in \mathbb{Z}} f^n(\mathrm{Cl}(A_n))\right) = 0.$$

Theorem 1.24. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. Then, a Borel probability measure μ is an expansive measure of f if and only if f has a μ-generator.*

Proof. First suppose that μ is expansive and let δ be its expansivity constant. Take $\mathcal{A}$ as the collection of the open δ-balls centered at $x \in X$. Then, for any bisequence $A_n \in \mathcal{A}$ one has

$$\bigcap_{n\in\mathbb{Z}} f^n(\mathrm{Cl}(A_n)) \subset \Gamma_\delta(x), \qquad \forall x \in \bigcap_{n\in\mathbb{Z}} f^n(\mathrm{Cl}(A_n)),$$

so

$$\mu\left(\bigcap_{n\in\mathbb{Z}} f^n(\mathrm{Cl}(A_n))\right) \leq \mu(\Gamma_\delta(x)) = 0.$$

Therefore, $\mathcal{A}$ is a μ-generator of f.

Conversely, suppose that f has a μ-generator $\mathcal{A}$ and let $\delta > 0$ be a *Lebesgue number* of $\mathcal{A}$. If $x \in X$, then for every $n \in \mathbb{Z}$ there is $A_n \in \mathcal{A}$ such that the closed δ-ball around $f^n(x)$ belongs to $\mathrm{Cl}(A_n)$. It follows that

$$\Gamma_\delta(x) \subset \bigcap_{n\in\mathbb{N}} f^{-n}(\mathrm{Cl}(A_n))$$

so $\mu(\Gamma_\delta(x)) = 0$ since $\mathcal{A}$ is a μ-generator. □

1.4 Properties

Consider any map $f : X \to X$ in a metric space X. We already defined the omega-limit set $\omega(z)$ of z. In the invertible case we also define the *alpha-limit set*

$$\alpha(z) = \left\{ y \in X : y = \lim_{k\to\infty} f^{n_k}(z) \text{ for some sequence } n_k \to -\infty \right\}.$$

Following [74] we say that z is a *point with converging semiorbits under* a bijective map $f : X \to X$ if both $\alpha(z)$ and $\omega(z)$ reduce to singleton.

Denote by $A(f)$ the set of points with converging semiorbits under f.

An useful tool to study $A(f)$ is as follows. For all $x, y \in X$, $n \in \mathbb{N}^+$ and $m \in \mathbb{N}$ we define $A(x,y,n,m)$ as the set of points $z \in X$ satisfying

$$A(x,y,n,m) = \left\{ z : \max\{d(f^{-i}(z),x), d(f^i(z),y)\} \leq \frac{1}{n}, \quad \forall i \geq m \right\}.$$

An useful property of this set is given by the following lemma.

Lemma 1.25. *For every bijective map $f : X \to X$ of a separable metric space X there is a sequence $x_k \in X$ satisfying*

$$A(f) \subset \bigcap_{n \in \mathbb{N}^+} \bigcup_{k,k',m \in \mathbb{N}^+} A(x_k, x_{k'}, n, m). \tag{1.6}$$

Proof. Since X is separable there is a dense sequence x_k. Take $z \in A(f)$ and $n \in \mathbb{N}^+$. As $z \in A(f)$ there are points x, y such that $\alpha(z) = x$ and $\omega(z) = y$. Then, there is $m \in \mathbb{N}^+$ such that

$$\max\{d(f^{-i}(z),x), d(f^i(z),y)\} \leq \frac{1}{2n}, \qquad \forall i \geq m.$$

Since x_k is dense there are $k, k' \in \mathbb{N}^+$ such that

$$\max\{d(x,x_k), d(y,x_{k'})\} \leq \frac{1}{2n}.$$

Therefore,

$$d(f^{-i}(z),x_k) \leq d(f^{-i}(z),x) + d(x,x_k) \leq \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n},$$

and, analogously,

$$d(f^i(z),x_k) \leq d(f^i(z),x) + d(x,x_{k'}) \leq \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n},$$

for all $i \geq m$ proving $z \in A(x_k, x_{k'}, n, m)$ so (1.6) holds. □

An old result by Reddy [74] is stated below. For completeness we include its proof here (for another proof see Theorem 2.2.22 in [5]).

Theorem 1.26. *The set of points with converging semiorbits under a expansive homeomorphism of a compact metric space is countable.*

Proof. Let $f : X \to X$ be the expansive homeomorphism in the statement. Since compact metric spaces are separable we can choose a sequence x_k as in Lemma 1.25. Suppose by contradiction that $A(f)$ is uncountable. Applying (1.6) we see that $\bigcup_{k,k',m\in\mathbb{N}^+} A(x_k, x_{k'}, n, m)$ is uncountable for all $n \in \mathbb{N}^+$. Fix an expansivity constant e of f and a positive integer n with $\frac{1}{n} \leq \frac{e}{2}$. Then, there are $k, k', m \in \mathbb{N}$ such that $A(x_k, x_{k'}, n, m)$ is uncountable (and so infinite). Therefore, as X is compact, there are distinct $z, w \in A(x_k, x_{k'}, n, m)$ such that

$$d(f^i(z), f^i(w)) < e, \qquad \forall |i| \leq m.$$

As $z, w \in A(x_k, x_{k'}, n, m)$ we also have

$$d(f^{-i}(z), f^{-i}(w)) \leq d(f^{-i}(z), x_k) + d(f^{-i}(w), x_k) \leq \frac{e}{2} + \frac{e}{2} = e$$

and

$$d(f^i(z), f^i(w)) \leq d(f^i(z), x_{k'}) + d(f^i(w), x_{k'}) \leq \\ \frac{e}{2} + \frac{e}{2} = e, \qquad \forall |i| \geq m.$$

Consequently $w \in \Gamma_e(z)$ contradicting that e is an expansivity constant of f. Therefore $A(f)$ is countable and the proof follows. □

In light of this result we can ask if there is a version of it for expansive measures. Since countable sets corresponds naturally to zero measure sets it seems natural to prove the following result. Its proof follows by adapting the aforementioned proof of Theorem 1.26 to the measure theoretical context.

Theorem 1.27. *The set of points with converging semiorbits under a homeomorphism of a separable metric space has zero measure with respect to any expansive measure.*

Proof. Let $f : X \to X$ the homeomorphism in the statement and x_k be a sequence as in Lemma 1.25. Suppose by contradiction that there is an expansive measure μ such that $\mu(A(f)) > 0$. Applying (1.6) we get

$$\mu\left(\bigcup_{k,k',m\in\mathbb{N}^+} A(x_k, x_{k'}, n, m)\right) > 0 \qquad \forall n \in \mathbb{N}^+.$$

Fix an expansivity constant e of μ and a positive integer $n \leq \frac{e}{2}$. By the previous inequality there are $k, k', m \in \mathbb{N}$ such that

$$\mu(A(x_k, x_{k'}, n, m)) > 0.$$

Let us prove that there is $z \in A(x_k, x_{k'}, n, m)$ and $\delta_0 > 0$ satisfying

$$\mu(A(x_k, x_{k'}, n, m) \cap B[z, \delta]) > 0, \qquad \forall 0 < \delta < \delta_0, \tag{1.7}$$

where $B[\cdot, \delta]$ indicates the closed δ-ball operation.

Otherwise, for every $z \in A(x_k, x_{k'}, n, m)$ we could find $\delta_z > 0$ such that

$$\mu(A(x_k, x_{k'}, n, m) \cap B[z, \delta_z]) = 0.$$

Clearly

$$\left\{ B\left(z, \frac{\delta_z}{2}\right) : z \in A(x_k, x_{k'}, n, m) \right\}$$

is an open covering of $A(x_k, x_{k'}, n, m)$. As X is a separable metric space we have that $A(x_k, x_{k'}, n, m)$ also does, and, since separable metric spaces are Lindelöf, we have that the above open covering has a countable subcover $\{B_i : i \in \mathbb{N}\}$ (say). Therefore,

$$\mu(A(x_k, x_{k'}, n, m)) \leq \sum_{i \in \mathbb{N}} \mu(A(x_k, x_{k'}, n, m) \cap B_i) = 0$$

which is absurd. This proves the existence of z and $\delta_0 > 0$ satisfying (1.7).

On the other hand, as f is continuous, and both z and m are fixed, we can also find $0 < \delta_1 < \delta_0$ satisfying

$$d(f^i(z), f^i(w)) \leq \frac{e}{2} \text{ whenever } |i| \leq m \text{ and } d(z, w) < \delta_1.$$

We claim that

$$A(x_k, x_{k'}, n, m) \cap B[z, \delta_1] \subset \Gamma_e(z).$$

Indeed, take $w \in A(x_k, x_{k'}, n, m) \cap B[z, \delta_1]$.

Since $w \in B[z, \delta_1]$ one has $d(z, w) < \delta_1$ so

$$d(f^i(w), f^i(z)) \leq e, \quad \forall -m \leq i \leq m.$$

Since $z, w \in A(x_k, x_{k'}, n, m)$ and $\frac{1}{n} \leq \frac{e}{2}$ one has

$$d(f^{-i}(w), f^{-i}(z)) \leq d(f^{-i}(w), x_k) + d(f^{-i}(z), x_k) \leq e$$

and

$$d(f^i(w), f^i(z)) \leq d(f^i(w), x_{k'}) + d(f^i(z), x_{k'}) \leq e, \qquad \forall |i| \geq m.$$

All this together yield $w \in \Gamma_e(z)$ and the claim follows. Therefore,

$$0 < \mu(A(x_k, x_{k'}, n, m) \cap B[z, \delta_1]) \leq \mu(\Gamma_e(z))$$

which is absurd since e is an expansivity constant. This ends the proof. □

Remark 1.28. *If f is an expansive homeomorphism of a compact metric space, then every non-atomic Borel probability measure is an expansive measure of f. Then, Theorem 1.27 implies that the set of points with converging semiorbits under f has zero measure with respect to any non-atomic Borel probability measure. From this and well-known measure-theoretical results [73] we obtain that the set of points with converging semiorbits under f is countable. This provides another proof of the Reddy's result [74].*

The following lemma will be useful in the next proof.

Lemma 1.29 (see Lemma 4 p. 72 in [16]). *If $f : X \to X$ is a continuous map of a compact metric space X and $\omega(x)$ is finite for some $x \in X$, then there is a periodic point $z \in X$ of f such that $d(f^n(x), f^n(z)) \to 0$ as $n \to \infty$.*

Proof. Take any nonempty proper closed subset $F \subset \omega(x)$. We claim that $F \cap \mathrm{Cl}(\omega(x) \setminus F) \neq \emptyset$. Otherwise there are open sets O_1, O_2 such that $\omega(x) \setminus F \subset O_1$, $F \subset O_2$ and $\mathrm{Cl}(O_2) \cap f(\mathrm{Cl}(O_1)) = \emptyset$. For n large, $f^n(x)$ belongs to O_1 or O_2 and in both for infinitely many n' s. Then, there is an infinite sequence n_k with $f^{n_k}(x) \in O_1$ and $f^{n_k+1}(x) \in O_2$. Any limit point y of $f^{n_k}(x)$ satisfies $y \in \mathrm{Cl}(O_1) \cap f^{-1}(\mathrm{Cl}(O_2))$ thus $\mathrm{Cl}(O_2) \cap f(\mathrm{Cl}(O_1)) \neq \emptyset$ which is absurd. This proves the claim.

Since $\omega(x)$ is finite there is a periodic orbit $P \subset \omega(x)$. If $P \neq \omega(x)$ we could apply the claim to the closed subset $F = \omega(x) \setminus P$ yielding $(\omega(x) \setminus P) \cap P \neq \emptyset$ which is absurd. Therefore, $P = \omega(x)$ from which the result easily follows. □

By *heteroclinic point* of a bijective map $f : X \to X$ on a metric space X we mean any point for which both the alpha and the omega-limit sets reduce to periodic orbits. The lemma below relates homoclinic and points with converging semiorbits. Denote by $Het(f)$ the set of heteroclinic points of f.

Lemma 1.30. *If $f : X \to X$ is a homeomorphism of a compact metric space X, then*

$$Het(f) \subset \bigcup_{n \in \mathbb{N}^+} A(f^n).$$

Proof. If $x \in Het(f)$, then both $\alpha(x)$ and $\omega(x)$ are finite sets. Applying Lemma 1.29 we get a periodic point y such that $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$. Denoting by n_y the period of y we get $d(f^{kn_y}(x), y) \to 0$ as $k \to \infty$ and so $\omega_{f^{n_y}}(x) = \{y\}$. Analogously, $\alpha_{f^{n_z}}(x) = \{z\}$ for some periodic point z of period n_z. Taking $n = n_y n_z$ we obtain $n \in \mathbb{N}^+$ such that $\alpha_{f^n}(x) = z$ and $\omega_{f^n}(x) = y$ so $x \in A(f^n)$ and the inclusion follows. □

Theorem 1.27 and Lemma 1.30 have the following consequence.

Theorem 1.31. *The set of heteroclinic points of a homemorphism in a compact metric space has measure zero with respect to any expansive measure.*

Proof. Let $f : X \to X$ be a homeomorphism of a compact metric space. By Lemma 1.30 we have that the set of heteroclinic points satisfies the inclusion $Het(f) \subset \bigcup_{n \in \mathbb{N}^+} A(f^n)$. Now, take any expansive measure μ of f. By Lemma 1.22 we have that μ is also an expansive measure of f^n, and so, $\mu(A(f^n)) = 0$ for all $n \in \mathbb{N}^+$ by Theorem 1.27. Then, the inclusion above implies

$$\mu(Het(f)) \leq \sum_{n \in \mathbb{N}^+} \mu(A(f^n)) = 0$$

proving the result. □

A consequence of the above result is given below.

Corollary 1.32. *A homeomorphism with finite nonwandering set of a compact metric space has no expansive measures.*

Proof. This follows from Theorem 1.31 since every point for such homeomorphisms is heteroclinic. □

(In the probability case this corollary is a particular case of Corollary 1.19).

Another consequence is the following version of Theorem 3.1 in [86]. Denote by $\mathrm{Per}(f)$ the set of periodic points of f.

Corollary 1.33. *The set of periodic points of a homeomorphism of a compact metric space has measure zero with respect to any expansive measure.*

Proof. Let μ be an expansive measure of a homeomorphism f of a compact metric space. Denoting by $\mathrm{Fix}(f) = \{x \in X : f(x) = x\}$ the set of fixed points of a map f we have $\mathrm{Per}(f) = \cup_{n\in\mathbb{N}^+}\mathrm{Fix}(f^n)$. Now, μ is an expansive measure of f^n by Proposition 1.22 and every element of $\mathrm{Fix}(f^n)$ is a heteroclinic point of f^n thus $\mu(\mathrm{Fix}(f^n)) = 0$ for all n by Theorem 1.27. Therefore, $\mu(\mathrm{Per}(f)) \leq \sum_{n\in\mathbb{N}^+} \mu(\mathrm{Fix}(f^n)) = 0$. □

We finish this section by describing the expansive measures in dimension one. To start with we prove that there are no such measures for homeomorphisms of compact intervals.

Theorem 1.34. *A homeomorphism of a compact interval has no expansive measures.*

Proof. Suppose by contradiction that there is an expansive measure μ for some homeomorphism f of I. Since f is continuous we have that $\mathrm{Fix}(f) \neq \emptyset$. Such a set is also closed since f is continuous, so, its complement $I\backslash\mathrm{Fix}(f)$ in I consists of countably many open intervals J. It is also clear that every point in J is a point with converging semi-orbits therefore $\mu(I\backslash \mathrm{Fix}(f)) = 0$ by Theorem 1.27. But $\mu(\mathrm{Fix}(f)) = 0$ by Corollary 1.33 so $\mu(I) = \mu(\mathrm{Fix}(f)) + \mu(I\backslash \mathrm{Fix}(f)) = 0$ which is absurd. □

Next, we shall consider the circle S^1. Recall that an orientation-preserving homeomorphism of the circle S^1 is *Denjoy* if it is not topologically conjugated to a rotation [40].

Theorem 1.35. *A circle homeomorphism has expansive measures if and only if it is Denjoy.*

Proof. Let f be a Denjoy homeomorphism of S^1. As is well known f has no periodic points and exhibits a unique minimal set Δ which is a Cantor set [40]. In particular, Δ is compact without isolated points thus it exhibits a non-atomic Borel probability meeasure ν (c.f. Corollary 6.1 in [73]). On the other hand, one sees as in Example 1.2 of [25] that f/Δ is expansive so ν is an expansive measure of f/Δ. Then, we are done by Example 1.12.

Conversely, let μ be an expansive measure of a homeomorphism $f : S^1 \to S^1$ and suppose by contradiction that f is not Denjoy. Then, either f has periodic points or is conjugated to a rotation (c.f. [40]). In the first case we can assume by Proposition 1.22 that f has a fixed point. Then, we can cut open S^1 along the fixed point to obtain an expansive measure for some homeomorphism of I which contradicts Theorem 1.34. In the second case we have that f is conjugated to a rotation. Since μ is expansive it would follow from Example 1.13 that there are circle rotations with expansive measures. However, such rotations cannot exist by Example 1.9 since they are isometries. This contradiction proves the result. ☐

In particular, there are no expansive measures for C^2 diffeomorphisms of S^1. Similarly, there are no such measures for diffeomorphisms of S^1 with derivative of bounded variation.

1.5 Probabilistic proofs in expansive systems

The goal of this short section is to present the proof of some results in expansive systems using the ours.

To start with we shall prove the following result.

Proposition 1.36. *The set of periodic points of a measure-expansive homeomorphism $f : X \to X$ of a compact metric space X is countable.*

Proof. Since $\mathrm{Per}(f) = \cup_{n\in\mathbb{N}^+}\mathrm{Fix}(f^n)$ it suffices to prove that $\mathrm{Fix}(f^n)$ is countable for all $n \in \mathbb{N}^+$. Suppose by contradiction that $\mathrm{Fix}(f^n)$ is uncountable for some n. Since f is continuous we have that $\mathrm{Fix}(f^n)$ is also closed, so, it is complete and separable with respect to the induced topology. Thus, by Corollary 6.1 p. 210 in [73], there is a non-atomic Borel probability measure ν in $\mathrm{Fix}(f^n)$. Taking $\mu(A) = \nu(Y \cap A)$ for all borelian A of X we obtain a non-atomic Borel probability measure μ of X satisfying $\mu(\mathrm{Fix}(f^n)) = 1$. Since $\mathrm{Fix}(f^n) \subset \mathrm{Per}(f)$ we conclude that $\mu(\mathrm{Per}(f)) = 1$. However, μ is an expansive measure of f thus $\mu(\mathrm{Per}(f)) = 0$ by Corollary 1.33, a contradiction. This contradiction yields the result. □

Since every expansive homeomorphism of a compact metric space is measure-expansive the above proposition yields another proof of the following result due to Utz (see Theorem 3.1 in [86]).

Corollary 1.37. *The set of periodic points of an expansive homeomorphism of a compact metric space is countable.*

A second result is as follows.

Proposition 1.38. *Measure-expansive homeomorphisms of compact intervals do not exist.*

Proof. Suppose by contradiction that there is a measure-expansive homeomorphism of a compact interval I. Since the Lebesgue measure *Leb* of I is non-atomic we obtain that *Leb* is an expansive measure of f. However, there are no such measures for such homeomorphisms by Theorem 1.34. □

From this we obtain another proof of the following result by Jacobson and Utz [46] (details in [19]).

Corollary 1.39. *There are no expansive homeomorphisms of a compact interval.*

The following lemma is motivated by the well known property that for every homeomorphism f of a compact metric space X one has that $\mathrm{supp}(\mu) \subset \Omega(f)$ for all invariant Borel probability measure μ of f. Indeed, we shall prove that this is true also for all expansive measure oif every homeomorphism of S^1 even in the noninvariant case.

Lemma 1.40. *If $f : S^1 \to S^1$ is a homeomorphism, then $supp(\mu) \subset \Omega(f)$ for every expansive measure μ of f.*

Proof. Suppose by contradiction that there is $x \in \text{supp}(\mu) \setminus \Omega(f)$ for some expansive measure μ of f. Let δ be an expansivity constant of μ. Since $x \notin \Omega(f)$ we can assume that the collection of open intervals $f^n(B(x, \delta))$ as n runs over $\mathbb{Z}$ is disjoint. Therefore, there is $N \in \mathbb{N}$ such that the length of $f^n(B(x, \delta))$ is less than δ for $|n| \geq N$.

From this and the continuity of f we can arrange $\epsilon > 0$ such that $B(x, \epsilon) \subset \Gamma_\delta(x)$ therefore $\mu(\Gamma_\delta(x)) \geq \mu(B(x, \epsilon)) > 0$ as $x \in \text{supp}(\mu)$. This contradicts the expansiveness of μ and the result follows. □

A direct consequence of this lemma is the following.

Corollary 1.41. *A homeomorphism of S^1 has no expansive measures supported on S^1.*

Proof. Suppose by contradiction that there is a homeomorphism $f : S^1 \to S^1$ exhibiting an expansive measure μ with $supp(\mu) = S^1$. By Theorem 1.35 we have that f is Denjoy, and so, $\Omega(f)$ is nowhere dense. However, we have by Lemma 1.40 that $supp(\mu) \subset \Omega(f)$ so S^1 is nowhere dense too which is absurd. □

This corollary implies immediately the following one.

Corollary 1.42. *There are no measure-expansive homeomorphisms of S^1.*

Proof. If there were such homeomorphisms in S^1, then the Lebesgue measure would be an expansive measure of some homeomorphism of S^1 contradicting Corollary 1.41. □

From this we obtain the following classical fact due to Jacobsen and Utz [46]. Classical proofs can be found in Theorem 2.2.26 in [5], Subsection 2.2 of [25], Corollary 2 in [74] and Theorem 5.27 of [89].

Corollary 1.43. *There are no expansive homeomorphisms of S^1.*

1.6 Exercices

Exercice 1.44. Prove that the set of heteroclinic points of a homeomorphism of a compact metric space is Borel measurable.

Exercice 1.45. Are there homeomorphisms of compact metric spaces exhibiting a *unique* expansive measure? (Just in case prove that such a measure is invariant).

Exercice 1.46. Are there measure-expansive homeomorphisms of compact non-atomic metric spaces which are not countably-expansive?

Exercice 1.47. Prove (or disprove) that every homeomorphism possessing an expansive probability measure on a compact metric space also possesses ergodic expansive invariant probability measures.

Exercice 1.48. Are there measure-expansive homeomorphisms of S^2?

Exercice 1.49. It is well known that, for expansive homeomorphisms f on compact metric spaces, the *entropy map* $\mu \mapsto h_\mu(f)$ is uppersemicontinuous [89]. Are the expansive measures for homeomorphisms on compact metric space uppersemicontinuity points of the corresponding entropy map?

Exercice 1.50. It is well known that every compact metric space supporting expansive homeomorphisms has finite topological dimension [62]. Is the support of an expansive measure of a homeomorphisms of a compact metric space finite dimensional?

Exercice 1.51. A bijective map $f : X \to X$ of a metric space X is *distal* if

$$\inf_{n\in\mathbb{Z}} d(f^n(x), f^n(y)) > 0, \qquad \forall x \in X.$$

It is well known that a distal homeomorphism has zero topological entropy ([8],[36], [68]). Are there distal homeomorphisms with expansive measures of compact metric spaces.

Exercice 1.52. A generalization of the previows problem can be stated as follows. A *Li-Yorke pair* of a continuous map $f : X \to X$ is a pair $(x, y) \in X \times X$ which is *proximal* (i.e. $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$) but not *asymptotic* (i.e. $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$). We say that f is *almost distal* if it has no Li-Yorke pairs. Every distal homeomorphism is almost distal but not conversely. On the other hand, almost distal maps on compact metric space have some similarities with the distal ones as, in particular, all of them have zero topological entropy [15]. Prove (or disprove) that every almost distal homeomorphism of a compact metric space has expansive measures.

Exercice 1.53. Prove that the space of expansive measures $\mathcal{M}_{exp}(f)$ of a measurable map $f : X \to X$ on a metric space X is a *cone*, i.e., $\alpha\mu + \rho \in \mathcal{M}_{exp}(f)$ whenever $\alpha \in \mathbb{R}^+$ and $\mu, \rho \in \mathcal{M}_{exp}(f)$. Furthermore, if $\phi : X \to Y$ is a conjugation between f and another measurable map $g : Y \to Y$ of a metric space Y, then

$$f_*(\mathcal{M}_{exp}(f)) = \mathcal{M}_{exp}(g).$$

Exercice 1.54. Prove that if $f : S^1 \to S^1$ is a local homeomorphism of the circle S^1, then the Lebesgue measure is expansive for f if and only if f is expansive.

Exercice 1.55. Given metric spaces (X, d^X) and (Y, d^Y) we define the metric $d^{X\times Y}((x_1, y_1), (x_2, y_2)) = \max\{d^X(x_1, y_1), d^Y(x_2, y_2)\}$ in $X \times Y$. With respect to this metric prove that if μ and ν are expansive measures of the homeomorphisms $f : X \to X$ and $g : Y \to Y$, then so is the product measure $\mu \times \nu$ of $X \times Y$ for the product map $f \times g$. Give a counterexample for the converse (see Exercice 1.56).

Exercice 1.56. Let $f : X \to X$ be a homeomorphism of a metric space X. Prove that a Borel measure μ of X is expansive for f if and only if the product measure $\mu \times Leb$ of μ with the Lebesgue measure Leb of $[0, 1]$ is expansive for the product map $f \times Id : X \times [0, 1] \to X \times [0, 1]$, where Id is the identity map of $[0, 1]$.

Exercice 1.57. Prove that a homeomorphim $f : D \to D$ of the closed unit 2-disk $D \subset \mathbb{R}^2$ for which the alpha-limit set $\alpha(x) = (0, 0)$ for all $x \in Int(D)$ has no expansive measures.

Exercice 1.58. We say that a homeomorphism $f : X \to X$ of a metric space X is *proximal* if $\inf_{n\in\mathbb{Z}} d(f^n(x), f^n(y)) = 0$ for every $x, y \in X$. Find examples of proximal homeomorphisms of compact metric spaces with and without expansive measures.

Exercice 1.59. Motivated by [75] we call a non-trivial Borel measure μ of a metric space X *pointwise expansive* for a homeomorphism $f : X \to X$ if for every $x \in X$ there is $\delta_x > 0$ such that $\mu(\Gamma_{\delta_x}(x)) = 0$. Investigate the vality (or not) of the results of this chapter for pointwise expansive measures instead of expansive ones.

Exercice 1.60. Prove that the property of being an expansive measure is a metric invariant in the following sense: If $f : X \to X$ is a homeomorphism of a metric space (X, d) and d' is a metric of X equivalent to d, then a Borel measure is expansive for f if and only if it does for $f : (X, d') \to (X, d')$.

Chapter 2

Finite expansivity

2.1 Introduction

We already seem that every expansive homeomorphism of a non-atomic metric space is measure-expansive (i.e. it satisfies that every non-atomic Borel probability measure is an expansive measure). It is then natural to ask if the converse property holds, i.e., is a measure-expansive homeomorphism of a non-atomic metric space expansive? The results of this chapter will provide negative answer for this question even on compact metric spaces (see Exercice 3.44).

2.2 Preliminaries

In this section we establish some topological preliminaries. Let X a set and n be a nonnegative integer. Denote by $\#A$ the cardinality of A. The set of metrics of X (including ∞-metrics [32]) will be denoted by $\mathbb{M}(X)$. Sometimes we say that $\rho \in \mathbb{M}(X)$ has a certain property whenever its underlying metric space (X, ρ) does. For example, ρ is compact whenever (X, ρ) is, a point a is *ρ-isolated* in $A \subset X$ if it is isolated in A with respect to the metric space (X, ρ), etc.. The closure operation in (X, ρ) will be denoted by $Cl_\rho(\cdot)$. A map $f : X \to X$ is a *ρ-homeomorphism* if it is a homeomorphism of the metric space (X, ρ). If $x \in X$ and $\delta > 0$ we denote by $B^\rho[x, \delta]$ the closed δ-ball around x (or $B[x,\delta]$ if there is no confusion).

Given $\rho \in \mathbb{M}(X)$ and $A \subset X$ we say that ρ is *n-discrete on A* if there is $\delta > 0$ such that $\#(B[x,\delta] \cap A) \leq n$ for all $x \in A$. Equivalently, if there is $\delta > 0$ such that $\#(B[x,\delta] \cap A) \leq n$ for all $x \in X$. When necessary we emphasize δ by saying that *ρ is n-discrete on A with constant δ*. We say that ρ is n-discrete if it is n-discrete on X. Clearly ρ is n-discrete on A if and only if the *restricted metric $\rho/A \in \mathbb{M}(A)$* defined by $\rho/A(a,b) = \rho(a,b)$ for $a, b \in A$ is n-discrete.

Evidently, there are no 0-discrete metrics and the 1-discrete metrics are precisely the discrete ones. Since every n-discrete metric is m-discrete for $n \leq m$ one has that every discrete metric is n-discrete. There are however n-discrete metrics which are not discrete. Moreover, we have the following example ([1]).

Example 2.1. *Every infinite set X carries an n-discrete metric which is not $(n-1)$-discrete.*

Indeed, if $n = 1$ we simply choose ρ as the standard discrete metric $\delta(x,y)$ defined by $\delta(x,y) = 1$ whenever $x \neq y$. Otherwise, we can arrange n disjoint sequences $x^1_k, x^2_k \cdots, x^n_k$ in X and define ρ by $\rho(x,y) = \frac{1}{4+k}$ (if $(x,y) = (x^i_k, x^j_k)$ for some $k \in \mathbb{N}$ and $1 \leq i \neq j \leq n$) and $\rho(x,y) = \delta(x,y)$ (if not).

On the one hand, ρ is n-discrete with constant $\delta = \frac{1}{4}$ since $B\left[x, \frac{1}{4}\right]$ is either $\{x^1_k, \cdots, x^n_k\}$ or $\{x\}$ (depending on the case) and, on the other, ρ is not $(n-1)$-discrete since for all $\delta > 0$ the set of points x for which $\#B[x,\delta] = n$ is infinite (e.g. take $x = x^1_k$ with k large).

Remark 2.2. *None of the metrics in Example 2.1 can be compact for, otherwise, we could cover X with finitely many balls of radius $\delta = 1/4$ which would imply that X is finite.*

In the sequel we present some basic properties of n-discrete metrics. Clearly if ρ is n-discrete on A, then it is also n-discrete on B for all $B \subset A$. Moreover, if ρ is n-discrete on A and m-discrete on B, then it is $(n+m)$-discrete on $A \cup B$. A better conclusion is obtained when the distance between A and B is positive.

Lemma 2.3. *If ρ is n-discrete on A, m-discrete on B and $\rho(A,B) > 0$, then ρ is* $\max\{n,m\}$*-discrete on $A \cup B$.*

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[1] communicated by professors L. Florit and A. Iusem.

Proof. Choose $0 < \delta < \frac{\rho(A,B)}{2}$ such that $\#(B[x,\delta] \cap A) \leq n$ (for $x \in A$) and $\#(B[x,\delta] \cap B) \leq m$ (for $x \in B$). If $x \in A$ then $B[x,\delta] \cap B = \emptyset$ because $\delta < \frac{\rho(A,B)}{2}$ so $\#(B[x,\delta] \cap (A \cup B)) = \#(B[x,\delta] \cap A) \leq n \leq \max\{n,m\}$. If $x \in B$ then $B[x,\delta] \cap A = \emptyset$ because $\delta < \frac{\rho(A,B)}{2}$ so $\#(B[x,\delta] \cap (A \cup B)) = \#(B[x,\delta] \cap B) \leq m \leq \max\{n,m\}$. Then, ρ is $\max\{n,m\}$-discrete on $A \cup B$ with constant δ. □

Lemma 2.4. *If ρ is n-discrete on A, then A is ρ-closed and so $\rho(A,B) > 0$ for every ρ-compact subset B with $A \cap B = \emptyset$.*

Proof. We only have to prove the first part of the lemma. By hypothesis there is $\delta > 0$ such that $\#(B[x,\delta] \cap A) \leq n$ for all $x \in A$. Let x_k be a sequence in A converging to some $y \in X$. It follows that there is $k_0 \in \mathbb{N}^+$ such that $x_k \in B[y, \delta/2]$ for all $k \geq k_0$. Triangle inequality implies $\{x_k : k \geq k_0\} \subseteq B[x_{k_0}, \delta] \cap A$ and so $\{x_k : k \geq k_0\}$ is a finite set. As $x_k \to y$ we conclude that $y \in A$ hence A is closed. □

Now we prove that n-discreteness is preserved under addition of finite subsets.

Proposition 2.5. *If ρ is n-discrete on A, then ρ is n-discrete on $A \cup F$ for all finite $F \subset X$.*

Proof. We can assume that $A \cap F = \emptyset$. As F is finite (hence compact) we can apply Lemma 2.4 to obtain $\rho(A,F) > 0$. As F is finite one has that ρ is 1-discrete on F so ρ is n-discrete on $A \cup F$ by Lemma 2.3. □

For the next result we introduce some basic definitions. Let $f : X \to X$ be a map. We say that $A \subset X$ is *invariant* if $f(A) = A$. If f is bijective and $x \in X$ we denote by $O_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$ the orbit of x. An *isometry* (or *ρ-isometry* to emphasize ρ) is a bijective map f satisfying $\rho(f(x), f(y)) = \rho(x,y)$ for all $x, y \in X$.

The following elementary fact will be useful later one: If f is a ρ-isometry and $a \in X$ satisfies that a is ρ-isolated in $O_f(a)$, then ρ is discrete on $O_f(a)$. Indeed, if ρ were not discrete on $O_f(a)$, then there are integer sequences $n_k \neq m_k$ such that $\rho(f^{n_k}(a), f^{m_k}) \to 0$ as $k \to \infty$. As f is an isometry one has that $\rho(f^{n_k}(a), f^{m_k}(a)) = \rho(a, f^{l_k}(a))$, where $l_k = m_k - m_k$, so $\rho(a, f^{l_k}(a)) \to 0$ for some sequence $l_k \in \mathbb{Z} \setminus \{0\}$ thus a is not ρ-isolated in $O_f(a)$.

Given $d, \rho \in \mathbb{M}(X)$ we write $d \leq \rho$ whenever $d(x,y) \leq \rho(x,y)$ for all $x, y \in X$. We write $\rho \preceq d$ to indicate lower semicontinuity of the map $\rho : X \times X \to [0, \infty]$ with respect to the product metric $d \times d$ in $X \times X$. Equivalently, the following property holds for all sequences x_k, y_k in X and all $\delta > 0$, where $x_k \overset{d}{\to} x$ indicates convergence in (X, d):

$$x_k \overset{d}{\to} x, \quad y_k \overset{d}{\to} y \quad \text{and} \quad y_k \in B^\rho[x_k, \delta] \quad \Longrightarrow \quad y \in B^\rho[x, \delta]. \tag{2.1}$$

Hereafter we denote by $Fix(f) = \{x \in X : f(x) = x\}$ the set of fixed points of f, and by $Per(f) = \bigcup_{m \in \mathbb{N}^+} Fix(f^m)$ the set of periodic points of f.

The following proposition is inspired on Lemma 2 p. 176 of [89].

Proposition 2.6. *Let $d, \rho \in \mathbb{M}(X)$ be such that d is compact and $d \leq \rho \preceq d$. Let $f : X \to X$ be a map which is simultaneously a d-homeomorphism and a ρ-isometry. If A is an invariant set with countable complement which is n-discrete with respect to ρ and $Per(f) \cap A$ is countable, then ρ is n-discrete on $A \cup O_f(a)$ for all $a \in X$.*

Proof. We can assume $a \notin A$ (otherwise $A \cup O_f(a) = A$) so $\rho(A, a) > 0$ by Lemma 2.4. Since f is a ρ-isometry and A is invariant one has $\rho(A, f^i(a)) = \rho(A, a)$ so $\rho(A, O_f(a)) > 0$. Then, by Lemma 2.3, it suffices to prove that ρ is n-discrete on $O_f(a)$.

Suppose that it is not so. Then, as previously remarked, a is non ρ-isolated in $O_f(a)$. Since $d \leq \rho$ we have that a is also non ρ-isolated in $O_f(a)$. As f is a d-homeomorphism we conclude that $O_f(a)$ is a nonempty ρ-perfect set. As d is compact (and so F_{II}) we obtain that $Cl_d(O_f(a))$ is uncountable. As $X \setminus A$ is countable we conclude that $Cl_d(O_f(a)) \cap A$ is uncountable. Choose $x \in Cl_d(O_f(a)) \cap A$. Then, there is a sequence $l_k \in \mathbb{Z}$ such that

$$f^{l_k}(a) \overset{d}{\to} x. \tag{2.2}$$

Let $\delta > 0$ be such that ρ is n-discrete on A with constant δ. Since ρ is not n-discrete on $O_f(a)$ we can arrange different integers $N_1, \cdots, N_{n+1}$ satisfying

$$f^{N_j}(a) \in B^\rho[f^{N_1}(a), \delta], \qquad \forall j \in \{1, \cdots, n+1\}.$$

On the other hand, f is a ρ-isometry so the above inclusions yield

$$f^{N_j}(f^{l_k}(a)) \in B^\rho[f^{N_1}(f^{l_k}(a)), \delta], \qquad \forall j \in \{1, \cdots, n+1\}, \quad \forall k \in \mathbb{N}.$$

By taking limit as $k \to \infty$ in the above inclusion, keeping j fixed and applying (2.1) and (2.2) to obtain

$$f^{N_j}(x) \in B^\rho[f^{N_1}(x), \delta], \qquad \forall j \in \{1, \cdots, n+1\}.$$

Now observe that $f^{N_j}(x) \in A$ for all $j \in \{1, \cdots, n+1\}$ because A is invariant. Therefore,

$$\{f^{N_1}(x), \cdots, f^{N_{n+1}}(x)\} \subset B^\rho[f^{N_1}(x), \delta] \cap A.$$

But $\#(B^\rho[f^{N_1}(x), \delta] \cap A) \leq n$ by the choice of δ so the above inclusion implies $f^{N_j}(x) = f^{N_r}(x)$ for some different indexes $j, r \in \{1, \cdots, n+1\}$. As the integers $N_1, \cdots, N_{n+1}$ are different we conclute that $x \in Per(f)$ and so $x \in Per(f) \cap A$. Therefore,

$$Cl_d(O_f(a)) \cap A \subset Per(f) \cap A.$$

As $Cl_d(O_f(a)) \cap A$ is uncountable we conclude that $Per(f) \cap A$ also is thus we get a contradiction. This proves the result. □

Corollary 2.7. *Let $d, \rho \in \mathbb{M}(X)$ be such that d is compact and $d \leq \rho \preceq d$. Let $f : X \to X$ be a map which is simultaneously a d-homeomorphism and a ρ-isometry. If $Per(f)$ is countable and there are $a_1, \cdots, a_l \in X$ such that ρ is n-discrete on $X \setminus \bigcup_{i=1}^{l} O_f(a_i)$, then ρ is n-discrete.*

Proof. Define the invariant sets $A_j = X \setminus \bigcup_{i=j}^{l} O_f(a_i)$ for $1 \leq j \leq l$. As $X \setminus A_j = \bigcup_{i=j}^{l} O_f(a_i)$ one has that A_j has countable complement for all $1 \leq j \leq l$. On the other hand, ρ is n-discrete on A_1 by hypothesis and $Per(f) \cap A_1$ is countable (since $Per(f)$ is) so ρ is n-discrete on $A_2 = A_1 \cup O_f(a_1)$ by Proposition 2.6. By the same reasons if ρ is n-discrete on A_j, then ρ also is on $A_{j+1} = A_j \cup O_f(a_i)$. Then, the result follows by induccion. □

2.3 n-expansive systems

In this section we define and study the class of n-expansive systems. To motivate the definition we recall some classical definitions. Let (X, d) be a metric space and $A \subset X$. A map $f : X \to X$ is *positively expansive on* A if there is $\delta > 0$ such that for every $x, y \in A$ with $x \neq y$ there is $i \in \mathbb{N}$ such that $d(f^i(x), f^i(y)) > \delta$, or, equivalently, if $\{y \in A : d(f^i(x), f^i(y)) \leq \delta, \forall i \in \mathbb{N}\} = \{x\}$ for all $x \in A$. On the other hand, a bijective map $f : X \to X$ is *expansive on* A if there is $\delta > 0$ such that $\{y \in A : d(f^i(x), f^i(y)) \leq \delta, \forall i \in \mathbb{Z}\} = \{x\}$ for all $x \in A$. If $A = X$ we recover the notions of *positively expansive* and *expansive* maps respectively. These definitions suggest the following one.

Definition 2.8. *Given $n \in \mathbb{N}^+$ a bijective map (resp. map) f is* n-expansive *(resp.* positively n-expansive*) on A if there is $\delta > 0$ such that*

$$\#\{y \in A : d(f^i(x), f^i(y)) \leq \delta, \forall i \in \mathbb{Z}\} \leq n$$

$$(\textit{resp. } \#\{y \in A : d(f^i(x), f^i(y)) \leq \delta, \forall i \in \mathbb{N}\} \leq n)$$

$\forall x \in A$. *If case $A = X$ we say that f is* n-expansive *(resp.* positively n-expansive*).*

Clearly the 1-expansive bijective maps are precisely the expansive ones (which in turn are n-expansive for all $n \in \mathbb{N}^+$).

In the sequel we introduce two useful operators. For every $f : X \to X$ and $d \in \mathbb{M}(X)$ we define the pull-back metric

$$f_*(d)(x, y) = d(f(x), f(y))$$

(clearly $f_*(d) \in \mathbb{M}(X)$ if and only if f is 1-1). Using it we can define the operator $\mathcal{L}_f^+ : \mathbb{M}(X) \to \mathbb{M}(X)$ by

$$\mathcal{L}_f^+(d) = \sup_{i \in \mathbb{N}} f_*^i(d), \qquad \forall d \in \mathbb{M}(X).$$

If f is bijective we can define $\mathcal{L}_f : \mathbb{M}(X) \to \mathbb{M}(X)$ by

$$\mathcal{L}_f(d) = \sup_{i \in \mathbb{Z}} f_*^i(d), \qquad \forall d \in \mathbb{M}(X).$$

Lemma 2.9. *If f is bijective, then $d \leq \mathcal{L}_f(d)$ and f is a $\mathcal{L}_f(d)$-isometry. If, in addition, f is a d-homeomorphism, then $\mathcal{L}_f(d) \preceq d$.*

Proof. The first inequality is evident. As

$$f_*(\mathcal{L}_f(d))(x,y) =$$
$$\sup_{i\in\mathbb{Z}} d(f^{i+1}(x), f^{i+1}(y)) = \sup_{i\in\mathbb{Z}} d(f^i(x), f^i(y)) = \mathcal{L}_f(d)(x,y)$$

$(\forall x, y \in X)$ one has $f_*(\mathcal{L}_f(d)) = \mathcal{L}_f(d)$ hence f is an $\mathcal{L}_f(d)$-isometry. Now we prove $\mathcal{L}_f(d) \preceq d$ whenever f is a d-homeomorphism. Suppose that $x_k \overset{d}{\to} x$, $y_k \overset{d}{\to} y$ and $\mathcal{L}_f(d)(x_k, y_k) \leq \delta$ for all $k \in \mathbb{N}$. Fixing $i \in \mathbb{Z}$ the latter inequality implies $d(f^i(x_k), f^i(y_k)) \leq \delta$ for all k. As f is a d-homeomorphism one can take the limit as $k \to \infty$ in the last inequality to obtain $d(f^i(x), f^i(y)) \leq \delta$. As $i \in \mathbb{Z}$ is arbitrary we obtain $\mathcal{L}_f(d)(x,y) \leq \delta$ which together with (2.2) implies the result. □

These operators give the link between discreteness and expansiveness by the following result. Hereafter we shall write f is (positively) n-expansive (on A) *with respect to* d in order to emphazise the metric d in Definition 2.8.

Lemma 2.10. *The following properties hold for all $f : X \to X$, $A \subset X$ and $d \in \mathbb{M}(X)$:*

1. *f is positively n-expansive on A with respect to d if and only if $\mathcal{L}^+_f(d)$ is n-discrete on A.*
2. *If f is bijective, f is n-expansive on A with respect to d if and only if $\mathcal{L}_f(d)$ is n-discrete on A.*

Proof. Clearly for all $x \in X$ and $\delta > 0$ one has

$$B^{\mathcal{L}^+_f(d)}[x,\delta] \cap A = \{y \in A : d(f^i(x), f^i(y)) \leq \delta, \quad \forall i \in \mathbb{N}\},$$

so

$$\#(B^{\mathcal{L}^+_f(d)}[x,\delta] \cap A) \leq n \quad \Longleftrightarrow$$
$$\#(\{y \in A : d(f^i(x), f^i(y)) \leq \delta, \quad \forall i \in \mathbb{N}\}) \leq n$$

which proves the equivalence (1). The proof of the equivalence (2) is analogous. □

As a first application of the above equivalence we shall exhibit non-trivial examples of positively n-expansive maps. More precisely, we prove that every bijective map $f : X \to X$ with at least n non-periodic points ($n \geq 2$) carries a metric ρ making it continuous positively n-expansive but not positively $(n-1)$-expansive. Indeed, by hypothesis there are $x^1, \cdots, x^n \in X$ such that $f^i(x^j) \neq f^k(x^j)$, for all $1 \leq j \leq n$ and $i \neq k \in \mathbb{N}$, and $f^i(x^j) \neq f^i(x^k)$ for all $i \in \mathbb{N}$ and $1 \leq j \neq k \leq n$. Define the sequences $x^1_k, \cdots, x^n_k$ in X by $x^i_k = f^k(x^i)$ for $1 \leq i \leq n$ and $k \in \mathbb{N}$. Clearly these sequences are disjoint thus they induce a metric ρ in X which is n-discrete but not $(n-1)$-discrete as in Example 2.1. On the other hand, a straightforward computation yields $\mathcal{L}^+_f(\rho) = \rho$ thus f is continuous (in fact Lipschitz) for ρ. Since ρ is n-discrete and $\rho = \mathcal{L}^+_f(\rho)$ one has that $\mathcal{L}^+_f(\rho)$ is n-discrete so f is positively n-expansive by Lemma 2.10. Since ρ is not $(n-1)$-discrete and $\rho = \mathcal{L}^+_f(\rho)$ the same lemma implies that f is not positively $(n-1)$-expansive.

Notice however that none of the above metrics is compact (see for instance Remark 2.2). This fact leads the question as to whether a bijective map can carry a compact metric making it positively n-expansive but not positively $(n-1)$-expansive. Indeed, the following result gives a partial positive answer for this question.

Proposition 2.11. *For every $k \in \mathbb{N}^+$ there is a homeomorphism f_k of a compact metric space (X_k, ρ_k) which is positively 2^k-expansive but not positively $(2^k - 1)$-expansive.*

Proof. To start with we recall that a *Denjoy map* of the circle S^1 is a nontransitive homeomorphism of S^1 with irrational rotation number. As is well known [40] every Denjoy map h exhibits a unique minimal set E_h which is also a Cantor set.

Hereafter we fix the standard Riemannian metric l of S^1. We shall prove that h/E_h is positively 2-expansive with respect to l/E_h. Let α be half of the length of the largest interval I in the complement $S^1 \setminus E_h$ and $0 < \delta < \alpha$.

We claim that $Int(B^{\mathcal{L}^+_h(l)}[x, \delta]) \cap E_h = \emptyset$ for all $x \in E_h$. Otherwise, there is some $z \in Int(B^{\mathcal{L}^+_h(l)}[x, \delta]) \cap E_h$. Pick $w \in \partial I$ (thus $w \in E_h$). Since E_h is minimal there is a sequence $n_k \to \infty$ such that $h^{-n_k}(w) \to z$. Now, the interval sequence $\{h^{-n}(I) : n \in \mathbb{N}\}$

is disjoint so we have that the length of the intervals $h^{-n_k}(I) \to 0$ as $k \to \infty$. It turns out that there is some integer k such that $h^{-n_k}(I) \subset B^{\mathcal{L}_h^+(l)}[x,\delta]$. From this and the fact that $h(B^{\mathcal{L}_h^+(l)}[x,\delta]) \subset B^{\mathcal{L}_h^+(l)}[h(x),\delta]$ one sees that $I \subset B^{\mathcal{L}_h^+(l)}[h^{n_k}(x),\delta]$ which is clearly absurd because the length of I is greather than $\alpha > 2\delta$. This contradiction proves the claim.

Since $B^{\mathcal{L}_h^+(l)}[x,\delta]$ reduces to closed interval (possibly trivial) the claim implies that $B^{\mathcal{L}_h^+(l)}[x,\delta] \cap E_h$ consists of at most two points. It follows that $\mathcal{L}_h^+(l)$ is 2-discrete on E_h (with constant δ), so, h/E_h is positively 2-expansive with respect to l/E_h by Lemma 2.10. Since there are no positively expansive homeomorphisms on infinite compact metric spaces (e.g. [28]) one sees that h/E_h cannot be positively expansive with respect to l/E_h. Taking $X_1 = E_h$, $\rho_1 = l/E_h$ and $f_1 = h/E_h$ we obtain the result for $k = 1$. To obtain the result for $k \geq 2$ we shall proceed according to the following straightforward construction.

Take copies E_1, E_2 of E_h and recall the map

$$\max\{\cdot,\cdot\} : \mathbb{M}(E_1) \times \mathbb{M}(E_2) \to \mathbb{M}(E_1 \times E_2)$$

defined by

$$\max\{d_1, d_2\}(x,y) = \max\{d_1(x_1,y_1), d_2(x_2,y_2)\}$$

for all $x = (x_1,x_2)$ and $y = (y_1,y_2)$ in $E_1 \times E_2$. One clearly sees that

$$B^{\max\{d_1,d_2\}}[x,\delta] = B^{d_1}[x_1,\delta] \times B^{d_2}[x_2,\delta], \qquad \forall x \in E_1 \times E_2, \forall \delta > 0.$$

Afterward, take copies h_1, h_2 of h/E_h and define the product $h_1 \times h_2 : E_1 \times E_2 \to E_1 \times E_2$, $(h_1 \times h_2)(x) = (h_1(x_1), h_2(x_2))$. It turns out that

$$(h_1 \times h_2)_*(\max\{d_1,d_2\}) = \max\{h_{1*}(d_1), h_{2*}(d_2)\}$$

so

$$\mathcal{L}^+_{h_1\times h_2}(\max\{d_1,d_2\}) = \max\{\mathcal{L}^+_{h_1}(d_1), \mathcal{L}^+_{h_2}(d_2)\}$$

thus

$$B^{\mathcal{L}^+_{h_1\times h_2}(\max\{d_1,d_2\})}[x,\delta] = B^{\mathcal{L}^+_{h_1}(d_1)}[x_1,\delta] \times B^{\mathcal{L}^+_{h_2}(d_2)}[x_2,\delta].$$

Finally, take copies d_1, d_2 of the metric l/E_h each one in E_1, E_2 respectively. As h_i is positively 2-expansive with respect to d_i one has that $\mathcal{L}^+_{h_i}(d_i)$ is 2-discrete for $i = 1, 2$. We can choose the same constant for $i = 1, 2$ (δ say) thus,

$$\#(B^{\mathcal{L}^+_{h_1\times h_2}(\max\{d_1,d_2\})}[x,\delta]) =$$

$$\#(B^{\mathcal{L}^+_{h_1}(d_1)}[x_1,\delta]) \cdot \#(B^{\mathcal{L}^+_{h_2}(d_2)}[x_2,\delta]) \leq 2^2, \quad \forall x \in E_1 \times E_2. \quad (2.3)$$

Now, consider the compact metric space $(E_1 \times E_2, \max\{d_1, d_2\})$. It follows from (2.3) and Lemma 2.10 that $h_1 \times h_2$ (which is clearly a homeomorphism) is positively 2^2-expansive map with respect to $\max\{d_1, d_2\}$. One can see that $\#(B^{\mathcal{L}^+_{h_1\times h_2}(\max\{d_1,d_2\})}[x,\delta]) = 2^2$ for infinitely many x's and arbitrarily small δ thus $h_1 \times h_2$ cannot be positively $2^2 - 1$-expansive. Taking $X_2 = E_1 \times E_2$, $\rho_2 = \max\{d_1, d_2\}$ and $f_2 = h_1 \times h_2$ we obtain the result for $k = 2$.

By repeating this argument we obtain the result for arbitrary $k \in \mathbb{N}^+$ taking $X_2 = E_1 \times \cdots \times E_k$, $\rho_k = \max\{d_1, \cdots, d_k\}$ and $f_k = h_1 \times \cdots \times h_k$. $\square$

As a second application of the equivalence in Lemma 2.10 we establish the following lemma which is well-known among expansive systems (e.g. Lemma 1 in [89]).

Lemma 2.12. *If a homeomorphism f of a metric space (X, d) is n-expansive on A, then $Per(f) \cap A$ is countable.*

Proof. It follows from the hypothesis and Lemma 2.10 that there is $\delta > 0$ such that $\#(B^{\mathcal{L}_f(d)}[x,\delta] \cap A) \leq n$ for all $x \in X$.

First we prove that f^m is n-expansive on A, $\forall m \in \mathbb{N}^+$. Observe that f is continuous since d is compact so there is $\epsilon > 0$ such that $d(x, y) \leq \epsilon$ implies $d(f^i(x), f^i(y)) \leq \delta$ for all integer $-m \leq i \leq m$. Then, $B^{\mathcal{L}_{f^m}(d)}[x,\epsilon] \subset B^{\mathcal{L}_f(d)}[x,\delta]$ for all $x \in X$, so, $\#(B^{\mathcal{L}_{f^m}(d)}[x,\epsilon] \cap A) \leq \#(B^{\mathcal{L}_f(d)}[x,\delta] \cap A) \leq n$ for all $x \in A$. Therefore, $\mathcal{L}_{f^m}(d)$ is n-discrete on A (with constant ϵ) which implies that f^m is n-expansive on A by Lemma 2.10. This proves the assertion.

Since $Per(f) = \bigcup_{m\in\mathbb{N}^+} Fix(f^m)$ by the previous assertion we only have to prove that $Fix(f) \cap A$ is finite whenever f is n-expansive on A. To prove it suppose that there is an infinite sequence of fixed points $x_k \in A$. Since d is compact one can assume that $x_n \xrightarrow{d} x$ for some $x \in X$. On the other hand, one clearly has $\mathcal{L}_f(d) = d$ in $Fix(f)$ thus, by the triangle inequality on x, there is $n_0 \in \mathbb{N}$ such that $x_n \in B^{\mathcal{L}_f(d)}[x_{n_0}, \delta]$ for all $n \geq n_0$. Thus, $\#(B^{\mathcal{L}_f(d)}[x_{n_0}, \delta] \cap A) = \infty$ which contradicts the choice of δ above. This ends the proof. □

2.4 The results

In this section we state and prove our main results. The first one establishes that there are arbitrarily large values of n for which there are infinite compact metric spaces carrying positively n-expansive homeomorphisms. As is well known, this is not true in the positively expansive case (see for instance [28]).

Theorem 2.13. *For every $k \in \mathbb{N}^+$ there is an infinite compact metric space (X_k, ρ_k) carrying positively 2^k-expansive homeomorphisms which are not positively $(2^k - 1)$-expansive.*

Proof. Take X_k, ρ_k and f_k as in Proposition 2.11. As f_k is not positively $(2^k - 1)$-expansive one has that X_k is infinite. □

From this we obtain the following corollary.

Corollary 2.14. *There are compact metric spaces without isolated points exhibiting homeomorphism which are not positively expansive but for which every non-atomic Borel probability measure is positively expansive.*

Our second result generalizes the one in [12].

Theorem 2.15. *A map (resp. bijective map) of a metric space (X, d) is positively n-expansive (resp. n-expansive) if and only if it is positively n-expansive (resp. n-expansive) on $X \setminus F$ for some finite subset F.*

Proof. Obviously we only have to prove the if part. We do it in the positively n-expansive case as the n-expansive case follows analogously. Suppose that a map f of X is positively n-expansive on $X \setminus F$ for some finite subset F. Then, $\mathcal{L}_f^+(d)$ is n-discrete on $A = X \setminus F$ by Lemma 2.10. Since F is finite Proposition 2.5 implies that $\mathcal{L}_f^+(d)$ is n-discrete so f is positively n-expansive by Lemma 2.10. □

Finally we state our last result which extends a well-known property of expansive homeomorphisms (c.f. [85],[89]).

Theorem 2.16. *A necessary and sufficient condition for a homeomorphism f of a compact metric space (X, d) to be n-expansive is that f is n-expansive on $X \setminus \bigcup_{i=1}^{l} O_f(a_i)$ for some $a_1, \cdots, a_l \in X$.*

Proof. We only have to prove the if part. By hypothesis f is a d-homeomorphism so f is an $\mathcal{L}_f(d)$-isometry and $d \leq \mathcal{L}_f(d) \preceq d$ by Lemma 2.9. Since f is n-expansive on $A = X \setminus \bigcup_{i=1}^{l} O_f(a_i)$ one has that $Per(f) \cap A$ is countable by Lemma 2.12. As $X \setminus A = \bigcup_{i=1}^{l} O_f(a_i)$ is clearly countable we conclude that $Per(f)$ is countable. On the other hand, f is n-expansive on $X \setminus \bigcup_{i=1}^{l} O_f(a_i)$ so $\mathcal{L}_f(d)$ is n-discrete on $X \setminus \bigcup_{i=1}^{l} O_f(a_i)$ by Lemma 2.10. Then, $\mathcal{L}_f(d)$ is n-discrete by Corollary 2.7 and so f is n-expansive by Lemma 2.10. □

2.5 Exercices

Exercice 2.17. Prove the assertion in Remark 2.2.

Exercice 2.18. Prove that for every integer $n \geq 2$ there is an n-expansive homeomorphism of a compact metric space which is not $(n-1)$-expansive.

Exercice 2.19. Prove that every compact metric space with n-expansive homeomorphisms (for some $n \in \mathbb{N}^+$) has finite topological dimension and that the minimal sets of such a homeomorphism are zero-dimensional (for $n = 1$ see Mañé [62]).

Exercice 2.20. Prove that every n-expansive homeomorphism $f : X \to X$ of a metric space is *pointwise expansive*, i.e., for every $x \in X$ there is $\delta_x > 0$ such that $\Gamma_{\delta_x}(x) = \{x\}$ (see [75]).

Exercice 2.21. Prove that the non-expansive pointwise expansive homeomorphisms defined by Reddy in Section 3 of [75] are 2-expansive. Modify these examples to find for all $n \geq 2$ an n-expansive homeomorphism of a compact metric space which is not pointwise expansive.

Exercice 2.22. Are there differentiable manifolds supporting n-expansive homeomorphisms which are not $(n-1)$-expansive?

Exercice 2.23. Prove that an n-expansive homeomorphism $f : X \to X$ of a compact metric space X has *measures of maximal entropy*, i.e., a Borel measure μ satisfying the identity $h_\mu(f) = h(f)$ where $h_\mu(f)$ and $h(f)$ denotes the metric and topological entropies of f (for $n = 1$ see [38]).

Exercice 2.24. Prove (or disprove) that every n-expansive Axiom A diffeomorphism of a closed manifold is expansive.

Exercice 2.25. Are there n-expansive homeomorphisms of S^2? (the answer is negative for $n = 1$, see Hiraide [42] and Lewowicz [59]).

Chapter 3

Positively expansive measures

3.1 Introduction

Ergodic measures with positive entropy for continuous maps on compact metric spaces have been studied in the recent literature. For instance, [14] proved that the set of points belonging to a proper asymptotic pair (i.e. points whose stable classes are not singleton) constitute a full measure set. Moreover, [43] proved that if f is a homeomorphism with positive entropy $h_\mu(f)$ with respect to one of such measures μ, then there is a full measure set A such that for all $x \in A$ there is a closed subset $A(x)$ in the stable class of x satisfying $h(f^{-1}, A(x)) \geq h_\mu(f)$, where $h(\cdot, \cdot)$ is the Bowen's entropy operation [10]. We can also mention [27] which proved that every ergodic endomorphism on a Lebesgue probability space having positive entropy on finite measurable partitions formed by continuity sets is pairwise sensitive (see also Exercice 3.48).

In this chapter we introduce the notion of positively expansive measure and prove that every ergodic measure with positive entropy on a compact metric space is positively expansive. Using this result we will prove that, on compact metric spaces, every stable class has measure zero with respect to any ergodic measure with positive entropy (this seems to be new as far as we know). We also prove through the use of positively expansive measures that every continuous map on a compact metric space exhibiting countably many stable classes has zero topological entropy (a similar result with different techniques has been obtained in [45] but in the *transitive* case). Still in the compact case we prove that every continuous map which is Lyapunov stable on its recurrent set has zero topological entropy too (this is known but for *one-dimensional maps* [35], [81], [92]). Finally we use expansive measures to give necessary conditions for a continuous map on a complete separable metric space to be chaotic in the sense of Li and Yorke [60]. Most results in this chapter were obtained in [6] and [7].

3.2 Definition

In this chapter we introduce the notion of positively expansive measure. First we recall the following definition.

Definition 3.1. *A continuous map $f : X \to X$ of a metric space X is* positively expansive *(c.f. [33]) if there is $\delta > 0$ such that for every pair of distinct points $x, y \in X$ there is $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \delta$. Equivalently, f is positively expansive if there is $\delta > 0$ such that $\Phi_\delta(x) = \{x\}$, where*

$$\Phi_\delta(x) = \{y \in X : d(f^i(x), f^i(y)) \leq \delta, \forall i \in \mathbb{N}\}$$

(again we write $\Phi_\delta^f(x)$ to indicate dependence on f).

This motivates the following definition

Definition 3.2. *A* positively expansive measure *of a measurable map $f : X \to X$ is a Borel probability measure μ for which there is $\delta > 0$ such that $\mu(\Phi_\delta(x)) = 0$ for all $x \in X$. The constant δ will be referred to an* positive expansivity constant *of μ.*

As in the invertible case we have that a measure μ is a positively expansive measure of f if and only if there is $\delta > 0$ such that $\mu(\Phi_\delta(x)) = 0$ for μ-a.e. $x \in X$. An atomic measures μ cannot be an expansive measure of any map and every non-atomic Borel probability measure is a positively expansive measure of any is positively expansive map.

Example 3.3. *There are nonexpansive continuous maps on certain compact metric spaces for which every non-atomic measure is expansive (e.g. an n-expansive homeomorphism with $n \geq 2$). The homeomorphism $f(x) = 2x$ in $\mathbb{R}$ exhibits positively expansive measures (e.g. the Lebesgue measure) but not positively expansive* invariant *ones.*

Contrary to what happen in the expansive case ([79], [77]), there are infinite compact metric spaces supporting homeomorphisms with positively expansive measures (extreme cases will be discussed in Exercice 3.43). On the other hand, a necessary and sufficient for a measure to be positively expansive is given as in the homeomorphism case.

We shall need a previous result stated as follows. Let $f : X \to X$ be a measurable map of a metric space X. Given $x \in X$, $n \in \mathbb{N}^+$ and $\delta > 0$ we define

$$B[x, n, \delta] = \bigcap_{i=0}^{n-1} f^{-i}(B[f^i(x), \delta]). \tag{3.1}$$

A basic property of these sets is given below.

$$\Phi_\delta(x) = \bigcap_{n=1}^{\infty} B[x, n, \delta]. \tag{3.2}$$

Since, in addition, $B[x, m, \delta] \subset B[x, n, \delta]$ for $n \leq m$, we obtain

$$\mu(\Phi_\delta(x)) = \lim_{n\to\infty} \mu(B[x, \delta, n])$$

for every $x \in X$ and every Borel probability measure μ of X.

From this we obtain the pointwise convergence

$$\mu_\delta = \lim_{n\to\infty} \mu_{\delta,n} \tag{3.3}$$

where $\mu_\delta, \mu_{\delta,n} : X \to \mathbb{R}^+$ are the functions defined by

$$\mu_\delta(x) = \mu(\Phi_\delta(x)) \quad \text{and} \quad \mu_{\delta,n}(x) = \mu(B[x, \delta, n]). \tag{3.4}$$

Moreover, μ is positively expansive if and only if there is $\delta > 0$ such that

$$\liminf_{n\to\infty} \mu(B[x, n, \delta]) = 0, \qquad \text{for all } x \in X. \tag{3.5}$$

It follows that if $n \in \mathbb{N}^+$, then μ is a positively expansive measure of f, if and only if it is a positively expansive measure of f^n. The proof of these assertions is analogous to the corresponding results for homeomorphisms (see exercices 3.34 and 3.35).

Next we present the following lemma dealing with the measurability of the map μ_δ.

Lemma 3.4. *If $f : X \to X$ is a continuous map of a compact metric space X and μ is a finite Borel measure of X, then μ_δ is a measurable map for every $\delta > 0$.*

Proof. Fix $\delta > 0$, $n \in \mathbb{N}^+$ and define

$$D_n = \{(x, y) \in X \times X : d(f^i(x), f^i(y)) \leq \delta, \quad \forall 0 \leq i \leq n-1\}.$$

Denote by $\mathcal{B}(Y)$ the Borel σ-algebra associated to a topological space X. Since f is continuous we have that D_n is closed in $X \times X$ with respect to the product topology. From this we obtain $D_n \in \mathcal{B}(X \times X)$. But since X is compact the product σ-algebra $\mathcal{B}(X) \otimes \mathcal{B}(X)$ satisfies $\mathcal{B}(X) \otimes \mathcal{B}(X) = \mathcal{B}(X \times X)$ (e.g. Lemma 6.4.2 in [17]). Therefore $D_n \in \mathcal{B}(X) \otimes \mathcal{B}(X)$. This allows us to apply the Fubini Theorem (e.g. Theorem 3.4.1 in [17]) to conclude that the map $x \mapsto \int_X \chi_{D_n}(x, y) d\mu(y)$ is measurable, where χ_{D_n} denotes the characteristic function of D_n. But it follows from the definition of D_n that

$$\mu_{\delta,n}(x) = \int_X \chi_{D_n}(x, y) d\mu(y)$$

so $\mu_{\delta,n}$ is measurable, $\forall n \in \mathbb{N}^+$. It follows from (3.3) that μ_δ is the pointwise limit of measurable functions and so measurable. □

As in the expansive case we have the following observation for bijective maps $f : X \to X$, namely,

$$f(\Phi_\delta(x)) \subset \Phi_\delta(f(x)), \qquad \forall (x, \delta) \in X \times \mathbb{R}^+.$$

Using it we obtain the elementary lemma below.

Lemma 3.5. *Let $f : X \to X$ be a homeomorphism of a metric space X. If μ is an expansive measure with expansivity constant δ of f, then so is $f_*^{-1}\mu$.*

Following the proof of Lemma 1.17 and using (3.5) (instead of Lemma 1.16) we can obtain the following result.

Lemma 3.6. *If $f : X \to X$ is a homeomorphism of a metric space X, then every invariant measure of f which is the limit (with respect to the weak-* topology) of a sequence of positively expansive probability measures with a common expansivity constant of f is positively expansive for f.*

Using this lemma we obtain the following result closely related to Example 3.3.

Theorem 3.7. *A homeomorphism of a compact metric space has positively expansive probability measures if and only if it has positively expansive* invariant *probability measures.*

Proof. Let μ be a positively expansive measure with positive expansivity constant δ of a homeomorphism $f : X \to X$ of a compact metric space X. By Lemma 3.5 we have that $f_*^{-1}\mu$ is a positively expansive measure with positive expansivity constant δ of f. Therefore, $f_*^{-i}\mu$ is a positively expansive measure with positively expansivity constant δ of f $(\forall i \in \mathbb{N})$, and so,

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} f_*^{-i}\mu, \qquad n \in \mathbb{N}^+$$

is a sequence of positively expansive probability measures of f with common expansivity constant δ. As X is compact there is a subsequence $n_k \to \infty$ such that μ_{n_k} converges to a Borel probability measure μ. Since μ is clearly invariant for f^{-1} and f is a homeomorphism we have that μ is also an invariant measure of f. Then, we can apply Lemma 3.6 to this sequence to obtain that μ is a positively expansive measure of f. ☐

An equivalent condition for positively μ-expansiveness is given using the idea of positive generators as in Lemma 3.3 of [26]. Call a finite open covering $\mathcal{A}$ of X *positive μ-generator* of f if for every sequence $\{A_n : n \in \mathbb{N}\} \subset \mathcal{A}$ one has

$$\mu\left(\bigcup_{n\in\mathbb{N}} f^n(\mathrm{Cl}(A_n))\right) = 0.$$

As in the homeomorphism case we obtain the following proposition.

Proposition 3.8. *Let $f : X \to X$ be a continuous map of a compact metric space X. Then, a Borel probability measure of X is a positively invariant measure of f if and only if if f has a positive μ-generator.*

We shall use this proposition to obtain examples of positively expansive measures. If M is a closed manifold we call a differentiable map $f : M \to M$ *volume expanding* if there are constants $K > 0$ and $\lambda > 1$ such that $|\det(Df^n(x))| \geq K\lambda^n$ for all $x \in M$ and $n \in \mathbb{N}$. Denoting by *Leb* the Lebesgue measure we obtain the following proposition.

Proposition 3.9. *The Lebesgue measure Leb is a positively expansive measure of every volume expanding map of a closed manifold.*

Proof. If f is volume expanding there are $n_0 \in \mathbb{N}$ and $\rho > 1$ such that $g = f^{n_0}$ satisfies $|\det(Dg(x))| \geq \rho$ for all $x \in M$. Then, for all $x \in M$ there is $\delta_x > 0$ such that

$$Leb(g^{-1}(B[x,\delta])) \leq \rho^{-1} Leb(B[x,\delta]), \qquad \forall x \in M, \forall 0 < \delta < \delta_x. \tag{3.6}$$

Let δ be half of the Lebesgue number of the open covering $\{B(x,\delta_x) : x \in M\}$ of M. By (3.6) any finite open covering of M by δ-balls is a positive *Leb*-generator, so, *Leb* is positively expansive for g by Proposition 3.8. Since $g = f^{n_0}$ we conclude that *Leb* is a positively expansive measure of f (see the remark after (3.5)). □

As in the homeomorphism case we obtain an equivalent condition for positively expansiveness using the diagonal. Given a map g of a metric space Y and a Borel probability ν in Y we say that $I \subset Y$ is a *ν-repelling set* if there is a neighborhood U of I satisfying

$$\nu(\{z \in Y : g^n(z) \in U, \forall n \in \mathbb{N}\}) = 0.$$

As in the homeomorphism case we can prove the following.

Proposition 3.10. *Let $f : X \to X$ be a continuous map of a compact metric space X. Then, a Borel probability measure of X is a positively expansive for f if and only if the diagonal Δ is a μ^2-repelling set of $f \times f$.*

We shall use the following useful characterization of positively expansive measures which is analogous to the expansive case (c.f. Lemma 1.20).

Lemma 3.11. *A Borel probability measure μ is positively expansive for a measurable map f if and only if there is $\delta > 0$ such that*

$$\mu(\Phi_\delta(x)) = 0, \qquad \forall \mu\text{-a.e. } x \in X. \tag{3.7}$$

This lemma together with the corresponding definition for expansive maps suggests the following.

Definition 3.12. *A* positively expansive constant *of a Borel probability measure μ is a constant $\delta > 0$ satisfying (3.7).*

3.3 Properties

In this section we select the properties of positively expansive measures we shall use later one. For the first one we need the following definition.

Definition 3.13. *Given a map $f : X \to X$ and $p \in X$ we define $W^s(p)$, the* stable set *of p, as the set of points x for which the pair (p, x) is* asymptotic, *i.e.,*

$$W^s(p) = \left\{ x \in X : \lim_{n\to\infty} d(f^n(x), f^n(p)) = 0 \right\}.$$

By a stable class *we mean a subset equals to $W^s(p)$ for some $p \in X$.*

The following shows that every stable class is negligible with respect to any expansive *invariant* measure.

Proposition 3.14. *The stable classes of a measurable map have measure zero with respect to any positively expansive* invariant *measure.*

Proof. Let $f : X \to X$ a measurable map and μ be a positively expansive invariant measure. Denoting by $B[\cdot,\cdot]$ the closed ball operation one gets

$$W^s(p) = \bigcap_{i\in\mathbb{N}^+} \bigcup_{j\in\mathbb{N}} \bigcap_{k\geq j} f^{-k}\left(B\left[f^k(p), \frac{1}{i} \right] \right).$$

As clearly

$$\bigcup_{j\in\mathbb{N}}\bigcap_{k\geq j} f^{-k}\left(B\left[f^k(p),\frac{1}{i+1}\right]\right)\subseteq\bigcup_{j\in\mathbb{N}}\bigcap_{k\geq j} f^{-k}\left(B\left[f^k(p),\frac{1}{i}\right]\right),$$

$(\forall i\in\mathbb{N}^+)$ we obtain

$$\mu(W^s(p))\leq\lim_{i\to\infty}\sum_{j\in\mathbb{N}}\mu\left(\bigcap_{k\geq j} f^{-k}\left(B\left[f^k(p),\frac{1}{i}\right]\right)\right). \tag{3.8}$$

On the other hand,

$$\bigcap_{k\geq j} f^{-k}\left(B\left[f^k(p),\frac{1}{i}\right]\right)=f^{-j}\left(\Phi_{\frac{1}{i}}(f^j(p))\right)$$

so

$$\mu\left(\bigcap_{k\geq j} f^{-k}\left(B\left[f^k(p),\frac{1}{i}\right]\right)\right)=$$

$$\mu\left(f^{-j}\left(\Phi_{\frac{1}{i}}(f^j(p))\right)\right)=\mu\left(\Phi_{\frac{1}{i}}(f^j(p))\right)$$

since μ is invariant. Then, taking i large, namely, $i>\frac{1}{\epsilon}$ where ϵ is a expansivity constant of μ (c.f. Definition 3.12) we obtain $\mu\left(\Phi_{\frac{1}{i}}(f^j(p))\right)=0$ so

$$\mu\left(\bigcap_{k\geq j} f^{-k}\left(B\left[f^k(p),\frac{1}{i}\right]\right)\right)=0.$$

Replacing in (3.8) we get the result. □

For the second property we will use the following definition [35].

Definition 3.15. *A map $f:X\to X$ is said to be* Lyapunov stable *on $A\subset X$ if for any $x\in A$ and any $\epsilon>0$ there is a neighborhood $U(x)$ of x such that $d(f^n(x),f^n(y))<\epsilon$ whenever $n\geq 0$ and $y\in U(x)\cap A$.*

(Notice the difference between this definition and the corresponding one in [81].) The following implies that measurable sets where the map is Lyapunov stable are negligible with respect to any expansive measure (invariant or not).

Proposition 3.16. *If a measurable map of a separable metric space is Lyapunov stable on a measurable set A, then A has measure zero with respect to any positively expansive measure.*

Proof. Fix a measurable map $f : X \to X$ of a separable metric space X, a positively expansive measure μ and $\Delta > 0$. Since μ is regular there is a closed subset $C \subset A$ such that

$$\mu(A \setminus C) \leq \Delta.$$

Let us compute $\mu(C)$.

Fix a positive expansivity constant ϵ of μ (c.f. Definition 3.12). Since f is Lyapunov stable on A and $C \subset A$ for every $x \in C$ there is a neighborhood $U(x)$ such that

$$d(f^n(x), f^n(y)) < \epsilon \qquad \forall n \in \mathbb{N}, \forall y \in U(x) \cap C. \tag{3.9}$$

On the other hand, C is separable (since X is) and so Lindelöf with the induced topology. Consequently, the open covering $\{U(x) \cap C : x \in C\}$ of C admits a countable subcovering $\{U(x_i) \cap C : i \in \mathbb{N}\}$. Then,

$$\mu(C) \leq \sum_{i \in \mathbb{N}} \mu\left(U(x_i) \cap C\right). \tag{3.10}$$

Now fix $i \in \mathbb{N}$. Applying (3.9) to $x = x_i$ we obtain $U(x_i) \cap C \subset \Phi_\epsilon(x_i)$ and then $\mu\left(U(x_i) \cap C\right) \leq \mu(\Phi_\epsilon(x)) = 0$ since ϵ is a positive expansivity constant. As i is arbitrary we obtain $\mu(C) = 0$ by (3.10).

To finish we observe that

$$\mu(A) = \mu(A \setminus C) + \mu(C) = \mu(A \setminus C) \leq \Delta$$

and so $\mu(A) = 0$ since Δ is arbitrary. This ends the proof. □

From these propositions we obtain the following corollary. Recall that the *recurrent set* of $f : X \to X$ is defined by $R(f) = \{x \in X : x \in \omega_f(x)\}$, where

$$\omega_f(x) = \left\{ y \in X : y = \lim_{k \to \infty} f^{n_k}(x) \text{ for some sequence } n_k \to \infty \right\}.$$

Corollary 3.17. *A measurable map of a separable metric space which either has countably many stable classes or is Lyapunov stable on its recurrent set has no positively expansive invariant measures.*

Proof. First consider the case when there are countably many stable classes. Suppose by contradiction that there exists a positively expansive invariant measure. Since the collection of stable classes is a partition of the space it would follow from Proposition 3.14 that the space has measure zero which is absurd.

Now consider the case when the map f is Lyapunov stable on $R(f)$. Again suppose by contradiction that there is a positively expansive invariant measure μ. Since μ is invariant we have $\mathrm{supp}(\mu) \subset R(f)$ by Poincaré recurrence. However, since f is Lyapunov stable on $R(f)$ we obtain $\mu(R(f)) = 0$ from Proposition 3.16 so $\mu(\mathrm{supp}(\mu))) = \mu(R(f)) = 0$ which is absurd. This proves the result. □

3.4 Applications

We start this section by proving that positive entropy implies expansiveness among ergodic invariant measures for continuous maps on compact metric spaces. Afterward we include some short applications.

To star with we introduce the following basic result due to Brin and Katok [18]. Let μ be an invariant measure of a measurable map $f : X \to X$ of a metric space X. The *entropy* of μ with respect to f is defined by

$$h_\mu(f) = \sup\{h_\mu(f, P) : P \text{ is a finite measurable partition of } X\},$$

where

$$h_\mu(f, P) = -\lim_{n\to\infty} \frac{1}{n} \sum_{\xi \in P_{n-1}} \mu(\xi) \log \mu(\xi)$$

and P_n is the pullback partition of P under f^n.

Theorem 3.18 (Brin-Katok Theorem). *If μ is a non-atomic ergodic invariant measure of a continuous map $f : X \to X$ of a compact metric space, then*

$$\sup_{\delta>0} \liminf_{n\to\infty} -\frac{\log(\mu(B[x, n, \delta]))}{n} = h_\mu(f), \qquad \mu\text{-a.e. } x \in X.$$

Next we state the main result of this section.

Theorem 3.19. *Every ergodic invariant probability measure with positive entropy of a continuous map on a compact metric space is positively expansive.*

Proof. Let μ be an ergodic invariant measure μ with positive entropy $h_\mu(f) > 0$ of a continuous map $f : X \to X$ on a compact metric space X. Fix $\delta > 0$ and define

$$X_\delta = \{x \in X : \mu(\Phi_\delta(x)) = 0\}.$$

Clearly $X_\delta = \mu_\delta^{-1}(0)$ (where μ_δ is defined in (3.4) and so X_δ is measurable by Lemma 3.4. Then, we are left to prove by Lemma 3.11 that there is $\delta > 0$ such that $\mu(X_\delta) = 1$.

Fix $x \in X$. It follows from the definition of $\Phi_\delta(x)$ that $\Phi_\delta(x) \subset f^{-1}(\Phi_\delta(f(x)))$ so

$$\mu(\Phi_\delta(x)) \leq \mu(\Phi_\delta(f(x)))$$

since μ is invariant. Then, $\mu(\Phi_\delta(x)) = 0$ whenever $x \in f^{-1}(X_\delta)$ yielding

$$f^{-1}(X_\delta) \subset X_\delta.$$

Denote by $A\Delta B$ the symmetric difference of the sets A, B. Since $\mu(f^{-1}(X_\delta)) = \mu(X_\delta)$ the above implies that X_δ is *essentially invariant*, i.e., $\mu(f^{-1}(X_\delta)\Delta X_\delta) = 0$. Since μ is ergodic we conclude that $\mu(X_\delta) \in \{0, 1\}$ for all $\delta > 0$. Then, we are left to prove that there is $\delta > 0$ such that $\mu(X_\delta) > 0$. To find it we proceed as follows.

For all $\delta > 0$ we define the map $\phi_\delta : X \to I\!R \cup \{\infty\}$,

$$\phi_\delta(x) = \liminf_{n\to\infty} -\frac{\log \mu(B[x, n, \delta])}{n}.$$

Take $h = \frac{h_\mu(f)}{2}$ (thus $h > 0$) and define

$$X^m = \left\{x \in X : \phi_{\frac{1}{m}}(x) > h\right\}, \qquad \forall m \in I\!N^+.$$

Notice that $\phi_\delta(x) \geq \phi_{\delta'}(x)$ whenever $0 < \delta < \delta'$.

From this it follows that $X^m \subset X^{m'}$ for $m \leq m'$ and further

$$\left\{x \in X : \sup_{\delta>0} \phi_\delta(x) = h_\mu(f)\right\} \subset \bigcup_{m\in I\!N^+} X^m.$$

Then,

$$\mu\left(\left\{x \in X : \sup_{\delta>0} \phi_\delta(x) = h_\mu(f)\right\}\right) \le \lim_{m\to\infty} \mu(X^m).$$

But μ is non-atomic for it is ergodic invariant with positive entropy. So, the Brin-Katok Theorem implies

$$\mu\left(\left\{x \in X : \sup_{\delta>0} \phi_\delta(x) = h_\mu(f)\right\}\right) = 1$$

yielding

$$\lim_{m\to\infty} \mu(X^m) = 1.$$

Consequently, we can fix $m \in I\!N^+$ such that

$$\mu(X^m) > 0.$$

We shall prove that $\delta = \frac{1}{m}$ works.

Let us take $x \in X^m$. It follows from the definition of X^m that $\mu(B[x,n,\delta]) < e^{-hn}$ for all n large. Since $h > 0$ we conclude that $\lim_{n\to\infty} \mu(B[x,n,\delta]) = 0$. Since $\mu_{\delta,n}(x) = \mu(B[x,n,\delta])$ we conclude from (3.3) that $\mu(\Phi_\delta(x)) = 0$ thus $x \in X_\delta$. As $x \in X^m$ is arbitrary we obtain $X^m \subset X_\delta$ whence

$$0 < \mu(X^m) \le \mu(X_\delta)$$

and the proof follows. □

The converse of the above theorem is false, i.e., a positively expansive measure may have zero entropy even in the ergodic invariant case. A counterexample is as follows.

Example 3.20. *There are continuous maps in the circle exhibiting ergodic invariant measures with zero entropy which, however, are positively expansive.*

Proof. Since all circle homeomorphisms have zero topological entropy it remains to prove that every Denjoy map h exhibits positively expansive measures. As is well-known h is uniquely ergodic and the support of its unique invariant measure μ is a *minimal set*, i.e., a set which is minimal with respect to the property of being compact invariant. We shall prove that this measure is positively expansive. Denote by E the support of μ. It is well known that E is a Cantor set. Let α be half of the length of the biggest interval I in the complement $S^1 - E$ of E and take $0 < \delta < \alpha/2$. Fix $x \in S^1$ and denote by $Int(\cdot)$ the interior operation. We claim that $Int(\Phi_\delta(x)) \cap E = \emptyset$. Otherwise, there is some $z \in Int(\Phi_\delta(x)) \cap E$. Pick $w \in \partial I$ (thus $w \in E$). Since E is minimal there is a sequence $n_k \to \infty$ such that $h^{-n_k}(w) \to z$. Since μ is a finite measure, the interval sequence $\{h^{-n}(I) : n \in I\!N\}$ is disjoint, we have that the length of the intervals $h^{-n_k}(I) \to 0$ as $k \to \infty$. It turns out that there is some integer k such that $h^{-n_k}(I) \subset \Phi_\delta(x)$.

From this and the fact that $h(\Phi_\delta(x)) \subset \Phi_\delta(h(x))$ one sees that $I \subset B[h^{n_k}(x), \delta]$ which is clearly absurd because the length of I is greather than $\alpha > 2\delta$. This contradiction proves the claim. Since $\Phi_\delta(x)$ is either a closed interval or $\{x\}$ the claim implies that $\Phi_\delta(x) \cap E = \Phi_\delta(x) \cap E$ consists of at most two points. Since μ is clearly non-atomic we conclude that $\mu(\Phi_\delta(x)) = 0$. Since $x \in S^1$ is arbitrary we are done. $\square$

A first application of Theorem 3.19 is as follows.

Theorem 3.21. *The stable classes of a continuous map of a compact metric space have measure zero with respect to any ergodic invariant measure with positive entropy.*

Proof. In fact, since these measures are positively expansive by Theorem 3.19 we obtain the result from Proposition 3.14. $\square$

We can also use Theorem 3.19 to compute the topological entropy of certain continuous maps (for the related concepts see [3] or [89]). As a motivation let us mention the known facts that both *transitive* continuous maps with countably many stable classes on compact metric spaces and continuous maps of the *interval* or the *circle* which are Lyapunov stable on their recurrent sets have zero topological entropy (see Corollary 2.3 p. 263 in [45], [35], Theorem B in [81] and [92]). Indeed we improve these result in the following way.

Theorem 3.22. *A continuous map of a compact metric space which either has countably many stable classes or is Lyapunov stable on its recurrent set has zero topological entropy.*

Proof. If the topological entropy were not zero the variational principle [89] would imply the existence of ergodic invariant measures with positive entropy. But by Theorem 3.19 these measures are positively expansive against Corollary 3.17. □

Example 3.23. *An example satisfying the first part of Theorem 3.22 is the classical pole North-South diffeomorphism on spheres. In fact, the only stable sets of this diffeomorphism are the stable sets of the poles. The Morse-Smale diffeomorphisms [40] are basic examples where these hypotheses are fulfilled.*

Now we use positively expansive measures to study the chaoticity in the sense of Li and Yorke [60]. Recall that if $\delta \geq 0$ a *δ-scrambled set* of $f : X \to X$ is a subset $S \subset X$ satisfying

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > \delta \tag{3.11}$$

for all different points $x, y \in S$. The following result relates scrambled sets with positively expansive measures.

Theorem 3.24. *A continuous map of a Polish metric space carrying an uncountable δ-scrambled set for some $\delta > 0$ also carries positively expansive probability measures.*

Proof. Let X a Polish metric space and $f : X \to X$ be a continuous map carrying an uncountable δ-scrambled set for some $\delta > 0$. Then, by Theorem 16 in [13], there is a *closed* uncountable δ-scrambled set S. As S is closed and X is Polish we have that S is also a Polish metric space with respect to the induced metric. As S is uncountable we have from [73] that there is a non-atomic Borel probability measure ν in S. Let μ be the Borel probability induced by ν in X, i.e., $\mu(A) = \nu(A \cap S)$ for all Borelian $A \subset X$. We shall prove that this measure is positively expansive. If $x \in S$ and $y \in \Phi_{\frac{\delta}{2}}(x) \cap S$ we have that $x, y \in S$ and $d(f^n(x), f^n(y)) \leq \frac{\delta}{2}$ for all $n \in \mathbb{N}$ therefore $x = y$ by the second inequality in (3.11). We conclude that $\Phi_{\frac{\delta}{2}}(x) \cap S = \{x\}$ for all $x \in S$.

As ν is non-atomic we obtain $\mu(\Phi_{\frac{\delta}{2}}(x)) = \nu(\Phi_{\frac{\delta}{2}}(x) \cap S) = \nu(\{x\}) = 0$ for all $x \in S$. On other hand, it is clear that every open set which does not intersect S has μ-measure 0 so μ is supported in the closure of S. As S is closed we obtain that μ is supported on S. We conclude that $\mu(\Phi_{\frac{\delta}{2}}(x)) = 0$ for μ-a.e. $x \in X$, so, μ is positively expansive by Lemma 3.11. □

Corollary 3.25. *Every homeomorphism of a compact metric space carrying an uncountable δ-scrambled set for some δ also carries positively expansive invariant probability measures.*

Proof. Every compact metric space is Polish so Theorem 3.24 yields positively expansive probability measures. Now apply Theorem 3.7. □

Now recall that a continuous map is *Li-Yorke chaotic* if it has an uncountable 0-scrambled set.

Until the end of this section M will denote either the interval $I = [0, 1]$ or the unit circle S^1.

Corollary 3.26. *Every Li-Yorke chaotic map in M carries positively expansive measures.*

Proof. Theorem in p. 260 of [31] together with theorems A and B in [57] imply that every Li-Yorke chaotic map in M has an uncountable δ-scrambled set for some $\delta > 0$. Then, we obtain the result from Theorem 3.24. □

It follows from Example 3.20 that there are continuous maps with zero topological entropy in the circle exhibiting positively expansive *invariant* measures. This leads to the question whether the same result is true on compact intervals. The following consequence of the above corollary gives a partial positive answer for this question.

Example 3.27. *There are continuous maps with zero topological entropy in the interval carrying positively expansive measures.*

Indeed, by [47] there is a continuous map of the interval, with zero topological entropy, exhibiting a δ-scrambled set of positive Lebesgue measures for some $\delta > 0$. Since sets with positive Lebesgue measure are uncountable we obtain a positively expansive measure from Theorem 3.24.

Another interesting example is the one below.

Example 3.28. *The Lebesgue measure is an ergodic invariant measure with positive entropy of the tent map $f(x) = 1 - |2x - 1|$ in I. Therefore, this measure is positively expansive by Theorem 3.19.*

It follows from this example that there are continuous maps in I carrying positively expansive measures μ *with full support* (i.e. $supp(\mu) = I$). These maps also exist in S^1 (e.g. an expanding map). Now, we prove that Li-Yorke and positive topological entropy are equivalent properties among these maps in I. But previously we need a result based on the following well-known definition.

A *wandering interval* of a map $f : M \to M$ is an interval $J \subset M$ such that $f^n(J) \cap f^m(J) = \emptyset$ for all different integers $n, m \in \mathbb{N}$ and no point in J belongs to the stable set of some periodic point.

Lemma 3.29. *If $f : M \to M$ is continuous, then every wandering interval has measure zero with respect to every positively expansive measure.*

Proof. Let J a wandering interval and μ be a positively expansive measure with expansivity constant ϵ (c.f. Definition 3.12). To prove $\mu(J) = 0$ it suffices to prove $Int(J) \cap supp(\mu) = 0$ since μ is non-atomic. As J is a wandering interval one has $\lim_{n\to\infty} |f^n(J)| = 0$, where $|\cdot|$ denotes the length operation.

From this there is a positive integer n_0 satisfying

$$|f^n(J)| < \epsilon, \qquad \forall n \geq n_0. \tag{3.12}$$

Now, take $x \in Int(J)$. Since f is clearly uniformly continuous and n_0 is fixed we can select $\delta > 0$ such that $B[x, \delta] \subset Int(J)$ and $|f^n(B[x, \delta])| < \epsilon$ for $0 \leq n \leq n_0$. This together with (3.12) implies $|f^n(x) - f^n(y)| < \epsilon$ for all $n \in \mathbb{N}$ therefore $B[x, \delta] \subset \Phi_\epsilon(x)$ so $\mu(B[x, \delta]) = 0$ since ϵ is an expansivity constant. Thus $x \notin supp(\mu)$ and we are done. □

From this we obtain the following corollary.

Corollary 3.30. *A continuous map carrying positively expansive measures with full support of the circle or the interval has no wandering intervals. Consequently, a continuous map of the interval carrying positively expansive measures with full support is Li-Yorke chaotic if and only if it has positive topological entropy.*

Proof. The first part is a direct consequence Lemma 3.29 while, the second, follows from the first since a continuous interval map without wandering intervals is Li-Yorke chaotic if and only if it has positive topological entropy [82]. □

3.5 The smooth case

Now we turn our attention to smooth ergodic theory. The motivation is the well-known fact that a diffeomorphism restricted to a hyperbolic basic set is expansive. In fact, it is tempting to say that every hyperbolic ergodic measures of a diffeomorphism is positively expansive (or at least expansive) but the Dirac measure supported on a hyperbolic periodic point is a counterexample. This shows that some extra hypotheses are necessary for a hyperbolic ergodic measure to be positively expansive. Indeed, by the results above, we only need to recognize which conditions imply positive entropy. Let us state some basic definitions in order to present our result.

Assume that X is a compact manifold and that f is a C^1 diffeomorphism. We say that point $x \in X$ is a *regular point* whenever there are positive integers $s(x)$ and numbers $\{\lambda_1(x), \cdots, \lambda_{s(x)}(x)\} \subset \mathbb{R}$ (called *Lyapunov exponents*) such that for every $v \in T_xM \setminus \{0\}$ there is $1 \leq i \leq s(x)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\| = \lambda_i(x).$$

An invariant measure μ is called *hyperbolic* if there is a measurable subset A with $\mu(A) = 1$ such that $\lambda_i(x) \neq 0$ for all $x \in A$ and all $1 \leq i \leq s(x)$.

On the other hand, the Eckmann-Ruelle conjecture [9] asserts that every hyperbolic ergodic measure μ is *exac-dimensional*, i.e., the limit below

$$d(x) = \lim_{r\to 0^+} \frac{\mu(B(x,r))}{r}$$

exists and is constant μ-a.e. $x \in X$. This constant is the so-called *dimension* of μ.

With these definitions we can state the following result.

Theorem 3.31. *Let f be a C^2 diffeomorphism of a compact manifold.*

1. *Every hyperbolic ergodic measure of f which either has positive dimension or is absolutely continuous with respect to Lebesgue is positively expansive.*

2. *If f has a non-atomic hyperbolic ergodic measure, then f also has a positively expansive ergodic invariant measure.*

Proof. Let us prove (1). First assume that the measure has positive dimension. As noticed in [9] p. 761 Theorem C′ p. 544 in [58] implies that if the entropy vanishes, then the stable and unstable dimension of the measure also do. In such a case we have from Theorem F p. 548 in [58] that the measure has zero dimension, a contradiction. Therefore, the measure has positive entropy and then we are done by Theorem 3.19.

Now assume that the measure is absolutely continuous with respect to the Lebesgue measure. Then, it is non-atomic so the argument in the proof of Theorem 4.2 p. 167 in [56] implies that it has at least one positive Lyapunov exponent. Therefore, the Pesin formula (c.f. p. 139 in [52]) implies positive entropy so we are done by Theorem 3.19.

To prove (2) we only have to see that Corollary 4.2 in [52] implies that every diffeomorphism as in the statement of (2) has positive topological entropy. Then, we are done by the variational principle and Theorem 3.19 (see Exercice 3.39). □

3.6 Exercices

Exercice 3.32. Prove that the Lebesgue measure of S^2 is an expansive measure of the Bernoulli diffeomorphism in S^2 found in [53] (therefore Corollary 1.41 is false for S^2 instead of S^1). Is such a diffeomorphism measure-expansive?

Exercice 3.33. Is it true that every continuous map $f : X \to X$ exhibiting positively expansive probability measures of a compact metric space also exhibits positively expansive *invariant* measures?

Exercice 3.34. Let $f : X \to X$ be a measurable map of a metric space X. Prove that a Borel probability measure μ of X is positively expansive for f if and only if if there is $\delta > 0$ such that

$$\liminf_{n\to\infty} \mu(B[x,n,\delta]) = 0, \qquad \text{for all } x \in X,$$

where $B[x,n,\delta]$ is defined in 3.1.

Exercice 3.35. Prove the equivalence of the following properties for every continuous map $f : X \to X$ of compact metric space and every Borel probabiltity measure μ of X:

- μ is positively expansive for f;
- there is $n \in \mathbb{N}^+$ such that μ is positively expansive for f^n;
- μ is positively expansive for f^n, $\forall n \in \mathbb{N}^+$.

Exercice 3.36. Prove that the constant map cannot have positively expansive measures.

Exercice 3.37. Prove lemmas 3.5, 3.11, 3.6 and Proposition 3.10.

Exercice 3.38. Prove that e Borel probability measure μ is positively expansive for a measurable map $f : X \to X$ of a metric space X if and only if there are $\delta > 0$ and a negligible set X_0 of X such that $\mu(\Phi_\delta(x)) = 0$ for every $x \in X_0$ (*negligible* means that $\mu(A) = 0$ for every measurable subset $A \subset X_0$).

Exercice 3.39. Prove that every continuous map of a compact metric space $f : X \to X$ satisfies the variational principle,

$$h(f) = \sup_{\mu\in\mathcal{M}^*_{exp}(f)} h_\mu(f),$$

where $\mathcal{M}^*_{exp}(f)$ denotes the space of expansive invariant probability measures of f (of course, with the supremum being zero if $\mathcal{M}^*_{exp}(f) = \emptyset$).

Exercice 3.40. Following [21] we say that a Borel measure μ of a metric space X is *almost expansive* for a Borel isomorphism $f : X \to X$ if there is $\delta > 0$ such that $\Gamma_\delta(x) = \{x\}$ for μ-a.e. $x \in X$. Find examples of homeomorphisms of compact metric spaces exhibiting expansive ergodic invariant measures which are not almost expansive.

Exercice 3.41. Prove that a circle homeomorphism exhibits positively expansive measures if and only if it is Denjoy.

Exercice 3.42. Investigate the parameter values $0 \leq \beta \leq 1$ for which the Lebesgue measure is positively expansive for the map $g_\beta(x) = \beta(1 - |2x-1|)$ of the unit interval I. Analogously for the family $f_\lambda(x) = \lambda x(1-x)$, $0 \leq \lambda \leq 4$.

Exercice 3.43. Call a continuous map $f : X \to X$ of a non-atomic metric space X *positively measure-expansive* if every non-atomic Borel measure is positively expansive for f. Find examples of positively measure-expansive homeomorphisms of non-atomic compact metric spaces.

Exercice 3.44. Find a homeomorphism of a compact non-atomic metric space which is positively measure-expansive (and so measure-expansive) but not expansive.

Exercice 3.45. Prove that there are no Li-Yorke chaotic homeomorphisms of the circle. Conclude that there are continuous maps of compact metric spaces with positively expansive measures which are not Li-Yorke chaotic.

Exercice 3.46. Does every Li-Yorke chaotic map of a compact metric space carry positively expansive measures?

Exercice 3.47. Are there diffeomorphisms of closed manifolds exhibiting non-atomic hyperbolic measure which are neither expansive nor positively expansive?

Exercice 3.48. A measurable map $f : X \to X$ of a metric is called *pairwise sensitive* for a Borel measure μ if there is $\delta > 0$ such that

$$\mu^2\left(\{(x,y) \in X \times X : \exists n \in \mathbb{N} \text{ such that } d(f^n(x), f^n(y)) \geq \delta\}\right) = 1$$

(c.f. [27]). Prove that a Borel probability measure μ of X is positively expansive for f if and only if f is pairwise sensitive for μ.

Chapter 4

Measure-sensitive maps

4.1 Introduction

In this chapter we will try to extend the notion of measure expansivity from metric to measurable spaces. For this we introduce the auxiliary definition of *measure-sensitive partitions* and *measure-sensitive spaces*. We prove that every non-atomic standard probability spaces is measure-sensitive and that every measure-sensitive probability spaces is non-atomic. With this concept we introduce the notion of *measure-sensitive partition* which will play a role similar to the expansivity constant for expansive maps. We prove that in a non-atomic probability space every strong generator is a measure-sensitive partition but not conversely (results about strong generators can be found in [41], [48], [69], [70] and [71]). We exhibit examples of measurable maps in non-atomic probability spaces carrying measure-sensitive partitions which are not strong generators. Motivated by these examples we shall study the *measure-sensitive maps*([1]) i.e. measurable maps on measure spaces carrying measure-sensitive partitions. Indeed, we prove that every measure-sensitive map is aperiodic and also, in the probabilistic case, that its corresponding space is non-atomic.

From this we obtain a characterization of nonsingular countable

[1]Called measure-expansive maps in [64]

to one measure-sensitive mappings on non-atomic Lebesgue probability spaces as those having strong generators. Furthermore, we prove that every ergodic measure-preserving map with positive entropy is a probability space is measure-sensitive (thus extending a result in [27]). As an application we obtain some properties for ergodic measure-preserving maps with positive entropy (c.f. corollaries 4.14 and 4.20). A reference for the results in this chapter is [64].

4.2 Measure-sensitive spaces

Hereafter the term *countable* will mean either finite or countably infinite.

A *measure space* is a triple $(X, \mathcal{B}, \mu)$ where X is a set, $\mathcal{B}$ is a σ-algebra of subsets of X and μ is a positive measure in $\mathcal{B}$. A *probability space* is one for which $\mu(X) = 1$.

A *partition* is a disjoint collection P of nonempty measurable sets whose union is X. We allow $\mu(\xi) = 0$ for some $\xi \in P$. Given partitions P and Q we write $P \leq Q$ to mean that each member of Q is contained in some member of P (mod 0). A sequence of partitions $\{P_n : n \in \mathbb{N}\}$ (or simply P_n) is *increasing* if $P_i \leq P_j$ for $i \leq j$.

Motivated by the concept of *Lebesgue sequence of partitions* (c.f. p. 81 in [61]) we introduce the following definition.

Definition 4.1. *A* measure-sensitive sequence of partitions *of a measure space* $(X, \mathcal{B}, \mu)$ *is an increasing sequence of countable partitions* P_n *such that*

$$\mu\left(\bigcap_{n\in\mathbb{N}} \xi_n\right) = 0$$

for all sequence of measurable sets ξ_n *satisfying* $\xi_n \in P_n$, $\forall n \in \mathbb{N}$. *A* measure-sensitive space *is a measure space carrying measure-sensitive sequences of partitions.*

Let us present a sufficient condition for sequences of partitions to be measure-sensitive. Recall that the join of finitely many partitions $P_0, \cdots, P_n$ is the partition defined by

$$\bigvee_{k=0}^{n} P_k = \left\{\bigcap_{k=0}^{n} \xi_k : \xi_k \in P_k, \forall 0 \leq k \leq n\right\}.$$

Certainly

$$P_n = \bigvee_{k=0}^{n} f^{-k}(P), \qquad n \in \mathbb{N}, \tag{4.1}$$

defines an increasing sequence of countable partitions satisfying

$$P_n(x) = \bigcap_{k=0}^{n} f^{-k}(P(f^k(x)), \qquad \forall x \in X.$$

Since for all $x \in X$ one has

$$\{y \in X : f^n(y) \in P(f^n(x)), \quad \forall n \in \mathbb{N}\} =$$

$$\bigcap_{n=0}^{\infty} f^{-n}(P(f^n(x))) = \bigcap_{n=0}^{\infty} P_n(x),$$

we obtain that the identity below

$$\lim_{n\to\infty} \sup_{\xi \in P_n} \mu(\xi) = 0 \tag{4.2}$$

is sufficient condition for an increasing sequence P_n of countable partitions to be measure-sensitive. It is also necessary in probability spaces (see Exercice 4.27).

Let us state basic properties of the measure-sensitive spaces. For this recall that a measure space is *non-atomic* if it has no *atoms*, i.e., measurable sets A of positive measure satisfying $\mu(B) \in \{0, \mu(A)\}$ for every measurable set $B \subset A$. Recall that a *standard probability space* is a probability space $(X, \mathcal{B}, \mu)$ whose underlying measurable space $(X, \mathcal{B})$ is isomorphic to a Polish space equipped with its Borel σ-algebra (e.g. [1]).

The class of measure-sensitive spaces is broad enough to include all non-atomic standard probability spaces. Precisely we have the following proposition.

Proposition 4.2. *Every non-atomic standard probability spaces is measure-sensitive.*

Proof. It is well-known that if $(X, \mathcal{B}, \mu)$ is a non-atomic standard probability space, then there are a measurable subset $X_0 \subset X$ with

$\mu(X \setminus X_0) = 0$ and a sequence of countable partitions Q_n of X_0 such that $\bigcap_{n\in\mathbb{N}} \xi_n$ contains at most one point for every sequence of measurable sets ζ_n in X_0 satisfying $\zeta_n \in Q_n$, $\forall n \in \mathbb{N}$ (c.f. [61] p. 81). Defining $P_n = \{X \setminus X_0\} \cup Q_n$ we obtain an increasing sequence of countable partitions of $(X, \mathcal{B}, \mu)$. It suffices to prove that this sequence is measure-sensitive. For this take a fixed (but arbitrary) sequence of measurable sets ξ_n of X with $\xi_n \in P_n$ for all $n \in \mathbb{N}$. It follows from the definition of P_n that either $\xi_n = X \setminus X_0$ for some $n \in \mathbb{N}$, or, $\xi_n \in Q_n$ for all $n \in \mathbb{N}$. Then, the intersection $\bigcap_{n\in\mathbb{N}} \xi_n$ either is contained in $X \setminus X_0$ or reduces to a single measurable point. Since both $X \setminus X_0$ and the measurable points have measure zero (for non-atomic spaces are diffuse [10]) we obtain $\mu\left(\bigcap_{n\in\mathbb{N}} \xi_n\right) = 0$. As ξ_n is arbitrary we are done. □

Although measure-sensitive probability spaces need not be standard (Exercice 4.26) we have that all of them are non-atomic. Indeed, we have the following result of later usage.

Proposition 4.3. *Every measure-sensitive probability spaces is non-atomic.*

Proof. Suppose by contradiction that a measure-sensitive probability space $(X, \mathcal{B}, \mu)$ has an atom A. Take a measure-sensitive sequence of partitions P_n. Since A is an atom one has that $\forall n \in \mathbb{N}\ \exists! \xi_n \in P_n$ such that $\mu(A \cap \xi_n) > 0$ (and so $\mu(A \cap \xi_n) = \mu(A)$). Notice that $\mu(\xi_n \cap \xi_{n+1}) > 0$ for, otherwise, $\mu(A) \geq \mu(A \cap (\xi_n \cup \xi_{n+1})) = \mu(A \cap \xi_n) + \mu(A \cap \xi_{n+1}) = 2\mu(A)$ which is impossible in probability spaces. Now observe that $\xi_n \in P_n$ and $P_n \leq P_{n+1}$, so, there is $L \subset P_{n+1}$ such that

$$\mu\left(\xi_n \triangle \bigcup_{\zeta\in L} \zeta\right) = 0. \tag{4.3}$$

If $\xi_{n+1} \cap \left(\bigcup_{\zeta\in L} \zeta\right) = \emptyset$ we would have $\xi_n \cap \xi_{n+1} = \xi_n \cap \xi_{n+1} \setminus \bigcup_{\zeta\in L} \zeta$ yielding

$$\mu(\xi_n \cap \xi_{n+1}) = \mu\left(\xi_n \cap \xi_{n+1} \setminus \bigcup_{\zeta\in L} \zeta\right) \leq \mu\left(\xi_n \setminus \bigcup_{\zeta\in L} \zeta\right) = 0$$

which is absurd. Hence $\xi_{n+1} \cap \left(\bigcup_{\zeta \in L} \zeta\right) \neq \emptyset$ and then $\xi_{n+1} \in L$ for P_{n+1} is a partition and $\xi_{n+1} \in P_{n+1}$. Using (4.3) we obtain $\xi_{n+1} \subset \xi_n$ (mod 0) so $A \cap \xi_{n+1} \subset A \cap \xi_n$ (mod 0) for all $n \in \mathbb{N}^+$.

From this and well-known properties of probability spaces we obtain

$$\mu\left(A \cap \bigcap_{n \in \mathbb{N}} \xi_n\right) = \mu\left(\bigcap_{n \in \mathbb{N}} (A \cap \xi_n)\right) = \lim_{n \to \infty} \mu(A \cap \xi_n) = \mu(A) > 0.$$

But P_n is measure-sensitive and $\xi_n \in P_n$, $\forall n \in \mathbb{N}$, so $\mu\left(\bigcap_{n \in \mathbb{N}} \xi_n\right) = 0$ yielding $\mu\left(A \cap \bigcap_{n \in \mathbb{N}} \xi_n\right) = 0$ which contradicts the above expression. This contradiction yields the proof. □

4.3 Measure-sensitive maps

Let $(X, \mathcal{B})$ be a measure space. If $f : X \to X$ is measurable and $k \in \mathbb{N}$ we define for every partition P the pullback partition $f^{-k}(P) = \{f^{-k}(\xi) : \xi \in P\}$ which is countable if P is.

Definition 4.4. *A* measure-sensitive partition *of a measurable map $f : X \to X$ is a countable partition P satisfying*

$$\mu(\{y \in X : f^n(y) \in P(f^n(x)), \quad \forall n \in \mathbb{N}\}) = 0, \quad \forall x \in X, \tag{4.4}$$

where $P(x)$ stands for the element of P containing $x \in X$.

The basic examples of measure-sensitive partitions are given as follows. A *strong generator* of a measurable map $f : X \to X$ is a countable partition P for which the smallest σ-algebra of $\mathcal{B}$ containing $\bigcup_{k \in \mathbb{N}} f^{-k}(P)$ equals $\mathcal{B}$ (mod 0) (see [69]).

The result below is the central motivation of this chapter.

Theorem 4.5. *Every strong generator of a measurable map f in a non-atomic probability space is a measure-sensitive partition of f.*

Proof. Let P be a strong generator of a measurable map $f : X \to X$ in a non-atomic probability space $(X, \mathcal{B}, \mu)$. Then, the sequence (4.1) generates $\mathcal{B}$ (mod 0).

From this and Lemma 5.2 p. 8 in [61] we obtain that the set of all finite unions of elements of these partitions is everywhere dense in the measure algebra associated to $(X, \mathcal{B}, \mu)$. Consequently, Lemma 9.3.3 p. 278 in [10] implies that the sequence (4.1) satisfies (4.2) and then (4.4) holds. □

We shall see in Example 4.13 that the converse of this theorem is false, i.e., there are certain measurable maps in non-atomic probability spaces carrying measure-sensitive partitions which are not strong generators. These examples motivates the study of measure-sensitive partitions for measurable maps in measure spaces.

The following equivalence relates both measure-sensitive partitions for maps and measure-sensitive sequences of partitions of measurable spaces

Lemma 4.6. *The following properties are equivalent for measurable maps $f : X \to X$ and countable partitions P on measure spaces $(X, \mathcal{B}, \mu)$:*

(i) The sequence P_n in (4.1) is measure-sensitive for X.

(ii) The partition P is measure-sensitive for f.

(iii) The partition P satisfies

$$\mu(\{y \in X : f^n(y) \in P(f^n(x)), \quad \forall n \in \mathbb{N}\}) = 0, \forall \mu\text{-a.e. } x \in X.$$

Proof. Previously we state some notation. Given a partition P and $f : X \to X$ measurable we define

$$P_\infty(x) = \{y \in X : f^n(y) \in P(f^n(x)), \forall n \in \mathbb{N}\}, \qquad \forall x \in X.$$

Notice that

$$P_\infty(x) = \bigcap_{n \in \mathbb{N}^+} P_n(x) \tag{4.5}$$

and

$$P_n(x) = \bigcap_{i=0}^{n} f^{-i}(P(f^i(x))) \tag{4.6}$$

so each $P_\infty(x)$ is a measurable set. For later use we keep the following identity

$$\left(\bigvee_{i=0}^{n} f^{-i}(P)\right)(x) = P_n(x), \qquad \forall x \in X. \tag{4.7}$$

Clearly (4.4) (resp. (iii)) is equivalent to $\mu(P_\infty(x)) = 0$ for every $x \in X$ (resp. for μ-a.e. $x \in X$).

First we prove that (i) implies (ii). Suppose that the sequence (4.1) is measure-sensitive and fix $x \in X$. By (4.5) and (4.7) we have $P_\infty(x) = \bigcap_{n\in\mathbb{N}} \xi_n$ where $\xi_n = P_n(x) \in P_n$. As the sequence P_n is measure-sensitive we obtain $\mu(P_\infty(x)) = \mu\left(\bigcap_{n\in\mathbb{N}} \xi_n\right) = 0$ proving (ii). Conversely, suppose that (ii) holds and let ξ_n be a sequence of measurable sets with $\xi_n \in P_n$ for all n. Take $y \in \bigcap_{n\in\mathbb{N}} \xi_n$. It follows that $y \in P_n(x)$ for all $n \in \mathbb{N}$ whence $y \in P_\infty(x)$ by (4.1). We conclude that $\bigcap_{n\in\mathbb{N}} \xi_n \subset P_\infty(x)$ therefore $\mu\left(\bigcap_{n\in\mathbb{N}} \xi_n\right) \leq \mu(P_\infty(x)) = 0$ proving (i).

To prove that (ii) and (iii) are equivalent we only have to prove that (iii) implies (i). Assume by contradiction that P satifies (iii) but not (ii). Since μ is a probability and (3) holds the set $X' = \{x \in X : \mu(P_\infty(x)) = 0\}$ has measure one. Since (ii) does not hold there is $x \in X$ such that $\mu(P_\infty(x)) > 0$. Since μ is a probability and X' has measure one we would have $P_\infty(x) \cap X' \neq \emptyset$ so there is $y \in P_\infty(x)$ such that $\mu(P_\infty(y)) = 0$. But clearly the collection $\{P_\infty(x) : x \in X\}$ is a partition (for P is) so $P_\infty(x) = P_\infty(y)$ whence $\mu(P_\infty(x)) = \mu(P_\infty(y)) = 0$ which is a contradiction. This ends the proof. □

Recall that a measurable map $f : X \to X$ is *measure-preserving* if $\mu \circ f^{-1} = \mu$. Moreover, it is *ergodic* if every measurable invariant set A (i.e. $A = f^{-1}(A)$ (mod 0)) satisfies either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$; and *totally ergodic* if f^n is ergodic for all $n \in \mathbb{N}^+$.

Example 4.7. *If f is a totally ergodic measure-preserving map of a probability space, then every countable partition P with $0 < \mu(\xi) < 1$ for some $\xi \in P$ is measure-sensitive with respect to f (this follows from the equivalence (iii) in Lemma 4.6 and Lemma 1.1 p. 208 in [61]).*

Hereafter we fix a measure space $(X, \mathcal{B}, \mu)$ and a measurable map $f : X \to X$. We shall not assume that f is measure-preserving unless otherwise stated.

Using the *Kolmogorov-Sinai's entropy* we obtain sufficient conditions for the measure-sensitivity of a given partition. Recall that the *entropy of a finite partition* P is defined by

$$H(P) = -\sum_{\xi \in P} \mu(\xi) \log \mu(\xi).$$

The entropy of a finite partition P with respect to a measure-preserving map f is defined by

$$h(f, P) = \lim_{n \to \infty} \frac{1}{n} H(P_{n-1}).$$

Then, we have the following lemma.

Lemma 4.8. *A finite partition with finite positive entropy of an ergodic measure-preserving map f in a probability space is a measure-sensitive partition of f.*

Proof. Since f is ergodic, the Shannon-Breiman Theorem (c.f. [61] p. 209) implies that the partition P (say) satisfies

$$-\lim_{n \to \infty} \frac{1}{n} \log(\mu(P_n(x))) = h(f, P), \qquad \mu\text{-a.e. } x \in X, \tag{4.8}$$

where $P_n(x)$ is as in (4.6). On the other hand, $P_{n+1}(x) \subset P_n(x)$ for all n so (4.5) implies

$$\mu(P_\infty(x)) = \lim_{n \to \infty} \mu(P_n(x)), \qquad \forall x \in X. \tag{4.9}$$

But $h(f, P) > 0$ so (4.8) implies that $\mu(P_n(x))$ goes to zero for μ-a.e. $x \in X$. This together with (4.9) implies that P satisfy the equivalence (iii) in Lemma 4.6 so P is measure-sensitive. □

It follows at once from Lemma 4.6 that measure-sensitive maps only exist on measure-sensitive spaces. Consequently we obtain the following result from Proposition 4.3.

Theorem 4.9. *Every probability space carrying measure-sensitive maps is non-atomic.*

A simple but useful example is as follows.

Example 4.10. *The irrational rotations in the circle are measure-sensitive maps with respect to the Lebesgue measure. This follows from Example 4.7 since all such maps are measure-preserving and totally ergodic.*

On the other hand, it is not difficult to find examples of measure-sensitive measure-preserving maps which are not ergodic. These examples together with Example 4.10 suggest the question whether an ergodic measure-preserving map is measure-sensitive. However, the answer is negative by the following example.

Example 4.11. *If $(X, \mathcal{B}, \mu)$ is a measure space with $\mathcal{B} = \{X, \emptyset\}$, then no map is measure-sensitive although they are all ergodic measure-preserving.*

In spite of this we can give conditions for the measure-expansivity of ergodic measure-preserving maps as follows.

Recall that the *entropy* (c.f. [61], [89]) of f is defined by

$$h(f) = \sup\{h(f, Q) : Q \text{ is a finite partition of } X\}.$$

We obtain a result closely related to Theorem 3.19 and Theorem 3.1 in [27].

Theorem 4.12. *Every ergodic measure-preserving maps with positive entropy of a probability space is measure-sensitive.*

Proof. Let f be one of such a map with entropy $h(f) > 0$. We can assume that $h(f) < \infty$. It follows that there is a finite partition Q with $0 < h(f, Q) < \infty$. Taking $P = \bigvee_{i=0}^{n-1} f^{-i}(Q)$ with n large we obtain a finite partition with finite positive entropy since $h(f, P) = h(f, Q) > 0$. It follows that P is measure-sensitive by Lemma 4.8 whence f is measure-sensitive by definition. □

A first consequence of the above result is that the converse of Theorem 4.5 is false.

Example 4.13. *Let $f : X \to X$ be a homeomorphism with positive topological entropy of a compact metric space X. By the variational principle [89] there is a Borel probability measures μ with respect to which f is an ergodic measure-preserving map with positive entropy. Then, by Theorem 4.12, f carries a measure-sensitive partition which, by Corollary 4.18.1 in [89], cannot be a strong generator. Consequently, there are measurable maps in certain non-atomic probability spaces carrying measure-sensitive partitions which are not strong generators.*

On the other hand, it is also false that ergodic measure-sensitive measure-preserving maps on probability spaces have positive entropy. The counterexamples are precisely the irrational circle rotations (c.f. Example 4.10). Theorems 4.9 and 4.12 imply the probably well-known result below.

Corollary 4.14. *Every probability spaces carrying ergodic measure-preserving maps with positive entropy is non-atomic.*

4.4 Aperiodicity

In this section we analyse the aperiodicity of measure-sensitive maps. According to [69] a measurable map f is *aperiodic* whenever for all $n \in \mathbb{N}^+$ if $n \in \mathbb{N}^+$ and $f^n(x) = x$ on a measurable set A, then $\mu(A) = 0$. Let us extend this definition in the following way.

Definition 4.15. *We say that f is* eventually aperiodic *whenever the following property holds for every $(n, k) \in \mathbb{N}^+ \times \mathbb{N}$: If A is a measurable set such that for every $x \in A$ there is $0 \leq i \leq k$ such that $f^{n+i}(x) = f^i(x)$, then $\mu(A) = 0$.*

It follows easily from the definition that an eventually periodic map is aperiodic. The converse is true for invertible maps but not in general (e.g. the constant map $f(x) = c$ where c is a measurable point of zero mass).

With this definition we can state the following result.

Theorem 4.16. *Every measure-sensitive map is eventually aperiodic (and so aperiodic).*

Proof. Let f be a measure-sensitive map of X. Take $(n,k) \in \mathbb{N}^+ \times \mathbb{N}$ and a measurable set A such that for every $x \in A$ there is $0 \leq i \leq k$ such that $f^{n+i}(x) = f^i(x)$. Then,

$$A \subset \bigcup_{i=0}^{k} f^{-i}(Fix(f^n)), \tag{4.10}$$

where $Fix(g) = \{x \in X : g(x) = x\}$ denotes the set of fixed points of a map g. Let P be a measure-sensitive partition of f. Then, $\bigvee_{m=0}^{k+n} f^{-m}(P)$ is a countable partition. Fix $x, y \in A \cap \xi$. In particular

$$\xi = \left(\bigvee_{m=0}^{k+n} f^{-m}(P)\right)(x)$$

whence

$$y \in \left(\bigvee_{m=0}^{k+n} f^{-m}(P)\right)(x).$$

This together with (4.6) and (4.7) yields

$$f^m(y) \in P(f^m(x)), \qquad \forall 0 \leq m \leq k+n. \tag{4.11}$$

But $x, y \in A$ so (4.10) implies $f^i(x), f^j(y) \in Fix(f^n)$ for some $i, j \in \{0, \cdots, k\}$. We can assume that $j \geq i$ (otherwise we interchange the roles of x and y in the argument below).

Now take $m > k+n$. Then, $m > j+n$ so $m - j = pn + r$ for some $p \in \mathbb{N}^+$ and some integer $0 \leq r < n$. Since $0 \leq j + r < k + n$ (for $0 \leq j \leq k$ and $0 \leq r < n$) one gets

$$\begin{aligned} f^m(y) = f^{m-j}(f^j(y)) &= f^{pn+r}(f^j(y)) \\ &= f^r(f^{pn}(f^j(y))) \\ &= f^{j+r}(y) \\ &\overset{(4.11)}{\in} P(f^{j+r}(x)). \end{aligned}$$

But

$$\begin{aligned} P(f^{j+r}(x)) = P(f^{j+r-i}(f^i(x))) &= P(f^{j+r-i}(f^{pn}(f^i(x)))) \\ &= P(f^{m-i}(f^i(x))) \\ &= P(f^m(x)) \end{aligned}$$

so

$$f^m(y) \in P(f^m(x)), \qquad \forall m > k+n.$$

This together with (4.11) implies that $f^m(y) \in P(f^m(x))$ for all $m \in \mathbb{N}$ whence $y \in P_\infty(x)$. Consequently $A \cap \xi \subset P_\infty(x)$. As P is measure-sensitive, Lemma 4.6 implies

$$\mu(A \cap \xi) = 0, \qquad \forall \xi \in \bigvee_{i=0}^{k+n} f^{-i}(P).$$

On the other hand, $\bigvee_{i=0}^{k+n} f^{-i}(P)$ is a partition so

$$A = \bigcup_{\xi \in \bigvee_{i=0}^{k+n} f^{-i}(P)} (A \cap \xi)$$

and then $\mu(A) = 0$ since $\bigvee_{i=0}^{k+n} f^{-i}(P)$ is countable. This ends the proof. □

By Lemma 4.5 we have that, in non-atomic probability spaces, every measurable map carrying strong generators is measure-sensitive. This motivates the question as to whether every measure-sensitive map has a strong generator. We give a partial positive answer for certain maps defined as follows. We say that f is *countable to one (mod 0)* if $f^{-1}(x)$ is countable for μ-a.e. $x \in X$. We say that f is *nonsingular* if a measurable set A has measure zero if and only if $f^{-1}(A)$ also does. All measure-preserving maps are nonsingular. A *Lebesgue probability space* is a complete measure space which is isomorphic to the completion of a standard probability space (c.f. [1], [10]).

Corollary 4.17. *The following properties are equivalent for non-singular countable to one (mod 0) maps f on non-atomic Lebesgue probability spaces:*

1. *f is measure-sensitive.*
2. *f is eventually aperiodic.*
3. *f is aperiodic.*

4. f has a strong generator.

Proof. Notice that (1) $\Rightarrow$ (2) by Theorem 4.16 and (2) $\Rightarrow$ (3) follows from the definitions. On the other hand, (3) $\Rightarrow$ (4) by a Parry's Theorem (c.f. [69], [71], [70]) while (4) $\Rightarrow$ (1) by Lemma 4.5. ☐

Denote by $Fix(g) = \{x \in X : g(x) = x\}$ the set of fixed points of a mapping g.

Corollary 4.18. *If $f^k = f$ for some integer $k \geq 2$, then f is not measure-sensitive.*

Proof. Suppose by contradiction that it does. Then, f is eventually aperiodic by Theorem 4.16. On the other hand, if $x \in X$ then $f^k(x) = f(x)$ so $f^{k-1}(f^k(x)) = f^{k-1}(f(x)) = f^k(x)$ therefore $f^k(x) \in Fix(f^{k-1})$ whence $X \subset f^{-k}(Fix(f^{k-1}))$. But since f is eventually aperiodic, $n = k-1 \in \mathbb{N}^+$ and X measurable we obtain from the definition that $\mu(X) = 0$ which is absurd. This ends the proof. ☐

Example 4.19. *By Corollary 4.18 neither the identity $f(x) = x$ nor the constant map $f(x) = c$ are measure-sensitive (for they satisfy $f^2 = f$). In particular, the converse of Theorem 4.16 is false for the constant maps are eventually aperiodic but not measure-sensitive.*

It is not difficult to prove that an ergodic measure-preserving map of a non-atomic probability space is aperiodic. Then, Corollary 4.14 implies the well-known fact that *all ergodic measure-preserving maps with positive entropy on probability spaces are aperiodic*. However, using theorems 4.12 and 4.16 we obtain the following stronger result.

Corollary 4.20. *All ergodic measure-preserving maps with positive entropy on probability spaces are eventually aperiodic.*

Now we study the following variant of aperiodicity introduced in [41] p. 180.

Definition 4.21. *We say that f is* HS-aperiodic *([2]) whenever for every measurable set of positive measure A and $n \in \mathbb{N}^+$ there is a measurable subset $B \subset A$ such that $\mu(B \setminus f^{-n}(B)) > 0$.*

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[2] called aperiodic in [41].

Notice that HS-aperiodicity implies the aperiodicity used in [48] or [83] (for further comparisons see p. 88 in [56]).

On the other hand, a measurable map f is *negative nonsingular* if $\mu(f^{-1}(A)) = 0$ whenever A is a measurable set with $\mu(A) = 0$. Some consequences of the aperiodicity on negative nonsingular maps in probability spaces are given in [56]. Observe that every measure-preserving map is negatively nonsingular.

Let us present two technical (but simple) results for later usage. We call a measurable set A satisfying $A \subset f^{-1}(A)$ (mod 0) a *positively invariant set (mod* 0*)*. For completeness we prove the following property of these sets.

Lemma 4.22. *If A is a positively invariant set (mod 0) of finite measure of a negative nonsingular map f, then*

$$\mu\left(\bigcap_{n=0}^{\infty} f^{-n}(A)\right) = \mu(A). \tag{4.12}$$

Proof. Since $\mu(A) = \mu(A \setminus f^{-1}(A)) + \mu(A \cap f^{-1}(A))$ and A is positively invariant (mod 0) one has $\mu(A) = \mu(A \cap f^{-1}(A))$, i.e.,

$$\mu\left(\bigcap_{n=0}^{1} f^{-n}(A)\right) = \mu(A).$$

Now suppose that $m \in \mathbb{N}^+$ satisfies

$$\mu\left(\bigcap_{n=0}^{m} f^{-n}(A)\right) = \mu(A).$$

Since

$$\mu\left(\bigcap_{n=0}^{m+1} f^{-n}(A)\right) =$$

$$\mu\left(\bigcap_{n=0}^{m} f^{-n}(A)\right) - \mu\left(\left(\bigcap_{n=0}^{m} f^{-n}(A)\right) \setminus f^{-m-1}(A)\right)$$

and

$$\begin{aligned}\mu\left(\left(\bigcap_{n=0}^{m} f^{-n}(A)\right)\setminus f^{-m-1}(A)\right) &\leq \mu(f^{-m}(A)\setminus f^{-m-1}(A))\\ &= \mu(f^{-m}(A\setminus f^{-1}(A)))\\ &= 0\end{aligned}$$

because f is negative nonsingular and A is positively invariant (mod 0), one has $\mu\left(\bigcap_{n=0}^{m+1} f^{-n}(A)\right) = \mu(A)$. Therefore

$$\mu\left(\bigcap_{n=0}^{m} f^{-n}(A)\right) = \mu(A), \qquad \forall m \in \mathbb{N}, \tag{4.13}$$

by induction. On the other hand,

$$\bigcap_{n=0}^{\infty} f^{-n}(A) = \bigcap_{m=0}^{\infty}\bigcap_{n=0}^{m} f^{-n}(A)$$

and $\bigcap_{n=0}^{m+1} f^{-n}(A) \subset \bigcap_{n=0}^{m} f^{-n}(A)$. As $\mu(A) < \infty$ we conclude that

$$\mu\left(\bigcap_{n=0}^{\infty} f^{-n}(A)\right) = \lim_{m\to\infty}\mu\left(\bigcap_{n=0}^{m} f^{-n}(A)\right) \overset{(4.13)}{=} \lim_{m\to\infty}\mu(A) = \mu(A)$$

proving (4.12). □

We use the above lemma only in the proof of the proposition below.

Proposition 4.23. *Let P be a measure-sensitive partition of a negative nonsingular map f. Then, no $\xi \in P$ with positive finite measure is positively invariant (mod 0).*

Proof. Suppose by contradiction that there is $\xi \in P$ with $0 < \mu(\xi) < \infty$ which is positively invariant (mod 0). Taking $A = \xi$ in Lemma 4.22 we obtain

$$\mu\left(\bigcap_{n=0}^{\infty} f^{-n}(\xi)\right) = \mu(\xi). \tag{4.14}$$

As $\mu(\xi) > 0$ we conclude that $\bigcap_{n=0}^{\infty} f^{-n}(\xi) \neq \emptyset$, and so, there is $x \in \xi$ such that $f^n(x) \in \xi$ for all $n \in \mathbb{N}$. As $\xi \in P$ we obtain $P(f^n(x)) = \xi$ and so $f^{-n}(P(f^n(x))) = f^{-n}(\xi)$ for all $n \in \mathbb{N}$. Using (4.6) we get

$$P_m(x) = \bigcap_{n=0}^{m} f^{-n}(\xi).$$

Then, (4.5) yields

$$P_\infty(x) = \bigcap_{m=0}^{\infty} P_m(x) = \bigcap_{m=0}^{\infty} \bigcap_{n=0}^{m} f^{-n}(\xi) = \bigcap_{n=0}^{\infty} f^{-n}(\xi)$$

and so $\mu(P_\infty(x)) = \mu(\xi)$ by (4.14). Then, $\mu(\xi) = 0$ by Lemma 4.6 since P is measure-sensitive which is absurd. This contradiction proves the result. $\square$

We also need the following lemma resembling a well-known property of the expansive maps.

Lemma 4.24. *If $k \in \mathbb{N}^+$, then f is measure-sensitive if and only if f^k is.*

Proof. The notation $P_\infty^f(x)$ will indicate the dependence of $P_\infty(x)$ on f.

First of all suppose that f^k is an measure-sensitive with measure-sensitive partition P. Then, $\mu(P_\infty^{f^k}(x)) = 0$ for all $x \in X$ by Lemma 4.6. But by definition one has $P_\infty^f(x) \subset P_\infty^{f^k}(x)$ so $\mu(P_\infty^f(x)) = 0$ for all $x \in X$. Therefore, f is measure-sensitive with measure-sensitive partition P. Conversely, suppose that f is measure-sensitive with expansivity constant P. Consider $Q = \bigvee_{i=0}^{k} f^{-i}(P)$ which is a countable partition satisfying $Q(x) = \bigcap_{i=0}^{k} f^{-i}(P(f^i(x)))$ by (4.7). Now, take $y \in Q_\infty^{f^k}(x)$. In particular, $y \in Q(x)$ hence $f^i(y) \in P(f^i(x))$ for every $0 \leq i \leq k$. Take $n > k$ so $n = pk + r$ for some nonnegative integers p and $0 \leq r < k$. As $y \in Q_\infty^{f^k}(x)$ one has $f^{pk}(y) \in Q(f^{pk}(x))$ and then $f^n(y) = f^{pk+i}(y) = f^i(f^{pk}(y)) \in P(f^i(f^{pk}(x)) = P(f^n(x))$ proving $f^n(y) \in P(f^n(x))$ for all $n \in \mathbb{N}$. Then, $y \in P_\infty(x)$ yielding $Q_\infty^{f^k}(x) \subset P_\infty(x)$. Thus $\mu(Q_\infty^{f^k}(x)) = 0$ for all $x \in X$ by the equivalence (ii) in Lemma 4.6 since P is measure-sensitive. It follows that f^k is measure-sensitive with measure-sensitive partition Q. $\square$

With these definitions and preliminary results we obtain the following.

Theorem 4.25. *Every measure-sensitive negative nonsingular map in a probability space is HS-aperiodic.*

Proof. Suppose by contradiction that there is a measure-sensitive map f which is negative nonsingular but not HS-aperiodic. Then, there are a measurable set of positive measure A and $n \in \mathbb{N}^+$ such that $\mu(B \setminus f^{-n}(B)) = 0$ for every measurable subset $B \subset A$. It follows that every measurable subset $B \subset A$ is positively invariant (mod 0) with respect to f^n. By Lemma 4.24 we can assume $n = 1$.

Now, let P be a measure-sensitive partition of f. Clearly, since $\mu(A) > 0$ there is $\xi \in P$ such that $\mu(A \cap \xi) > 0$. Taking $\eta = A \cap \xi$ we obtain that η is positively invariant (mod 0) with positive measure. In addition, consider the new partition $Q = (P \setminus \{\xi\}) \cup \{\eta, \xi \setminus A\}$ which is clearly measure-sensitive (for P is). Since this partition also carries a positively invariant (mod 0) member of positive measure (say η) we obtain a contradiction by Proposition 4.23. The proof follows. □

4.5 Exercices

Exercice 4.26. Find non-standard measure-sensitive probability spaces.

Exercice 4.27. Prove that the condition (4.2) for a sequence of partitions to be measure-sensitive is also necessary in probability spaces.

Exercice 4.28. Is the converse of Proposition 4.3 true among probability spaces, namely, is every non-atomic probability space measure-sensitive?

Exercice 4.29. Prove the assertion in Example 4.7.

Exercice 4.30. Prove that if P_n is a measure-sensitive sequence of partitions of a probability space $(X, \mathcal{B}, \mu)$, then $\lim_{n\to\infty} h(f, P_n)$ exists for every measure-preserving map $f : X \to X$. Prove that this limit may depend on the measure-sensitive sequence P_n.

Exercice 4.31. Prove that every measurable map of a separable metric space which is pairwise sensitive with respect to a Borel probability measure μ is measure-sensitive with respect to μ. Find a counterexample for the converse of this statement.

Exercice 4.32. Prove that every expansive map of a separable non-atomic metric space is measure-sensitive with respect to any non-atomic Borel probability measure.

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