Some Recent Developments in the Theory of Minimal Surfaces in 3-Manifolds

Publicações Matemáticas

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24º Colóquio Brasileiro de Matemática

Estrada Dona Castorina, 110
22460-320 Rio de Janeiro, RJ

Impresso no Brasil / Printed in Brazil

Capa: Noni Geiger / Sérgio R. Vaz

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Distribuição:
IMPA
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e-mail: ddic@impa.br
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ISBN: 85-244-0210-5

Some recent developments in the theory of minimal surfaces.

Harold Rosenberg

September 10, 2003

I suspect that each person can recall some moments from his childhood when he was captived by the beauty of soap bubbles. Their form can be truly marvelous. As adults, we do not tend to stare at soap bubble clusters in amazement, but as a professional mathematician, it happens that once one begins their study, one can remain seduced for a lifetime.

My first contact with minimal surfaces happened some twenty years ago. I met Andre Haefliger while walking in Paris, we continued walking together and he asked me: "Is there a foliation of $\mathbb{R}^3$ by minimal surfaces, other than a foliation by parallel planes?" This (apparently easy) question made me think about the nature of a minimal surface; I have not stopped thinking about it since then. We will see in section 7, that Haefligers' question is unsolved for minimal laminations of $\mathbb{R}^3$.

Certainly soap films and soap bubbles seem a specialized subject; yet analysis, geometry, and topology are fundamental to their understanding.

Soap bubble clusters are modelled on surfaces of constant mean curvature H, meeting along a curve $\mathcal{S}$ in a precise manner. At a smooth point of $\mathcal{S}$, there are three smooth surfaces (of the cluster) meeting at equal angles (i.e., 120^o) and $\mathcal{S}$ has isolated singular points where six surfaces of the cluster meet at angles approximately 109^o. For example, the barycenter of a regular tetrahedron, and the triangles meeting at the barycenter which are formed by the edges of the tetrahedra and the barycenter.

These properties of soap bubble clusters were observed by Plateau in the years 1870 and were established only relatively recently. Some excellent references for this subject are [28], [29], [33], [41].

We state a mathematical interpretation of soap bubbles. Consider positive real numbers $V_1, ..., V_n$ and surface configurations that separate space $\mathbb{R}^3$ into regions having volumes $V_1, ..., V_n$. A soap bubble is such a configuration such that the area of the surfaces in the cluster is a (local) minimum among all such surfaces bounding the volumes $V_1, ..., V_n$. When

$n = 1$, it has been known since antiquity that a round sphere is the solution to this problem. For $n = 2$, it has only recently been established that the standard double-bubble is the solution [11].

The theory of soap bubbles in Riemannian manifolds other than $\mathbb{R}^3$ is an active subject of research today. Some references are [2] and [27].

This paper is organized as follows. In section 1 we present the equations of minimal surfaces in both parametric and non-parametric form and the first variation formula. In section 2 we apply the equations to obtain information about minimal submanifolds of $\mathbb{R}^n$: the coordinate functions are harmonic and the monotonocity formula. Section 3 discusses the Plateau problem and how its solution can be used to construct properly embedded minimal surfaces by taking limits. Stable surfaces are introduced and some applications are mentionned.

In section 4 we state the second variation formula for hypersurfaces and give some easy applications in terms of Ricci curvature.

We describe Schoen's curvature estimate for stable surfaces as derived by Colding and Minicozzi. We conclude this section with the theorem of D. F. Colbrie [12]: a stable orientable minimal surface in $\mathbb{R}^3$ with compact boundary (and complete) has finite total curvature.

Section 5 gives a proof of Pascal Collin's theorem [8] (the Nitsche conjecture) due to Colding and Minicozzi. The proof supposes their theorem on one sided curvature estimates.

In section 6, we discuss minimal laminations of $\mathbb{R}^3$. This is work done by W. Meeks and the author.

Section 7 discusses minimal surfaces in $M \times \mathbb{R}$, M a Riemannian surface. Particular attention is given to the case M is a 2-sphere of constant curvature. Many examples are discussed. References for this section are [36], [24], [30], [10].

In section 8, we develop the theory of minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$, and we show how the Abresch family of constant mean curvature $H = 1/2$ surfaces in $\mathbb{R}^3$, yields a two-parameter family of minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$ foliated by circles as level curves.

In this section 8, we also describe minimal surfaces of higher genus in $\mathbb{S}^2 \times \mathbb{R}$.

In section 9, we describe some of the main theorems established by Meeks and me concerning minimal surfaces in $M \times \mathbb{R}$, M a compact Riemannian surface [24].

In section 10, we discuss constant mean curvature H surfaces in $M \times \mathbb{R}$, $H \neq 0$. We obtain height estimates and derive some applications.

Finally in section 11, we return to minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ of finite topology, and we show they have linear area growth when their curvature is bounded.

1 The classical equations of minimal surface theory.

We present the minimal surface equation in non parametric form in $\mathbb{R}^n$ and the parametric form in a Riemannian manifold.

Let $\mathcal{U} \subset \mathbb{R}^n$ be an open domain and $u : \mathcal{U} \to \mathbb{R}$ a C^1-function. Denote by $G(u)$ the graph of u. The area (volume) of $G(u)$ is

$$\int_{\mathcal{U}} \sqrt{1+|\nabla u|^2}.$$

Let $\eta \in C_0^1(\mathcal{U}) = \{C^1 - \text{functions on } \mathcal{U} \text{ of compact support}\}$. The area of

$$G(u+t\eta) = \int_{\mathcal{U}} \sqrt{1+|\nabla u + t\nabla\eta|^2}.$$

One has

$$\left.\frac{d}{dt}\right|_{t=0} G(u+t\eta) = \int_{\mathcal{U}} \frac{\langle \nabla u, \nabla \eta\rangle}{W},$$

where $W = \sqrt{1+|\nabla u|^2}$. Thus $G(u)$ is a critical point for the area functional means

$$\int_{\mathcal{U}} \frac{\langle \nabla u, \nabla \eta\rangle}{W} = 0, \forall \eta \in C_0^1(\Omega).$$

Integrating by parts, this last equality is equivalent to

$$\int_{\mathcal{U}} \eta \operatorname{div}\left(\frac{\nabla u}{W}\right) = 0, \quad \forall \eta \in C_0^1(\Omega).$$

Consequently $\operatorname{div}\left(\frac{\nabla u}{W}\right) = 0$, which is the non-parametric form of the minimal surface equation.

Now let $\Sigma = \Sigma^k$ and $M = M^n$ be Riemannian manifolds. Let $F : \Sigma \times [-\varepsilon, \varepsilon] \to M$ be an immersion defining a deformation of compact support of $\Sigma(0) = F(\Sigma \times (0))$ i.e., $F(x,0) = x$ ($\Sigma = \Sigma(0)$ considered as a submanifold of M) and $F(x,t) = x$ for all t and x outside of some compact set. Let $Y(x) = \left.\frac{d}{dt}\right|_{t=0} F(x,t)$ denote the variation vector field of this variation. Then

$$\left.\frac{d}{dt}\right|_{t=0} Vol(\Sigma(t)) = -\int_{\Sigma} \left\langle Y, \vec{H} \right\rangle d\Sigma,$$

where $\vec{H}$ is the mean curvature vector field of Σ. This is the first variation formula for volume and the proof can be found in [17].

Consequently a submanifold Σ is a critical point for the volume functional under compactly supported variations means $\vec{H} \equiv 0$, and this defines a minimal submanifold.

There is another useful form of the minimal surface equation which applies to submanifolds Σ which may not be smooth ($\vec{H}$ not defined). This equation is

$$\int_\Sigma \mathrm{div}_\Sigma Y = 0,$$

for all vector fields Y along Σ with compact support. To derive this equation, let $e_1, ..., e_k$ be an orthonormal frame for $T\Sigma$ and $Y^\perp$ denote the component of Y in the normal bundle to $T\Sigma$ in TM. Then

$$\begin{aligned} 0 &= -\int_\Sigma \left\langle \vec{H}, Y \right\rangle = -\sum_i \int_\Sigma \left\langle (\nabla_{e_i} e_i)^\perp, Y^\perp \right\rangle \\ &= -\sum_i \int_\Sigma \left\langle \nabla_{e_i} e_i, Y^\perp \right\rangle = \sum_i \int_\Sigma \left\langle e_i, \nabla_{e_i} Y^\perp \right\rangle \\ &= \sum_i \int_\Sigma \langle e_i, \nabla_{e_i} Y\rangle - \left\langle e_i, \nabla_{e_i} Y^t \right\rangle \\ &= \int_\Sigma \mathrm{div}_\Sigma Y. \end{aligned}$$

where Y^t denotes the tangent part of Y. Stokes' theorem yields $\int_\Sigma \mathrm{div}_\Sigma (Y^t) = 0$, since Y has compact support.

2 Some applications of the equations.

Here are some classical applications of the divergence characterization of minimal submanifolds.

Let $\Sigma \subset \mathbb{R}^n$ be a minimal submanifold. Then the coordinate functions $x_1, ..., x_n$ of $\mathbb{R}^n$, are harmonic functions on Σ. To see this let $e_1, ..., e_n$ be the canonical basis of $\mathbb{R}^n$, and let $\eta \in C_0^1(\Sigma)$. Let $Y = \eta e_i$. Then

$$\mathrm{div}_\Sigma(Y) = \langle \nabla_\Sigma \eta, e_i \rangle + \eta \mathrm{div}_\Sigma(e_i) = \langle \nabla_\Sigma \eta, e_i \rangle,$$

$$\begin{aligned} 0 &= \int_\Sigma \langle \nabla_\Sigma \eta, e_i \rangle = \int_\Sigma \left\langle \nabla_\Sigma \eta, e_i^t \right\rangle \\ &= \int_\Sigma \langle \nabla_\Sigma \eta, \nabla_\Sigma x_i \rangle . \end{aligned}$$

Now

$$\mathrm{div}(\eta \nabla_\Sigma x_i) = \eta \mathrm{div}(\nabla_\Sigma x_i) + \langle \nabla_\Sigma \eta, \nabla_\Sigma x_i \rangle ,$$

and since η has compact support:

$$0 = \int_\Sigma \langle \nabla_\Sigma \eta, \nabla_\Sigma x_i \rangle = -\int_\Sigma \eta \mathrm{div}(\nabla_\Sigma x_i),$$

so $\triangle_\Sigma x_i = \mathrm{div}(\nabla_\Sigma x_i) = 0$.

Another simple application is the monotonocity formula for the growth of area of minimal surfaces.

Let $x_0 \in \Sigma \subset \mathbb{R}^n$ be a minimal submanifold and B_s denote the Euclidean ball of $\mathbb{R}^n$, centered at x_0, of radius s. Then

$$\frac{\mathrm{Vol}(\Sigma \cap B_s)}{s^k}$$

is a monotone non decreasing function. Moreover the value of this quotient tends to the volume of B_1 (assuming Σ is embedded near x_0)as $s \to 0$.

To see this, let $r^2 = x_1^2 + ... + x_n^2$, so $\triangle(r^2) = 2\sum_{i=1}^n x_i \triangle x_i + 2\sum_{i=1}^n |\nabla x_i|^2 = 2k$, where the laplacian and gradient are calculated on Σ. Integrating this last inequality yields:

$$\begin{aligned} 2k\mathrm{Vol}(\Sigma \cap B_s) &= \int_{\Sigma \cap B_s} \triangle(r^2) = \int_\partial \frac{\partial}{\partial \nu}(r^2) \\ &= 2s \int_\partial |\nabla r| \leq 2s \int_\partial \frac{1}{|\nabla r|} \\ &= 2s \frac{d}{ds}(\mathrm{Vol}(\Sigma \cap B_s)). \end{aligned}$$

The last equality uses the coarea formula at regular values. Finally the above inequality may be written

$$\frac{d}{ds}\left(\frac{(\mathrm{Vol}(\Sigma \cap B_s))}{s^k}\right) \geq 0,$$

which proves the monotonocity formula.

3 The Plateau problem and properly embedded surfaces

The classical Plateau problem for Jordan curves Γ in $\mathbb{R}^n$ concerns the existence of disks Σ in $\mathbb{R}^n$ with boundary equal to Γ, which minimize area among all disks with boundary Γ. Douglas and Rado proved that this problem has a solution provided Γ is rectifiable and the solution is an immersion except at isolated branch points. R. Osserman proved that there are no interior branch points when $n = 3$. We now know (using geometric measure theory) that there is an embedded surface of least area with boundary Γ, which is not simply connected in general. In higher dimensions, solutions to the Plateau problem exist for minimal submanifolds and the singularity set is of codimension seven. This theory also works in complete Riemannian manifolds.

Solutions to a Plateau problem are useful in studying the existence and geometry of properly embedded minimal submanifolds.

To construct examples of properly embedded surfaces one chooses larger and larger Jordan curves $\Gamma(n)$ and let $\Sigma(n)$ be a least area surface with boundary $\Gamma(n)$. Then one looks for a subsequence of $\Sigma(n)$ that converges to a complete minimal surface $\Sigma(\infty)$. When this works, the limit surface $\Sigma(\infty)$ has the important property of minimizing area on compact subsets K, up to second order. That is for any compact subset K of $\Sigma(\infty)$, and any variation of K, fixed on ∂K, the second derivative of the area of the variation (at K) is non negative. Minimal surfaces satisfying this condition are called stable. The study of stable minimal submanifolds is important. One can often use the knowledge of stable minimal submanifolds to obtain information about arbitrary minimal submanifolds. For example, a beautiful theorem of Hoffman and Meeks says that two properly immersed minimal surfaces M_1 and M_2, in $\mathbb{R}^3$, that are disjoint, are parallel planes [15]. The proof of this goes as follows. Solving Plateau problems and passing to a limit, one constructs a complete stable minimal surface Σ between M_1 and M_2. A theorem of R. Schoen [37], and independently, M. do Carmo and Peng [4], says that the only stable complete minimal surface in $\mathbb{R}^3$ is a plane. Thus Σ is a plane and M_1 and M_2 are in half-spaces determined by Σ. Finally, one proves that a properly immersed minimal surface in a half-space, is a plane.

4 The second variation formula

We state the second variation formula for codimension one minimal submanifolds Σ of a Riemannian manifold N. Assume Σ has a trivial normal bundle in N and n is a unit normal vector field to Σ in N.

Let $Y = fn$ be a vector field along Σ with $f \in C_0^2(\Sigma)$. Let $F : \Sigma \times [-t, t] \to N$ be a deformation of Σ with compact support whose variation vector field is Y. Let $A(t)$ be the volume of $F(\Sigma \times (t))$. Then

$$\begin{aligned} \left.\frac{d^2A(t)}{dt^2}\right|_{t=0} &= \int_\Sigma |\nabla f|^2 - \mathrm{Ric}(n,n)f^2 - |A|^2 f^2 \\ &= -\int_\Sigma f\left(\triangle f + \mathrm{Ric}(n,n)f + |A|^2 f\right) \\ &= -\int_\Sigma fL(f). \end{aligned}$$

Here $L = \triangle + \mathrm{Ric} + |A|^2$, where A is the second fundamental form of Σ in N, and $\triangle$ and ∇ are calculated on Σ.

Thus Σ stable means $\displaystyle\int_\Sigma f(Lf) \le 0$, for all $f \in C_0^2(\Sigma)$.

An application of this formula proves there are no stable closed (i.e., compact and without boundary) minimal hypersurfaces Σ in an N with Ric > 0; let $f \equiv 1$ on Σ, $Y = fn$, so stability would yield the contradiction:

$$0 \le -\int_S (\mathrm{Ric}(n) + |A|^2) < 0.$$

For example, S^3 with the canonical metric, has no closed stable minimal submanifolds. Notice that this yields also that a convex surface N has no stable closed geodesics.

Remark that for orientable surface in 3-manifolds, the stability operator can be written:

$$L = \triangle + S + \frac{1}{2}|A|^2 - K.$$

Here S is the scalar curvature of N and K the intrinsic curvature of Σ. This formula follows easily from the Gauss equation for $\Sigma \subset N$.

When the scalar curvature S of N is positive, one obtains information about the geometry of stable $\Sigma \subset N$ by putting interesting test functions into the stability inequality.

We prove now a very useful inequality due to Colding-Minicozzi [5].

Theorem 1. *Let Σ be a riemannian surface, $s > 0$, and $B_s(x_0)$ the geodesic disk in Σ centered at x_0 of radius s. Assume $B_s = B_s(x_0)$ is disjoint from $\partial\Sigma$. Let $L = \triangle + V - cK$ be a differential operator on Σ where $V \ge 0$, $c > \frac{1}{2}$. Suppose $-L$ is nonnegative on $H_0^1(B_s(x_0))$. Then one has*

$$\frac{Area(B_s(x_0))}{s^2} + \frac{1}{(2c-1)} \int_{B_s(x_0)} V\left(1 - \frac{r}{s}\right)^2 \le \frac{2\pi c}{2c-1}.$$

Corollary 1. *If Σ is a stable oriented minimal surface in N, and N has non negative scalar curvature, then Σ has quadratic area growth (take $c = 1$ and $V = S + \frac{|A|^2}{2}$).*

Proof of the Theorem. Let $l(r)$ be the length of $\partial B_r(x_0)$ and $K(r) = \displaystyle\int_{B_r(x_0)} K$. By the first variation formula for length and Gauss-Bonnet, we have,

$$l'(s) = \int_{\partial B(s)} k_g(s)ds \le 2\pi\chi(B(s)) - \int_{B(s)} K \le 2\pi - K(s).$$

Remark that $l(s)$ is not even continuous in general, however it is true that l is differentiable almost everywhere and for almost all s one has [39]

$$l'(s) \leq 2\pi\chi(B(s)) - K(s),$$

and for $s > s_0 \geq 0$,

$$l(s) - l(s_0) \leq \int_{s_0}^{s} l'(s)ds.$$

Now choose cut-off functions. Let $\eta : [0, s] \to \mathbb{R}^+$ be smooth and satisfy $\eta(0) = 1, \eta(s) = 0$, $\eta' \leq 0$ and $(\eta^2)' \leq 0$. Take $f = \eta(r)$ in the inequality:

$$0 \leq -\int_{B_s} fL(f) = \int_{B_s} \left(|\nabla f|^2 - f^2 V + cf^2 K\right),$$

so

$$\begin{aligned} \int_{B_s} f^2 V &\leq \int_{B_s} |\nabla f|^2 + c\int_{B_s} f^2 K \\ &= \int_{t=0}^{s} \left[(\eta')^2 \int_{\partial B_t} 1\right] + c\int_0^s \eta^2 \int_{\partial B_t} K. \end{aligned}$$

Integrating by parts and using $K(s) \leq 2\pi - l'(s)$, we have

$$\begin{aligned} \int_0^s \eta^2 \int_{\partial B_t} K dt &= \int_0^s \eta^2 K'(t) = -\int_0^s (\eta^2)' K(t) \\ &\leq \int_0^s (\eta^2)'(l'(t) - 2\pi). \end{aligned}$$

Thus

$$\int_{B_s} f^2 V \leq \int_0^s (\eta')^2 l(t) + c\int_0^s (\eta^2)' l'(t) + 2\pi c.$$

Choose $\eta = 1 - t/s$, so $(\eta')^2 = \dfrac{1}{s^2}$, $(\eta^2)' = 2\left(1 - \dfrac{t}{s}\right)\left(-\dfrac{1}{s}\right)$. Then

$$\int_{B_s} f^2 V \leq \frac{a(s)}{s^2} - 2c\frac{a(s)}{s^2} + 2\pi c$$

where $a(s) = \text{Area}(B(s))$. Thus

$$\frac{a(s)}{s^2}(2c - 1) + \int_{B(s)} \left(1 - \frac{t}{s}\right)^2 V \leq 2\pi c.$$

$\square$

Corollary 2. *Let Σ be a complete oriented stable minimal surface in N^3 with zero scalar curvature. Then Σ has quadratic area growth and $\int_\Sigma |A|^2$ is finite. When $N^3 = \mathbb{R}^3$, Σ is a plane [37],[4].*

Proof. By the Colding-Minicozzi inequality, Σ has quadratic area growth and $\int_\Sigma |A|^2$ is finite, (use the fact that $1 - t/s \geq 1/2$ if $t \leq s/2$). Now in $\mathbb{R}^3$, $|A|^2 = -2K$, so Σ has finite total curvature. Thus

$$l(s) \leq \int_0^s l'(s)ds \leq \int_0^s (2\pi - K(s)) \leq cs.$$

Recall that the stability inequality says $\int_{B(s)} |A|^2 f^2 \leq \int_{B(s)} |\nabla f|^2$, for any test function f. Let f be the test function $f = 1$ for $s \leq 1$, $f = 1 - \dfrac{\ln(r)}{\ln(s)}$, for $1 \leq r \leq s$, and $f = 0$ elsewhere. Then

$$\int_{B(s)} |\nabla f|^2 = \frac{1}{(\ln(s))^2} \int_1^s \frac{l(r)}{r^2} dr \leq \frac{c\ln(s)}{(\ln(s))^2}.$$

The last inequality uses $l(r) \leq cr$. Since $\dfrac{1}{\ln(s)} \to 0$, as $s \to \infty$, we conclude $\int_{B(1)} |A|^2 = 0$, hence $|A| = 0$ in $B(1)$ and Σ is a plane. □

Remark 1. [12]: *Let Σ be a stable orientable complete minimal surface in $\mathbb{R}^3$ with compact boundary. Then Σ has finite total curvature. This is proved just as in theorem 1; there is a fixed boundary term that arises.*

Notice that the ends of Σ (assuming Σ is embedded) are asymptotic to catenoid or planar ends, when Σ has finite total curvature.

5 The Nitsche conjecture

In [31], Nitsche conjectured that a minimal surface M in $\mathbb{R}^3$ that meets every plane x_3=constant, in a Jordan curve, is a catenoid. This problem was solved affirmatively by Pascal Collin [8].

Pascal's proof of the Nitsche conjecture is a "work of art". It is also a great deal of work to go through the proof. I will present a proof here, due to Colding and Minicozzi [7], which is quite different. It uses a very deep and usefull theorem they have discovered (and proved) called the "one-sided curvature estimates". Here is the statement.

Theorem 2. [6] *There exists $\varepsilon_0 > 0$ such that the following holds. Let $y \in \mathbb{R}^3$, $r_0 > 0$ and*

$$\Sigma \subset B_{2r_0}(y) \cap \{x_3 > x_3(y)\} \subset \mathbb{R}^3$$

be a compact embedded minimal disk with $\partial\Sigma \subset \partial B_{2r_0}(y)$. For any connected component Σ' of $B_{r_0}(y) \cap \Sigma$ with $B_{\varepsilon_0 r_0}(y) \cap \Sigma' \neq \emptyset$,

$$\sup_{\Sigma} |A_{\Sigma'}|^2 \leq r_0^{-2}.$$

Notice that the statement of this theorem is scale invariant; it is often stated for y the origin and $r_0 = 1$. Also the catenoid shows that one needs the simply connected hypothesis for the theorem to hold. The proof of Colding and Minicozzi of this theorem is difficult. There have been several important applications of this result [23],[24].

I now show how it implies the Nitsche conjecture. More generally, they prove [7]:

Theorem 3. *There exists $\varepsilon > 0$ such that any complete properly embedded minimal annular end $E \subset C(-\varepsilon)$ has finite total curvature ($C(t) = \{x_3 > \tau\sqrt{x_1^2 + x_2^2}\}$).*

Proof. For simplicity, we will assume $E \subset \{x_3 > 0\}$; the proof is the same, but the notation is simpler.

First we prove that for any $\delta > 0$, there exists $y_j \in E - C(\delta)$ with $|y_j| \to \infty$ ([9]).

Suppose on the contrary, that for some cone $C(\delta)$, $\delta > 0$, $E \subset C(\delta)$. E is properly embedded hence $E \cap \{x_3 = \mathit{const}\}$ is compact. By the maximun principle and elementary topology (E is an annulus) it follows that E meets each plane sufficiently high, in one Jordan curve. So assume this holds for $x_3 \geq 1$, $\partial E \subset \{x_3 < 1\}$. Let $\beta(t) = E \cap \{x_3 = t\}$, and $\Omega(t) = C(\delta) \cap \{1 \leq x_3 \leq t\}$. Denote by $W(t)$ the component of $\Omega(t) - E$ in which $\beta(1)$ is not contractible. Then for each integer $n > 1$, $\beta(1)$ and $\beta(n)$ are homologous in $W(n)$ and neither $\beta(1)$ nor $\beta(n)$ is homologous to zero in $W(n)$. By [26], there exists a least area connected minimal surface $\Sigma(n)$ in $W(n)$ with $\partial\Sigma(n) = \beta(1) \cup \beta(n)$. By standard compactness techniques, a subsequence $\Sigma(n_i)$ of $\Sigma(n)$ converges to a stable complete minimal surface Σ with $\partial\Sigma = \beta(1)$. Σ has finite total curvature so the ends of Σ are catenoid or planar type ends. But Σ is contained in the cone $C(\delta)$ so this is impossible. Thus, for every $\delta > 0$, there exists $y_j \in E - C(\delta)$ with $|y_j| \to \infty$.

Now let β be a curve in $\{x_3 \geq 0\}$ joining the origin to a point of ∂E, whose interior is disjoint from E. For $\varepsilon > 0$, let $y_j \in E - C(\varepsilon)$ with $4 < |y_j| \to \infty$. Choose r_j such that

$$|y_j| - 2 \leq 6r_j \leq |y_j| - 1.$$

An application of the maximum principle shows the connected component Σ_j of $B_{2r_j}(\widetilde{y}_j) \cap E$, containing y_j, is topologically a disk [7]; $\widetilde{y}_j = \pi(y_j), \pi(x, y, z) = (x, y)$.

Choose ε so that $26\varepsilon \leq \varepsilon_0$, ε_0 the constant of the one-sided curvature estimates. Since $x_3(y_j) \leq 12\varepsilon r_j$ it follows Σ_j satisfies the one-sided curvature estimates, i.e.,

$$\sup_{\Sigma_j^1} |A_{\Sigma_j^1}|^2 \leq r_j^{-2},$$

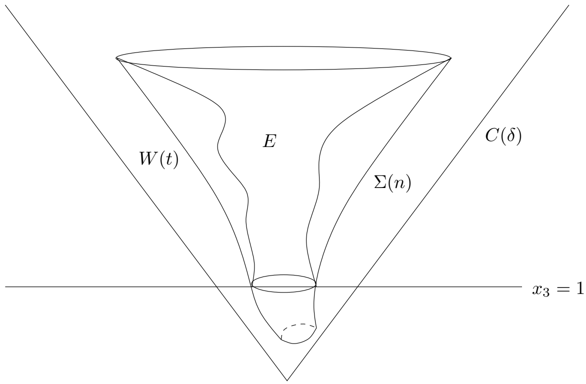

Figure 1

where Σ_j^1 is the connected component of $B_{r_j}(\widetilde{y}_j) \cap \Sigma_j$ containing y_j.

This curvature estimate on Σ_j^1 allows one to apply the Harnack inequality, [3], to the positive harmonic function x_3, on the intrinsec disk $D_{r_j}(y_j) \subset \Sigma_j^1$. Then

$$\sup_{D_{\frac{3}{4}r_j}(y_j)} (x_3) \leq C y_j \leq 26 C \varepsilon r_j,$$

where C comes from the Harnack estimate. From this one obtains a gradient estimate for x_3 on a slightly smaller disk, [3];

$$\sup_{D_{\frac{5}{8}r_j}(y_j)} |\nabla x_3| \leq C_1(26 C \varepsilon r_j),$$

where ∇ is the intrinsic gradient. Thus $D_{\frac{5}{8}r_j}(y_j)$ is a vertical graph with small gradient. Let y_j^1 be a point in

$$\partial B_{|y_j|} \cap \{(x_1 - x_1(y_j))^2 + (x_2 - x_2(y_j))^2 = \frac{r_j^2}{4}\} \cap D_{\frac{5}{8}r_j}(y_j)$$

so that if $26 C\varepsilon \leq \varepsilon_0$, one can apply the preceding argument with y_j^1 in place of y_j.

Repeating this $48\pi + 1$ times, one can go once around the surface

$$K = \partial B_{|y_j|} \cap \{0 \leq x_3 \leq \bar{C} 6 \varepsilon |y_j|\}, \bar{C} = C^m, m = 48\pi + 1.$$

For ε sufficiently small, the one-sided curvature estimates hold in K so one obtains a curve $\gamma_j \subset K \cap E$ which is almost horizontal. Since γ_j is embedded, it is either a Jordan curve or a spiral. It can not be a spiral since E is properly embedded and γ_j lies above $x_3 = 0$.

Assume $|y_j|$ chosen so that E meets $\partial B_{|y_j|}$ transversally. Notice that the connected component E_j of $E \cap B_{|y_j|}$ that contains ∂E is topologically an annulus, $\partial E_j = \partial E \cup \sigma(j)$, $\sigma(j)$ a Jordan curve on $\partial B_{|y_j|}$. This follows since there are no compact components of E, outside $B_{|y_j|}$ whose boundary is on $\partial B_{|y_j|}$. All the other components (if they exist) of $E \cap B_{|y_j|}$ are disks.

Observe that $\sigma(j) \subset K$. Otherwise $\sigma(j) \neq \gamma_j$ and γ_j would bound a disk $D \subset E \cap B_{|y_j|}$, $D \cap E_j = \emptyset$. But D is in the convex hull of γ_j, so D separates in $B_{|y_j|}$, the origin from points in $B_{|y_j|}$ whose height is above the height of K. This implies $D \cap (E_j \cup \beta) \neq \emptyset$, a contradiction.

Since $\sigma(j) \subset K$, the previous discussion shows $\sigma(j)$ is a vertical graph with controlled gradient, hence its length is at most $2\pi(1 + C_0)|y_j|$ for some constant C_0.

R. Osserman has proved an isoperimetric inequality for annular minimal surfaces [32],

$$Area(E_j) \leq \frac{C_1}{4\pi}|y_j|^2,$$

where $C_1 < (2\pi + 2C\pi + |\partial E|)^2$ does not depend on j. Then this quadratic area growth yields E has finite total curvature, which finishes the proof.

□

6 Minimal lamination of $\mathbb{R}^3$

In the introduction I mentionned the question André Haefliger posed to me: "Is there a foliation of $\mathbb{R}^3$ by minimal surfaces other than a foliation by parallel planes?" A leaf of a foliation by minimal surfaces of $\mathbb{R}^3$ is stable hence it is a plane. So minimal foliations of $\mathbb{R}^3$ are linear.

What are the minimal laminations of $\mathbb{R}^3$ is a mystery at the present time. I know of two types of examples: a lamination with exactly one leaf - a properly embedded minimal surfaces, and a lamination consisting of a closed set of parallel planes. Are there others?

In this chapter I will describe what Bill Meeks and I know about minimal laminations of $\mathbb{R}^3$ [23]. First, a definition is in order.

Definition 1. *A closed set $\mathcal{L}$ in $\mathbb{R}^3$ is called a minimal lamination if $\mathcal{L}$ is the union of pairwise disjoint connected complete injectively immersed minimal surfaces. Locally we require that there are $C^{1,\alpha}$ coordinate charts $f\colon D \times (0,1) \to \mathbb{R}^3, 0 < \alpha < 1$, with $\mathcal{L}$ in $f(D \times (0,1))$ the image of the $D \times \{t\}$, t varying over a closed subset of $(0,1)$. The minimal surfaces in $\mathcal{L}$ are called the leaves of $\mathcal{L}$.*

The leaves L of a minimal lamitation $\mathcal{L}$ are smooth (even analytic), and if K is a compact set of L which is a limit leaf of $\mathcal{L}$, then the leaves of $\mathcal{L}$ converge smoothly to L over K; the convergence is uniform in the C^k-topology for any k.

A consequence of the one-side curvature estimates of Colding and Minicozzi is the following. Let Σ be any compact smooth surface passing through the origin with boundary contained in the boundary of the ball $B(1)$ of radius one centered at the origin. There is an ε and a constant c such that if D is an embedded minimal disk in $B(1)$, disjoint from Σ, and with boundary contained in the boundary of $B(1)$, then in $B(\varepsilon)$, the curvature of D is bounded by c. This can be seen by homothectically expanding Σ; the ε depends on the norm of the second fundamental form of Σ in the ball $B(1/2)$. In our applications Σ will be a stable minimal disk for which one always has a bound on the norm of the second fundamental form in $B(1/2)$ by curvature estimates for stable surfaces.

The only known examples of minimal laminations of $\mathbb{R}^3$ with more than one leaf are closed sets of parallel planes in $\mathbb{R}^3$ and we conjecture that these are the only ones. In fact, we will prove that in the case $\mathcal{L}$ has more than one leaf, then every leaf of $\mathcal{L}$ with finite topology is a plane.

Every leaf L of a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ has locally bounded Gaussian curvature in the sense that the intersection of L with any ball has Gaussian curvature bounded from below by a constant that only depends on the ball. The reason that the curvature is locally bounded is that the intersection of $\mathcal{L}$ with a closed ball is compact and the Gaussian curvature function is continuous.

We now state the main theorem concerning minimal laminations of $\mathbb{R}^3$.

Theorem 4. *Suppose $\mathcal{L}$ is a minimal lamination in $\mathbb{R}^3$. If $\mathcal{L}$ has one leaf, then this leaf is a properly embedded surface in $\mathbb{R}^3$. If $\mathcal{L}$ has more than one leaf, then $\mathcal{L}$ consists of the disjoint union of a nonempty closed set of parallel planes $\mathcal{P} \subset \mathcal{L}$ together with a collection of complete minimal surfaces of unbounded Gaussian curvature that are properly embedded in the open slabs and halfspaces of $\mathbb{R}^3 - \mathcal{P}$ and each of these open slabs and half-spaces contains at most one leaf of $\mathcal{L}$. In this case every plane, parallel to but different from the planes in $\mathcal{P}$, intersects at most one of the leaves of $\mathcal{L}$ and separates such a leaf into two components. Furthermore, in the case $\mathcal{L}$ contains more than one leaf, the leaves of $\mathcal{L}$ of finite topology are planes.*

The proof of this theorem will follow from the following lemmas.

Lemma 1. *Suppose M is a complete connected embedded minimal surface in $\mathbb{R}^3$ with locally bounded Gaussian curvature. Then one of the following holds:*

1. *M is properly embedded in $\mathbb{R}^3$;*

2. *M is properly embedded in an open half-space of $\mathbb{R}^3$ with limit set the boundary plane of this half-space;*

3. *M is properly embedded in an open slab of $\mathbb{R}^3$ with limit set consisting of the boundary planes.*

Proof. Let x_n be any sequence of points in M, converging to some x in $\mathbb{R}^3$. Since M has locally bounded curvature, there is a $\delta = \delta(x)$ such that for n sufficiently large, M is a graph F_n over the disk $D_\delta(x_n)$ in the tangent plane to M at x_n, of radius δ and centered at x_n. Moreover each such local graph F_n has bounded geometry.

Choose a subsequence of the x_n so that the tangent planes to M at the subsequence converge to some plane P at x. Then the F_n of this subsequence will be graphs (for n large) over the disk D of radius $\delta/2$ in P centered at x. By compactness of minimal graphs, a subsequence of the F_n will converge to a minimal graph F_∞ over $D, x \in F_\infty$.

Notice that F_∞ at x does not depend on the subsequence of the x_n. If $y_n \in M$ is a sequence converging to x with the tangent planes of M at y_n converging to a plane Q at x. Then $P = Q$ and the local graphs G_n of M at y_n converge to F_∞ as well. If this were not the case then F_∞ and G_∞ would cross each other near x (i.e, $x \in F_\infty \cap G_\infty$ and the maximum principle implies there are points of $F_\infty \cap G_\infty$ near x where they meet transversely). Now F_∞ is the uniform limit of the graphs F_n and G_∞ is the uniform limit of the graphs G_n so near a point of transverse intersection of F_∞ and G_∞ we would have F_i intersecting G_j transversely for i, j large. This contradicts M embedded. Notice also that each F_n is disjoint from F_∞; this follows by the same reasoning as above. Thus we have a local lamination of $\bar{M}$ near x.

Each point $y \in \partial F_\infty$ is also an accumulation point of M so there is a limit graph $F_\infty(y)$ over a disk of radius $\delta(y)$ centered at y. By uniqueness of limits, $F_\infty(y) = F_\infty$ where they intersect. Thus F_∞ may be continued analytically to obtain a complete minimal surface in $\bar{M}$. The lamination $\mathcal{L}$ is obtained by taking the closure of all the limit surfaces so obtained.

Next we will prove that any limit leaf of $\mathcal{L}$ is a plane.

Let L be a limit leaf and $\widehat{L}$ the universal covering space of L. The exponential map of L is a local diffeormorphism and there is a normal bundle ν over $\widehat{L}$, of varying radius, that submerses in $\mathbb{R}^3$. Give ν the flat metric induced by the submersion; $\widehat{L}$ is the zero section of ν.

Let $\widehat{D}$ be a compact simply connected domain of $\widehat{L}$, D its projection into L. Each point of D has a neighborhood that is a uniform limit of (pairwise disjoint) local graphs of M. The usual holomony construction allows one to lift these local graphs along the lifting of paths in D to obtain $\widehat{D}$ as a uniform limit of pairwise-disjoint embedded minimal surfaces E_n in ν.

It is known that any compact domain F (here $F = \widehat{D}$) that is a limit of disjoint minimal domains E_n is stable. Here is a proof. If F were unstable, the first eigenvalue λ_1 of the stability operator L of the minimal surface F (here $L = \Delta - 2K$) is negative. Let $\vec{n}$ denote the unit vector field along F in ν and f the eigenfunction of $\lambda_1, L(f) + \lambda_1 f = 0, f > 0$ in F and $f = 0$ on ∂F.

Consider the variation of F: $F(t) = \{x + tf(x)\vec{n}(x) \mid x \in F\}$. The first variation $\dot{H}(0)$ of the mean curvature of $F(t)$ at $t = 0$ is given by $L(f)$. Since $\lambda_1 < 0$, and $f(x) > 0$ for $x \in \mathrm{Int}(F)$, it follows that the mean curvature vector of $F(t)$, for t small, points away from F, i.e, $< \vec{H}_t(x), \vec{n}(x) >> 0$.

Now for t_0 small, choose n large so that $E(n)$ is close enough to F so there is a a non empty intersection of $F(t_0)$ and $E(n)$. As t decreases from t_0 to 0, the $F(t)$ go from $F(t_0)$ to F. So there will be a smallest positive t so that $D(t)$ has a non-empty intersection with $E(n)$. Let $y \in F(t) \cap E(n)$. Near y, $E(n)$ is on the mean convex side of $F(t)$. Since $E(n)$ is a minimal surface, this is impossible.

Then by Fischer-Colbrie and Schoen's theorem [13], $\widehat{L}$ is a plane, hence L as well, and each limit leaf of $\mathcal{L}$ is a plane. □

Remark 2. *F. Xavier [42] proved that a complete nonflat immersed minimal surface of bounded curvature in $\mathbb{R}^3$ is not contained in a halfspace.*

Lemma 2. *Suppose M is a complete connected embedded minimal surface in $\mathbb{R}^3$ with locally bounded curvature. If M is not proper and P is a limit plane of M, then, for any $\varepsilon > 0$, the closed ε-neighborhood of P intersects M in a connected set.*

Proof. Suppose P is a limit plane of M and, to be concrete, suppose P is the x_1x_2-plane and that M lies above P. Let $P(\varepsilon)$ be the plane at height ε and suppose that M intersects the closed slab S between P and $P(\varepsilon)$ in at least two components $M(1)$, $M(2)$. By Sard's Theorem, we may assume that $P(\varepsilon)$ intersects M transversely. We know that M is proper in the open slab between P and $P(\varepsilon)$ since through any accumulation point of M in the open slab there would pass a limit plane of M.

Let R be region of S bounded by $M(1) \cup M(2)$. Consider a smooth compact exhaustion $\Sigma(1), \Sigma(2), \ldots, \Sigma(n), \ldots$ of $M(1)$. Let $\tilde{\Sigma}(i) \subset R$ with $\partial\tilde{\Sigma}(i) = \partial\Sigma(i)$ be least-area surfaces $\mathbb{Z}_2$-homologous to $\Sigma(i)$ in R. Standard curvature estimates and local area bounds imply that a subsequence of the $\tilde{\Sigma}(i)$ converges to a properly embedded stable minimal surface Σ in R with boundary $\partial M(1)$. Since S is simply-connected, Σ separates S. Therefore, Σ is orientable and the curvature estimates of Schoen [37] then imply curvature estimates at any uniform distance from $P(\varepsilon)$.

By the Halfspace Theorem in [15], or rather its proof, Σ can not be proper in S. As in the previous lemma, the limit set of Σ is a plane $P' \subset S$. Since Σ has curvature estimates near P', there exists a δ, $0 < \delta < \varepsilon/2$, such that the normal line to $\Sigma(\delta) = \Sigma \cap \{(x_1, x_2, x_3) \mid 0 < x_3 < \delta\}$ is close to a vertical line. Hence, the orthogonal projection $\pi\colon \Sigma(\delta) \to P'$ is a submersion onto its image. Furthermore, given any compact disk $D \subset P'$, every component of $\pi^{-1}(D)$ is compact. Using this compactness property, and a slight variation of the following lemma, it follows that π is injective. Therefore, Σ is proper in S, which we observed before can not occur. This contradiction proves the lemma.

$\square$

Lemma 3. *Suppose M and N are smooth connected manifolds of the same dimension such that N is simply-connected and M may have boundary. If $\pi\colon M \to N$ is a proper submersion onto its image and $\pi|\partial M$ is injective on each boundary component of M, then π is injective. In particular, if M is a smooth immersed surface with boundary in $\mathbb{R}^3$, the projection $\pi\colon M \to \mathbb{R}^2$ to the x_1x_2-plane is a proper submersion onto its image and $\pi|\partial M$ is injective, then M is a graph over $\pi(M) \subset \mathbb{R}^2 \times \{0\}$.*

Proof. If M has no boundary, then $\pi\colon M \to N$ is a connected covering space and the lemma follows since N is simply-connected.

If ∂ is a boundary component of M, then $\pi(\partial)$ is a properly embedded codimension-one submanifold of N. Since N is simply-connected, $\pi(\partial)$ separates N into two open components. We label these components of $N - \pi(\partial)$ by $C(M)$ and $C(\partial)$, where $C(M)$ is the component such that the closure of $\pi^{-1}(C(M))$ contains ∂ as boundary component. Now consider the quotient space $\hat{M}$ obtained from the disjoint union of M with all the closures of $\overline{C}(\partial_\alpha)$, ∂_α a boundary component of M, with identification map π, $\pi\colon \partial M \to \cup\overline{C}(\partial_\alpha)$. Let $\hat{\pi}\colon \hat{M} \to N$ be the natural projection that extends π on $M \subset \hat{M}$. It is straighforward to check that $\hat{\pi}$ is a connected covering space of N. Since N is simply-connected, $\hat{\pi}$ is injective which proves the lemma.

$\square$

Lemma 4. *If M is a complete embedded minimal surface in $\mathbb{R}^3$ with finite topology and locally bounded curvature, then M is properly embedded in $\mathbb{R}^3$.*

Proof. Suppose now that M has finite topology and lies in the upper half-space of $\mathbb{R}^3$ with limit set the x_1x_2-plane P. If M has bounded curvature in some ε-neighborhood of P, then it was proved above that M is proper in this neighborhood and has a plane in its closure.

This is impossible by the half-space theorem. It remains to prove that M has bounded curvature in an ε-neighborhood of P.

Arguing by contradiction, assume that M does not have bounded curvature. In this case, there is an annular end $E \subset M$ whose Gaussian curvature is not bounded in the slab $S = \{(x_1, x_2, x_3) \mid 0 \leq x_3 \leq 1\}$. After a homothety of M, we may assume that ∂E is contained in the ball B_0 of radius one centered at the origin. Since M has locally bounded curvature, the part of E inside $B(0)$ has bounded curvature.

Since $E \cap S$ does not have bounded curvature, there exists a sequence $p(1), \ldots, p(i), \ldots$ in $E \cap S$ with $\|p(i)\| \geq i$ and $|K(p(i))| \geq i$. After possibly rotating M around the x_3-axis and choosing a subsequence, we may assume that the sequence $(5/\|p(i)\|) \cdot p(i)$ converges to the point $(5, 0)$ in the x_1x_2-plane. Let B be the ball of radius one in the x_1x_2-plane and centered at $(5, 0)$. Notice that there is no compact connected minimal surface with one boundary curve in B_0 and the other boundary curve in B (pass a catenoid between B_0 and B). Using the convex hull property of a compact minimal surface, it is easy to check that $[(5/\|p(i)\|)E] \cap B$ consists only of simply-connected components which are disjoint from the boundary of $(5/\|p(i)\|)E$. The curvature estimates $\mathcal{C}$ imply that, as $i \to \infty$, the curvature of $(5/\|p(i)\|)E$ at $(5/\|p(i)\|) \cdot p(i)$ converges to 0. But the Gaussian curvature at such points approaches $-\infty$ as $i \to \infty$. This contradiction proves the lemma. $\square$

7 Examples of minimal surfaces in $M \times \mathbb{R}$

We will now discuss properly embedded minimal surfaces Σ in $M \times \mathbb{R}$ where M is a complete Riemannian surface. When M is the flat $\mathbb{R}^2$, this is the classical theory of minimal surfaces in $\mathbb{R}^3$. We will see that there are many interesting examples and theorems, in particular when M is a space-form (constant curvature). The references for this section are [36], [30], [10], [21], [22], [24], [25].

When M is a flat 2-torus (quotient of $\mathbb{R}^2$ by two independant translations), one obtains doubly-periodic minimal surfaces. The connected non-flat minimal surfaces in $M \times \mathbb{R}$ are precisely the quotient of connected minimal surfaces in $\mathbb{R}^3$ by the translation group [21]. For example, Scherk's doubly periodic minimal surface is a four punctured sphere in the quotient. One way of constructing these periodic minimal surfaces is by choosing a suitable polygon Γ in $\mathbb{R}^3$, and let $\Sigma(\Gamma)$ be a least area disk with boundary Γ obtained as a solution to the Plateau problem. Then extend $\Sigma(\Gamma)$ to a complete surface by rotating about all the line segment in Γ and continue the rotations about all the new segments that appear.

If Γ is well chosen this will produce a properly embedded minimal surface with symmetries coming from Γ. Here is an example. Consider the polygon $\Gamma(n)$ of figure 2.

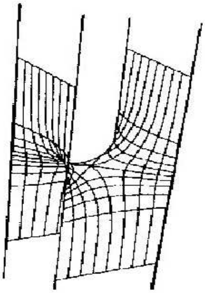

Figure 2-a

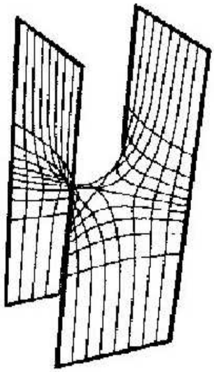

Figure 2-b

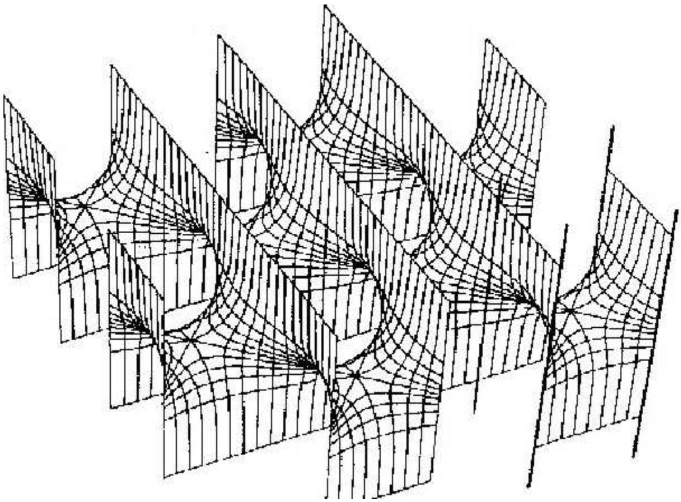

Figure 2-c

The vertical sides go from $x_3 = -n$ to $x_3 = +n$, and assume the vertical projection of $\Gamma(n)$ is a rhombus. Let $\Sigma(n)$ be a least area disk with boundary $\Gamma(n)$. The interior of $\Sigma(n)$ is a vertical graph over the interior of the rhombus. Also, the symmetries of the rhombus imply there is exactly one point of $\Sigma(n)$ with a horizontal tangent plane (at height 0) and this point does not depend on n. As $n \to \infty$, a subsequence of the $\Sigma(n)$ converge to a minimal surface $\Sigma(\infty)$ and $\partial\Sigma(\infty)$ equals the four vertical lines over the vertices of the rhombus. Rotating by π about the line boundaries of $\Sigma(\infty)$ (and continuing these rotations about all the line boundaries arising) extends $\Sigma(\infty)$ to a complete embedded minimal surface Σ in $\mathbb{R}^3$. Σ is invariant by the group Γ generated by the translations $2v_1$,$2v_2$, v_1, v_2 the (vector) sides of the rhombus. Then Σ/Γ is a doubly-periodic minimal surface in $\mathbb{T}^2 \times \mathbb{R}$, modelled on the 4-punctured sphere.

We will see this technique works in many manifolds.

In this paper, we often solve Plateau problems, finding least area surfaces with fixed boundary in a given isotopy class. Some references for doing this in 3-manifolds are [14] and [26]. A reference for finding minimax surfaces of controlled topology is [35].

7.1 Surfaces in $\mathbb{S}^2 \times \mathbb{R}$; unduloids

Let $\mathbb{S}^2$ denote the 2-sphere of curvature one, and $S(t) = \mathbb{S}^2 \times (t)$. We refer to $S(t)$ as the horizontal sphere at height t, and we denote by h the height function on $\mathbb{S}^2 \times \mathbb{R}$, which is the $\mathbb{R}$-coordinate of a point. In a very interesting paper concerning isoperimetric-hypersurfaces in $Q \times \mathbb{R}$, Q an n-dimensional simply connected space-form, Pedrosa and Ritore found and studied the minimal hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ invariant under the group of rotations of the first factor [34]. When $n = 2$, they call these surfaces unduloids (embedded) and nodoids, in analogy with the Delaunay surfaces. They are foliated by circles $C(t)$ in the sphere $S(t)$, of radius $r(t)$; the radius function determines the surface.

Before writing the equations of these surfaces found by Pedrosa and Ritore, We describe their existence by Plateau techniques.

Let p denote a fixed point of $\mathbb{S}^2$ (e.g., the north pole) and let r denote distance to p on $\mathbb{S}^2$. Denote by $C(0)$ the geodesic $r = \pi/2$ of $\mathbb{S}^2$. Then $\Sigma = C(0) \times \mathbb{R}$ is a totally geodesic minimal annulus in $N = \mathbb{S}^2 \times \mathbb{R}$; we will refer to this as a vertical flat annulus. Let D_1, D_2 be the two disks of $S(0)$ bounded by $C(0)$. For $T > 0$, T small, $\Sigma(T)$ (the part of Σ between heights 0 and T) is a stable minimal surface with boundary $C(0) \cup C(T)$. Also, $C(0) \cup C(T)$ bounds another stable surface in $D_1 \times \mathbb{R}$, the union of the two horizontal disks. So there is an unstable surface in $D_1 \times \mathbb{R}$ with boundary $C(0) \cup C(T)$. It is a connected annulus since the only compact minimal surface bounded by a $C(h)$ is a horizontal disk. This annulus can then be extended to an embedded complete minimal annulus by rotation by π about the geodesic boundaries. This rotation about $C(0)$ is the composition of the isometries $(x,t) \mapsto (x,-t)$, and the isometry which is reflection of each $S(t)$ by the geodesic $C(t)$.

Calculations of M. Ritore show that as T tends to 0, these unstable annuli converge to a double cover of the horizontal disk, with a singularity at the center of the disk, just as a catenoid converges to a doubly-covered plane by contraction.

Pedrosa and Ritore call these surfaces unduloids. They deform the flat vertical annulus Σ in the same manner as the Delaunay mean curvature one surfaces in $\mathbb{R}^3$ deform the cylinder of mean curvature one.

They derive the equations for rotationally invariant constant mean curvature H hypersurfaces in $\mathbb{R} \times \mathbb{S}^2$; more generally, in $\mathbb{R} \times$ (a space form) (this is the only place in this paper we change the order of the factors in $\mathbb{S}^2 \times \mathbb{R}$. We do this until the end of this section to respect the order chosen by Pedrosa and Ritore). Here is their solution.

Identify the orbit space with $[0,\pi] \times \mathbb{R}$. An invariant hypersurface is determined by its generating curve γ in the quotient space. Parametrize γ by arc length s and write

$$(x'(s), y'(s)) = (\cos\sigma(s), \sin\sigma(s)).$$

Then Σ has mean curvature H in $\mathbb{S}^n \times \mathbb{R}$ is equivalent to γ satisfying the system:

$$\begin{aligned} x' &= \cos\sigma \\ y' &= \sin\sigma \\ \sigma' &= H + (n-1)\cot(y)\cos(\sigma). \end{aligned}$$

In addition to the embedded unduloid solutions, they show there are immersed solutions as well; they call them nodoids.

The unduloids are invariant under vertical translation by $2\pi T$, hence yield embedded minimal tori in $\mathbb{S}^1(r) \times S^2$.

7.2 Helicoids in $\mathbb{S}^2 \times \mathbb{R}$

We obtain a helicoid by rotating the geodesic $C(0)$ at a constant speed as one rises on the vertical axis at a constant speed. This yields a complete minimal annulus Σ in $\mathbb{S}^2 \times \mathbb{R}$ and by passing to the quotient by a suitable vertical translation, an embedded minimal torus in $\mathbb{S}^2 \times \mathbb{S}^1$.

A conformal parametrization of a helicoid can be obtained as follows. Let

$$X(u,v) = (\cos f(u)\cos v, \cos f(u)\sin v, \sin f(u), v).$$

Then X is conformal in terms of $z = u + iv$, if f satisfies the equation (an elliptic function)

$$f'(u)^2 = 1 + \cos^2 f(u).$$

The mean curvature vector of Σ in $\mathbb{R}^4$ $(\mathbb{S}^2 \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4)$ is easily calculated in terms of

$$\frac{\partial^2 X}{\partial u^2} + \frac{\partial^2 X}{\partial v^2}.$$

A simple calculation then shows the projection of this mean curvature vector onto the unit normal of Σ in $\mathbb{S}^2 \times \mathbb{R}$, is zero. Hence Σ is minimal in $\mathbb{S}^2 \times \mathbb{R}$.

7.3 Riemann type minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$

Just as we obtained unduloids in $\mathbb{S}^2 \times \mathbb{R}$ by a minimax technique applied to a small stable compact vertical minimal annulus, we can obtain surfaces starting with stable pieces of a helicoid. The surfaces obtained are the analogue of Riemann's minimal surface in $\mathbb{R}^3$ [20].

Let $C(0) \subset S(0)$ and $C(t) \subset S(t)$ be two geodesic circles on a helicoid in $\mathbb{S}^2 \times \mathbb{R}$ and suppose t sufficiently small so that the part $\Sigma(t)$ of the helicoid bounded by $C(0) \cup C(t) = \Gamma$ is stable. Then Γ is the boundary of another stable minimal surface, the union of two horizontal disks bounded by Γ. Thus there is an index one minimal annulus $A(t)$ with boundary Γ obtained by the minimax technique [35]. $A(t)$ meets each horizontal sphere $S(\tau), 0 \leq \tau \leq t$, in a circle by [10]. One extends $A(t)$ to a properly embedded minimal annulus A in $\mathbb{S}^2 \times \mathbb{R}$ by the natural symmetries in the boundary geodesics. This is what we call a Riemann-type minimal surface. Notice that $A(t)$ inherits the symmetries of Γ in $\mathbb{S}^2 \times \mathbb{R}$, so A is invariant by screw-motion isometry of $\mathbb{S}^2 \times \mathbb{R}$ and yields an embedded torus in the quotient. We will see another way to obtain these surfaces in a later section.

8 Properties of minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$

Let A be a properly immersed minimal annulus in $\mathbb{S}^2 \times \mathbb{R}$; A is topologically $D^* = \{z \in \mathbb{C} \mid 0 < |z| \leq 1\}$, with ∂A corresponding to $\{|z| = 1\}$. We will see that A behaves in the same way as when the ambient space is $\mathbb{T} \times \mathbb{R}$, $\mathbb{T}$ a flat 2-torus [21]; i.e., we will see that A is conformally D^* and a subend of A meets each horizontal sphere transversaly and in at most one Jordan curve. First we assure the height function is harmonic on A.

Lemma 5. *Let Σ be a minimal hypersurface of a Riemannian manifold N. Let X be a Killing vector field on N. If $X = \nabla f$, is the gradient of some function f on N, then f is harmonic on Σ.*

Proof. Let e_1 , e_2 ..., e_k , n be an orthonormal frame in a neighborhood of a point of Σ, where n is normal to Σ. Since X is a Killing vector field on N, we have:

$$\mathrm{div}(X) = 0 = \langle \nabla_n X, n \rangle.$$

Thus

$$0 = \sum_{i=1}^{k} \langle \nabla_{e_i} X, e_i \rangle + \langle \nabla_n X, n \rangle = \sum_{i=1}^{k} \langle \nabla_{e_i} X, e_i \rangle.$$

Write $X = X^\perp + \nabla_\Sigma f$, $X^\perp$ the normal component of X. Then

$$\begin{aligned} 0 &= \textstyle\sum_{i=1}^{k} \langle \nabla_{e_i} \nabla_\Sigma f, e_i \rangle + \sum_{i=1}^{k} \left\langle \nabla_{e_i} X^\perp, e_i \right\rangle \\ &= \Delta_\Sigma f - \textstyle\sum_{i=1}^{k} \left\langle X^\perp, \nabla_{e_i} e_i \right\rangle \\ &= \Delta_\Sigma f - \left\langle X^\perp, H \right\rangle = \Delta_\Sigma f. \end{aligned}$$

□

Corollary 3. *The only compact minimal surfaces (with no boundary) in $\mathbb{S}^2 \times \mathbb{R}$ are the $S(t)$.*

Proposition 1. *. Let A be a properly immersed minimal annulus in $M \times \mathbb{R}$, M a compact surface. Then A is conformally the punctured disk D^*, and a subend of A can be conformally parametrized by D^* so that $h = c\ell n|z|$. In particular, this subend meets each $M(t)$ transversally in at most one Jordan curve.*

Proof. We proceed as in [21]. Since A is proper, A must go up or go down, but not both. So we can suppose A goes up, zero is a regular value of h, and $h/\partial A < 0$. Then $\Delta = h^{-1}(-\infty, 0]$ is a smooth compact surface; one component of the boundary of Δ is below zero (namely ∂A), and the others are smooth Jordan curves in $M(0)$. There is no compact minimal surface with boundary in $M(0)$ (other than a part of $M(0)$) since the harmonic function h would have an interior extremum on such a surface. A is an annulus, so it follows there is exactly one component of the boundary of Δ in $M(0)$. By the same reasoning, A meets each $M(t)$, $t > 0$, transversally and in one Jordan curve. Now it is easy to parametrize the subend $h^{-1}[0, \infty)$ so that $h = \ell n|z|$. Use the facts that any compact annulus is conformally $\mathbb{S}^1 \times [1, R]$, and a harmonic function on this annulus, constant on each boundary circle, is of the form $a \log |z| + b$, for some contants a, and b. □

8.1 Abresch surfaces in $\mathbb{R}^3$ yield minimal annuli foliated by circles in $\mathbb{S}^2 \times \mathbb{R}$.

Given a torus M of constant mean curvature in $\mathbb{R}^3$, its Gauss map f to the unit sphere $\mathbb{S}^2$ is a harmonic map. Its holomorhic Hopf quadratic differential is:

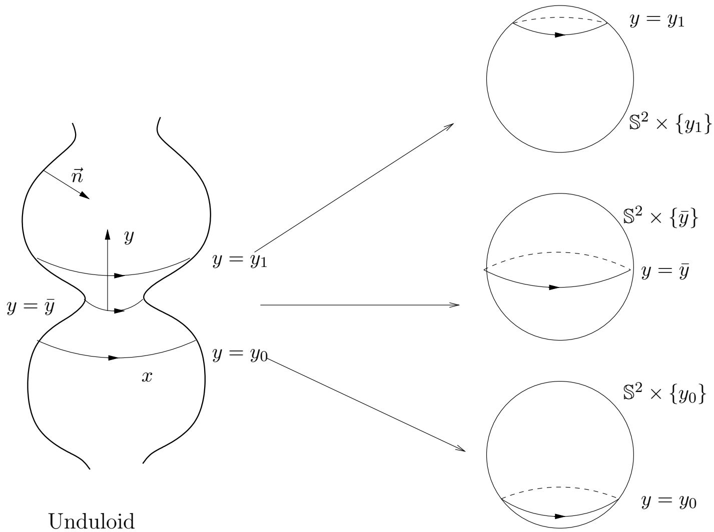

Figure 3:Unduloid

$$Q(f) = \left[\left| \frac{\partial f}{\partial x} \right|^2 - \left| \frac{\partial f}{\partial y} \right|^2 - 2i \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \right] dz^2.$$

Since M is a torus, this is constant:

$$Q(f) = cdz^2.$$

After a linear change of coordinates we can assume the constant c is one. Then the map:

$$F : \mathbb{S}^1 \times \mathbb{R} \longrightarrow \mathbb{S}^2 \times \mathbb{R}, F(x, y) = (f(x, y), y)$$

is a conformal harmonic map, i.e., a minimal surface. The Delaunay surfaces yield the unduloids and the nodoids yield the helicoids in $\mathbb{S}^2 \times \mathbb{R}$ (see figure 3 and 4).

Abresch studied the family of Wente tori (constant mean curvature $\frac{1}{2}$ in $\mathbb{R}^3$) whose principle curvature lines are planar [1]. More generally, he studied constant mean curvature surfaces parametrized by $\mathbb{R}^2$, with the coordinate axes x and y yielding the (smaller and larger) principle lines of curvature.

Those tori whose small curvature lines, λ_1, are planar were found among the solutions of the system:

$$\Delta\omega + \sinh(\omega) \cdot \cosh(\omega) = 0$$

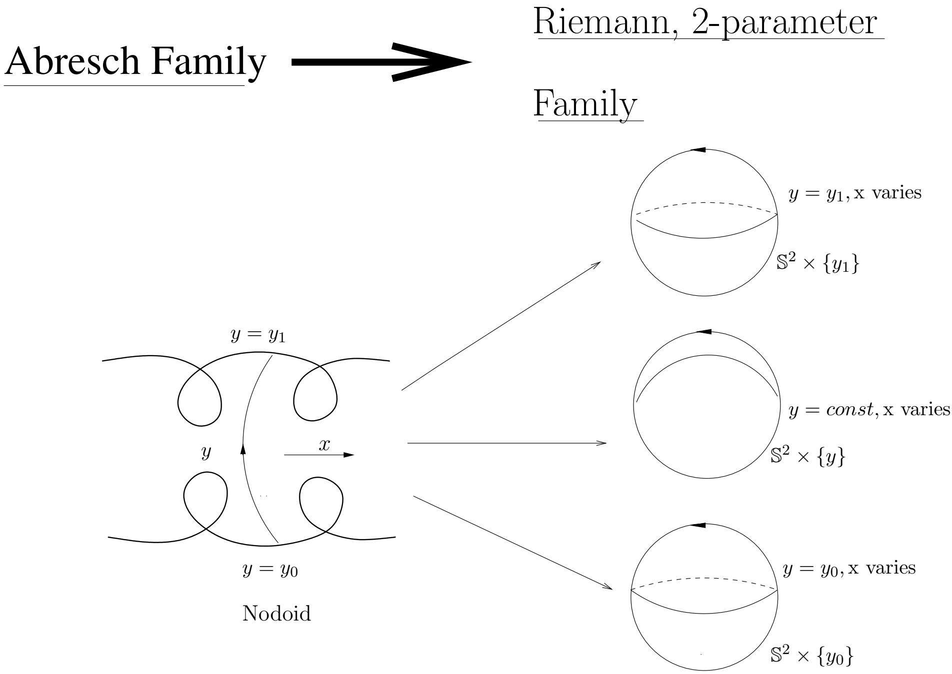

Figure 4: Helicoid

$$\sinh(\omega)\cdot \dot{\omega}' - \cosh(\omega)\cdot \dot{\omega}\, \omega' = 0.$$

Here $'$ and $\dot{}$ denote the derivatives with respect to x and y, respectively. We remark that $Q(f) = dz^2$ for these surfaces.

Abresch classified all real-analytic solutions $\omega\colon \mathbb{R}^2 \to \mathbb{R}$ of the above system in terms of elliptic functions. He then went on to study the system which corresponds to the larger curvature lines λ_2 being planar curves. It follows that the Gaussian image of each such λ_2 line is contained in a circle of $\mathbb{S}^2$. These examples are solutions of the following system I:

$$\text{(I)} \qquad \Delta\omega + \sinh(\omega)\cdot\cosh(\omega) = 0$$

$$\cosh(\omega)\cdot \dot{\omega}' - \sinh(\omega)\cdot \dot{\omega}\, \omega' = 0.$$

We remark that $Q(f) = -dz^2$ when f is the Gauss map of this Abresch family. Abresch classifies all solutions of this system as well (it is analogous to the first solution space), and shows that the associated constant mean curvature surfaces do not close up in $\mathbb{R}^3$ and so do not yield Wente tori. However, the solutions of system (I) do yield minimal immersions $F\colon \mathbb{S}^1 \times \mathbb{R} \longmapsto \mathbb{S}^2 \times \mathbb{R}$. Laurent Hauswirth [10] observed that the second equation of system (I) is precisely the condition that the level curves of F be circles. We define $\mathcal{A}$ to be the family of these minimal surfaces, induced by solutions of (I) or which are helicoids in $\mathbb{S}^2 \times \mathbb{R}$. If f denotes the Gauss map of a constant mean curvature surface solution of (I), then the minimal surface $(x, y) \longmapsto (f(y, x), y)$ in $\mathbb{S}^2 \times \mathbb{R}$ is foliated by circles in level set spheres.

Hauswirth [10] also observed that the classical Shiffman Jacobi function for minimal surfaces in $\mathbb{R}^3$ (transverse to the planes $\mathbb{R}^2 \times \{t\}$) generalizes to $\widetilde{M} \times \mathbb{R}$ where $\widetilde{M}$ is a simply-connected space form. This yields another way to generate Riemann-type examples of minimal surfaces in $\widetilde{M} \times \mathbb{R}$.

Let A be a compact minimal annulus in $\widetilde{M} \times \mathbb{R}$ with boundary curves in $\widetilde{M} \times \{t_1\}$ and $\widetilde{M} \times \{t_2\}$ with $t_1 < t_2$. Let $S(r)$ denote the circle of circumference r. Assume $S(r) \times [t_1, t_2]$ is a conformal parametrization of A with $h(x, t) = t$ and where r is the flux of h. Let $\kappa(\theta, t)$ be the corresponding geodesic curvature function of the level set curve at height t. Then the Shiffman function is:

$$S = \lambda \frac{\partial \kappa}{\partial \theta}.$$

This is a Jacobi function on A where $\lambda(\theta, t)$ is the conformal factor or speed of the level set curve at the parameter values (θ, t). If the boundary curves of A are chosen to be circles and A is strictly stable, then the Shiffmann function has zero boundary values on A and so vanishes on A. This means that A is foliated by circles. In the case $\widetilde{M}$ is $\mathbb{S}^2$, then by analytic continuation, we obtain a periodic minimal surface. In fact we obtain all the examples in $\mathcal{A}$ in this way. In the case $\widetilde{M} = \mathbb{R}^2$ we obtain $\widetilde{A}$ which is a catenoid or one of the Riemann examples.

Now suppose one considers a small stable part of a catenoid in $\mathbb{H}^2 \times \mathbb{R}$ bounded by a circle in $\mathbb{H}^2 \times \{0\}$ and a circle in $\mathbb{H}^2 \times \{t\}$. Translate sideways slightly the circle in $\mathbb{H}^2 \times \{t\}$. Then there is a new annulus $A(t, s)$ bounded by the two new circles and, using the Shiffman function, we see that it is foliated by circles. Notice that $A(t, s)$ has a vertical plane of symmetry coming from the symmetry of its boundary circles. Now $A(t, s)$ can be extended to a minimal surface $B(t, s)$ in some open neighborhood of $A(t, s)$ since $\partial A(t, s)$ is analytic. Clearly $B(t, s)$ is also foliated by circles. Thus, there is a maximal open minimal annulus $\widetilde{A}(s)$ foliated by circles, and containing $A(t, s)$. A simple maximum principle argument shows the asymptotic boundary of $\widetilde{A}(s)$ consists of two horocycles C_1 and C_2. Assuming C_2 higher than C_1, $\widetilde{A}(s)$ extends above C_2 by level curves of constant curvature less than one; i.e., equidistant curves. These curves eventually become a geodesic γ_2. Similarly extending $\widetilde{A}(s)$ below C_1, by equidistant curves, we arrive at a geodesic γ_1. Now rotate about γ_1, and

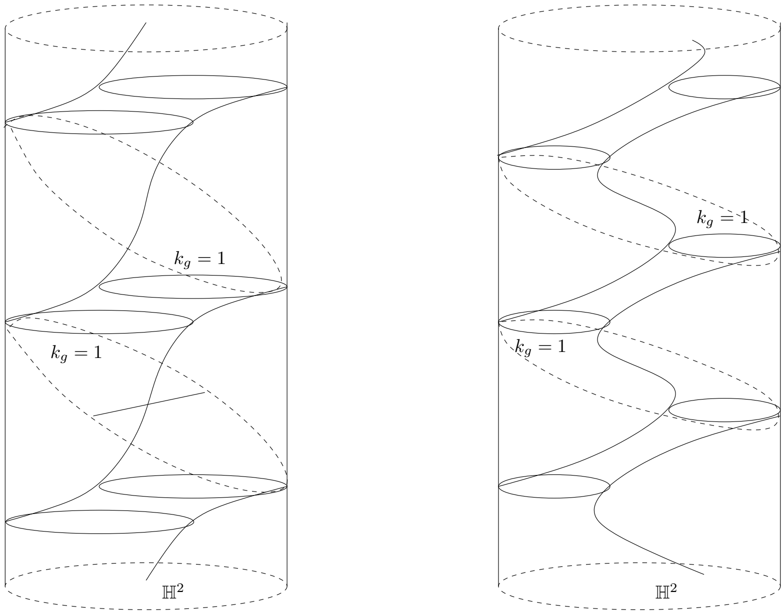

Figure 5: Two examples of Riemann's type in $\mathbb{H}^2 \times \mathbb{R}$

γ_2 and continue to obtain a complete embedded "Riemann-type" minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ (see figure 5).

In a very interesting paper, L. Hauswirth has described all the minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ whose level curves have constant curvature [10]. This yields many more beautiful examples in $\mathbb{H}^2 \times \mathbb{R}$. He parametrizes these surfaces by a 2-parameter family of elliptic functions which generalizes to the space form, what Abresch did in $\mathbb{R}^2 \times \mathbb{R}$. He generalizes the Shiffman function to minimal annuli in $\mathbb{H}^2 \times \mathbb{R}$ (or $\mathbb{S}^2 \times \mathbb{R}$). He then proves that a minimal annulus A bounded by circles or geodesics in $M \times \{t_1\}$ and $M \times \{t_2\}$ ($M = \mathbb{S}^2$ or $\mathbb{H}^2$), and A of index one, is foliated by circles and geodesics at each level.

8.2 Some surfaces of higher genus

We will now construct properly embedded minimal surfaces of finite topology in $\mathbb{S}^2 \times \mathbb{R}$. They will be conformally equivalent to a compact Riemann surface of genus g with two punctures. They have one top end, one bottom end and each is asymptotic to a flat vertical annulus. We will then prove that any properly embedded minimal surface, has exactly one

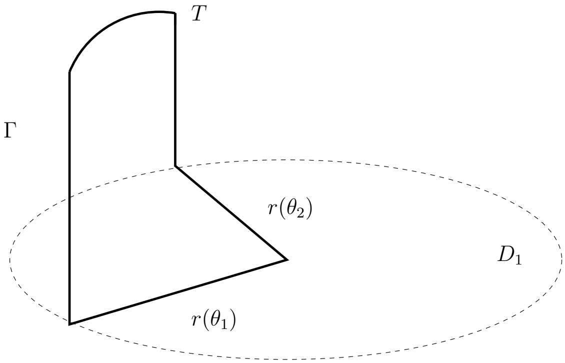

Figure 6

top end and one bottom end (or is some $S(t)$).

Recall that $C(0)$ is a fixed geodesic of $S(0)$. Let D_1 , D_2 be the two disks of $S(0)$ bounded by $C(0)$. Consider $\mathbb{S}^2 \times \mathbb{R}$ as the union of the two vertical solid cylinders $(D_1 \times \mathbb{R}) \cup (D_2 \times \mathbb{R})$, identified along their common boundary; the flat vertical annulus $C(0) \times \mathbb{R}$.

Consider geodesic coordinates (r, θ) in D_1, where the center of D_1 is $r = 0$, and $C(0)$ is $r = \pi/2$. The rays $r(\theta) = \{\theta = \text{ constant}, 0 \leq r \leq \pi/2\}$ are geodesics, and the circular arcs of $C(0)$ between two θ values, θ_1 and θ_2 , we denote by $C(\theta_1, \theta_2)$. When $|\theta_1 - \theta_2| < \pi$, $C(\theta_1, \theta_2)$ denotes the arc of $C(0)$ of length less than π. In these coordinates, the end points of $C(\theta_1, \theta_2)$ are $(r = \pi/2, \theta_1)$ and $(r = \pi/2, \theta_2)$. Let T be a fixed number, and denote by $\Gamma(T, \theta_1, \theta_2)$ the geodesic polygon in $S \times \mathbb{R}$, with the five sides, $r(\theta_1)$, $r(\theta_2)$, the two vertical geodesic segments $\{(\pi/2, \theta_1, t) \mid 0 \leq t \leq T\}$, $\{(\pi/2, \theta_2, t) \mid 0 \leq t \leq T\}$ and the arc $(C(\theta_1, \theta_2), T)$ of $C(T)$; cf. figure 6

Let Σ be a least area compact minimal surface with boundary $\Gamma = \Gamma(T, \theta_1, \theta_2)$. We claim Σ is an embedded disk and $\mathrm{int}(\Sigma)$ is a vertical graph over the domain in D_1 bounded by $r(\theta_1) \cup r(\theta_2) \cup C(\theta_1, \theta_2)$.

To see this notice that Rado's theorem is true for minimal surfaces in $D_1 \times \mathbb{R}$: if $\Gamma \subset D_1 \times \mathbb{R}$ is a Jordan curve which has a convex projection to D_1 then any compact minimal surface in $D_1 \times \mathbb{R}$ bounded by Γ is an embedded disk and its interior is a vertical graph over the domain in D_1 bounded by the projection of Γ. Vertical translation in $D_1 \times \mathbb{R}$ is an isometry, and the height function is harmonic on a minimal surface, so the usual proof of Rado's theorem goes through.

In our case, $\partial D_1 \times \mathbb{R}$ is a good barrier for solving the Plateau problem in $D_1 \times \mathbb{R}$ so

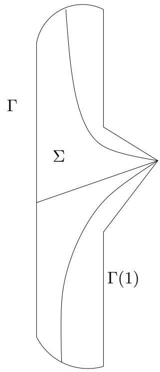

Figure 7

$\Sigma \subset D_1 \times \mathbb{R}$ and Rado's theorem applies.

Next observe that Σ can be continued by rotation by π about each edge in its boundary. Given a geodesic C in some $S(t)$, rotation by π about C is the ambient isometry which is the composition of symmetry of $S(t)$ by C, and symmetry of $\mathbb{S}^2 \times \mathbb{R}$ by $S(t)$. Given a vertical geodesic B of $\mathbb{S}^2 \times \mathbb{R}$, rotation by π about B is the symmetry of each $S(t)$ through the point $S(t) \cap B$. Notice that when $B \subset \partial D_1 \times \mathbb{R}$, the rotation by π about B permutes $D_1 \times \mathbb{R}$ and $D_2 \times \mathbb{R}$. On the other hand, rotation about an $r(\theta)$ in $D_1 \times \mathbb{R}$ sends Σ into $D_1 \times \mathbb{R}$. Consider the rotation of Σ about $r(\theta_1)$. The polygon Γ has image a polygon $\Gamma(1)$ as depicted in figure 7, and the image of Σ is easy to understand.

Continuing to reflect across the rays, the resulting surface will close-up after $2k$ successive reflections when $\theta_2 - \theta_1 = \pi/k$ (for some integer $k = 1, 2, \ldots$). Then the $(2k)$ images of Σ yield an embedded minimal disk $\Sigma(k)$ whose interior is a smooth vertical graph over D_1, and whose boundary is a geodesic polygon on $\partial D_1 \times \mathbb{R}$, composed of vertical and horizontal geodesics. The vertical geodesics go from height $-T$ to height T. Below is the case $k = 2$; figure 8.

Notice that the height function on $\Sigma(k)$ has a critical point at the center of D_1, which is on $\Sigma(k)$, of index $1 - k$, and no critical points elsewhere on $\Sigma(k)$.

There are two natural ways to proceed now to obtain properly embedded minimal surfaces from $\Sigma(k)$. We can let $T \to \infty$, or we can do all symmetries of $\Sigma(k)$ across the geodesic boundaries.

First consider the surface obtained by fixing T and doing all the symmetries in the sides of $\Sigma = \Sigma(k, T)$. This yields a properly embedded minimal surface in $\mathbb{S}^2 \times \mathbb{R}$ which is invariant by vertical translation by $2T$. In the quotient $\mathbb{S}^2 \times \mathbb{R}/2T = \mathbb{S}^2 \times \mathbb{S}^1$, one obtains a compact

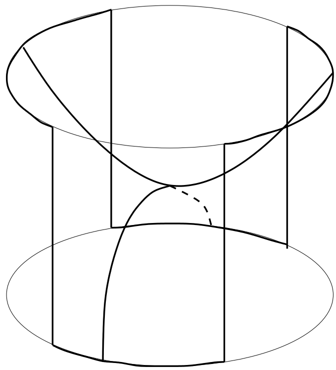

Figure 8

surface of genus k (just count the indices of the critical points of h). Notice that the surface $k = 1$ is an embedded minimal annulus in $\mathbb{S}^2 \times \mathbb{R}$; in fact, it is the helicoid we introduced previously.

Next consider letting $T \to \infty$. Recall that the Plateau solution $\Sigma(T)$ with boundary $\Gamma(T, \theta_1, \theta_2)$ is a vertical graph over the geodesic triangle $\Delta = r(\theta_1) \cup r(\theta_2) \cup C(\theta_1, \theta_2)$, of the function $u(T)$ with boundary values zero on the sides $r(\theta_1) \cup r(\theta_2)$ and the value T on $C(\theta_1, \theta_2)$. The function $u(T)$ is continuous at all points of Δ except the two endpoints of $C = C(\theta_1, \theta_2)$. We will prove shortly that the functions $u(T)$ converge uniformly (on compact subsets of $\Delta - C$), to a function $u(\infty)$, defined on $\Delta - C$, provided $\theta_1 - \theta_2$ is strictly less then π. The graph of $u(\infty)$ is a minimal surface with boundary $\Gamma(\infty, \theta_1, \theta_2)$, and its gradient approaches infinity as one approaches C from the interior of Δ (cf. theorem 5).

Thus, when $\theta_2 - \theta_1 = \pi/k$ (for some integer $k = 2, 3, \dots$), The surfaces $\Sigma(k)$ converge to a minimal surface $\Sigma(\infty)$ bounded by the complete vertical geodesics $B(i\,\pi/k)$, $i = 1, \dots, 2k$; the vertical lines over the points on ∂D_1, given by ($r = \pi/2$, $\theta_i = i\pi/k$). Clearly $\Sigma(\infty) = \Sigma(k, \infty)$ is a graph over D_1 (i.e. its interior).

Now do rotation by π about the vertical geodesic $B(\pi/k)$. This induces a diffeomorphism from $\partial\Sigma(k, \infty)$ to itself and extends $\Sigma(k, \infty)$ to a complete properly embedded surface M with no boundary. The reader can verify that there is one top end, one bottom end, and each of these ends is asymptotic to a flat vertical annulus. The height function has exactly two critical points, each of index $1 - k$. They are the centers of D_1 and D_2. Since the top and bottom ends each give rise to a critical point of index one at the punctures, it follows that M is conformally diffeomorphic to a closed Riemann surface of genus $k - 1$ punctured in two points.

The Gauss-Bonnet theorem yields the total curvature of $\Sigma(k,T)$ to be $2\pi(1-k)$. Since this does not depend on T, the total curvature of $\Sigma(k,\infty)$ is also $2\pi(1-k)$. Hence the total curvature of M is $4\pi(1-k)$, which is $2\pi\chi(M)$.

Now we will prove the existence of the Scherk-type surface we discussed.

Assume $0<\theta_1<\theta_2<\pi$, and $\Sigma(T)$ is the least area Plateau solution with boundary $\Gamma(T,\theta_1,\theta_2)$. We know that $\Sigma(T)$ is the minimal graph of a function $u(T)$ with boundary values equal to zero on the two sides of the triangle $\{\theta=\theta_1,\ 0\le r<1\}$, $\{\theta=\theta_2,\ 0\le r<1\}$ and equal to T on the third side of the triangle $C=C(\theta_1,\theta_2)=\{r=1,\ \ \theta_1\le\theta\le\theta_2\}$.

Theorem 5. *As $T\to\infty$, $u(T)$ converges to the function $u(\infty)$ defined on the triangle with boundary values zero on the sides of the triangle $r(\theta_1)$ and $r(\theta_2)$ and the value infinity on the third side $C=C(\theta_1,\theta_2)$. Moreover the gradient of $u(\infty)$ diverges as one approaches the third side from the interior of the triangle.*

Proof. To show that $u(\infty)$ exists we will prove that for any compact set K of the triangle minus the third side C, the functions $u(T)$ are all bounded above on K; the bound independent on T. We will construct a barrier over the graph of the $u(T)$ on K.

Let ε and δ be small positive numbers (to be determined) and define a geodesic quadrilateral in D_1 with sides $A(\delta)$, $B(\delta)$, $C(\delta)$, $D(\delta)$ defined as follows.

$$A(\delta)=\{(r,\theta)\ \mid\ \varepsilon\le r\le 1,\ \ \theta=\theta_1-\delta\},$$
$$B(\delta)=\{(r,\theta)\ \mid\ \varepsilon\le r\le 1,\ \ \theta=\theta_2+\delta\},$$
$$C(\delta)=\{(r,\theta)\ \mid\ r=1,\ \ \theta_1-\delta\le\theta\le\theta_2+\delta\},$$

and $D(\delta)$ is the minimizing geodesic joining $(\varepsilon,\theta_1-\delta)$ to $(\varepsilon,\theta_2+\delta)$, whose length we call ε_1. Let F denote the convex domain on $D(1)$ bounded by this quadrilateral.

Let $h>0$ and denote by $R(1,h)$ and $R(2,h)$ the Jordan curves which are the boundary of $A(\delta)\times[0,h]$ and $B(\delta)\times[0,h]$ respectively. The area of each of these disks is $(\pi/2-\varepsilon)h$. Consider the piecewise smooth annulus with boundary $R(1,h)\cup R(2,h) : F\cup F(h)\cup(C\times[0,h])\cup(D\times[0,h])$ (we omitted δ in C and D). The area of this annulus is at most $\pi+\pi+l(\delta)h+\varepsilon_1 h$, where $l(\delta)=(\theta_2+\delta)-(\theta_1-\delta)$. Clearly one can choose ε small so that for all δ sufficiently small and h sufficiently large, this annulus has less area than the two disks $R(1,h)\cup R(2,h)$. By the Douglas criteria for the Plateau problem, there exists a least area minimal annulus $a(\delta,h)$ with boundary $R(1,h)\cup R(2,h)$. Henceforth, we assume h large enough so that $a(\delta,h)$ exists.

Observe that for each $T>0$, the surface $a(\delta,h)$ is above the graph of $u(T)$, in the following sense. Vertically translate $a(\delta,h)$ a height T (so every point of $a(\delta,h)$ is then above height T). Now continuously lower the translated $a(\delta, h)$ back to height zero. By the maximum principle there is no point of contact between the surfaces as one goes from height T to height zero; we chose $\delta > 0$, so the boundary of $a(\delta, h)$ never touches the graph of $u(T)$. Thus $a(\delta)$ is above $u(T)$ in the sense that if a vertical line meets both surfaces then, the point of $u(T)$ is below the points of $a(\delta, h)$. Now we can let δ tend to zero to conclude $a(h) = a(0, h)$ is also above the graph of $u(T)$, and by the boundary maximum principle, at each interior point of the vertical lines on $\Gamma(T, \theta_1, \theta_2)$, the tangent plane to $a(h)$ is "outside" the tangent plane to the graph of $u(T)$.

This barrier $a(h)$ shows that $u(T)$ is uniformly bounded over some compact domain of $\Delta \setminus C$: the domain covered by $a(h)$. The idea is now to show that these compact domains exhaust $\Delta \setminus C$ as $h \to \infty$.

For $h_2 > h_1$, one can use $a(h_1)$ as a barrier to solve the Plateau problem to find a least area annulus $a(h_2)$ with boundary $R(1, h_2) \cup R(2, h_2)$. So as one translates $a(h_1)$ vertically a height $h_2 - h_1$, there is no point where the two surfaces touch; interior or boundary. Thus as $h_2 \to \infty$, the angle the tangent plane of $a(h_2)$ makes along the vertical boundary segments, is controlled by that of $a(h_1)$.

For each positive integer n, let $M(n)$ be the surface $a(2n)$ translated down a distance n. A subsequence of the $M(n)$ converges to a minimal surface $M(\infty)$. Notice that $a(h_1)$ can be translated up to $+\infty$, and down to $-\infty$, without ever touching $M(\infty)$. So there is some component M of $M(\infty)$ whose boundary equals the vertical lines L_1, L_2 passing through the endpoints of C; $L_1 \cup L_2 = \partial(C \times \mathbb{R})$. Moreover the maximum distance between M and $C \times \mathbb{R}$ is strictly less than $\pi/2$. To complete the proof of theorem 5, it suffices to show $M = C \times \mathbb{R}$.

Recall that D_1 is the hemisphere of $S(0)$ containing the spherical triangle Δ. Choose a point $p \in \partial D_1 \setminus C$ and let $\alpha(t)$ denote the family of geodesic arcs of D_1 joining p to $-p$, such that $\alpha(0) = \alpha$, $\alpha(1)$ is the geodesic arc of ∂D_1 joining p to $-p$ that contains the arc C, and let $\alpha(t)$ foliate the half-disk E of D_1 between $\alpha(0)$ and $\alpha(1)$, $0 \le t \le 1$. Denote by $F(t)$ the minimal surfaces $\alpha(t) \times \mathbb{R}$, $0 \le t \le 1$. The $F(t)$ foliate the region $E \times \mathbb{R}$; the foliation is singular at $\{p\} \times \mathbb{R}$ and $\{-p\} \times \mathbb{R}$.

Now $M \subset \Delta \times \mathbb{R} \subset E \times \mathbb{R}$. As t goes from 0 to 1, the family of surfaces $F(t)$ can not touch M at some first $t < 1$, since M would then equal $F(t)$ by the maximum principle. So either $M = C \times \mathbb{R}$ or there is a smallest positive $t < 1$ such that M is asymptotic to $F(t)$ at infinity. The latter case is impossible. If not, let $x_n \in M$ be such that $dist(x_n, F(t))$ tends to zero as $n \to \infty$. Let $\Sigma(n)$ be the minimal surface M vertically translated so the height of x_n becomes zero. A subsequence of $\Sigma(n)$ converges to a minimal surface Σ that touches $F(t)$ at some point (at height zero) so $\Sigma = F(t)$.

Consider a compact domain K of $F(t)$, K a positive distance from $\partial F(t)$, and choose K so that the vertical projection on D_1 contains points of $E \setminus \Delta$. Domains of $M(n)$ converge uniformly to K as $n \to \infty$, so there are also points of $M(n)$ whose vertical projection is in $E \setminus \Delta$. This is impossible since $M(n)$ and M have the same vertical projection. This completes the proof of theorem 5. $\square$

Now we will see that the end structure of the surfaces we constructed is typical.

Theorem 6. *Let M be a properly embedded minimal surface in $\mathbb{S}^2 \times \mathbb{R}$ of finite topology. Then M has exactly one top end and one bottom end, or $M = S(t)$ for some t.*

Proof. M of finite topology means M is homeomorhic to a compact surface minus a finite number of points. A neighborhood of each such point in M can be chosen homeomorhic to an annulus. If M is bounded above or below, then the height function would have an extremum on M and then M equals some $S(t)$ by the maximum principle for harmonic functions. So we can assume M has at least one annular end going up and another annular end going down. It suffices to prove that there can not be more than one end (going up say). Suppose on the contrary that A_1 and A_2 are annular ends going up.

By Proposition 1, we can assume A_1 and A_2 both meet each $S(t)$ transversally in exactly one Jordan curve $C_1(t)$ and $C_2(t)$ respectively, for each $t \geq 0$.

Denote by $E(t)$ the annular region of $S(t)$ bounded by $C_1(t) \cup C_2(t)$. For each integer n, let $B(n)$ be the union of the $E(t)$, $\quad 0 \leq t \leq n$. Notice that $\partial B(n)$ is a good barrier for solving a Plateau problem in $B(n)$. Also, $C_1(0)$ and $C_1(n)$ are homologous in $B(n)$ but neither $C_1(0)$ nor $C_1(n)$ is homologous to zero in $B(n)$. Thus there is a least area connected annulus $\Sigma(n)$ in $B(n)$ with boundary $C_1(0) \cup C_1(n)$.

By standard curvature estimates, a subsequence of the $\Sigma(n)$ converges to a complete stable minimal annulus Σ, with $\partial\Sigma = C_1(0)$.

As before, we can assume Σ meets each $S(t)$ transversally in one Jordan curve $\gamma(t)$.

Now we observe that the area of Σ is infinite. Let ν be the upward pointing conormal vector along $\gamma(t)$. The height function h is harmonic on Σ hence has a constant non zero flux across each $\gamma(t)$. This flux is

$$\int_{\gamma(t)} |\nabla h| \, ds,$$

where s is arc length along $\gamma(t)$.

By the coarea formula, the area of Σ is

$$\int_{t=0}^{\infty} \left(\int_{\gamma(t)} \frac{ds}{|\nabla h|} \right) dt \geq \int_{t=0}^{\infty} \left(\int_{\gamma(t)} |\nabla h| ds \right) dt = \infty.$$

However by the work of Doris-Fischer Colbrie [12] and Silveira [40], there is no stable minimal annulus in $\mathbb{S}^2 \times \mathbb{R}$ of infinite area. The stability operator is $L = \Delta - K + q$, where K is the intrinsic curvature of Σ, $\quad q = T + \dfrac{|A|^2}{2}$, $\quad T$ the scalar curvature of $\mathbb{S}^2 \times \mathbb{R}$ (which is one) and A the second fundamental form. Stability yields a positive function u satisfying $L(u) = 0$. The metric $u\,ds = d\tilde{s}$ is then a complete metric on Σ whose curvature $\widetilde{K}$ is non negative and given by

$$\widetilde{K} = \frac{1}{u^2}\left(q + \frac{|\nabla u|^2}{u^2}\right).$$

Then

$$\int_\Sigma q dA \leq \int_\Sigma \widetilde{K}\, d\widetilde{A} < \infty,$$

so $\int\limits_\Sigma T dA < \infty$, which contradicts infinite area.

□

The techniques used in Theorem 6 also give information about intersection of minimal submanifolds.

Theorem 7. . *Let Σ_1 , Σ_2 be properly embedded minimal submanifolds of $\mathbb{S}^2 \times \mathbb{R}$. Then $\Sigma_1 \cap \Sigma_2 \neq \emptyset$ or $\Sigma_1 = S(t_1)$, $\quad \Sigma_2 = S(t_2)$ for some t_1 , t_2 .*

Proof. We know that a properly immersed minimal submanifold is either some $S(t)$, or meets each $S(t)$ in a non empty compact set. We can assume the latter case holds for both Σ_1 and Σ_2 . We will assume $\Sigma_1 \cap \Sigma_2 = \emptyset$ and arrive at a contradiction.

Elementary separation properties imply $\Sigma_1 \cup \Sigma_2 = \partial B, \quad B$ a domain of $\mathbb{S}^2 \times \mathbb{R}$. Then $\Sigma_1(t) \cup \Sigma_2(t) = \partial B(t)$ for each t such that Σ_1 and Σ_2 meet $S(t)$ transversally. $\Sigma_1(t)$ is homologous to $\Sigma_1(0)$ in $B \cap [0,t]$ and $\Sigma_1(t)$ is not homologous to zero in $B \cap [0,t]$. Thus $\Sigma_1(t) \cup \Sigma_1(0)$ bounds a connected least area minimal surface $\Sigma(t)$ in $B \cap [0,t]$. A subsequence of the $\Sigma(t)$ as $t \to \infty$, converges to a stable minimal surface Σ with $\partial\Sigma = \Sigma_1(0)$. As in the proof of Theorem 6, no such stable surface exists; which proves the theorem. □

Remark 3. *Notice that the above argument shows one need not assume finite topology in theorem 6.*

We can say something for properly immersed surfaces.

Theorem 8. *Let Σ be a properly immersed minimal surface in $\mathbb{S}^2 \times \mathbb{R}$. Then Σ meets every flat vertical annulus.*

Proof. Let $A = C(0) \times \mathbb{R}$ be a flat vertical annulus and assume $A \cap \Sigma = \emptyset$. We can assume (after a possible rotation of the S factor) that $\operatorname{dist}(A, \Sigma) = 0$; so some sequence of points in Σ is converging to A at infinity. Let F be a (small) compact piece of an unduloid, chosen so that $\partial F \subset A$ and $F \cap \Sigma = \emptyset$. Such an F can be found since Σ is properly immersed and unduloids exist arbitrarily close to A.

Now translate F vertically. Since Σ is asymptotic to A at infinity, there will be a first point of contact of the translated F with Σ. Then Σ equals this translated unduloid by the maximum principle. This contradicts $\Sigma \cap A = \emptyset$. $\square$

We finish this section with a conjecture: a properly embedded minimal annulus in $\mathbb{S}^2 \times \mathbb{R}$ meets each $S(t)$ in a circle. There is a 2-parameter family of such annuli, and each properly embedded minimal annular end is asymptotic to the end of a surface in this family.

9 Theory of minimal surfaces in $M \times \mathbb{R}$

In section 8, we discussed minimal surfaces Σ properly embedded in $\mathbb{S}^2 \times \mathbb{R}$. We proved that Σ has exactly two ends or Σ equals $\mathbb{S}^2 \times \{t\}$ for some real t. Bill Meeks and the author discovered a very general property concerning ends [24].

Theorem 9. *A properly embedded minimal surface Σ in $M \times \mathbb{R}$, where M is a compact Riemannian surface, has a finite number of ends.*

This is the only completely general result we know in this subject. It implies that a compact Riemannian surface, punctured in an infinite set of points, can not be realized as a properly embedded minimal surface in $M \times \mathbb{R}$. Another important property of minimal surfaces Σ in $M \times \mathbb{R}$ is their area growth. We proved [24]:

Theorem 10. *Let Σ be a properly embedded minimal surface in $M \times \mathbb{R}$, M compact, and assume Σ has bounded curvature. Then Σ has linear area growth. That is the area of $\Sigma \cap (M \times [-t, t])$ is at most ct, where the constant c depends only upon M, a lower bound of the vertical flux of Σ, and an upper bound on the absolute value of the Gaussian curvature of Σ.*

This theorem is important since when one has uniform curvature and area bounds for a family of properly embedded minimal surfaces then one has precompactness. More precisely, one obtains:

Corollary 4. *If Σ is a properly embedded non compact minimal surface of bounded curvature in $M \times \mathbb{R}$, M compact, then any sequence of vertical translations of Σ contains a convergent subsequence to another properly embedded minimal surface with the same bound on its curvature.*

Of fundamental importance is deciding when a minimal surface is obliged to have bounded curvature. Using the theory developed by Colding and Minicozzi, together with the techniques developed in [23], we are able to prove:

Theorem 11. *Let Σ be a properly embedded minimal surface in $M \times \mathbb{R}$, M compact, and assume the genus of Σ is finite. Then*

1. *Σ has bounded curvature,*
2. *If M has nonpositive curvature then Σ has finite index with respect to the stability operator and the total curvature of Σ equals $2\pi\chi(\Sigma)$;*
3. *If M has nonpositive curvature and M is not a torus then each end of Σ is asymptotic to $\gamma \times \mathbb{R}$, where γ is a stable embedded geodesic of M.*

We also studied the topological type of properly embedded minimal surfaces and obtained an unknotting theorem. Recall that a *handlebody* is a three-manifold with boundary that is homeomorphic to a closed regular neighborhood of a connected properly embedded one-dimensional CW-complex in $\mathbb{R}^3$ and that a surface Σ in a three-manifold N^3 is a *Heegaard* surface if it separates N^3 into closed complements which are handlebodies.

Theorem 12. *(Unknotted Theorem) Suppose $\mathbb{S}^2$ is a two sphere endowed with a Riemannian metric with no stable simple closed geodesics. Then:*

1. *If Σ is a noncompact properly embedded minimal surface in $S^2 \times \mathbb{R}$, then Σ is a Heegaard surface for $\mathbb{S}^2 \times \mathbb{R}$;*
2. *Every Heegaard surface for $\mathbb{S}^2 \times \mathbb{R}$ has two ends and if Σ is a connected orientable surface with two ends, then Σ embedds in $\mathbb{S}^2 \times \mathbb{R}$ as a Heegaard surface;*
3. *Heegaard surfaces of $\mathbb{S}^2 \times \mathbb{R}$ are unknotted in the sense that if two such surfaces are diffeomorphic, then there exists an orientation preserving diffeomorphism of $\mathbb{S}^2 \times \mathbb{R}$ which interchanges them.*

10 Constant mean curvature surfaces in $M \times \mathbb{R}$

In this section we will discuss some work in progress by myself, Jorge Lira and David Hoffman. This concerns H-surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, with $H \neq 0$. We do not have many examples now. There are the rotational examples (about a vertical geodesic) discussed by Pedrosa and Ritoré in [34]. There are H-surfaces that are compact and embedded arising from isoperimetric problems. That is, fix a volume and minimize area among all surfaces in $M \times \mathbb{R}$ bounding the prescribed volume. Then an embedded solution always exists and has constant mean curvature. However in general we know neither the solution, nor how its mean curvature depends on the volume. One way to construct H-surfaces of given topology is by the glueing techniques of Mazzeo and Pacard [18], [19].

We will not discuss existence of H-surfaces here. A useful tool to study H-surfaces is the following height estimate.

Theorem 13. *Let Σ be a compact embedded H-surface in $M \times \mathbb{R}$, $H \neq 0$ and M a Riemannian surface. Assume $\partial\Sigma \neq \emptyset$ and $\partial\Sigma \subset M(0) = M \times \{0\}$. Then*

1. *If M has non-negative curvature, one has $|h| \leq 2/H$, where h is the height function on Σ.*
2. *If $K_M \geq 2\tau, (\tau < 0)$ and $H^2 > |\tau|$, then $|h| \leq \dfrac{2H}{H^2 - |\tau|}$ on Σ.*

Proof. Without loss of generality, we may suppose $h \geq 0$ on Σ. Let $M(t) = M \times \{t\}$, and notice that vertical symmetry in $M(t)$ is an isometry of $M \times \mathbb{R}$. Now consider doing Alexandrov reflection of Σ through the $M(t)$, coming down to Σ from above. Fot t large, $M(t) \cap \Sigma = \emptyset$ and there is a largest T such that $M(T) \cap \Sigma \neq \emptyset$. For t slightly smaller than T, the part of Σ above, $M(t)$ is a vertical graph over part of $M(t)$ and its symmetry through $M(t)$ is above the part of Σ below $M(t)$; meeting Σ precisely along its part in $M(t)$. Then as usual in the technique of Alexandrov reflection one decreases t until a first t_0 where an accident occurs. Either $M(t_0)$ is a symmetry surface of Σ or $t_0 \leq \dfrac{a}{2}$ where a is the maximum height of Σ above $M(0)$. In any case the part of Σ above height $a/2$ is a vertical graph. Thus it suffices to prove the theorem when Σ is a vertical graph, and the 2 does not appear on the right side of the height estimate.

So assuming Σ is a vertical graph with zero boundary values. Let $\vec{n}$ be the downward pointing unit normal vector to Σ and define $n = \left\langle \vec{n}, \dfrac{\partial}{\partial t} \right\rangle$. Then $n \leq 0$ on Σ. One has 2 equations:

$$\begin{aligned} \triangle h &= 2Hn \\ \triangle n &= -\left(|A|^2 + \mathrm{Ric}(\vec{n})\right) n, \end{aligned}$$

where $\triangle$ is the laplacian on Σ and A is the second fundamental form of Σ.

Let c be a positive constant to be determined and define

$$\phi = ch + n$$

on Σ. On $\partial\Sigma$, $\phi = n \leq 0$, so if $\triangle\phi \leq 0$, it follows that $\phi \leq 0$ on Σ, i.e., $h + \dfrac{n}{c} \leq 0$ on Σ, so $h \leq -\dfrac{n}{c} \leq \dfrac{1}{c}$.

Thus we want to choose c so that $\triangle\phi \geq 0$. We calculate

$$\triangle\phi = \left(2cH - \left(|A|^2 + \mathrm{Ric}(\vec{n})\right)\right) n,$$

so we want

$$\begin{aligned} 2cH &\leq |A|^2 + \mathrm{Ric}(\vec{n}) = 4H^2 - 2K_e + \mathrm{Ric}(\vec{n}) \\ &= 2H^2 + 2(H^2 - K_e) + \mathrm{Ric}(\vec{n}). \end{aligned}$$

Here K_e is the extrinsic curvature of Σ, i.e;, the determinant of A. Since $H^2 - K_e \geq 0$, we want to choose c so that

$$cH \leq H^2 + \frac{\mathrm{Ric}(\vec{n})}{2}.$$

Clearly if $\mathrm{Ric}(\vec{n}) \geq 0$ on $M \times \mathbb{R}$, then $c = H$ works. Hence $h \leq \dfrac{1}{H}$ as desired when $K_M \geq 0$. If $\mathrm{Ric}(\vec{n}) \geq 2\tau$, $\tau < 0$, then

$$H^2 - cH + \frac{\mathrm{Ric}(\vec{n})}{2} \geq H^2 - cH + \tau \geq 0$$

when $H^2 - cH \geq |\tau|$. Since one has equality when $c = \dfrac{H^2 - |\tau|}{H}$, this proves the theorem. □

Corollary 5. *Let M be a closed Riemannian surface whose curvature is bounded below by 2τ, τ a real constant. Let Σ be a properly embedded, non compact, H-surface in $M \times \mathbb{R}$, $H \neq 0$. Assume $H^2 \geq |\tau|$ if $\tau < 0$; Then Σ has at least two ends.*

Proof. Assume the contrary, that Σ has exactly one end E. Since M is compact and Σ is proper, E must go up or down in $M \times \mathbb{R}$, but not both. So assume E goes down. Then there is a largest T such that $\Sigma \cap M(T) \neq \emptyset$ and $\Sigma \cap M(t) = \emptyset$ for $t > T$.

Now do Alexandrov reflection of Σ with the horizontal surfaces $M(s)$, $s \leq T$. As s decreases from T to $-\infty$, the part of Σ above $M(s)$ is a vertical graph with zero boundary values (in $M(s)$) as long as no accident occurs in Alexandrov reflection. By our height estimates an accident must occur for a certain s. But then Σ is compact which is a contradiction. □

10.1 Flux Fromulae

An important tool for studying H-surfaces are the formulae for the flux of appropriately chosen ambient vector fields across the surface. An important reference for this material is [16].

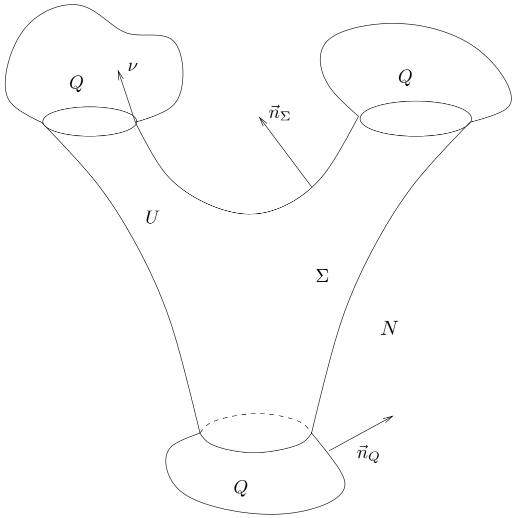

Figure 9

Let N be a Riemannian manifold and Σ an H-hypersurface of N, Σ compact and with boundary which may be empty. Let Q be a compact orientable hypersurface of N with $\partial Q = \partial\Sigma$. Assume that $Q \cup \Sigma$ is the oriented homological boundary of a domain U of N, where $Q \cup \Sigma$ are oriented by unit vector fields $\vec{n}_Q$ and $\vec{n}_\Sigma$, which point out of U. Denote by ν the unit conormal to Σ along $\partial\Sigma$, pointing out of Σ along $\partial\Sigma$; (see figure 9).

Let Y be a vector field on N, Div$Y$, and div$Y$, denote the divergence of Y in N and along Σ respectively.

The first variation of the volume of U by the vector field Y is given by

$$\begin{aligned} \delta_Y(|U|) &= \int_U \mathrm{Div}(Y) = \int_{\partial U} \langle Y, \vec{n}_{\partial U}\rangle \\ &= \int_\Sigma \langle Y, \vec{n}_\Sigma\rangle + \int_Q \langle Y, \vec{n}_Q\rangle \,. \end{aligned}$$

The first variation of area of Σ under Y is given by

$$\begin{aligned} \delta_Y(|\Sigma|) &= \int_\Sigma \mathrm{div}(Y) = \int_\Sigma \mathrm{div}(Y^t) + \int_\Sigma \mathrm{div}(Y^\perp) \\ &= \int_{\partial\Sigma} \langle Y, \nu\rangle - \int_\Sigma \left\langle Y, \vec{H}_\Sigma \right\rangle \\ &= \int_{\partial\Sigma} \langle Y, \nu\rangle - H \int_\Sigma \langle Y, \vec{n}_\Sigma\rangle \,. \end{aligned}$$

In the last equality, notice that the sign of H is determined by $\vec{H}_\Sigma = H\vec{n}_\Sigma$; so that $H > 0$ if $\vec{H}_\Sigma$ points out of U.

This yields

$$\delta_Y(|\Sigma| + H\,|U|) \;=\; \int_{\partial\Sigma} \langle Y, \nu\rangle \;+ H \int_Q \langle Y, \vec{n}_Q\rangle.$$

In the geometric situation we will encounter, it will often be the case that $\vec{H}_\Sigma$ points into U so that $H < 0$ in the above formula. For convenience we will take $H > 0$ when $\vec{H}_\Sigma$ points into U so the above flux formula then becomes:

$$\delta_Y(|\Sigma| - H\,|U|) \;=\; \int_{\partial\Sigma} \langle Y, \nu\rangle \;- H \int_Q \langle Y, \vec{n}_Q\rangle,$$

with $H > 0$ and $\vec{H}_\Sigma$ pointing into U.

An important consequence of this is when Y is a Killing vector field of N (so that the area and volume do not change) is:

$$0 \;=\; \int_{\partial\Sigma} \langle Y, \nu\rangle \;- H \int_Q \langle Y, \vec{n}_Q\rangle.$$

We now apply this flux formula to properly embedded H-surface Σ in $M \times \mathbb{R}$, that separate $M \times \mathbb{R}$. Let U be the mean convex domain of $M \times \mathbb{R}$ bounded by Σ. For fixed t, let $U(t) = U \cap (M \times [0,t])$, $Q(s) = U \cap (M \times \{s\})$ and $Q = Q(0) \cap Q(t)$. Apply the flux formula to the killing field $Y = \frac{\partial}{\partial t}$:

$$\begin{aligned} 0 \;&=\; \int_{\Sigma(0)\cup\Sigma(t)} \langle Y, \nu\rangle \;- H \int_Q \langle Y, \vec{n}_Q\rangle \\ &=\; \int_{\Sigma(0)\cup\Sigma(t)} \langle Y, \nu\rangle \;- H(|Q(t)| - |Q(0)|), \end{aligned}$$

where $\Sigma(s) = \Sigma \cap (M \times \{s\})$. Notice that when M is compact, $|Q(t)|$ is bounded by the area of M, so the vertical flux- $\int_{\Sigma(t)} \langle \frac{\partial}{\partial t}, \nu\rangle$ is uniformly bounded.

Theorem 14. *(Linear area growth) Let Σ be a properly embedded H-surface in $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$, contained in a cylinder $C \times \mathbb{R}$. Assume C is a circle of geodesic radius less than $\frac{\pi}{2}$ when $\Sigma \subset \mathbb{S}^2 \times \mathbb{R}$, and C is any geodesic circle of $\mathbb{H}^2$ when $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$. Then Σ has linear area growth, i.e., there exists a constant $c > 0$ such that*

$$|\,\Sigma \cap (M \times [0,t])| \leq ct,$$

for all $t \in \mathbb{R}^+$ and $M = \mathbb{S}^2$ or $\mathbb{H}^2$.

Proof. We will give the proof for $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$, the other case being similar. Let U be the mean convex domain bounded by Σ, $Q(t) = U \cap (\mathbb{H}^2 \times \{t\})$, and $\Sigma[0,t] = \Sigma \cap (\mathbb{H}^2 \times [0,t])$. For any vector field X on $\mathbb{H}^2 \times \mathbb{R}$, the flux formula on $U[0,t] = U \cap (\mathbb{H}^2 \times [0,t])$ can be written

$$\int_{\Sigma[0,T]} divX \; - H \int_{U[0,T]} DivX \; = \int_{\partial\Sigma[0,T]} \langle X, \nu\rangle \; - H \int_{Q(0)\cup Q(T)} \langle X, \vec{n}_Q\rangle,$$

where ν is the unit conormal along $\partial\Sigma[0,t]$ pointing out of Σ, and $\vec{n}_Q$ the unit normal on $Q(0) \cup Q(t)$ pointing out of $U[0,t]$. Note again that the minus sign in the formula appears in order to reconcile our assumption that H is a positive number with the fact that that the mean curvature vector $\vec{H}$ (which points *into* U) is given by $\vec{H} = H\vec{n}$, where $\vec{n}$ is the *outward*-pointing unit normal along Σ.

Choose polar coordinates (r,θ) on $\mathbb{H}^2$ such that $0 < r_1 < r < r_2$ and $0 < \delta \leq \theta < \pi$ on the circle C. Define vector fields $Y = t\partial_t$ and $Z = t\theta\partial_\theta$. If we denote $n_t = \langle \vec{n}, \partial_t\rangle$, $n_\theta = \langle \vec{n}, \partial_\theta\rangle$, then straight-forward calculations yield

$$\begin{aligned} &DivY = 1, \;\; divY = 1 - n_t^2, \\ &DivZ = t, \;\; divZ = t(1 - \frac{n_\theta^2}{sh^2(r)}) - \theta n_t n_\theta. \end{aligned}$$

Combining the first pair of equations with the flux formula applied to Y, we have

$$\int_{\Sigma[0,T]} (1 - n_t^2) \; - \; H\,|U[0,T]| \; = \int_{\partial\Sigma[0,T]} \langle t\partial_t, \nu\rangle \; - \; H \int_{Q(0)\cup Q(T)} \langle t\partial_t, \vec{n}_Q\rangle.$$

The right-hand side becomes

$$T\,\Big[\int_{\partial\Sigma(T)} \langle \partial_t, \nu\rangle \; - H\,|Q(T)|\,\Big],$$

where $\Sigma(t) = \Sigma \cap (\mathbb{H}^2 \times \{t\})$. Note that since ∂_t is a Killing field, then (again by the flux formula) the term in brackets is independent of T, while $|U[\,0,T]|$ is bounded by $|D|\,T$, where D is the disk bounded by the circle C. We conclude that for some constant c,

$$\int_{\Sigma[0,T]} 1 - n_t^2 \; \leq \; cT.$$

Now write $\Sigma = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1 = \{p \in \Sigma \mid n_t^2 \leq 1 - \delta\}$, $\Sigma_2 = \{p \in \Sigma \mid n_t^2 > 1 - \delta\}$. By the above inequality, we see that

$$\delta\,|\Sigma_1 \cap \Sigma[0,T]| \; \leq \int_{\Sigma_1\cap\Sigma[0,T]} 1 - n_t^2 \; \leq \; cT,$$

and thus we need only prove linear area growth for Σ_2.

To this end, consider the surface S defined by the equation $t\theta = T$. Since $\delta \leq \theta \leq \pi$, this surface remains within height range $\frac{T}{\pi} \leq t \leq \frac{T}{\delta}$. Consider the part $\widetilde{\Sigma}$ of Σ lying between $Q(0)$ and S. Let $\widetilde{Q}(T)$ be the compact region in S bounded by $\Sigma \cap S$, and $\widetilde{U}$ the domain bounded by $\widetilde{\Sigma}$, $Q(0)$, and $\widetilde{Q}(T)$; cf. figure 10.

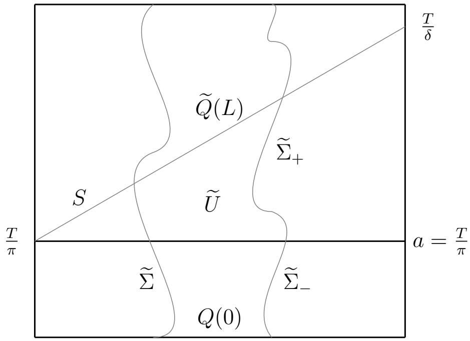

Figure 10

Applying the flux formula to $Z = t\theta\partial_\theta$ in $\widetilde{U}$ yields

$$\int_{\widetilde{\Sigma}} divZ \ - H\int_{\widetilde{U}} DivZ = \int_{\partial\widetilde{\Sigma}} \langle t\theta\partial_\theta, \nu\rangle \ - H\int_{Q(0)\cup\widetilde{Q}(T)} \langle t\theta\partial_\theta, \vec{n}_{\widetilde{Q}}\rangle.$$

By the formulas above, the left-hand side of this is

$$\int_{\widetilde{\Sigma}} \left[t(1 - \frac{n_\theta^2}{sh^2(r)}) - \theta n_t n_\theta\right] \ - H\int_{\widetilde{U}} t,$$

while the right-hand side is

$$T\ \Big[\int_{\partial\widetilde{\Sigma}} \langle\partial_\theta, \nu\rangle \ - H\int_{\widetilde{Q}(T)} \langle\partial_\theta, \vec{n}_{\widetilde{Q}}\rangle\Big].$$

Again, ∂_θ is Killing, so the term in brackets is independent of T. If the area of Σ is finite, then the theorem is evident, so we can assume the area is infinite. Since Σ is properly embedded, the intersection $\Sigma \cap (\mathbb{H}^2 \times [n, n+1])$ is compact for all n. Thus by choosing δ sufficiently small we can assume that the area of $\widetilde{\Sigma}_- = \widetilde{\Sigma}\cap\Sigma[0,a]$ is less than the area of $\widetilde{\Sigma}_+ = \widetilde{\Sigma}\cap\Sigma[a,L]$, where $a = \frac{T}{\pi}$ and $L = \frac{T}{\delta}$. Note that if this holds for some $T > 0$ and $\delta > 0$, it holds for all larger T.

We have $\|\vec{n}\|^2 = 1 = n_t^2 + \langle\vec{n}, \partial_r\rangle^2 + \langle\vec{n}, \frac{\partial_\theta}{|\partial_\theta|}\rangle^2$, and $|\partial_\theta| = sh(r)$. So $\frac{n_\theta^2}{sh^2(r)} < 1$ and $1 - \frac{n_\theta^2}{sh^2(r)} = n_t^2 + \langle\vec{n}, \partial_r\rangle^2 \geq n_t^2$. Thus on $\widetilde{\Sigma}_2 = \Sigma_2 \cap \widetilde{\Sigma}_+$, we have $n_t^2 \geq 1 - \delta$, and $1 - \delta \leq 1 - \frac{n_\theta^2}{sh^2(r)}$. Hence

$$(1-\delta)\frac{T}{\pi} \ \leq\ t(1 - \frac{n_\theta^2}{sh^2(r)}) \quad \text{on } \widetilde{\Sigma}_2, \text{ and}$$
$$2n_t n_\theta \ \leq\ n_t^2 + n_\theta^2 \ \leq\ 1 + sh^2(r) \ \leq\ 1 + sh^2(r_2) \quad \text{on all of } \widetilde{\Sigma}.$$

The former inequality implies

$$(1-\delta)\frac{T}{\pi}|\widetilde{\Sigma}_2| \leq \int_{\widetilde{\Sigma}_2} t(1-\frac{n_\theta^2}{sh^2(r)}) \leq \int_{\widetilde{\Sigma}} t(1-\frac{n_\theta^2}{sh^2(r)}).$$

Employing the flux formula for Z, this last integral is at most

$$c_1T + \int_{\widetilde{\Sigma}} \theta n_t n_\theta + H\int_{\widetilde{U}} t \ \leq \ c_1T + c_2|\widetilde{\Sigma}| + c_3(\frac{T}{\delta})^2,$$

where $c_2 = \frac{\pi(1+sh^2(r_2))}{2}$. We know that $|\widetilde{\Sigma}| = |\widetilde{\Sigma}_-| + |\widetilde{\Sigma}_+| \leq 2|\widetilde{\Sigma}_+|$, and so

$$(1-\delta)\frac{T}{\pi}|\widetilde{\Sigma}_2| \leq c_1T + 2c_2|\widetilde{\Sigma}_+| + \frac{c_3}{\delta^2}T^2.$$

Write $|\widetilde{\Sigma}_+| = |\widetilde{\Sigma}_1| + |\widetilde{\Sigma}_2|$, where $\widetilde{\Sigma}_1 = \widetilde{\Sigma}_+ \cap \Sigma_1$, $\widetilde{\Sigma}_2 = \widetilde{\Sigma}_+ \cap \Sigma_2$. Since $|\widetilde{\Sigma}_1| \leq \frac{cT}{\delta}$, the previous inequality becomes

$$\begin{aligned}(1-\delta)\frac{T}{\pi}|\widetilde{\Sigma}_2| \ &\leq \ c_1T + 2c_2|\widetilde{\Sigma}_1| + 2c_2|\widetilde{\Sigma}_2| + c_3(\frac{T}{\delta})^2 \\ &\leq \ c_1T + \frac{2c_2cT}{\delta} + 2c_2|\widetilde{\Sigma}_2| + c_3(\frac{T}{\delta})^2.\end{aligned}$$

Hence

$$((1-\delta)\frac{T}{\pi} - 2c_2)|\widetilde{\Sigma}_2| \leq c_4T + c_3(\frac{T}{\delta})^2,$$

where $c_4 = c_1 + \frac{2c_2c}{\delta}$. Finally, choose $T_0 > 0$ and $k > 0$ so that for $T \geq T_0$, $(1-\delta)\frac{T}{\pi} - 2c_2 \geq kT$. Then for $T \geq T_0$, $|\widetilde{\Sigma}_2| \leq c_5T$ and $|\Sigma[0,a]| \leq c_6T$. This completes the proof.

$\square$

Theorem 15. *Let Σ be a properly embedded H-surface in $D \times \mathbb{R}$ of finite topology. Then Σ has bounded curvature.*

Proof. The proof follows the same lines as the proof of the analogous theorem in $\mathbb{R}^3$ [16].

Σ has finite topology so it suffices to show each annular end of Σ has bounded curvature. Let E be such an annular end and we may assume E goes up. First one proves $\int_E |A|^2$ grows linearly, i.e., $\int_E(L, L+1)\,|A|^2$ is bounded independently of L.

We recall that $E(a,b) = E \cap (\mathbb{S}^2 \times [a,b])$ and $E(t) = E \cap (\mathbb{S}^2 \times \{a,b\})$. By the Gauss equation, we have:

$$\begin{aligned}|A|^2 \ &= \ 4H^2 - 2\det A \\ K_E \ &= \ K_N + \det A\end{aligned}$$

where K_N is the sectional curvature of the 2-plane tangent to E, $N = \mathbb{S}^2 \times \mathbb{R}$. Hence

$$|A|^2 = (4H^2 + 2K_N) - 2K_E.$$

Since $4H^2 + 2K_N$ is bounded and E has linear area growth, it suffices to show $\int_{E(L,L+1)} K_E$ is uniformly bounded, to reach the same conclusion for $|A|^2$.

For convenience, we assume $H = 1$. Let $a, b \in \mathbb{R}^+$ with $b - a > 4$. Assume E is transverse to $S(a)$ and $S(b)$ and $\partial E \subset \mathbb{S}^2(0)$. Each lacet in $\mathbb{S}(a) \cap E$ is essential (generates $\pi_1(E)$) or non-essential. A non-essential loop C in $\mathbb{S}(a) \cap E$ bounds a compact domain F in Σ and our height estimates tell us F can go at most a distance 2 from $\mathbb{S}(a)$. Similarly for non-essential loops in $E(b)$.

Let $\phi : \mathbb{S}^1 \times \mathbb{R}^+ \to E$ be a parametrization such that $\phi(S^1 \times \{0\}) \subset \mathbb{S}(0)$. Then ϕ orders the essential loops in $E(T)$ for $T > 0$. So in $E(a)$, there is a first essential loop α_1 and a first essential loop β in $E(b)$. The annulus in E bounded by α_1 and β may not be in $\mathbb{S}[a, b]$ but it is in $\mathbb{S}[a-2, b+2]$. Notice that after β (in $\mathbb{S}^1 \times \mathbb{R}^+$) there can be no essential loops going into $E(a)$ since this would contradict the height estimates. Let β_1 be the last essential loop in $E(b)$. Then α_1 and β_1 bound an annulus $\widetilde{E}(a, b)$ in E with

$$E(a, b) \subset \widetilde{E}(a, b) \subset E(a-2, b+2).$$

Since E has linear area growth, there is a constant c such that $|E(L, L+1)| \leq c$ for all L. hence by the coarea formula, for each k, there exists $a_k \in [L+k, L+k+1]$, such that $|E(a_k)| \leq c$ (here $|E(a_k)|$ is the length of $E(a_k)$). For notational convenience, suppose $|E(L+k)|) \leq c$, for all k.

Choose an integer $k_0 > 2c - 1$. Consider $\widetilde{E}(L, L+k_0+4)$, bounded by an essential curve α_1 in $E(L)$ and β_1 in $E(L+k_0+4)$; each of length at most c. We know that

$$\widetilde{E}(L, L+k_0+4) \subset E(L-2, L+k_0+6).$$

Now choose a point x on α_1 and y on β_1. Let $\bar{\alpha}_1$ be the minimizing geodesic on E that is homotopic to α_1 relative to x, and similarly define $\bar{\beta}_1$, the minimizing geodesic on E homotopic to β_1 rel y.

Then $\bar{\alpha}_1$ and $\bar{\beta}_1$ have length at most c. Denote by $\bar{E}$ the annulus on E bounded by $\bar{a}_1$ and $\bar{\beta}_1$. The exterior angle of $\bar{\alpha}_1$ (and of $\bar{\beta}_1$) at x (and y) is at most π, so the total curvature of $\bar{E}$ is at most 2π by Gauss-Bonnet. By construction

$$E(L+c, L+k_0+4-c) \subset \bar{E} \subset E(L-2-c, L+k_0+6+c)$$

Both the left and right side of these inclusions have uniformly bounded area hence $\int |A|^2$ is uniformly bounded on $E(L+c, L+k_0+4-c)$. We chose $k_0 > 2c-1$, so this yields that $\int_E |A|^2$ has linear growth.

The rest of the proof that E has bounded curvature proceeds as in [16]. We refer the reader to this paper for the details $\square$

We finish this section with some questions.

1) Assume M is a compact Riemannian surface and Σ is a properly embedded H-surface in $M \times \mathbb{R}$. Does Σ have a finite number of ends? As we indicated in section 9, this is true when $H = 0$.

2) Let Σ be a finite topology H-surface properly embedded in $\mathbb{S}^2 \times \mathbb{R}$. Does Σ have linear area growth or bounded curvature?

3) Suppose Σ is a properly embedded H-surface in $\mathbb{S}^2 \times \mathbb{R}$, contained in $D \times \mathbb{R}$, D a hemisphere of $\mathbb{S}^2$. Is Σ one of the rotational surfaces described by Pedrosa and Ritoré?

4) If Σ is a properly embedded minimal annulus in $\mathbb{S}^2 \times \mathbb{R}$, does Σ meet each $S(t)$ in a round circle? This question comes from W. Meeks and the author [24].

5) If Σ is an entire minimal graph in $\mathbb{H}^2 \times \mathbb{R}$, does Σ have the conformal type of the unit disk in $\mathbb{C}$? Were there an example whose conformal type is $\mathbb{C}$, the vertical projection would produce a harmonic surjective diffeomorphisme $\mathbb{C} \to \{0 \leq |z| < 1, z \in \mathbb{C}\}$ (where the disk has the hyperbolic metric). The existence of such a map is an open question.

11 Return to minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$

We stated in theorem 10, that properly embedded minimal surfaces of bounded curvature in $M \times \mathbb{R}$ have linear area growth. Then theorem 11 says that one indeed has bounded curvature when Σ has finite genus. This is quite different from H-surfaces in a hemisphere$\times\mathbb{R}$, $H \neq 0$. There one shows first that Σ has linear area growth and using this one proves Σ has bounded curvature if Σ has finite topology. The conclusions are the same in both cases but the worlds are quite different.

We now prove an easy case of theorem 10; the general case is more difficult.

Theorem 16. *Let Σ be a properly embedded minimal surface of bounded curvature in $\mathbb{S}^2 \times \mathbb{R}$, then Σ has linear area growth.*

Proof. Since $\mathbb{S}^2 \times \mathbb{R}$ has two ends, it is sufficient to assume that $\Sigma \subset \mathbb{S}^2 \times [0, \infty)$ and $\partial\Sigma \subset \mathbb{S}^2 \times \{0\} = \partial(\mathbb{S}^2 \times [0, \infty))$. For every $a \in (0, \infty]$, let $T_a \colon \mathbb{S}^2 \times [0, \infty) \to \mathbb{S}^2 \times \mathbb{R}$ be the translational isometry $T_a((p, t)) = (p, t - a)$. Suppose now that Σ fails to have linear area growth in $\mathbb{S}^2 \times [0, \infty)$. In this case there exist a sequence of $b_n \to \infty$ such that the area of $T_{b_n}(\Sigma) = \Sigma(n)$ in $\mathbb{S}^2 \times [-\frac{1}{n}, \frac{1}{n}]$ is greater than n. Since the curvature of $\Sigma(n)$ in $\mathbb{S}^2 \times \mathbb{R}$ is bounded and Σ is minimal and embedded, a subsequence of the $\Sigma(n)$ converges to a minimal lamination $\mathcal{L}$ of $\mathbb{S}^2 \times \mathbb{R}$ (see for example [23]).

We assert that $\mathcal{L}$ contains $S^2 \times \{0\}$ as a leaf. Suppose for the moment that there is a leaf of $\mathcal{L}$ which intersects $\mathbb{S}^2 \times \{0\}$ and is not equal to $\mathbb{S}^2 \times \{0\}$. Every such leaf of $\mathcal{L}$ intersects $\mathbb{S}^2 \times \{0\}$ transversely at some point by the maximum principle for minimal surfaces. Since the area of $\Sigma(n) \cap \mathbb{S}^2 \times [-\frac{1}{n}, \frac{1}{n}]$ goes to infinity, we may assume, after possibly going to a subsequence, that there exists a leaf L of $\mathcal{L}$ which is either a limit leaf or has infinite area multiplicity as a limit of the $\Sigma(n)$. Furthermore, L can be chosen so that there is a point $p \in L \cap (\mathbb{S}^2 \times \{0\})$ where the tangent plane to L is not horizontal. For some small geodesic ball B in $\mathbb{S}^2 \times \mathbb{R}$ centered at p of radius r, the tangent planes to $\mathcal{L} \cap B$ make a positive angle of at least θ_0 with the horizontal. The point p can also be chosen so that, after choosing a subsequence, the area of $\Sigma(n)$ in B is at least n and the tangent planes to $\Sigma(n)$ in B make an angle of at least $\theta_0/2$ with the horizontal. It follows that the fluxes of the $\Sigma(n)$ across $\mathbb{S}^2 \times \{0\}$ are unbounded. But the flux of $\Sigma(n)$ is equal to the flux of Σ which gives a contradiction and thereby proves our assertion. Now a standard holonomy argument shows that each of the leaves $\Sigma(n)$ is compact for n large, a contradiction. That is $\mathbb{S}^2 \times \{0\}$ is simply connected and compact, so for n large one can lift $\mathbb{S}^2 \times \{0\}$ into the leaves $\Sigma(n)$. The image of this lifting is open and closed in $\Sigma(n)$, hence $\Sigma(n)$ is a sphere; a contradiction.

□

References

[1] U. Abresch, *Constant mean curvature tori in terms of elliptic functions,* J. Reine Angew. Math., **37**(1987),169-192.

[2] F. Almgren and J. Taylor, *Geometry of soap films*, Sci. Am., **235**(1976), 82-93.

[3] S.Y. Cheng and S.T. Yau, *Differential equations on Riemannnian manifolds and their geometric applications*, Comm. Par. Appl. Math., **28**(1975), 333-354.

[4] M. do Carmo and C.K. Peng, *Stable minimal surfaces in $\mathbb{R}^3$ are planes*, Bulletin of A.M.S., **1**(1979), 903-906.

[5] Th. Colding and W.P. Minicozzi II, *Estimates for parametric elliptic integrands,* Int. Mat. Res. Not., **6**(2002), 291-297.

[6] Th. Colding and W.P. Minicozzi II, *Convergence and compactness of minimal surfaces without density bounds; I. Partial regularity and convergence*, preprint.

[7] Th. Colding and W.P. Minicozzi II, *Complete properly embedded minimal surfaces in* $\mathbb{R}^3$, preprint.

[8] P. Collin, *Topologie et courbure des surfaces minimale de* $\mathbb{R}^3$, Annals of Math. 2nd Series, **145**(1997), 1-31.

[9] Y. Fang and W. H. Meeks III, *Some Global Properties of Complete Minimal Surfaces of Finite Topology in* $\mathbb{R}^3$, Topology, **30**,no1,(1991), 9-20.

[10] L. Hauswirth, *Generalized Riemann examples in three-dimensional manifolds*, preprint.

[11] M. Hutchings, F. Morgan, M. Ritoré, A. Ros, *Proof of the double bubble conjecture,* Annals of Math., **155**(2002),no 2, 459-489.

[12] D. Fischer-Colbrie, *On Complete Minimal Surfaces with Finite Morse Index in 3-Manifolds*, Invent. Math., **82**(1985), 121-132.

[13] D. Fischer-Colbrie and R. Schoen, *The Structure of Complete Stable Minimal Surfaces in 3-Manifolds of Non-Negative Scalar Curvature*, Comm. on Pure and Appl. Math., **33**(1980), 199-211.

[14] M. Freedman and J. Hass and P. Scott, *Least area incompressible surfaces in 3-manifolds*, Invent. Math., **71**(1983), 609-642.

[15] D. Hoffman and Meeks III, W. H., *The strong halfspace theorem for minimal surfaces*, Invent. Math., **101**(1990), 373-377.

[16] N. Korevaar and R. Kusner and B. Solomon, *The Structure of Complete Embedded Surfaces with Constant Mean Curvature*, J. of Differential Geometry, **30**(1989), 465-503.

[17] H.B. Lawson Jr., *Lectures on Minimal Submanifolds*, Publish or Perish Press, Berkeley, (1971).

[18] R. Mazzeo and F. Pacard, *Constant mean curvature surfaces with Delaunay ends*, **9**(2001), no. 1, 169–237.

[19] F. Pacard and F. Pimentel, *Attaching handles to Bryant surfaces*, to appear.

[20] W. H. Meeks III, and J. Pérez and A. Ros, *Uniqueness of the Riemann minimal examples*, Invent. Math., **131**(1998), 107–132.

[21] W. H. Meeks III and H. Rosenberg, *The Global Theory of Doubly Periodic Minimal Surfaces*, Invent. Math., **97**(1989), 351-379.

[22] W. H. Meeks III and H. Rosenberg, *The Geometry of Periodic Minimal Surfaces*, Comment. Math. Helvetici, **68** (1993), 538-578.

[23] W.H. Meeks and H. Rosenberg, *The uniqueness of the helicoid and the asymptotic geometry of properly embedded minimal surfaces with finite topology*, preprint.

[24] W. H. Meeks III and H. Rosenberg, *The theory of minimal surfaces in $M \times \mathbb{R}$*, preprint.

[25] W. H. Meeks III and H. Rosenberg, *Stable minimal surfaces in $M \times \mathbb{R}$*, preprint.

[26] W.H. Meeks III and S.T. Yau, *The Existence of Embedded Minimal Surfaces and the Problem of Uniqueness*, Math. Z.,**179**(1982), 151-168.

[27] F. Morgan, *Geometric measure theory, a beginner's guide*, 2'nd edition, Academic Press, 1995.

[28] F. Morgan, *A Smooth Curve in $\mathbf{R^3}$ Bounding a Continuum of Minimal Manifolds*, Archive for Rational Mechanics and Analysis, **75**(1981), 193-197.

[29] F. Morgan and M. Ritoré, *Geometric measure theory and the proof of the double bubble conjecture,* to appear Clay Math. Proceedings on minimal surfaces, MSRI 2000.

[30] B. Nelli and H. Rosenberg, *Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, Bull. braz. Math. Soc., New series **33**(2)(2002), 263-292.

[31] J. C. C. Nitsche, *A Characterization of the Catenoid*, J. of Math. Mech., **11**(1962), 293-302.

[32] R. Osserman, *The Isoperimetric Inequality*, Bulletin of the AMS, **84**,no 6,(1978), 1182-1238.

[33] Joseph Plateau, *Statique expérimentale et théorique des liquides soumis aux seules forces moléculaires*, Gauthiers-Villars, Paris 1873.

[34] R. Pedrosa and M. Ritore, *Isoperimetric domains in the Riemannian product of a circle with a simply connected space form and applications to free boundary value problem*, Indiana Univ. Math. J., **48**(99), 1357-1394.

[35] J. Pitts and H. Rubenstein, *Applications of Minimax to Minimal Surfaces and the Topology of Three-manifolds*, Proceedings for the Center for Mathematical Analysis, Australian National University,Canberra, Australia, (1987), 137-170.

[36] H. Rosenberg, *Minimal surfaces in $M \times \mathbb{R}$*, to appear in Illinois Journal of Math.

[37] R. Schoen, *Estimates for Stable Minimal Surfaces in Three Dimensional Manifolds,* Seminar on Minimal Submanifolds, Annals of Math. Studies, **103**(1983), Princeton University Press, 111-126.

[38] M. Shiffmann, *On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes,* Annals of Math. (2), **63**(1956), 77-90.

[39] K. Shiokama and M. Tanaka, *The length function of geodesic parallel circles*, Adv. Stud. Pure Math., (1993), 299-305.

[40] A. da Silveira, *Stability of complete non compact surfaces with constant mean curvature*, Math. An., **277**(1987), 629-638.

[41] D'arcy Thompson, *On growth and form: the complete revised edition*, Dover Publications, Inc. New York, 1992.

[42] F. Xavier, *Convex hulls of complete minimal surfaces*, Math. Ann.,**269**(1984), 179-182.