Global Minimizers of Autonomous Lagrangians

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# Contents

1 Introduction. .............................................. 7
   1-1 Lagrangian Dynamics. .............................. 7
   1-2 The Euler-Lagrange equation. ...................... 9
   1-3 The Energy function. .............................. 12
   1-4 Hamiltonian Systems. .............................. 14
   1-5 Examples. .......................................... 16

2 Mañé’s critical value. .................................... 21
   2-1 The action potential and the critical value. ....... 21
   2-2 Continuity of the critical value. .................. 25
   2-3 Holonomic measures. ............................... 26
   2-4 Ergodic characterization of the critical value. .... 41
   2-5 The Aubry-Mather Theory. ......................... 44
      2-5.a Homology of measures. ......................... 44
      2-5.b The asymptotic cycle. ......................... 44
      2-5.c The alpha and beta functions. ................. 47
   2-6 Coverings. ......................................... 49

3 Globally minimizing orbits. ............................. 51
   3-1 Tonelli’s theorem. ................................ 51
   3-2 A priori compactness. .............................. 58
   3-3 Energy of time-free minimizers. .................... 61
3-4 The finite-time potential. .......................... 63
3-5 Global Minimizers. ................................. 65
3-6 Characterization of minimizing measures. ......... 68
3-7 The Peierls barrier. ............................ 71
3-8 Graph Properties. ............................... 74
3-9 Coboundary Property. ............................ 78
3-10 Covering Properties. ............................ 80
3-11 Recurrence Properties. .......................... 81

4 Dynamics on prescribed energy levels. ........... 89
4-1 The Hamilton-Jacobi equation. ................... 89
4-2 Subsolutions of the Hamilton-Jacobi equation. ... 91
4-3 Anosov energy levels. ............................ 100
4-4 Weak KAM Solutions. .............................. 102

5 Generic Lagrangians. ............................... 109
5-1 Generic Lagrangians. ............................. 110
5-2 Homoclinic Orbits. ............................... 122

Appendix. .............................................. 131
A Absolutely continuous functions. .................. 131
B Convex functions. .................................. 133
C The Frenshel and Legendre Transforms. .......... 134

Bibliography. ........................................ 139

Index. ............................................... 143
Chapter 1

Introduction.

1-1 Lagrangian Dynamics.

Let \( M \) be a boundaryless \( n \)-dimensional complete riemannian manifold. An (autonomous) Lagrangian on \( M \) is a smooth function \( L : TM \to \mathbb{R} \) satisfying the following conditions:

(a) Convexity: The Hessian \( \frac{\partial^2 L}{\partial v_i \partial v_j} (x, v) \), calculated in linear coordinates on the fiber \( T_x M \), is uniformly positive definite for all \( (x, v) \in TM \), i.e. there is \( A > 0 \) such that

\[
w \cdot L_{uv}(x, v) \cdot w \geq A|w|^2 \quad \text{for all} \ (x, v) \in TM \ \text{and} \ w \in T_x M.\]

(b) Superlinearity:

\[
\lim_{|v| \to +\infty} \frac{L(x, v)}{|v|} = +\infty, \quad \text{uniformly on} \ x \in M,
\]

equivalently, for all \( A \in \mathbb{R} \) there is \( B \in \mathbb{R} \) such that

\[
L(x, v) \geq A|v| - B \quad \text{for all} \ (x, v) \in TM.
\]
(c) Boundedness: For all $r > 0$,

$$\ell(r) = \sup_{(x,v) \in TM, \ |v| < r} L(x,v) < +\infty.$$  \hfill (1.1)

$$g(r) = \sup_{|w|=1 \atop |(x,v)| \leq r} w \cdot L_{vv}(x,v) \cdot w < +\infty.$$ \hfill (1.2)

The Euler-Lagrange equation associated to a lagrangian $L$ is (in local coordinates)

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{v}}(x, \dot{x}) = \frac{\partial L}{\partial x}(x, \dot{x}).$$ \hfill (E-L)

The condition (c) implies that the Euler-Lagrange equation (E-L) defines a complete flow $\varphi_t$ on $TM$ (proposition 1.3.2), called the Euler-Lagrange flow, by setting $\varphi_t(x_0, v_0) = (x_v(t), \dot{x}_v(t))$, where $x_v : \mathbb{R} \to M$ is the solution of (E-L) with $x_v(0) = x_0$ and $\dot{x}_v(0) = v_0$.

We shall be interested on coverings $p : N \to M$ of a compact manifold $M$ and the lifted Lagrangian $L = L \circ dp : TN \to \mathbb{R}$ of a convex superlinear lagrangian $L$ on $M$. The lagrangian $L$ then satisfies (a)-(c) and its flow $\psi_t$ is the lift of $\varphi_t$.

Observe that when we add a closed 1-form $\omega$ to the lagrangian $L$, the new lagrangian $L + \omega$ also satisfies the hypothesis (a)-(c) and has the same Euler-Lagrange equation as $L$. This can also be seen using the variational interpretation of the Euler-Lagrange equation.
1-2 The Euler-Lagrange equation.

The action of a differential curve \( \gamma : [0, T] \to M \) is defined by

\[
A(\gamma) = \int_0^T L(\gamma(t), \dot{\gamma}(t)) \, dt \tag{1.3}
\]

One of the main problems of Calculus of Variations is to find and to study the curves that minimize the action.

Denote by \( C^k(p, q, T) \) the set of \( k \)-differentiable curves \( \gamma : [0, T] \to M \) such that \( \gamma(0) = p \) and \( \gamma(T) = q \).

1-2.1. Proposition. Let \( (q_1, q_2, \ldots, q_n) \) a coordinate system in \( M \), then if a curve \( x \) in the space \( C^k(p, q, T) \) minimize the action among all the curves in \( C^k(p, q, T) \), then \( x \) satisfies the equation

\[
\frac{d}{dt} L_x(x(t), \dot{x}(t)) = L_{\nu}(x(t), \dot{x}(t)) \tag{1.4}
\]

This equation is called the Euler-Lagrange equation. Moreover if \( (Q_1, Q_2, \ldots, Q_n) \) is another coordinate system then \( x \) satisfies the Euler-Lagrange equation in the coordinates \( q_i \) if and only if it satisfies the Euler-Lagrange equation in the coordinates \( Q_i \).

Proof Let \( h(t) \) a differential curve such that \( h(0) = h(T) = 0 \), then for every \( \epsilon \), sufficiently small the curve \( y_\epsilon = x + \epsilon h \) is on \( C^k(p, q, T) \) and contained in the coordinate system. Define

\[
g(\epsilon) = A_L(y_\epsilon) \tag{1.5}
\]

Then \( g \) has a minimum in zero and
\[
\lim_{\epsilon \to 0} \frac{g(\epsilon) - g(0)}{\epsilon} = \lim_{\epsilon \to 0} \int_0^T \frac{L(x + \epsilon h, \dot{x} + \epsilon \dot{h}) - L(x, \dot{x})}{\epsilon} \, dt \\
= \int_0^T \lim_{\epsilon \to 0} \frac{\epsilon L_x h + \epsilon L_v \dot{h} + o(\epsilon)}{\epsilon} \, dt \\
= \int_0^T L_x h + L_v \dot{h} \, dt \\
= \int_0^T (L_x - \frac{d}{dt} L_v) h \, dt + L_v h \bigg|_0^T \\
= \int_0^T (L_x - \frac{d}{dt} L_v) h \, dt
\]

Hence

\[
0 = \int_0^T (L_x(x(t), \dot{x}(t)) - \frac{d}{dt} L_v(x(t), \dot{x}(t))) h \, dt \quad (1.6)
\]

For any function \( h \), this implies that \( x(t) \) satisfies the Euler Lagrange equation [1.4]

This proves shows that \( x(t) \) satisfies the Euler Lagrange equation if and only if \( x(t) \) is a critical point of the action functional \( A_L \) defined in the space \( C^k(p, q, T) \), this doesn’t depend on the coordinates and proves the second statement.

The Euler Lagrange equation is a second order differential equation on \( M \), but our hypothesis about the Lagrangian (\( L_{vv} \) invertible) implies that this equation can also be seen as a first order differential equation on \( TM \)

\[
\dot{x} = v \\
\dot{v} = (L_{vv})^{-1}(L_x - L_{vx}v)
\]

The associated vector field \( X_L \) on \( TM \) will be called the Lagrangian vector field and its flow \( \psi \), the Lagrangian flow.
1-2.2. Remark. It is possible to do the same thing in the space $C_T(p, q)$, the set of absolutely continuous curves $\gamma : [0, T] \to M$ such that $\gamma(0) = p$ and $\gamma(T) = q$. A priori minimizers do not have to be differentiable and there are examples where they are not, see Ball & Mizel [2]. However when the lagrangian flow is complete (cf. proposition 1-3.2), every absolutely continuous minimizers is $C^2$ and satisfies the Euler-Lagrange equation. See Mather [39].
1-3 The Energy function.

The energy function of the lagrangian \( L \) is \( E : TM \to \mathbb{R} \), defined by

\[
E(x,v) = \frac{\partial L}{\partial v}(x,v) \cdot v - L(x,v).
\] (1.7)

Observe that if \( x(t) \) is a solution of the Euler-Lagrange equation (E-L), then

\[
\frac{d}{dt} E(x, \dot{x}) = (\frac{d}{dt} L_v - L_x) \cdot \dot{x} = 0.
\]

Hence \( E : TM \to \mathbb{R} \) is an integral (i.e. invariant function\(^1\)) for the lagrangian flow \( \varphi_t \) and its level sets, called energy levels are invariant under \( \varphi_t \). Moreover, the convexity implies that

\[
\frac{d}{dt} E(x, sv)|_{s=1} = v \cdot L_{vv}(x,v) \cdot v > 0.
\]

Thus

\[
\min_{v \in T_xM} E(x,v) = E(x,0) = -L(x,0).
\]

Write

\[
e_0 := \max_{x \in M} E(x,0) = -\min_{x \in M} L(x,0) > -\infty,
\] (1.8)

by the superlinearity, then

\[
e_0 = \min \{ k \in \mathbb{R} \mid \pi : E^{-1}\{k\} \to M \text{ is surjective} \}.
\]

By the uniform convexity, and the boundedness condition,

\[
A := \inf_{(x,v) \in TM, |w|=1} w \cdot L_{vv}(x,v) \cdot w > 0,
\]

and then using (1.1) and (1.2),

\[
E(x,v) = E(x,0) + \int_0^{|v|} \frac{d}{ds} E(x, s\frac{v}{|v|}) \, ds \\
\geq -\ell(0) + A |v|.
\] (1.9)

\(^1\)The energy is invariant only for autonomous (i.e. time-independent) lagrangians.
Similarly,

$$E(x, v) \leq e_0 + g(|v|) |v|. \quad (1.10)$$

Hence

1-3.1. Remark. If $k \in \mathbb{R}$ and $K \subseteq M$ is compact, then $E^{-1}\{k\} \cap T_K M$ is compact.

1-3.2. Proposition. The Euler-Lagrange flow is complete.

Proof: Suppose that $]\alpha, \beta[ \,$ is the maximal interval of definition of $t \mapsto \varphi_t(v)$, and $-\infty < \alpha$ or $\beta < +\infty$. Let $k = E(v)$. Since $E(\varphi_t(v)) \equiv k$, by (1.9), there is $a > 0$ such that $0 \leq |\varphi_t(v)| \leq a$ for $\alpha \leq t \leq \beta$. Since $\varphi_t(v)$ is of the form $(\gamma(t), \dot{\gamma}(t))$, then $\varphi_t(v)$ remains in the interior of the compact set

$$Q := \{ (y, w) \in TM \mid d(y, x) \leq a [||\beta - \alpha|| + 1], \ |v| \leq a + 1 \},$$

where $x = \pi(v)$. The Euler-Lagrange vectorfield is uniformly Lipschitz on $Q$. Then by the theory of ordinary differential equations, we can extend the interval of definition $]\alpha, \beta[ \,$ of $t \mapsto \varphi_t(v)$. \qed
1-4 Hamiltonian Systems.

Let $T^*M$ be the cotangent bundle of $M$. Define the Liouville’s 1-form $\Theta$ on $T^*M$ as

$$\Theta_p(\xi) = p(d\pi \xi) \quad \text{for} \quad \xi \in T_p(T^*M),$$

where $\pi : T^*M \to M$ is the projection. The canonical symplectic form on $T^*M$ is defined as $\omega = d\Theta$.

A local chart $x = (x_1, \ldots, x_n)$ of $M$ induces a local chart $(x, p) = (x_1, \ldots, x_n; p_1, \ldots, p_n)$ of $T^*M$ writing $p \in T^*M$ as $p = \Sigma_i p_i \, dx_i$. In these coordinates the forms $\Theta$ and $\omega$ are written

$$\Theta = p \cdot dx = \sum_i p_i \, dx_i,$$

$$\omega = dp \wedge dx = \sum_i dp_i \wedge dx_i.$$

A hamiltonian is a smooth function $H : T^*M \to \mathbb{R}$. The hamiltonian vectorfield $X_H$ associated to $H$ is defined by

$$\omega(X_H, \cdot) = dH.$$

In local charts, the hamiltonian vectorfield defines the differential equation

$$\dot{x} = H_p, \quad \dot{p} = -H_x,$$  \hspace{1cm} (1.11)

where $H_x$ and $H_p$ are the partial derivatives of $H$ with respect to $x$ and $p$.

We shall be specially interested in hamiltonians obtained by the Frenshel transform of a lagrangian:

$$H(x, p) = \max_{v \in T_x M} p \cdot v - L(x, v).$$

Observe that $H = E \circ \mathcal{L}^{-1}$, where $E$ is the energy function (1.7) and $\mathcal{L}(x, v) = (x, L_v(x, v))$ is the Legendre transform of $L$. Moreover
1-4.1. Proposition. The Legendre transform $\mathcal{L} : TM \to T^*M$, $\mathcal{L}(x, v) = (x, L_v(x, v))$ is a conjugacy between the lagrangian flow and the hamiltonian flow.

Proof: By corollary C.2, the convexity and superlinearity hypothesis imply that $L = L^{**} = H^*$. So if $p = L_v(x, v)$ then $v = H_p(x, p)$. With this notation:

$$H(x, p) = v \cdot L_v(x, v) - L(x, v) = E \circ \mathcal{L}^{-1}$$
$$= p \cdot H_p(x, p) - L(x, H_p(x, p)).$$

Thus $H_x = -L_x$, and the Euler-Lagrange equation

$$\dot{x} = \frac{d}{dt} x = v = H_p,$$
$$\dot{p} = \frac{d}{dt} L_v = L_x = -H_x,$$

is the same as the hamiltonian equations. \qed
1-5 Examples.

We give here some basic examples of lagrangians.

Riemannian Lagrangians:
Given a riemannian metric $g = \langle \cdot, \cdot \rangle_x$ on $TM$, the riemannian lagrangian on $M$ is given by the kinetic energy

\[ L(x, v) = \frac{1}{2} \|v\|_x^2. \]  

(1.12)

Its Euler-Lagrange equation (E-L) is the equation of the geodesics of $g$:

\[ \frac{D}{dt} \dot{x} \equiv 0, \]  

(1.13)

and its Euler-Lagrange flow is the geodesic flow. Its corresponding hamiltonian is

\[ H(x, p) = \frac{1}{2} \|p\|_x^2. \]

Analogous to the riemannian lagrangian is the Finsler lagrangian, given also by formula (1.12), but where $\|\cdot\|_x$ is a Finsler metric, i.e. $\|\cdot\|_x$ is a (non necessarily symmetric\(^2\)) norm on $T_xM$ which varies smoothly on $x \in M$. The Euler-Lagrange flow of a Finsler lagrangian is called the geodesic flow of the Finsler metric $\|\cdot\|_x$.

Mechanic Lagrangians:
The mechanic lagrangian, also called natural lagrangian, is given by the kinetic energy minus the potential energy $U : M \to \mathbb{R}$,

\[ L(x, v) = \frac{1}{2} \|v\|_x^2 - U(x). \]  

(1.14)

Its Euler-Lagrange equation is

\[ \frac{D}{dt} \dot{x} = -\nabla U(x), \]

\(^2\text{i.e. } \|\lambda v\|_x = \lambda \|v\|_x \text{ only for } \lambda \geq 0\)
where $\nabla U$ is the gradient of $U$ with respect to the riemannian metric $g$, i.e.
\[
  d_x U(v) = \langle \nabla U(x), v \rangle_x \quad \text{for all } (x, v) \in TM.
\]

Its energy function and its hamiltonian are given by the kinetic energy plus potential energy:
\[
  E(x, v) = \frac{1}{2} \|v\|_x^2 + U(x),
  \\
  H(x, p) = \frac{1}{2} \|p\|_x^2 + U(x).
\]

**Symmetric Lagrangians.**

The symmetric lagrangians is a class of lagrangian systems which includes the riemannian and mechanic lagrangians. These are the lagrangians which satisfy
\[
  L(x, v) = L(x, -v) \quad \text{for all } (x, v) \in TM. \quad (1.15)
\]

Their Euler-Lagrange flow is reversible in the sense that $\varphi_{-t}(v) = -\varphi_t(-v)$.

**Magnetic Lagrangians.**

If one adds a closed 1-form $\omega$ to a lagrangian, $L(x, v) = L(x, v) + \omega_x(v)$, the Euler-Lagrange flow does not change. This can be seen by first observing that the solutions of the Euler-Lagrange equation are the critical points of the action functional on curves on $C(x, y, T)$ (with fixed time interval and fixed endpoints). Since $\omega$ is closed, the action functional of $L$ and $L$ on $C(x, y, T)$ differ by a constant and hence they have the same critical points.

But adding a non-closed 1-form to a lagrangian does change the Euler-Lagrange flow. We call a magnetic lagrangian a lagrangian of the form
\[
  L(x, v) = \frac{1}{2} \|v\|_x + \eta_x(v) - U(x), \quad (1.16)
\]
where $\| \cdot \|_x$ is a riemannian metric, $\eta$ is a 1-form on $M$ with $d\eta \neq 0$, and $U : M \to \mathbb{R}$ a smooth function. If $Y : TM \to TM$ is the bundle map such that

$$d\eta(u, v) = \langle Y(u), v \rangle.$$

then the Euler-Lagrange equation of (1.16) is

$$\frac{D}{dt} \dot{x} = Y_x(\dot{x}) - \nabla U(x). \quad (1.17)$$

This models the motion of a particle with unit mass and unit charge under the effect of a magnetic field with Lorentz force $Y$ and potential energy $U(x)$. The energy functional is the same as that of the mechanical lagrangian but its hamiltonian changes because of the change in the Legendre transform:

$$E(x, v) = \frac{1}{2} \| v \|_x^2 + U(x),$$

$$H(x, p) = \frac{1}{2} \| p - A(x) \|_x^2 + U(x),$$

where $A : M \to TM$ is the vectorfield given by $\eta_x(v) = \langle A(x), v \rangle_x$.

**Twisted geodesic flows.**

The twisted geodesic flows correspond to the motion of a particle under the effect of a magnetic field with no potential energy. This can be modeled as the Euler-Lagrange flow of a lagrangian of the form $L(x, v) = \frac{1}{2} \| v \|_x^2 + \eta_x(v)$, where $d\eta \neq 0$. But the Euler-Lagrange equations depend only on the riemannian metric and $d\eta$. A generalization of these flows can be made using a non-zero 2-form $\Omega$ instead of $d\eta$ and not requiring $\Omega$ to be exact. This is better presented in the hamiltonian setting.

Fix a riemannian metric $\langle \cdot, \cdot \rangle$ and a 2-form $\Omega$ on $M$. Let $K : TTM \to TM$ be the connection map $K\xi = \nabla_{\dot{x}} v$, where $\xi = \frac{d}{dt}(x(t), v(t))$. Let $\pi : TM \to M$ be the canonical projection. Let $\omega_0$ be the symplectic form in $TM$ obtained by pulling back the canonical symplectic form via the Legendre transform associated to the riemannian metric, i.e.

$$\omega_0(\xi, \zeta) = \langle d\pi \xi, K\zeta \rangle - \langle d\pi \zeta, K\xi \rangle.$$
The coordinates $T_b TM \ni \xi \leftrightarrow (d\pi \xi, K \xi) \in T_{\pi(\theta)} M \oplus T_{\pi(\theta)} M = H(\theta) \oplus V(\theta)$ are the standard way of writing the horizontal and vertical components of a vector $\xi \in T_b TM$ for a riemannian manifold $M$ (see Klingenberg [25]).

Define a new symplectic form $\omega_\Omega$ on $TM$ by

$$\omega_\Omega = \omega_0 + \pi^* \Omega.$$ 

This is called a twisted symplectic structure on $TM$. Let $H : TM \to \mathbb{R}$ be the hamiltonian

$$H(x, v) = \frac{1}{2} \|v\|^2_x.$$ 

Consider the hamiltonian vectorfield $X_F$ corresponding to $(H, \omega_\Omega)$, i.e.

$$\omega_\Omega(X_\Omega(\theta), \cdot) = dH.$$ (1.18)

Define $Y : TM \to TM$ as the bundle map such that

$$\Omega_x(u, v) = \langle Y(u), v \rangle_x.$$ (1.19)

The hamiltonian vectorfield $X_\Omega(\theta) \in T_b TM$ is given by $X_\Omega(\theta) = (\theta, Y(\theta)) \in H(\theta) \oplus V(\theta)$. Hence the hamiltonian equation is

$$\frac{d}{dt} \dot{x} = Y_x(\dot{x}),$$

recovering equation (1.17) with $U \equiv 0$, but where $\Omega$ doesn't need to be exact.

If $H^1(M, \mathbb{R}) = 0$, both approaches coincide, and any twisted geodesic flow is the lagrangian flow of a magnetic lagrangian of the form $L(x, v) = \frac{1}{2} \|v\|^2_x + \eta_x(v)$, with $d\eta = \Omega$. For example if $N$ is a compact manifold $\Omega$ is a 2-form in $N$ and $M$ is the abelian cover or the universal cover of $N$; if $\Omega$ is not exact, then the corresponding twisted geodesic flow is a lagrangian flow on $M$ but not on $N$ (where it is locally a lagrangian flow). This lagrangian flow on $M$ is actually the lift of the twisted geodesic flow on $N$. 

Embedding flows:

There is a way to embed the flow of any bounded vectorfield on a lagrangian system. Given a smooth bounded vectorfield $F : M \to TM$, let

$$L(x, v) = \frac{1}{2} \| v - F(x) \|^2_x.$$  \hspace{1cm} (1.20)

Since $F(x)$ is bounded, then the lagrangian $L$ is convex, superlinear and satisfies the boundedness condition. The lagrangian $L$ on a fiber $T_xM$ is minimized at $(x, F(x))$, hence the integral curves of the vectorfield, $\dot{x} = F(x)$, are solutions to the Euler-Lagrange equation.
Chapter 2

Mañe's critical value.

2-1 The action potential and the critical value.

We shall be interested on action minimizing curves with free time interval. Unless otherwise stated, all the curves will be assumed to be absolutely continuous. For $x, y \in M$, let

$$\mathcal{C}(x, y) = \{ \gamma : [0, T] \to M \mid T > 0, \gamma(0) = x, \gamma(T) = y \}.$$ 

For $k \in \mathbb{R}$ define the action potential $\Phi_k : M \times M \to \mathbb{R} \cup \{-\infty\}$, by

$$\Phi_k(x, y) = \inf_{\gamma \in \mathcal{C}(x, y)} A_{L+k}(\gamma).$$

Observe that if there exists a closed curve $\gamma$ on $N$ with negative $L + k$ action, then $\Phi_k(x, y) = -\infty$ for all $x, y \in N$, by going round $\gamma$ many times.

Define the critical level $c = c(L)$ as

$$c(L) = \sup\{ k \in \mathbb{R} \mid \exists \text{ closed curve } \gamma \text{ with } A_{L+k}(\gamma) < 0 \}.$$ 

Observe that the function $k \mapsto \Phi_k(x, y)$ is increasing. The superlinearity implies that $L$ is bounded below. Hence there is $k \in \mathbb{R}$ such that
\( L + k \geq 0 \). Thus \( c(L) < +\infty \). Since \( k \mapsto A_{L+k}(\gamma) \) is increasing for any \( \gamma \), we have that
\[
c(L) = \inf \{ k \in \mathbb{R} \mid A_{L+k}(\gamma) \geq 0 \ \forall \ \text{closed curve } \gamma \}.
\]

2-1.1. Proposition.

(1) (a) For \( k < c(L) \), \( \Phi_k(x, y) = -\infty \) for all \( x, y \in M \).

(b) For \( k \geq c(L) \), \( \Phi_k(x, y) \in \mathbb{R} \) for all \( x, y \in M \).

(2) For \( k \geq c(L) \), \( \Phi_k(x, z) \leq \Phi_k(x, y) + \Phi_k(y, z) \), \( \forall \ x, y, z \in M \).

(3) \( \Phi_k(x, x) = 0 \), \( \forall \ x \in M \).

(4) \( \Phi_k(x, y) + \Phi_k(y, x) \geq 0 \) \( \forall \ x, y \in M \).

For \( k > c(L) \), \( \Phi_k(x, y) + \Phi_k(y, x) > 0 \) if \( x \neq y \).

(5) For \( k \geq c(L) \) the action potential \( \Phi_k \) is Lipschitz.

2-1.2. Remark. The action potential \( \Phi_k \) is not symmetric in general, but items (2),(3),(4) imply that
\[
d_k(x, y) = \Phi_k(x, y) + \Phi_k(y, x)
\]
is a metric for \( k > c(L) \) and a pseudo-metric for \( k = c(L) \) [i.e. perhaps \( d_c(x, y) = 0 \) for some \( x \neq y \) and \( c = c(L) \)].

Proof:

(2) We first prove (2) for all \( k \in \mathbb{R} \). Since \( \Phi_k(x, y) \in \mathbb{R} \cup \{-\infty\} \), the inequality in item (2) makes sense for all \( k \in \mathbb{R} \). If \( \gamma \in C(x, y) \), \( \eta \in C(y, z) \), then \( \gamma \ast \eta \in C(x, z) \) and hence,
\[
\Phi_k(x, z) \leq A_{L+k}(\gamma \ast \eta) \leq A_{L+k}(\gamma) + A_{L+k}(\eta).
\]

Taking the infimums on \( \gamma \in C(x, y) \) and \( \eta \in C(y, z) \), we obtain (2).
(1) (a) If \( \gamma \) is a closed curve with \( A_{L+k}(\gamma) < 0 \) and \( \gamma(0) = z \), then

\[
\Phi_k(z, z) \leq \lim_{N \to \infty} A_{L+k}(\gamma * \cdots * \gamma) = \lim_N N A_{L+k}(\gamma) = -\infty.
\]

For \( x, y \in M \), item (2) implies that

\[
\Phi_k(x, y) \leq \Phi_k(x, z) + \Phi_k(z, z) + \Phi_k(z, y) = -\infty.
\]

Since the function \( k \mapsto \Phi_k(x, y) \) is increasing, then item (1)(a) follows.

(b) Conversely, if \( \Phi_k(x, y) = -\infty \) for some \( k \in \mathbb{R} \) and \( x, y \in M \), then

\[
\Phi_k(x, x) \leq \Phi_k(x, y) + \Phi_k(y, x) = -\infty.
\]

Thus there is \( \gamma \in \mathcal{C}(x, x) \) with \( A_{L+k}(\gamma) < 0 \). Then \( k \leq c(L) \). Observe that the set \( \{ k \in \mathbb{R} | A_{L+k}(\gamma) < 0 \text{ for some closed curve } \gamma \} \) is open. Hence \( \Phi_k(x, y) = -\infty \) actually implies that \( k < c(L) \).

This proves item (1)(b).

(3) Let \( k \in \mathbb{R} \) by the boundedness condition there exists \( Q > 0 \) be such that

\[
|L(x, v) + k| \leq Q \quad \text{for } |v| \leq 2. \tag{2.1}
\]

Now let \( \gamma : [0, 1] \to M \) be a differentiable curve with \( |\dot{\gamma}| \equiv 1 \) and \( \gamma(0) = x \). Then

\[
\Phi_k(x, x) \leq \Phi_k(x, \gamma(\varepsilon)) + \Phi_k(\gamma(\varepsilon), x) \\
\leq A_{L+k}(\gamma|_{[0, \varepsilon]}) + A_{L+k}(\gamma(t - \varepsilon)|_{[0, \varepsilon]}) \\
\leq 2Q \varepsilon.
\]

Letting \( \varepsilon \to 0 \) we get that \( \Phi_k(x, x) \leq 0 \). But the definition of \( c(L) \) and the monotonicity of \( k \mapsto \Phi_k(x, x) \) imply that \( \Phi_k(x, x) \geq 0 \) for all \( k \geq c(L) \).
(5) Let $k \geq c(L)$. Given $x_1, x_2 \in M$ we have that
\[
\Phi_k(x_1, x_2) \leq A_{L+k}(\gamma) \leq Q \, d_M(x_1, x_2),
\]
where $\gamma : [0, d(x_1, x_2)] \to N$ is a unit speed minimizing geodesic joining $x_1$ to $x_2$ and $Q > 0$ is from (2.1). If $y_1, y_2 \in M$, then the triangle inequality implies that
\[
\Phi_k(x_1, y_1) - \Phi_k(x_2, y_2) \leq \Phi_k(x_1, x_2) + \Phi_k(y_2, y_1) \\
\leq Q \, [d_M(x_1, x_2) + d_M(y_1, y_2)].
\]
Changing the roles of $(x_1, y_1)$ and $(x_2, y_2)$ we get item (5).

(4) The first part of item (4) follows from items (2) and (3). Now suppose that $k > c(L), x \neq y$ and $d_k(x, y) = 0$. Let $\gamma_n : [0, T_n] \to M, \gamma_n \in C(x, y)$ be such that $\Phi_k(x, y) = \lim_n A_{L+k}(\gamma_n)$. We claim that $T_n$ is bounded below.

Indeed, suppose that $\lim_n T_n = 0$. Let $A > 0$, from the superlinearity there is $B > 0$ such that $L(x, v) \leq A |v| - B, \forall (x, v) \in TM$. Then
\[
\Phi_k(x, y) = \lim_n \int_0^{T_n} L(\gamma_n, \dot{\gamma}_n) + k \\
\geq \lim_n A \int |\dot{\gamma}| + (k - B) T_n \\
= A \, d_M(x, y)
\]
Letting $A \to +\infty$ we get that $\Phi_k(x, y) = +\infty$ which is false.
Now let $\eta_n : [0, S_n] \to M, \eta_n \in C(y, x)$ with $\lim_n A_{L+k}(\eta_n) = \Phi_k(y, x)$. Choose $0 < T < \lim \inf_n T_n$ and $0 < S < \lim \inf_n S_n$. Then for $c = c(L) < k$,
\[
\Phi_c(x, x) \leq \lim_n A_{L+c}(\gamma_n \ast \eta_n) \\
\leq \lim_n A_{L+k}(\gamma_n) + (c - k)T + A_{L+k}(\eta_n) + (c - k)S \\
\leq \lim_n \Phi_k(x, y) + \Phi_k(y, x) + (c - k)(T + S) \\
\leq (c - k)(T + S) < 0,
\]
which contradicts item (3).
2-2 Continuity of the critical value.

2-2.1. Lemma. *The function* \( C^{\infty}(M, \mathbb{R}) \ni \psi \mapsto c(L + \psi) \) *is continuous in the topology induced by the supremum norm.*

**Proof:** Suppose that \( \psi_n \to \psi \) and let \( c_n := c(L + \psi_n) \) and \( c := c(L + \psi) \). We will prove that \( c_n \to c \).

Fix \( \varepsilon > 0 \). Since \( c - \varepsilon < c \), by the definition of critical value there exists a closed curve \( \gamma : [0, T] \to M \) such that \( A_{L + \psi + c - \varepsilon}(\gamma) < 0 \), hence for all \( n \) sufficiently large

\[
A_{L + \psi_n + c - \varepsilon}(\gamma) < 0.
\]

Therefore for \( n \) sufficiently large \( c - \varepsilon < c_n \), and thus \( c - \varepsilon \leq \lim \inf_n c_n \).

Since \( \varepsilon \) was arbitrary we have that \( c \leq \lim \inf_n c_n \).

We show now that \( \lim \sup_n c_n \leq c \). Suppose that \( c < \lim \sup_n c_n \).

Take \( \varepsilon \) such that

\[
c < c + \varepsilon < \lim \sup_n c_n.
\]

(2.2)

Since \( \psi_n \to \psi \), there exists \( n_0 \) such that for all \( n \geq n_0 \),

\[
-\varepsilon \leq \psi - \psi_n \leq \varepsilon.
\]

(2.3)

By (2.2), there exists \( m \geq n_0 \) such that

\[
c < c + \varepsilon < c_m.
\]

By the definition of critical value there exists a closed curve \( \gamma : [0, T] \to M \) such that

\[
A_{L + \psi_m + c + \varepsilon}(\gamma) < 0,
\]

and hence using (2.3) we have

\[
A_{L + \psi + c}(\gamma) \leq A_{L + \psi_m + c + \varepsilon}(\gamma) < 0,
\]

which yields a contradiction to the definition of the critical value \( c \).

\[\square\]

This proof also shows that \( L \mapsto c(L) \) is continuous if we endow the set of lagrangians \( L \) with the topology induced by the supremum norm.
2-3 Holonomic measures.

Let $C^0_\ell$ be the set of continuous functions $f : TM \to \mathbb{R}$ having linear growth, i.e.

$$\sup_{(x,v) \in TM} \frac{|f(x,v)|}{1 + ||v||} < +\infty.$$

Let $\mathcal{M}_\ell$ be the set of Borel probabilities $\mu$ on $TM$ such that

$$\int_{TM} ||v|| \, d\mu < +\infty,$$

endowed with the topology such that $\lim_n \mu_n = \mu$ if and only if

$$\lim_n \int f \, d\mu_n = \int f \, d\mu$$

for all $f \in C^0_\ell$.

Let $(C^0_\ell)'$ the dual of $C^0_\ell$. Then $\mathcal{M}_\ell$ is naturally embedded in $(C^0_\ell)'$ and its topology coincides with that induced by the weak* topology on $(C^0_\ell)'$.

We shall see that this topology is metrizable. Let $\{f_n\}$ be a sequence of functions with compact support on $C^0_\ell$ which is dense on $C^0_\ell$ in the topology of uniform convergence on compact sets of $TM$. Define a metric $d(\cdot, \cdot)$ on $\mathcal{M}_\ell$ by

$$d(\mu_1, \mu_2) = \left| \int |v| \, d\mu_1 - \int |v| \, d\mu_2 \right| + \sum_{n} \frac{1}{2^n} \frac{1}{c_n} \left| \int f_n \, d\mu_1 - \int f_n \, d\mu_2 \right|$$

(2.4)

where $c_n = \sup_{(x,v)} f_n(x,v)$.

2-3.1. Proposition.

The metric $d(\cdot, \cdot)$ induces the weak* topology on $\mathcal{M}_\ell \subset C^0_\ell$. 

Proof: We prove that $d(\cdot, \cdot)$ generates the weak* topology on $\mathcal{M}_\ell$. Suppose that

$$\int f \, d\mu_n \to \int f \, d\mu, \quad \forall f \in C_0^\ell.$$

Given $\varepsilon > 0$, choose $M > 0$ such that $\sum_{m \geq M} \frac{1}{2^n} \cdot 2 < \varepsilon$, and choose $N > 0$ such that

$$\left| \int f_m \, d\mu_n - \int f_m \, d\mu \right| < \varepsilon, \quad \text{for } 0 \leq m \leq M, \; n \geq N;$$

$$\left| \int |v| \, d\mu_n - \int |v| \, d\mu \right| < \varepsilon, \quad \text{for } n \geq N.$$

Since $\|f\|_{c_n} = 1$, then for $n > N$ we have that

$$d(\mu_n, \mu) \leq \varepsilon + \sum_{m=1}^{M} \frac{1}{2^n} \cdot \varepsilon + \sum_{m \geq M+1} \frac{1}{2^n} \cdot 2 \cdot \|f_m\|_{c_m} = 3 \varepsilon.$$

Thus $d(\mu_n, \mu) \to 0$.

Now suppose that $d(\mu_n, \mu) \to 0$. Let $K_m$ be compact sets such that $K_m \subset K_{m+1}$ and that $TM = \bigcup K_m$. Then

$$\int_{K_m} f \, d\mu_n \to \int_{K_m} f \, d\mu, \quad \forall f \in C_0^\ell, \quad \forall m;$$

$$\int |v| \, d\mu_n \to \int |v| \, d\mu.$$

This implies that

$$\lim_{n \to \infty} \int_{TM-K_m} |v| \, d\mu_n = \int_{TM-K_m} |v| \, d\mu, \quad \forall m. \quad (2.5)$$

Given $\varepsilon > 0$, choose $m(\varepsilon) > 0$ such that

$$\int_{TM-K_{m(\varepsilon)}} (1 + |v|) \, d\mu < \frac{\varepsilon}{4},$$
and \( N \) such that
\[
\int_{TM - K_{m(\varepsilon)}} (1 + |v|) \, d\mu_n < \frac{\varepsilon}{2}, \quad \forall n > N.
\]

Fix \( f \in C^0_{\ell} \). Choose \( N > 0 \) such that
\[
\left| \int_{K_{m(\varepsilon)}} f \, d\mu_n - \int_{K_{m(\varepsilon)}} f \, d\mu \right| < \varepsilon, \quad \forall n > N.
\]

Then
\[
\int_{TM - K_{m(\varepsilon)}} |f| \, d\mu_n \leq \|f\|_{\ell} \int_{TM - K_{m(\varepsilon)}} (1 + |v|) \, d\mu_n \leq \|f\|_{\ell} \frac{\varepsilon}{2}, \quad \forall n > N.
\]

Using a similar estimate for \( \mu \) we obtain that
\[
\left| \int f \, d\mu_n - \int f \, d\mu \right| \leq \varepsilon + \|f\|_{\ell} (\frac{\varepsilon}{2} + \frac{\varepsilon}{4}).
\]

\[\square\]

If \( \gamma : [0, T] \to M \) is a closed absolutely continuous curve, let \( \mu_{\gamma} \in \mathcal{M}_\ell \) be defined by
\[
\int f \, d\mu_{\gamma} = \frac{1}{T} \int_0^T f(\gamma(t), \dot{\gamma}(t)) \, dt
\]
for all \( f \in C^0_{\ell} \). Observe that \( \mathcal{C}(M) \subset \mathcal{M}_\ell \) because if \( \gamma \) is absolutely continuous then \( \int |\dot{\gamma}(t)| \, dt < +\infty \). Let \( \mathcal{C}(M) \) be the set of such \( \mu_{\gamma} \)'s and let \( \overline{\mathcal{C}(M)} \) be its closure in \( \mathcal{M}_\ell \). Observe that the set \( \overline{\mathcal{C}(M)} \) is convex. We call \( \overline{\mathcal{C}(M)} \) the set of holonomic measures on \( M \).
2-3.2. Theorem (Mañé [27], prop. 1.1, 1.3, 1.2).

1. $\mathcal{M}(L) \subseteq \overline{C(M)} \subseteq \mathcal{M}_\ell$.

2. If $\mu \in \overline{C(M)}$ satisfies
   \[ A_L(\mu) = \min \{ A_L(\nu) \mid \nu \in \overline{C(M)} \} , \]
   then $\mu \in \mathcal{M}(L)$.

3. If $M$ is compact and $a \in \mathbb{R}$, then the set $\{ \mu \in \overline{C(M)} \mid A_L(\mu) \leq a \}$ is compact.

The inclusion $\mathcal{M}(L) \subseteq \overline{C(M)}$ follows from Birkhoff’s ergodic theorem and the fact that $\overline{C(M)}$ is convex.

**Proof of item 2-3.2.3:**

Since $\overline{C(M)}$ is closed, it is enough to prove that the set
\[ \mathcal{A}(a) := \{ \mu \in \mathcal{M}_\ell \mid A_L(\mu) \leq a \} \]
is compact in $\mathcal{M}_\ell$. First we prove that $\mathcal{A}(a)$ is closed. Let $k > 0$ and define $L_k := \min\{L, k\}$. Let
\[ B_k := \{ \mu \in \mathcal{M}_\ell \mid \int L_k \, d\mu \leq a \} . \]
Since $L_k \in C^0_\ell$, then $B_k$ is closed in $\mathcal{M}_\ell$. Since $\mathcal{A}(a) = \cap_{k>0} B_k$, then $\mathcal{A}(a)$ is closed.

In order to prove the compactness, consider a sequence $\{\mu_n\} \subset \mathcal{A}(a)$. Applying the Riesz’ theorem, taking a subsequence we can assume that there exists a probability $\mu$ on the Borel $\sigma$-algebra of $TM$ such that
\[ \int f_i \, d\mu_n \rightarrow \int f_i \, d\mu , \]  \hspace{1cm} (2.6)
for every $f_i$ in the sequence used for the definition of $d(\cdot, \cdot)$. 
Approximating $L_k$ by functions $f_i$ we have that
\[ \int L_k \, d\mu = \lim_n L_k \, d\mu_n \leq \liminf_n L \, d\mu_n \leq a. \]

Letting $k \uparrow +\infty$, we get that
\[ A_L(\mu) \leq a. \]  \hspace{1cm} (2.7)

Let $B > 0$ be such that $|v| < L(x, v) + B$ for all $(x, v) \in TM$. Then
\[ \int |v| \, d\mu \leq A_L(\mu) + B \leq a + B < +\infty. \]  \hspace{1cm} (2.8)

So that $\mu \in \mathcal{M}_\ell$.

We now prove that $\lim_n \int |v| \, d\mu_n \longrightarrow \int |v| \, d\mu$. Let $\varepsilon > 0$. By adding a constant we may assume that $L > 0$. Choose $r > 0$ such that $L(x, v) > a \varepsilon^{-1} |v|$ for all $|v| > r$. Then
\[ \int_{|v|>r} |v| \, d\mu_n \leq \frac{\varepsilon}{a} \int_{|v|>r} L \, d\mu_n \leq \frac{\varepsilon}{a} \int L \, d\mu_n \leq \varepsilon. \]

Similarly, by (2.7),
\[ \int_{|v|>r} |v| \, d\mu \leq \varepsilon. \]

From (2.6) we obtain that there is $N > 0$ such that
\[ \left| \int_{|v|>r} |v| \, d\mu - \int_{|v|>r} |v| \, d\mu_n \right| < \varepsilon, \quad \text{for } n > N. \]

Adding these inequalities we get that
\[ \left| \int |v| \, d\mu_n - \int |v| \, d\mu \right| \leq 3 \varepsilon. \]

The prove item 2-3.2.2 requires some preliminary results which we present now. Item 2-3.2.2 is proved at the end of the section.
2-3.3. Proposition.

Given \( \mu \in \overline{\mathcal{C}(M)} \), there are \( \mu_{n} \in \mathcal{C}(M) \) such that \( \mu_{n} \to \mu \) and

\[
\lim_{n} \int L \, d\mu_{n} = \int L \, d\mu.
\]

2-3.4. Remark. The statement of proposition 2-3.3 is not trivial. It is easy to see that the function \( A_{L} : \overline{\mathcal{C}(M)} \to \mathbb{R} \) is always lower semicontinuous (see the last argument of the proof of 2-3.3), but in general it is not continuous. It is possible to give a sequence \( \mu_{n} \in \mathcal{C}(M) \) such that \( \mu_{n} \to \mu \) in \( \overline{\mathcal{C}(M)} \) but \( \liminf_{n} A_{L}(\mu_{n}) > A_{L}(\mu) \) for a quadratic lagrangian \( L \).

This can be made by calibrating the high speeds in \( \gamma_{n} \) so that \( \int_{|v| > R_{1}} |v| \, d\mu_{n} \to 0 \) but \( a := \liminf_{n} \int_{|v| > R_{1}} L \, d\mu_{n} > 0 \). Then the limit measure \( \mu \) will have support on \( [|v| \leq R] \) and "will not see" the remnant \( a \) of the action.

Proof: Let \( A > 1 \) and let \( \gamma : [0, T] \to M \) be a closed absolutely continuous curve. We reparametrize \( \gamma \) to a curve \( \eta : [0, S] \to M \) such that \( \dot{\eta} = \dot{\gamma} \) when \( |\gamma| < A \) and \( \dot{\eta} = \frac{\dot{\gamma}}{|\dot{\gamma}|} A \) when \( |\gamma| > A \). So that \( |\dot{\eta}| \leq A \). Write \( \eta(s(t)) = \gamma(t) \), \( w(s) = |\dot{\eta}(s)| \) and \( v(t) = |\dot{\gamma}(t)| \). We want

\[
\int_{0}^{s(t)} w(s) \, ds = \int_{0}^{t} v(t) \, dt,
\]

so that

\[
s'(t) = \frac{v(t)}{w(s(t))} = \begin{cases} 
1 & \text{when } v(t) \leq A, \\
\frac{v(t)}{A} & \text{when } v(t) > A.
\end{cases}
\]
Then
\[
S(T) = \int_{[v(t) \leq A]} dt + \int_{[v(t) \geq A]} \frac{v(t)}{A} dt,
\]
\[
S(T) \frac{T}{T} = \mu_\gamma(\{|v| \leq A\}) + \int_{\{|v| \geq A\}} \frac{|v|}{A} d\mu_\gamma,
\]
\[
\left| \frac{S(T)}{T} - 1 \right| \leq \int_{\{|v| > A\}} d\mu_\gamma + \int_{\{|v| \geq A\}} \frac{|v|}{A} d\mu_\gamma
\]
\[
\leq 2 \int_{\{|v| > A\}} |v| d\mu_\gamma. \tag{2.9}
\]

Suppose that \( f : TM \to \mathbb{R} \) is \( \mu_\gamma \)-integrable. Since \( \frac{ds}{dt} = \frac{v(t)}{A} \) when \( v(t) > A \) then
\[
\int_{\{|\dot{\gamma}(t(s))| > A\}} f(\eta(s), \frac{\dot{\gamma}(t(s))}{|\dot{\gamma}(t(s))|} A) \, ds = \int_{\{|\dot{\gamma}(t)| > A\}} f(\gamma(t), \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|} A) \frac{|\dot{\gamma}(t)|}{A} \, dt.
\]

Then
\[
\int f \, d\mu_\gamma = \frac{1}{S(T)} \int f(\eta(s), \dot{\eta}(s)) \, ds
\]
\[
= \frac{1}{S(T)} \left[ \int_{\{|\dot{\gamma}| \leq A\}} f(\gamma(t), \dot{\gamma}(t)) \, dt + \int_{\{|\dot{\gamma}(s(t))| > A\}} f(\eta(s), \frac{\dot{\gamma}(t(s))}{|\dot{\gamma}(t(s))|} A) \, ds \right]
\]
\[
= \frac{T}{S(T)} \left[ \int_{\{|v| \leq A\}} f(v) \, d\mu_\gamma(v) + \int_{\{|v| > A\}} f(\frac{v}{|v|} A) \frac{|v|}{A} d\mu_\gamma \right]
\]

For \( A > 1 \) big enough,
\[
\int_{\{|v| > A\}} |v| \, d\mu_\gamma < \varepsilon < \frac{1}{2}. \tag{2.10}
\]

Define
\[
f_A(v) := \begin{cases} 
    f(v) & \text{if } |v| \leq A, \\
    f(\frac{v}{|v|} A) \frac{|v|}{A} & \text{if } |v| > A.
\end{cases}
\]
Then

\[ \int f \, d\mu_\eta = \frac{T}{S(T)} \int f_A \, d\mu_\gamma. \]  

(2.11)

Observe that from (2.9) and (2.10), we have that

\[ \left| \frac{T}{S(T)} - 1 \right| \leq 4 \varepsilon. \]  

(2.12)

Then

\[ \left| \int f \, d\mu_\eta - \int f_A \, d\mu_\gamma \right| \leq \left| \frac{T}{S(T)} - 1 \right| \int |f_A| \, d\mu_\gamma \leq 4 \varepsilon \int |f_A| \, d\mu_\gamma. \]  

(2.13)

If \( ||f||_\infty \leq 1 \), then

\[ \left| \int f \, d\mu_\eta - \int f \, d\mu_\gamma \right| = \left| \frac{T}{S(T)} \int f_A \, d\mu_\gamma - \int f \, d\mu_\gamma \right| \]
\[ \leq \left| \frac{T}{S(T)} - 1 \right| \int |f_A| \, d\mu_\gamma + \int |f - f_A| \, d\mu_\gamma \]
\[ \leq \left| \frac{T}{S(T)} - 1 \right| + \int_{[|v| > A]} |v| \, d\mu_\gamma + \int_{[|v| > A]} \frac{|v|}{A} \, d\mu_\gamma \]
\[ \leq \left| \frac{T}{S(T)} - 1 \right| + 2 \int_{[|v| > A]} |v| \, d\mu_\gamma \]
\[ \leq 4 \varepsilon + 2 \varepsilon \leq 6 \varepsilon. \]

Also

\[ \left| \int |v| \, d\mu_\eta - \int |v| \, d\mu_\gamma \right| = \left| \frac{T}{S(T)} - 1 \right| \int |v| \, d\mu_\gamma \leq 4 \varepsilon \int |v| \, d\mu_\gamma. \]

Hence

\[ d_{C(M)}(\mu_\eta, \mu_\gamma) \leq 6 \varepsilon \int (|v| + 1) \, d\mu_\gamma. \]  

(2.14)
Now let $\mu \in \overline{C(M)}$. Let

$$K := \int (|v| + 1) \, d\mu + 1. \quad (2.15)$$

For $R > 0$, define

$$L_R(v) := \begin{cases} L(v) & \text{if } |v| \leq R. \\ L \left( \frac{v}{|v|} R \right) \frac{|v|}{R} & \text{if } |v| > R. \end{cases}$$

Given $N > 0$, choose $R = R(N) > 0$ such that

$$L(v) > 0 \text{ if } |v| > R \quad \text{and} \quad \int_{|v| > R} |v| \, d\mu < \frac{1}{N}. \quad (2.16)$$

Choose $\mu_{\gamma N} \in C(M)$ such that

$$d_{C(M)}(\mu_{\gamma N}, \mu) < \frac{1}{N}, \quad (2.17)$$

$$\int L_R(N) \, d\mu_{\gamma N} \leq \int L_R(N) \, d\mu + \frac{1}{N},$$

and

$$\int_{|v| \leq R(N)} |v| \, d\mu_{\gamma N} \geq \int_{|v| \leq R(N)} |v| \, d\mu - \frac{1}{N}. \quad (2.18)$$

Then

$$\int (|v| + 1) \, d\mu_{\gamma N} \leq K \quad \text{from (2.15) and (2.17)}, \quad (2.19)$$

$$\int_{|v| > R(N)} |v| \, d\mu_{\gamma N} < \frac{3}{N} \quad \text{from (2.16), (2.17) and (2.18)}. \quad (2.20)$$

Construct $\eta_N$ as above for $\gamma_N$ and $A = R(N)$. Then from (2.14), (2.19) and (2.20), $d_{C(M)}(\mu_{\eta N}, \mu_{\gamma N}) < \frac{18}{N} K$. From (2.17),

$$d_{C(M)}(\mu_{\eta N}, \mu) < \frac{18}{N} K + \frac{1}{N}.$$
Thus $\mu_{\eta_N} \xrightarrow{N} \mu$ in $\overline{C(M)}$. Moreover, from (2.11), (2.12) and (2.10),
\[
\int L \, d\mu_{\eta_N} = \frac{T_N}{s(T_N)} \int L_{R(N)} \, d\mu_{\eta_N} \leq \frac{T_N}{s(T_N)} \left[ \int L_{R(N)} \, d\mu + \frac{1}{4N} \right] \\
\leq \left( 1 + \frac{12}{N} \right) \left[ \int L \, d\mu + \frac{1}{4N} \right].
\]

Hence
\[
\limsup_N \int L \, d\mu_{\eta_N} \leq \int L \, d\mu.
\]

On the other hand, for any fixed $R > 0$ such that $L \geq 0$ on $|v| > R$,
\[
\liminf_N \int L \, d\mu_{\eta_N} \geq \liminf_N \int L_{R} \, d\mu_{\eta_N} = \int L_{R} \, d\mu.
\]

Letting $R \uparrow +\infty$, we get that
\[
\liminf_N \int L \, d\mu_{\eta_N} \geq \int L \, d\mu.
\]

\[\square\]

Given $x, y \in M$, define
\[
S(x, y; T) := \inf_{\gamma \in C_T(x, y)} A_{L}(\gamma).
\]
Observe that $S(x, y; T) > -\infty$ because $L$ is bounded below. If $\gamma \in C^{ac}([0, T], M)$, define
\[
S^+(\gamma) := A_{L}(\gamma) - S(\gamma(0), \gamma(T); T).
\]

The absolutely continuous curves $\gamma$ with $S^+(\gamma) = 0$ are called Tonelli minimizers. Observe that a Tonelli minimizer is a solution of (E-L).

Given $\gamma_1, \gamma_2 \in C^{ac}([0, T], M)$, the absolutely continuous distance $d_1(\gamma_1, \gamma_2)$ is defined by
\[
d_1(\gamma_1, \gamma_2) := \sup_{t \in [0, T]} d(\gamma_1(t), \gamma_2(t)) + \int_0^T d_{TM}([\gamma_1(t), \dot{\gamma}_1(t)], [\gamma_2(t), \dot{\gamma}_2(t)]) \, dt.
\]
2-3.5. Proposition. Given a compact subset $K \subset M$ and given $C, \varepsilon > 0$ there exist $\delta > 0$ such that if $\gamma : [0, T] \to M$ is absolutely continuous and satisfies

i. $1 \leq T \leq C$.

ii. $A_L(\gamma) \leq C$.

iii. $S^+(\gamma) \leq \delta$.

Then either $\gamma([0, T]) \cap K = \emptyset$ or there exists a Tonelli minimizer $\gamma_0 : [0, T] \to M$ such that $d_1(\gamma_0, \gamma) \leq \varepsilon$.

Proof: If such $\delta$ does not exists then there is a sequence $\gamma_n \in C^{ac}([0, T_n], M)$ such that $\gamma_n([0, T_n]) \cap K \neq \emptyset$, $1 \leq T_n \leq C$, $S^+(\gamma_n) \to 0$, $A_L(\gamma_n) \leq C$ and $d_1(\gamma_n, \eta) \geq \varepsilon$ for any Tonelli minimizer $\eta$.

Adding a constant we can assume that $L > 0$. Let $B > 0$ be such that $L(x, v) > |v| - B$ for all $(x, v) \in TM$. Choose $s_0 \in [0, T_n]$ such that $\gamma_n(s_0) \in K$. Then

\[
d(K, \gamma_n(t)) \leq d(\gamma_n(s_0), \gamma_n(t)) \leq \int_{[s_0, t]} |\dot{\gamma}_n|
\leq \int_{[s_0, t]} \left[ L(\gamma_n, \dot{\gamma}_n) + B \right] \leq C + BC.
\]

Let $Q := \{ y \in M \mid d(y, K) \leq C + BC \}$. Then we have that $\gamma_n([0, T_n]) \subseteq Q$.

We can assume that $T_n \to T$, $\gamma_n(0) \to x \in Q$ and $\gamma_n(T_n) \to y \in Q$. Moreover, we can assume that $T_n \equiv T$, $\gamma_n(0) \equiv x$ and $\gamma_n(T) \equiv y$. By theorem 3-1.2, the set $A[b] = \{ \gamma \in C_T(x, y) \mid A_L(\gamma) \leq b \}$ is compact in the $d_1$-topology. Then we can assume that there is $\gamma_0 \in C_T(x, y)$ such that $\gamma_n \to \gamma_0$ in the $d_1$-topology. Let $a := S(x, y; T)$. Then $\gamma_0 \in \cap_n A[a + S^+(\gamma_n)] = A[a]$, because $S^+(\gamma_n) \to 0$. Thus $\gamma_0$ is a Tonelli minimizer.

\[\square\]
Let
\[ \mathcal{H} := \{ h : TM \to \mathbb{R} \mid \|f\|_{\infty} \leq 1, [h]_{Lip} \leq 1, \text{h with compact support} \}, \]
where
\[ [h]_{Lip} = \sup_{(x,v) \neq (y,w)} \frac{|h(x,v) - h(y,w)|}{d_{TM}((x,v),(y,w))} \]
is the smallest Lipschitz constant for h.

Given \( h \in \mathcal{H} \) and \( C > 0 \) there exist \( \delta = \delta(C,h) > 0 \) such that if \( \gamma : [0,T] \to M \) satisfies conditions 2-3.5.i, 2-3.5.ii, 2-3.5.iii then
\[ \left| \oint_{\gamma} h - \oint_{\gamma_0} h \circ \varphi_1 \right| \leq 5. \tag{2.21} \]

Proof: Let \( K = \pi(\text{supp}(h) \cup \varphi_{-1}(\text{supp}(h))) \). Given \( C > 0 \) and \( \varepsilon > 0 \) let \( \delta = \delta(C,\varepsilon) > 0 \) and \( A > 0 \) be given by proposition 2-3.5 then if \( \gamma : [0,T] \to M \) satisfies conditions 2-3.5.i, 2-3.5.ii, 2-3.5.iii we have that either \( \gamma([0,T]) \cap K = \emptyset \), or we can take \( \gamma_0 \) minimizing such that \( d_1(\gamma_0,\gamma) \leq \varepsilon \).

Observe that if \( \gamma([0,T]) \cap K = \emptyset \), then \( h(\gamma,\dot{\gamma}) \equiv 0 \) and \( h \circ \varphi_1(\gamma,\dot{\gamma}) \equiv 0 \). This implies (2.21). Suppose then that \( d_1(\gamma_0,\gamma) \leq \varepsilon \).

We have that
\[ \left| \oint_{\gamma} h - \oint_{\gamma_0} h \right| \leq [h]_{Lip} d_1(\gamma,\gamma_0) \leq 1 \cdot 1 \cdot \varepsilon, \]
where \([h]_{Lip}\) is the smallest Lipschitz constant of \( h \). Let \( Q(h) := \varphi_{-1}(\text{supp}(h)) \), then
\[ \left| \oint_{\gamma} h \circ \varphi_1 - \oint_{\gamma_0} h \circ \varphi_1 \right| \leq [h]_{Lip} [\varphi_1|_{Q(h)}]_{Lip} d_1(\gamma,\gamma_0) \leq 1 \cdot [\varphi_1|_{Q(h)}]_{Lip} \cdot \varepsilon. \]
Since \( \gamma_0 \) is a solution of (E-L), we have that
\[
\left| \oint_{\gamma_0} h - \oint_{\gamma_0} h \circ \varphi_1 \right| = \left| \int_0^T h(\gamma_0(t), \dot{\gamma}_0(t)) - h(\gamma_0(t+1), \dot{\gamma}_0(t+1)) \, dt \right|
\leq \int_0^1 |h(\gamma_0, \dot{\gamma}_0)| \, dt + \int_T^{T+1} |h(\gamma_0, \dot{\gamma}_0)| \, dt \leq 2.
\]
Hence
\[
\left| \oint h - \oint h \circ \varphi_1 \right| \leq \varepsilon (1 + [\varphi_1|\gamma(h)]_{\text{Lip}}) + 2.
\]
\( \square \)

**Proof of item 2-3.2.2:**

Observe that to prove that \( \mu \) is invariant it is enough to prove that
\[
\int h \, d\mu = \int h \, d(\varphi_1^* \mu) \quad \text{for all } h \in \mathcal{H}.
\]  \hspace{1cm} (2.22)

By proposition 2-3.3, there exists a sequence \( \mu_{\gamma_n} \in C(M) \) such that \( \mu_{\gamma_n} \to \mu \) and
\[
\lim \limits_n A_L(\mu_{\gamma_n}) = A_L(\mu) = \min \{ \, A_L(\nu) \mid \nu \in \overline{C(M)} \, \} =: k.
\]  \hspace{1cm} (2.23)

Let \( T_n \) be a period of the curve \( \gamma_n : \mathbb{R} \to M \). Take an integer \( N > 0 \).

By joining a constant curve if necessary, we can assume that every \( T_n \) is a multiple of \( N \) and that \( \lim_{n \to \infty} T_n = +\infty \).

Given \( C > 0 \) let
\[
B_n(C) := \left\{ j \in \mathbb{N} \mid 1 \leq j \leq \frac{T_n}{N}, \ A_L(\gamma_{n,j}) \geq C \right\},
\]
where
\[
\gamma_{n,j} := \gamma_n|_{[jN,(j+1)N]}.
\]

By the superlinearity \( L \) is bounded below, adding a constant we can assume that \( L > 0 \). Then we can assume that
\[
A_L(\mu_{\gamma_n}) = \frac{1}{T_n} \int_0^{T_n} L(\gamma_n, \dot{\gamma}_n) \, dt \leq 2k \quad \forall n.
\]
Hence

\[ 2kT_n \geq \sum_{j \in B_n(C)} A_L(\gamma_{n,j}) \geq C \#B_n(C). \]

Thus

\[ \frac{\#B_n(C)}{T_n} \leq \frac{2k}{C}. \tag{2.24} \]

Given \( \delta > 0 \), let

\[ B'_n(\delta) := \{ j \in \mathbb{N} \mid 1 \leq j \leq \frac{T_n}{N} - 1, \quad S^+(\gamma_{n,j}) > \delta \}. \]

Then

\[ S^+(\gamma_n) \geq \sum_{j=1}^{(T_n/N)-1} S^+(\gamma_{n,j}) \geq \delta \#B'_n(\delta). \]

Moreover,

\[ k \leq \frac{1}{T_n} S(\gamma_n(0), \gamma(T_n); T_n) = A_L(\mu_{\gamma_n}) - \frac{1}{T_n} S^+(\gamma_n). \]

Hence

\[ S^+(\gamma_n) \leq T_n (A_L(\mu_{\gamma_n}) - k). \]

Therefore

\[ \frac{\#B'_n(\delta)}{T_n} \leq \frac{1}{\delta} (A_L(\mu_{\gamma_n}) - k). \tag{2.25} \]

Now fix \( h \in \mathcal{H} \). Then

\[ \left| \int h \, d\mu_{\gamma_n} - \int h \, d(\varphi_1^* \mu_{\gamma_n}) \right| \leq \frac{1}{T_n} \sum_{j=0}^{(T_n/N)-1} \left| \oint_{\gamma_{n,j}} h - \oint_{\gamma_{n,j}} h \circ \varphi_1 \right|. \]

Denote \( B''_n := B_n(C) \cup B'_n(\delta) \). Since \( \sup |h| \leq 1 \), then

\[ \left| \int h \, d\mu_{\gamma_n} - \int h \, d(\varphi_1^* \mu_{\gamma_n}) \right| \leq \frac{1}{T_n} \sum_{j \notin B''_n} \left| \oint_{\gamma_{n,j}} h - \oint_{\gamma_{n,j}} h \circ \varphi_1 \right| + \frac{1}{T_n} 2N \#B''_n. \]
Now choose $C \geq N^2$ and $\delta = \delta(C, h) > 0$ from corollary 2-3.6. Using equations (2.24), (2.25) and corollary 2-3.6 we obtain that

$$\left| \int h \, d\mu_{\gamma_n} - \int h \, d(\varphi_1^* \mu_{\gamma_n}) \right| \leq \frac{5}{T_n} \left( \frac{T_n}{N} - \#B''_n \right) + \frac{1}{T_n} 2N \#B''_n$$

$$\leq \frac{5}{N} + 2N \left( \frac{2k}{C} + \frac{1}{\delta} \left( A_L(\mu_{\gamma_n} - k) \right) \right).$$

Now let $n \to \infty$. Using equation (2.23) and that $C \geq N^2$, $\mu_{\gamma_n} \to \mu$ and $h, h \circ \varphi_1 \in C^0_\ell$ (because they have compact support), we obtain that

$$\left| \int h \, d\mu - \int h \, d(\varphi_1^* \mu) \right| \leq \frac{5}{N} + \frac{4k}{N}.$$

Since $N$ is arbitrary, this difference is zero and we get (2.22). \qed
2-4 Ergodic characterization of the critical value.

Given a Borel probability measure \( \mu \) in \( TM \) define its action by

\[
A_L(\mu) = \int_{TM} L \, d\mu.
\]

Since by the superlinearity the lagrangian \( L \) is bounded below, this action is well defined.

Let \( \mathcal{M}(L) \) be the set of \( \varphi_t \)-invariant probabilities on \( TM \).

2-4.1. Theorem (Mañé [28]). If \( M \) is compact, then

\[
c(L) = -\min \{ A_L(\mu) \mid \mu \in \mathcal{M}(L) \}.
\]

We will obtain theorem 2-4.1 from theorem 2-4.2 below, which also applies to the non-compact case.

Recall that if \( \gamma : [0, T] \to M \) is a closed absolutely continuous curve, the measure \( \mu_\gamma \in \mathcal{M}_\ell \) is defined by

\[
\int f \, d\mu_\gamma = \frac{1}{T} \int_0^T f(\gamma(t), \dot{\gamma}(t)) \, dt
\]

for all \( f \in C^0_\ell \), and that \( \overline{C(M)} \) is the closure of the set of such \( \mu_\gamma \)'s in \( \mathcal{M}_\ell \).

2-4.2. Theorem.

\[
c(L) = -\inf \{ A_L(\mu) \mid \mu \in \overline{C(M)} \}
= -\inf \{ A_L(\mu) \mid \mu \in C(M) \}, \quad (2.26)
\]
2. MANÉ'S CRITICAL VALUE.

2-4.3. Remarks.

1. The equality between the two infimums is non-trivial and follows from proposition 2-3.3.

2. Theorems 2-4.2 and 2-3.2 imply theorem 2-4.1.

3. If \( p : N \to M \) is a covering, \( M \) is compact and \( L = L \circ dp \) is the lifted lagrangian, then theorems 2-4.2 and 2-3.2 imply that

\[
c(L) = -\min\{ A_L(\mu) \mid \mu \in \mathcal{M}(L) \cap \overline{dp_*C(N)} \},
\]

by noticing that \( A_L(dp_*\nu) = A_L(\nu) \) for \( \nu \in \overline{C(N)} \). Here \( dp_*C(N) \) is the set of probabilities \( \mu_\gamma \) on \( TM \) where \( \gamma \) is a curve on \( M \) whose lifts to \( N \) are closed. The compactness property on theorem 2-3.2(3) allows to obtain a minimum on (2.27) instead of the infimum on (2.26) which may not be attained in the non-compact case.

4. The statement for coverings in equation (2.27) allows to obtain minimizing measures which don’t appear in the Mather’s theory. For example if \( c_u \) is the critical value of the universal cover \( \hat{M} \) of \( M \) and \( c_0 \) is the critical value of the abelian cover \( \hat{M} \) of \( M \); the minimizing measures on a fixed homology class (corresponding to Mather’s theory) all have action \( A(\mu) \geq -c_0 \), (see equation (2.30) and proposition 2-6.3), while the minimizing measures for \( \hat{M} \) have action \( c_u < c_0 \).

The measures for \( \hat{M} \) correspond to “minimizing in the zero homotopy class” while the measures for \( \hat{M} \) are minimizing in the zero homology class.

The drawback of this approach is that we obtain honest minimizing invariant measures on \( TM \) which may not lift to finite measures on the covering \( TN \).
Proof of theorem 2-4.2:

If \( \mu_\gamma \in C(M) \), then \( A_{L+c(L)}(\mu_\gamma) \geq 0 \). Hence \( A_{L+c(L)}(\mu) \geq 0 \). Thus

\[-c(L) \leq \inf \{ A_L(\mu) \mid \mu \in C(M) \} = \inf \{ A_L(\mu) \mid \mu \in \overline{C(M)} \},\]

where the last equality follows from proposition 2-3.3.

If \( k < c(L) \) then there is a closed absolutely continuous curve \( \gamma \) on \( M \) such that \( A_{L+k}(\gamma) < 0 \). Thus \( \mu_\gamma \in C(M) \) and

\[-k > A_L(\mu_\gamma) \geq \inf \{ A_L(\mu) \mid \mu \in \overline{C(M)} \} .\]

Now let \( k \uparrow c(L) \). \( \square \)
2-5 The Aubry-Mather Theory.

Through this section we shall assume that $M$ is compact.

2-5.a Homology of measures.

A holonomic probability $\mu \in \overline{C(M)}$ satisfies $\int_T |v| \, d\mu < +\infty$ and

$$\int_T df \, d\mu = 0 \quad \text{for all } f \in C^\infty(M, \mathbb{R}).$$

Then we can define its homology class as $\rho(\mu) \in H_1(M, \mathbb{R}) \approx H^1(M, \mathbb{R})^*$ by

$$\left\langle \rho(\mu), [\omega] \right\rangle = \int_T \omega \, d\mu,$$

(2.28)

for any closed 1-form $\omega$ on $M$, where $[\omega] \in H^1(M, \mathbb{R})$ is the cohomology class of $\omega$. Here we have used the identification\footnote{In fact, $H^1(M, \mathbb{R}) \approx \text{hom}(H_1(M, \mathbb{R}), \mathbb{R}) = H_1(M, \mathbb{R})^*$ by the universal coefficient theorem. Since $M$ is compact, then $H_1(M, \mathbb{R})$ is a finite dimensional vector space and hence it is naturally isomorphic to its double dual $H^1(M, \mathbb{R})^*$.} $H_1(M, \mathbb{R}) \approx H^1(M, \mathbb{R})^*$ and equation (2.28) shows how the homology class $\rho(\mu)$ acts on $H^1(M, \mathbb{R})$. Since $\mu$ is holonomic, the integral in (2.28) depends only on the cohomology class of $\omega$. The class $\rho(\mu)$ is called the homology of $\mu$ or the rotation of $\mu$ by analogy to the twist map theory.

Using a finite basis $\{ [\omega_1], \ldots, [\omega_k] \}$ for $H^1(M, \mathbb{R})$ and the topology of $\overline{C(M)}$, we have that

2-5.1. Lemma. The map $\rho : \overline{C(M)} \to H^1(M, \mathbb{R})$ is continuous.

2-5.b The asymptotic cycle.

Given a differentiable flow $\varphi_t$ on a compact manifold $N$ and a $\varphi_t$-invariant probability $\mu$, the Schwartzman's [56] asymptotic cycle of an
invariant probability $\mu$ is defined to be the homology class $A(\mu) \in H_1(N, \mathbb{R}) \approx H^1(N, \mathbb{R})^*$ such that

$$\langle A(\mu), [\omega] \rangle = \int_N \omega(X) \ d\mu,$$

for any closed 1-form $\omega$, where $[\omega] \in H^1(N, \mathbb{R})$ is the cohomology class of $\omega$ and $X$ is the vectorfield of $\varphi_t$. This integral depends only on the cohomology class of $\omega$ because the integral of a coboundary by an invariant measure is zero: in fact, if $df$ is an exact 1-form, then define

$$F(y) := \lim_{T \to +\infty} \frac{1}{T} \int_0^T df(X(\varphi_t y)) \ dt = \lim_{T \to +\infty} \frac{1}{T} \left[ f(\varphi_T y) - f(y) \right] \ = \ 0,$$

by Birkhoff's theorem,

$$\int_N df(X) \ d\mu = \int_N F \ d\mu = 0.$$

If $\mu$ is ergodic and $x \in N$ is a generic point \footnote{i.e. $\lim_{T \to +\infty} \frac{1}{T} \int_0^T f(\varphi_t x) \ dt = \int f \ d\mu$ for all $f \in C^0(N, \mathbb{R})$.} for $\mu$, then

$$\langle A(\mu), [\omega] \rangle = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \omega(X(\varphi_t x)) \ dt.$$

Applying this to a basis $\{\omega_1, \ldots, \omega_k\}$ for $H^1(N, \mathbb{R})$, we get that

$$A(\mu) = \lim_{T \to +\infty} \frac{1}{T} \left[ \gamma_T \ast \delta_T \right] \in H_1(N, \mathbb{R}),$$

where $\gamma_T(t) = \varphi_t(x)$, $t \in [0,T]$, the curve $\delta_T$ is a unit speed geodesic from $\varphi_T(x)$ to $x$, and the limit is on the finite dimensional vector space $H_1(N, \mathbb{R})$.

In the case of a lagrangian flow, the phase space $N = TM$ is not compact, but it has the same homotopy type as the configuration space $M$ because $M$ is a deformation retract of $TM$ (contracting $TM$ along the
fibers to the zero section $M \times 0$). Moreover, the ergodic components of an invariant measure of a lagrangian flow are contained in a unique energy level, which is a compact submanifold of $TM$ by remark 1-3.1.

We see that the homology of an invariant probability and its asymptotic cycle coincide under the identification $H_1(TM, \mathbb{R}) \cong H_1(M, \mathbb{R})$.

2-5.2. Proposition.

$$\pi_*(A(\mu)) = \rho(\mu) \quad \text{for all } \mu \in \mathcal{M}(L),$$

where $\pi_* : H_1(TM, \mathbb{R}) \to H_1(M, \mathbb{R})$ is the map induced by the projection $TM \to M$.

**Proof:** If $\omega$ is a closed 1-form on $M$, then

$$(\pi^* \omega)(X(x,v)) = \omega[d\pi(X(x,v))] = \omega_x(v),$$

because the lagrangian vectorfield $X$ has the form $X(x,v) = (v, \ast)$. Then

$$\left\langle \pi_*(A(\mu)), [\omega] \right\rangle = \left\langle A(\mu), \pi^* [\omega] \right\rangle = \int_{TM} (\pi^* \omega)(X) \ d\mu$$

$$= \int_{TM} \omega \ d\mu = \left\langle \rho(\mu), [\omega] \right\rangle.$$

\[\square\]

2-5.3. Lemma. The map $\rho : \mathcal{M}(L) \to H_1(M, \mathbb{R})$ is surjective.

**Proof:** Let $h \in H_1(M, \mathbb{Z})$ be an integer homology class. Let $\eta : [0, 1] \to M$ be a closed curve with homology class $h$. Let $\gamma$ be a minimizer of the action of $L$ among the set of absolutely continuous curves $[0, 1] \to M$ with the same homotopy class as $\eta$. Then by remark 1-2.2, $\gamma$ is a periodic orbit for the lagrangian flow with period 1. The invariant measure $\mu_\gamma$ satisfies $\rho(\mu_\gamma) = h$.

The map $\rho$ is affine and $\mathcal{M}(L)$ is convex; hence $\rho(\mathcal{M}(L))$ is convex and, in particular, it contains the convex hull of $H_1(M, \mathbb{Z})$. Thus, $H_1(M, \mathbb{R}) \subseteq \rho(\mathcal{M}(L))$. \[\square\]
2-5.c The alpha and beta functions.

The action functional $A_L : \mathcal{M}(L) \to \mathbb{R}$ is lower semicontinuous\(^3\) and the sets

$$\mathcal{M}(h) := \{ \mu \in \mathcal{M}(L) \mid \rho(\mu) = h \}$$

are closed. Hence we can define the Mather's beta function $\beta : H_1(M, \mathbb{R}) \to \mathbb{R}$, as

$$\beta(h) := \min_{\mu \in \mathcal{M}(h)} A_L(\mu).$$

We shall prove below that the $\beta$-function is convex. The Mather's alpha function $\alpha = \beta^* : H^1(M, \mathbb{R}) \to \mathbb{R}$ is the convex dual of the $\beta$-function:

$$\alpha([\omega]) = \max_{h \in H_1(M, \mathbb{R})} \left\{ \langle [\omega], h \rangle - \beta(h) \right\} \quad \text{by 2-5.3,}$$

$$= - \min_{\mu \in \mathcal{M}(L)} \left\{ A_L(\mu) - \langle [\omega], \rho(\mu) \rangle \right\}$$

$$= - \min_{\mu \in \mathcal{M}(L)} A_{L-\omega}(\mu)$$

$$= c(L - \omega), \quad \text{by 2-4.1.} \quad (2.29)$$

Observe that since $L - \omega$ is also a convex superlinear lagrangian, then $\alpha([\omega])$ is finite.

2-5.4. Theorem. The $\alpha$ and $\beta$ functions are convex and superlinear.

**Proof:** We first prove that $\beta$ is convex. Let $h_1, h_2 \in H_1(M, \mathbb{R})$ and $0 \leq \lambda \leq 1$. Let $\mu_1, \mu_2 \in \mathcal{M}(L)$ be such that $\rho(\mu_i) = h_i$ and $A_L(\mu_i) = \beta(h_i)$ for $i = 1, 2$. The probability $\nu = \lambda \mu_1 + (1 - \lambda) \mu_2$ satisfies $\rho(\nu) = \lambda h_1 + (1 - \lambda) h_2$. Hence

$$\beta(\lambda h_1 + (1 - \lambda) h_2) \leq A_L(\lambda \mu_1 + (1 - \lambda) \mu_2) = \lambda \beta(h_1) + (1 - \lambda) \beta(h_2).$$

\(^3\) $A_L$ is lower semicontinuous iff $\lim \inf_{n} A_L(\nu_n) \geq A_L(\mu)$ when $\nu_n \to \mu$. 

By proposition 2-5.3, \( \rho \) is surjective, and hence \( \beta \) is finite. By proposition C.1, \( \alpha \) is superlinear. By C.1, \( \alpha \) and \( \beta \) are convex. Formula (2.29), implies that \( \alpha \) is finite and then by C.1 \( \beta \) is superlinear.

For \( h \in H_1(M, \mathbb{R}) \) and \( \omega \in H^1(M, \mathbb{R}) \), write

\[
\mathcal{M}_h(L) := \{ \mu \in \mathcal{M}(L) \mid \rho(\mu) = h, A_L(\mu) = \beta(h) \},
\]

\[
\mathcal{M}^\omega(L) := \{ \mu \in \mathcal{M}(L) \mid A_{L-\omega}(\mu) = -c(L - \omega) \}.
\]

Since the \( \beta \)-function has a supporting hyperplane at each homology class \( h \), if \( \omega \in \partial \beta(h) \), then \( \mathcal{M}_h(L) \subseteq \mathcal{M}^\omega(L) \). Conversely, since by corollary C.2 \( \alpha^* = \beta \), then \( \mathcal{M}^\omega(L) \subseteq \mathcal{M}_h(L) \) if \( h \in \partial \alpha(\omega) \). Thus

\[
\bigcup_{h \in H_1(M, \mathbb{R})} \mathcal{M}_h(L) = \bigcup_{\omega \in H^1(M, \mathbb{R})} \mathcal{M}^\omega(L).
\]

We call these measures \textit{Mather minimizing} measures and the set

\[
\mathcal{M} := \bigcup_{\omega \in H^1(M, \mathbb{R})} \bigcup_{\mu \in \mathcal{M}^\omega(L)} \text{supp}(\mu) = \bigcup_{h \in H_1(M, \mathbb{R})} \bigcup_{\mu \in \mathcal{M}_h(L)} \text{supp}(\mu)
\]

the \textit{Mather set}.

Define the \textit{strict critical value} as

\[
c_0(L) := \min_{\omega \in H^1(M, \mathbb{R})} c(L - \omega) = \min_{\omega \in H^1(M, \mathbb{R})} \alpha(\omega) = -\beta(0).
\]

By corollary 3-6.3 the strict critical value is the lowest energy level which supports Mather minimizing measures and since \( c_0(L) = -\beta(0) \), these minimal energy Mather minimizing measures have trivial homology.
2-6 Coverings.

We shall deal mainly with compact manifolds $M$, but there are some important non-compact cases, for example the coverings of $M$. Particularly interesting are the abelian cover $\hat{M}$, the universal cover $\widetilde{M}$ and the finite coverings.

The abelian cover $\hat{M}$ of $M$ is the covering whose fundamental group is the kernel of the Hurewicz homomorphism $\pi_1(M) \to H_1(M, \mathbb{Z})$. Its deck transformation group is $H_1(M, \mathbb{Z})$ and $H^1(\hat{M}, \mathbb{Z}) = \{0\}$. When $\pi_1(M)$ is abelian, $\hat{M} = \widetilde{M}$. A closed curve in $\hat{M}$ projects to a closed curve in $M$ with trivial homology.

If $M_1 \rightarrow M$ is a covering, denote by $L_1 := L \circ dp : TM_1 \rightarrow \mathbb{R}$ the lifted lagrangian to $TM_1$.

2-6.1. Lemma. If $M_1 \rightarrow M$ is a covering, then $c(L_1) \leq c(L)$.

Proof: The lemma follows from the fact that closed curves on $N$ project to closed curves on $M$. \hfill \Box

2-6.2. Proposition. If $M_1$ is a finite covering of $M_2$ then $c(L_1) = c(L_2)$.

Proof: We know that $c(L_1) \leq c(L_2)$. Suppose that the strict inequality holds and let $k$ be such that $c(L_1) < k < c(L_2)$. Hence there exists a closed curve $\gamma$ in $M_2$ with negative $(L_2 + k)$-action. Since $M_1$ is a finite covering of $M_2$ some iterate of $\gamma$ lifts to a closed curve in $M_1$ with negative $(L_1 + k)$-action which contradicts $c(L_1) < k$. \hfill \Box

2-6.3. Proposition. [52]

\[ c_0(L) = c_a(L) = \text{critical value of the abelian cover.} \]

Then we have

\[ c_u(L) \leq c_a(L) = c_0(L) \leq c(L - \omega) \quad \forall [\omega] \in H^1(M, \mathbb{R}), \]
where $c_u$ is the critical value of the lift of the lagrangian to the universal cover. When $c_u(L) < c_0(L)$, the method in equation (2.27) gives some minimizing measures which are not Mather minimizing. For symmetric lagrangians $c(L) = e_0 = c_0(L) = c_u(L)$. Mañé [32] gives an example in which $e_0 < c_a(L) = c_0(L) < c(L)$. G. Paternain and M. Paternain [52] give an example in which $c_u(L) < c_a(L)$.

**Proof:** Let $\omega$ be a closed form in $M$. Since $H_1(M, \mathbb{R}) = \{0\}$, the lift $\hat{\omega}$ of $\omega$ to $\widehat{M}$ is exact, then

$$c_a(L) := c(\hat{L}) = c(\hat{L} - \hat{\omega}) \leq c(L - \omega).$$

Hence

$$c_a(L) \leq \min_{\omega \in H^1(M, \mathbb{R})} c(L - \omega) = c_0(L).$$

Moreover,

$$-c_a(L) = \inf \left\{ A_{\hat{E}}(\mu) \mid \mu \in \overline{C(M)} \right\} = \inf \left\{ A_L(\mu) \mid \mu \in \overline{C(M)}, \rho(\mu) = 0 \right\},$$

because if $\mu \in \overline{C(M)}$, by proposition 2-3.3 there are $\mu \gamma_n \in C(M)$ with $A_{\hat{E}}(\mu \gamma_n) \to A_{\hat{E}}(\mu)$. Since $\gamma_n$ is closed, its projection $p \circ \gamma_n$ has homology $[p \circ \gamma_n] = 0$. Then $p^*(\mu \gamma_n) = \mu \circ p \gamma_n \to p^*\mu$, $A_L(p^*\mu \gamma_n) \to A_L(p^*\mu) = A_{\hat{E}}(\mu)$ and by proposition 2-5.1 $\rho(p^*\mu) = \lim_n \rho(p^*\mu \gamma_n) = 0$.

Since $\mathcal{M}(L) \subseteq \overline{\mathcal{M}}$, we get that

$$-c_a(L) \leq \min \left\{ A_L(\mu) \mid \mu \in \mathcal{M}(L), \rho(\mu) = 0 \right\} = -\beta(0) = c_0(L).$$

The real abelian cover is the covering $\hat{M}$ of $M$ with $h : \pi(M) \to H_1(M, \mathbb{R})$ is the Hurewicz homomorphism. It is an intermediate covering $\widehat{M} \to \hat{M} \to M$ and the deck transformations of $\widehat{M} \to \hat{M}$ are given by the torsion$^4$ of $H_1(M, \mathbb{Z})$. Hence $\hat{M} \to \hat{M}$ is a finite cover so that they have the same critical value $c_a(L) = c_0(L)$.

$^4$i.e. the elements of finite order $\mathbb{Z}_n, \oplus \cdots \oplus \mathbb{Z}_n \subseteq H_1(M, \mathbb{Z})$. 
Chapter 3

Globally minimizing orbits.

3-1 Tonelli’s theorem.

Given $x, y \in M$ and $T > 0$, let

$$\mathcal{C}_T(x, y) := \{ \gamma \in C^{ac}([0, T], M) \mid \gamma(0) = x, \gamma(T) = y \}.$$ 

We say that $\gamma \in \mathcal{C}_T(x, y)$ is a Tonelli minimizer if

$$A_L(\gamma) = \min_{\eta \in \mathcal{C}_T(x, y)} A_L(\eta).$$

In this section we shall prove

3-1.1. Tonelli’s Theorem.

For all $x, y \in M$ and $T > 0$ there exists a Tonelli minimizer on $\mathcal{C}_T(x, y)$.

The only difference in the proof of this theorem when $M$ is noncompact is corollary 3-1.7. An independent proof of this corollary is given in remark 3-1.8.

The idea of Tonelli’s theorem is to prove that the sets

$$A(c) := \{ \gamma \in \mathcal{C}_T(x, y) \mid A_L(\gamma) \leq c \}$$

(3.1)
are compact in the $C^0$-topology. Then a Tonelli minimizer will be a curve in

$$\bigcap_{c \geq \alpha} A(c) \neq \emptyset,$$

where $\alpha = \inf_{\eta \in C_T(x, y)} A_L(\eta) \geq \inf L > -\infty$.

An addendum to Tonelli's theorem due to Mather [39] states that these sets are compact in the topology of absolutely continuous curves. Given $\gamma_1, \gamma_2 \in C^{ac}([0, T], M)$ define their absolutely continuous distance by

$$d_1(\gamma_1, \gamma_2) := \sup_{t \in [0, T]} d_M(\gamma_1(t), \gamma_2(t)) + \int_0^T d_{TM}([\gamma_1(t), \dot{\gamma}_1(t)], [\gamma_2(t), \dot{\gamma}_2(t)]) \, dt.$$

3.1.2. Theorem (Mather [39]). For any $x, y \in M, T > 0, b \in \mathbb{R}$, the set

$$A(b) := \{ \gamma \in C_T(x, y) \mid A_L(\gamma) \leq b \}$$

is compact in the $d_1$-topology.

This theorem follows from the fact that $A(b)$ is compact in the $C^0$ topology, which is proved in 3.1.11 and the following proposition (addendum on page 175 of Mather [39]).

3.1.3. Proposition. (Mather [39])

If $N \subseteq M$ is a compact subset and $\gamma_1, \gamma_2, \ldots$ is a sequence in $C^{ac}([a, b], N)$ which converges $C^0$ to $\gamma$ and $A_L(\gamma_i)$ converges to $A_L(\gamma)$, then $\gamma_1, \gamma_2, \ldots$ converges in the $d_1$-topology to $\gamma$.

We shall split the proof of Tonelli's theorem in several parts:

3.1.4. Definition.

A family $\mathcal{F} \subseteq C^0([a, b], M)$ is absolutely equicontinuous if $\forall \varepsilon > 0 \exists \delta > 0$ such that

$$\sum_{i=1}^N |t_i - s_i| < \delta \implies \sum_{i=1}^N d(x_{s_i}, x_{t_i}) < \varepsilon,$$

whenever $[s_1, t_1, \ldots, s_N, t_N]$ are disjoint intervals in $[a, b]$. 
3-1.5. Remark.

(i) An absolutely equicontinuous family is equicontinuous.

(ii) A uniform limit limit of absolutely equicontinuous functions is absolutely continuous.

3-1.6. Lemma. For all $c \in \mathbb{R}$ and $T > 0$, the family

$$\mathcal{F}(c) := \{ \gamma \in \mathcal{C}^{ac}([0, T], M) \mid A_L(\gamma) \leq c \}$$

is absolutely equicontinuous.

Proof: Since by the superlinearity, the lagrangian $L$ is bounded below; by adding a constant we may assume that $L \geq 0$. For $a > 0$ let

$$K(a) := \inf \left\{ \frac{L(x, v)}{|v|} \mid (x, v) \in TM, |v| \geq a \right\}. \quad (3.2)$$

The superlinearity implies that $\lim_{a \to +\infty} K(a) = +\infty$. Given $\varepsilon > 0$ let $a > 0$ be such that

$$\frac{c}{K(a)} < \frac{\varepsilon}{2}.$$

Let $0 \leq s_1 < t_1 \leq \cdots \leq s_N < t_N \leq T$, $J := \bigcup_{i=1}^N [s_i, t_i]$ and $E := J \cap [|\dot{x}| > a]$, then $L(x_s, \dot{x}_s) \geq K(a) |\dot{x}_s|$ for $s \in E$. We have that

$$K(a) \sum_{i=1}^N d(x_s, x_t) \leq K(a) \int_E |\dot{x}| + K(a) \int_{J \setminus E} |\dot{x}|$$

$$\leq \int_E L(x, \dot{x}) + a \cdot K(a) m(J)$$

$$\leq c + a \cdot K(a) m(J), \quad (\text{because } L \geq 0),$$

where $m$ is the Lebesgue measure on $[0, T]$

$$d(x_s, x_t) \leq \int_J |\dot{x}_s| \leq \frac{c}{K(a)} + a m(J) \leq \frac{\varepsilon}{2} + a m(J). \quad (3.3)$$

This implies the equicontinuity of $\mathcal{F}(c)$. 

In order to apply the Arzela-Ascoli theorem we need a compact range, for this we have:

3-1.7. Corollary. For all \( c \in \mathbb{R} \) and \( T > 0 \) there is \( R > 0 \) such that for all \( x, y \in M \),

\[
A(c) \subseteq C^{ac}([0, T], \overline{B}(x, R)),
\]

where \( \overline{B}(x, R) := \{ z \in M \mid d_M(x, z) \leq R \} \).

Proof: Inequality (3.3) for \( N = 1 \) and \( J = [s, t] \) is \( d(x_s, x_t) \leq \frac{\xi}{2} + a|t-s| \). It is enough to take \( R = \frac{\xi}{2} + aT \). \( \square \)

3-1.8. Remark.

Corollary 3-1.7 is the only difference for the proof of Tonelli’s theorem when \( M \) is non-compact. Another proof for corollary 3-1.7 is the following:

Adding a constant we may assume that \( L \geq 0 \). There is \( B > 0 \) such that \( L(x, v) \geq |v| - B \) for all \( (x, v) \in TM \). Then for \( 0 \leq s \leq t \leq T \), we have that

\[
d(x_s, x_t) \leq \int_s^t |\dot{x}| \leq BT + \int_s^t L(x, \dot{x}) \leq BT + c.
\]

\( \square \)

Recall that

\[
\mathcal{F}(c) := \{ \gamma \in C^{ac}([0, T], M) \mid A_L(\gamma) \leq c \}
\]

3-1.9. Theorem. (Cle)

If \( \gamma_n \in \mathcal{F}(c) \) and \( \gamma_n \xrightarrow{\mathcal{U}} \gamma \) in the uniform topology, then \( \gamma \in \mathcal{F}(c) \).

We shall need the following lemma. We may assume that \( M = \mathbb{R}^n \).

3-1.10. Lemma. Given \( K \) compact, \( a > 0 \) and \( \varepsilon > 0 \), there exists \( \eta > 0 \) such that if \( x \in K, \|x - y\| < \varepsilon, |v| \leq a \) and \( w \in \mathbb{R}^n \), then

\[
L(y, w) \geq L(x, v) + L_v(x, v) (w - v) - \varepsilon.
\]

(3.4)
Proof of Cle’s theorem:

It is immediate from the definition 3-1.4 that a uniform limit of absolutely equicontinuous curves is absolutely continuous. We may assume that \( \gamma_n([0, T]) \) is contained in a compact neighbourhood \( K \) of \( \gamma([0, T]) \). By the superlinearity we may assume that \( L \geq 0 \). Let \( \varepsilon > 0 \) and \( E = [|\dot{\gamma}| \leq a] \), then by lemma 3-1.10, for \( n \) large,

\[
\int_{[0, T] \setminus E} [L(\dot{\gamma}) + L_v(\dot{\gamma})(\dot{\gamma} - \dot{\gamma}_n) - \varepsilon] \leq \int_{E^c} L(\dot{\gamma}_n) \leq c \quad \text{(since } L \geq 0). \tag{3.5}
\]

Claim: \( \dot{\gamma}_n \to \dot{\gamma} \) in \( L^1 \).

Using the claim, since \( L_v(x, v) \) is bounded for \( x \in K \) and \( |v| \leq a \), then

\[
\int_{E^c} L_v(\dot{\gamma})(\dot{\gamma} - \dot{\gamma}_n) \xrightarrow{n \to \infty} 0.
\]

Letting \( n \to +\infty \) on (3.5), we have that

\[
\int_{E^c} L(\dot{\gamma}) - \varepsilon \, T \leq c.
\]

Since \( E^c \uparrow [0, T] \) when \( a \to +\infty \), then

\[
\int_0^T L(\dot{\gamma}) = \lim_{a \to +\infty} \int_{E^c} L(\gamma) \leq c + \varepsilon \, T.
\]

Now let \( \varepsilon \to 0 \).

We now prove the claim. By inequality (3.3), for \( \varepsilon > 0 \) there is \( \delta > 0 \) such that

\[
m(D) < \delta \quad \Rightarrow \quad \int_D |\dot{\gamma}_n| < \varepsilon. \tag{3.6}
\]

where \( m \) is the Lebesgue measure. If \( A = [s, t] \), then

\[
\lim_{n \to \infty} \int_A (\dot{\gamma}_n - \dot{\gamma}) = \lim_{n \to \infty} (\gamma_n - \gamma)|_s^t = 0. \tag{3.7}
\]
Equations (3.6) and (3.7) imply that (3.7) holds for any Borel set \( A \), approximating \( A \) by finite unions of intervals. Then
\[
\lim_{n} \int_{[0,T]} (\hat{\gamma}_n - \hat{\gamma}) \psi = 0,
\]
for any \( \psi \in L^\infty \). This implies that \( \hat{\gamma}_n \to \hat{\gamma} \) in \( L^1 \).

3-1.11. Proof of Tonelli’s theorem:

By lemma 3-1.6, the family \( \mathcal{A}(c) \) in (3.1) is equicontinuous, and by corollary 3-1.7, the curves in \( \mathcal{A}(c) \) have a uniform compact range. By Arzelà-Ascoli’s theorem and Cle’s theorem, \( \mathcal{A}(c) \) is compact. Then
\[
\gamma \in \bigcap_{c \geq \inf_{C_T(x,y)} A_L} \mathcal{A}(c)
\]
is a Tonelli’s minimizer on \( C_T(x,y) \).

Proof of lemma 3-1.10:

Let
\[
C_1 := \sup \{ L_v(x, v) \mid x \in K, \ |v| \leq a \}
\]
\[
C_2 := \sup \{ L(x, v) - L_v(x, v) \cdot v \mid x \in K, \ |v| \leq a \}.
\]

Let \( b > 0 \) be such that
\[
K(b) \cdot b \geq C_2 + C_1 r \quad \text{for all } r \geq b,
\]
where \( K(d) \) is from (3.2). Then if \( y \in M \) and \( |w| > b \),
\[
L(y, w) \geq K(d) |w| \\
\geq C_2 + C_1 |w| \\
\geq C_2 + L_v(x, v) \cdot w \\
\geq L(x, v) + L_v(x, v) \cdot (w - v) \quad \text{for } |w| \geq d.
\]
Since $L$ is convex,

$$L(x, w) \geq L(x, v) + L_v(x, v) \cdot (w - v) \quad \forall w \in \mathbb{R}^n.$$ 

Then there is $\delta > 0$ such that for $|x - y| < \delta$, $|v| \leq a$ and $|w| \leq b$ inequality (3.4) holds.
3.2 A priori compactness.

The following lemma, due to Mather [39] for Tonelli minimizers in the non-autonomous case, will be very useful. In the autonomous case its proof is very simple.

3.2.1. Lemma.  
For $C > 0$ there exists $A = A(C) > 0$ such that if $x, y \in M$ and $\gamma \in \mathcal{C}_T(x, y)$ is a solution of the Euler-Lagrange equation with $A_L(\gamma) \leq CT$, then $|\dot{\gamma}(t)| < A$ for all $t \in [0, T]$.

Proof: By the superlinearity there is $D > 0$ such that $L(x, v) \geq |v| - D$ for all $(x, v) \in TM$. Since $A_L(\gamma) \leq CT$, the mean value theorem implies that there is $t_0 \in ]0, T[$ such that

$$|\dot{\gamma}(t_0)| \leq D + C.$$ 

The conservation of the energy and the uniform bounds (1.10) and (1.9) imply that there is $A = A(C) > 0$ such that $|\dot{\gamma}| \leq A$.

For $k \geq c(L)$ and $x, y \in M$, define

$$\Phi_k(x, y; T) := \inf_{\gamma \in \mathcal{C}_T(x, y)} A_{L+k}(\gamma).$$

3.2.2. Corollary. Given $\varepsilon > 0$, there are constants $A(\varepsilon), B(\varepsilon), C(\varepsilon) > 0$ such that if $T \geq \varepsilon$ and $k \geq c(L)$, then

(i) $\Phi_k(x, y; T) \leq C(\varepsilon) T$.

(ii) If $\gamma \in \mathcal{C}_T(x, y)$ is a solution of the Euler-Lagrange equation such that $A_{L+k}(\gamma) \leq C(\varepsilon) T + 1$, then $|\dot{\gamma}| \leq A(\varepsilon)$ and $E(\gamma, \dot{\gamma}) \leq B(\varepsilon)$. 

Proof: Comparing with the action of a geodesic on $C_T(x, y)$, we get (i) with

$$C(\varepsilon) = \sup \{ L(v) \mid \|v\| \leq \frac{d(x, y)}{\varepsilon} \}.$$  

Using (i) and lemma 3-2.1, we obtain $A(\varepsilon)$. Using $A(\varepsilon)$ and inequality (1.10) we obtain $B(\varepsilon)$.

\[ \square \]

3-2.3. Lemma.

There exists $A > 0$ such that if $x, y \in M$ and $\gamma \in C_T(x, y)$ is a solution of the Euler-Lagrange equation with

$$A_{L+c}(\gamma) \leq \Phi_c(x, y) + d_M(x, y),$$

then (a) $T > \frac{1}{A} d_M(x, y)$.

(b) $|\dot{\gamma}(t)| < A$ for all $t \in [0, T]$.

Proof: Let $\eta : [0, d(x, y)] \to M$ be a minimal geodesic with $|\eta| \equiv 1$. Let $\ell(r)$ be from (1.1) and $D = \ell(1) + c + 2$. From the superlinearity condition there is $B > 0$ such that

$$L(x, v) + c > D |v| - B, \quad \forall (x, v) \in TM.$$  

Then

$$[\ell(1) + c] d(x, y) \geq A_{L+c}(\eta) \geq \Phi_c(x, y)$$

$$\geq A_{L+c}(\gamma) - d(x, t)$$

$$\geq \int_0^T (D |\dot{\gamma}| - B) \, dt - d(x, y)$$

$$\geq D d(x, y) - B T - d(x, y).$$

Hence

$$T \geq \frac{A - \ell - c - 1}{B} d(x, y) \geq \frac{1}{B} d(x, y).$$
From (3.8) and (3.9), we get that

\[ A_L(\gamma) \leq \left[ \ell(1) + c + 1 \right] d(x, y) - c T, \]
\[ \leq \left\{ B \left[ \ell(1) + c + 1 \right] - c \right\} T. \]

Then lemma 3-2.1 completes the proof.
3-3 Energy of time-free minimizers.

A curve $\gamma \in C(x, y)$ is a global minimizer or time free minimizer for $L + k$ if $k \geq c(L)$ and $A_{L+k}(\gamma) = \Phi_k(x, y)$.

3-3.1. Proposition. A time-free minimizer for $L+k$ has energy $E \equiv k$.

We need the following

3-3.2. Lemma. Let $x: [0, T] \rightarrow M$ be an absolutely continuous curve and $k \in \mathbb{R}$. For $\lambda > 0$, let $x_\lambda(t) := x(\lambda t)$ and $A(\lambda) := A_{L+k}(x_\lambda)$. Then

$$A'(1) = \int_0^T \left[ E(x, \dot{x}) - k \right] dt.$$ 

Proof: Since $\dot{x}_\lambda(t) = \lambda \dot{x}(\lambda t)$, then

$$A(\lambda) = \int_0^T \left[ L(x(\lambda t), \lambda \dot{x}(\lambda t)) + k \right] dt.$$ 

Differentiating $A(\lambda)$ and evaluating at $\lambda = 1$, we have that

$$A'(1) = -T \left[ L(x(T), \dot{x}(T)) + k \right] + \int_0^T \left[ L_x t \dot{x} + L_y (\dot{x} + t \ddot{x}) \right] dt.$$ 

Integrating by parts the term $(L_x \dot{x} + L_y \ddot{x}) t = (\frac{d}{dt} L) t$, we have that

$$A'(1) = -T \left[ L(x(T), \dot{x}(T)) + k \right] + L t \bigg|_0^T + \int_0^T (L_y \dot{x} - L) dt$$

$$= -T k + \int_0^T E(x, \dot{x}) dt = \int_0^T \left[ E(x, \dot{x}) - k \right] dt.$$ 

Proof of proposition 3-3.1.

Since $\gamma$ is a solution of the Euler-Lagrange equation its energy $E(\gamma, \dot{\gamma})$ is constant. Since it minimizes with free time, the derivative in lemma 3-3.2 must be zero. So that $E(\gamma, \dot{\gamma}) \equiv 0$. 

3-3.3. Corollary.

Let \( x \in C^a([0,1], M) \) and \( k > 0 \). For \( T > 0 \), write \( y_T(t) = x \left( \frac{t}{T} \right) \): \([0,T] \to M\) and \( B(T) = A_{L+k}(y_T) \). Then

\[
B'(T) = -\frac{1}{T} \int_0^T \left[ E(y_T, \dot{y}_T) - k \right] \, dt.
\]

Proof: Using \( \lambda = \frac{T}{S} \) on lemma 3-3.2, we have that \( \frac{d}{dS} = -\frac{T}{S^2} \frac{d}{d\lambda} \). Thus

\[
\left. \frac{d}{dS} \right|_{S=T} B = -\frac{1}{T} \left. \frac{d}{d\lambda} \right|_{\lambda=1} A = -\frac{1}{T} \int_0^T \left[ E(y_T, \dot{y}_T) - k \right] \, dt.
\]

\( \square \)
3-4 The finite-time potential.

Recall that if \( k \geq c(L) \), \( x, y \in M \) and \( T > 0 \),

\[
\Phi_k(x,y; T) := \inf_{\gamma \in C_T(x,y)} A_{L+k}^\gamma(\gamma).
\]

Here we shall prove

**3-4.1. Proposition.**

For \( k \geq c(L) \) and \( x, y \in M \), the function \( t \mapsto \Phi_k(x,y; t) \) is Lipschitz on \([\varepsilon, +\infty[\) for any \( \varepsilon > 0 \). Moreover,

\[
\lim_{\varepsilon \to 0^+} \Phi_k(x,y; \varepsilon) = +\infty, \quad \text{for} \quad k \geq c(L), \; x \neq y.
\]

\[
\lim_{T \to +\infty} \Phi_k(x,y; T) = +\infty, \quad \text{for} \quad k > c(L), \; x \neq y.
\]

**Proof of proposition 3-4.1**

We first compute the limits. Observe that if \( k > c(L) \),

\[
\lim_{T \to +\infty} \Phi_k(x,y; T) \geq \lim_{T \to +\infty} \left[ \Phi_c(x,y) + (k-c)T \right] = +\infty.
\]

Given \( A > 0 \), let \( B > 0 \) be such that \( L(x,v) > A |v| - B \). Then

\[
\Phi_k(x,y; \varepsilon) = \inf_{\gamma \in C_T(x,y)} A_{L+k}^\gamma(\gamma) \geq \inf_{\gamma} \int_0^\varepsilon A |\dot{\gamma}| - B + k \geq A d(x,y) + (k-B)\varepsilon.
\]

Thus, \( \liminf_{\varepsilon \to 0^+} \Phi_k(x,y; \varepsilon) \geq A d(x,y) \). Now let \( A \to +\infty \).

Fix \( \varepsilon > 0 \). If \( T > \varepsilon \) and \( \gamma \in C_T(x,y) \) is a Tonelli minimizer, from corollary 3-2.2 there exists \( C = C(\varepsilon) > 0 \) such that \( E(\gamma, \dot{\gamma}) - k \leq C(\varepsilon) - k \). Denote \( h(s) := \Phi_k(x,y; s) \). If \( \gamma_s(t) := \gamma(\frac{\varepsilon}{T} t) \), then \( h(s) \leq
\( A_{L+k}(\gamma_s) =: B(s) \). Using corollary 3.3.3 we have that

\[
f(T) := \limsup_{\delta \to 0} \frac{h(T + \delta) - h(T)}{\delta}
\leq B'(T) = \frac{1}{T} \int_0^T \left[ E(\gamma, \dot{\gamma}) - k \right] dt
\leq |C(\varepsilon) - k|.
\]

If \( S, T > \varepsilon \) we have that

\[
\Phi_k(x, y; S) \leq \Phi_k(x, y; T) + \int_T^S f(t) \, dt
\leq \Phi_k(x, y; T) + |C(\varepsilon) - k| |T - S|.
\]

Since we can reverse the roles of \( S \) and \( T \), this implies the Lipschitz condition for \( T \mapsto \Phi_k(x, y; T) \).

\( \square \)
3-5 Global Minimizers.

Here we construct curves that realize the action potential. For $k < c(L)$, $\Phi_k \equiv -\infty$, so there are no minimizers.

3-5.1. Proposition.

If $k > c(L)$ and $x, y \in M$, $x \neq y$, then there is $\gamma \in C(x, y)$ such that

$$A_{L+k}(\gamma) = \Phi_k(x, y).$$

Moreover, the energy of $\gamma$ is $E(\gamma, \gamma) \equiv k$.

Proof: Let $f(t) := \Phi_k(x, y; T)$. By proposition 3-4.1, $f(t)$ is continuous and $f(t) \to +\infty$ when $t \to 0^+$ or $t \to +\infty$. Hence it attains its minimum at some $T > 0$. Moreover, $\Phi_k(x, y) = \inf_{t > 0} \Phi_k(x, y; t) = \Phi_k(x, y; T)$. Now take a Tonelli minimizer $\gamma$ on $C_T(x, y)$. From lemma 3-3.2, the energy of $\gamma$ is $k$. \hfill \Box

We now study minimizers at $k = c(L)$. Observe that for $c = c(L)$ and any absolutely continuous curve $\gamma \in C(x, y)$, we have that

$$A_{L+c}(\gamma) \geq \Phi_c(x, y) \geq -\Phi_c(y, x).$$

(3.10)

3-5.2. Definition. Set $c = c(L)$.

An absolutely continuous curve $\gamma \in C(x, y)$ is said semistatic if

$$A_{L+c}(\gamma) = \Phi_c(x, y).$$

An absolutely continuous curve $\gamma \in C(x, y)$ is said static if

$$A_{L+c}(\gamma) = -\Phi_c(y, x).$$

By the triangle inequality for $\Phi_c$, the definition of semistatic curve $x: [a, b] \to M$ is equivalent to

$$A_{L+c}(x|_{[s,t]}) = \Phi_c(x(s), x(t)), \quad \forall a \leq s \leq t \leq b.$$

(3.11)

Inequality (3.10) implies that static curves are semistatic.
Moreover, a curve $\gamma \in C(x, y)$ is static if

(a) $\gamma$ is semistatic, and

(b) $d_c(x, y) = \Phi_c(x, y) + \Phi_c(y, x) = 0$.

3-5.3. Corollary. Semistatic curves have energy $E \equiv c(L)$.

3-5.4. Definition.

\[ \tilde{\mathcal{N}} = \Sigma(L) := \{ w \in TM \mid x_w : \mathbb{R} \to M \text{ is semistatic} \} \]

\[ A = \hat{\Sigma}(L) := \{ w \in TM \mid x_w : \mathbb{R} \to M \text{ is static} \} \]

\[ \Sigma^-(L) := \{ w \in TM \mid x_w : (\mathbb{R}, -\infty, 0] \to M \text{ is semistatic} \} \]

\[ \Sigma^+(L) := \{ w \in TM \mid x_w : [0, +\infty) \to M \text{ is semistatic} \} \]

We shall call $\tilde{\mathcal{N}}$ the Mañé set, $\mathcal{P} = \pi(\hat{\Sigma}(L))$ the Peierls set\(^1\) and $A = \hat{\Sigma}(L)$ the Aubry set.

Using the characterization of minimizing measures 3-6.1 and corollary 3-5.3 we have that\(^2\)

\[ \mathcal{M} \subseteq A \subseteq \tilde{\mathcal{N}} \subseteq \hat{\mathcal{E}} \]

where $\mathcal{M}$ is the Mather set, $A$ is the Aubry set, $\tilde{\mathcal{N}}$ is the Mañé set and $\hat{\mathcal{E}}$ is the energy level $\hat{\mathcal{E}} = [E \equiv c(L)]$. All these inclusions can be made proper constructing examples of embedded flows as in equation (1.20) and adding a properly chosen potential $\phi(x)$.

Denote by $\alpha(v)$ and $\omega(v)$ the $\alpha$ and $\omega$-limits of $v$ under the Euler-Lagrange flow.

3-5.5. Proposition.

A local static is a global static, i.e. if $x_v|_{[a, b]}$ is static then $v \in \hat{\Sigma}(L)$ (i.e. the whole orbit is static).

---

\(^1\)The name is justified by proposition 3-7.1.5.

\(^2\)The typographical relationship was observed by Albert Fathi.
Proof: Let $\eta(t) = \pi \varphi_t(v)$ and let $\gamma_n \in C_{T_n}(\eta(b), \eta(a))$ be solutions of (E-L) with

$$A_{L+c}(\gamma_n) < \Phi_c(\eta(b), \eta(a)) + \frac{1}{n}.$$ 

By the apriori bounds 3-2.3 $\dot{\gamma}_n < A$. We can assume that $\dot{\gamma}_n(0) \to w$. Let $\xi(s) = \pi \varphi_s(w)$. If $w \neq \dot{\eta}(b)$ then the curve $\eta_{[b-\epsilon,b]} \ast \xi_{[0,\epsilon]}$ is not $C^1$, and hence by remark 1-2.2, can not be a Tonelli minimizer. Thus

$$\Phi_c(\eta(b-\epsilon), \xi(\epsilon)) < A_{L+c}(\eta_{[b-\epsilon,b]}) + A_{L+c}(\xi_{[0,\epsilon]}).$$

$$\Phi_c(\eta(a), \eta(a)) \leq \Phi_c(\eta(a), \eta(b-\epsilon)) + \Phi_c(\eta(b-\epsilon), \xi(\epsilon)) + \Phi_c(\xi(\epsilon), \eta(a))$$

$$< A_{L+c}(\eta_{[a,b-\epsilon]}) + A_{L+c}(\eta_{[b-\epsilon,b]}) + A_{L+c}(\xi_{[0,\epsilon]}) + \liminf_n A_{L+c}(\gamma_n_{[\epsilon,T_n]})$$

$$\leq A_{L+c}(\eta_{[a,b]}) + \lim_n \left( \gamma_n_{[0,\epsilon]} \ast \gamma_n_{[\epsilon,T_n]} \right)$$

$$\leq -\Phi_c(\gamma(b), \gamma(a)) + \Phi_c(\gamma(a), \gamma(b)) = 0,$$

which contradicts proposition 2-1.1(3). Thus $w = \dot{\eta}(b)$ and similarly $\lim_n \dot{\gamma}_n(T_n) = \dot{\eta}(a)$.

If $\limsup T_n < +\infty$, we can assume that $\tau = \lim_n T_n > 0$ exists. In this case $\eta$ is a (semistatic) periodic orbit of period $\tau + b - a$ and then it is static.

Now suppose that $\lim_n T_n = +\infty$. If $s > 0$, we have that

$$A_{L+c}(\eta_{[a-s,b+s]}) + \Phi_c(\eta(b+s), \eta(a-s)) \leq$$

$$\leq \lim_n \left\{ A_{L+c}(\gamma_n_{[T_n-s,T_n]}) + A_{L+c}(\eta) + A_{L+c}(\gamma_n_{[0,s]}) \right\}$$

$$+ \Phi_c(\eta(b+s), \eta(a-s))$$

$$\leq \Phi_c(\eta(a), \eta(b))$$

$$+ \lim_n \left\{ A_{L+c}(\gamma_n_{[0,s]}) + A_{L+c}(\gamma_n_{[s,T_n-s]}) + A_{L+c}(\gamma_n_{[T_n-s,T_n]}) \right\}$$

$$\leq \Phi_c(\eta(a), \eta(b)) + \Phi_c(\eta(b), \eta(a)) = 0.$$

Thus $\eta_{[a-s,b+s]}$ is static. \qed
3-6 Characterization of minimizing measures.

3-6.1. Theorem (Mañé [32]).

\[ \mu \in \mathcal{M}(L) \text{ is a minimizing measure if and only if } \text{supp}(\mu) \subseteq \hat{\Sigma}(L). \]

**Proof:** Since \( \hat{\Sigma} \) is closed, it is enough to prove the theorem for ergodic measures. Suppose that \( \mu \in \mathcal{M}(L) \) is ergodic and \( \text{supp}(\mu) \subseteq \Sigma(L) \). Since \( \mu \) is finite, by Birkhoff's theorem there is a set of total \( \mu \)-measure \( A \) such that if \( \theta \in A \) then \( \liminf_{T \to +\infty} d_{TM}(\theta, \varphi_{T} \theta) = 0 \) and

\[
\int_{M} L + c \, d\mu = \lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} L(\varphi_{t} \theta) + c \, dt \\
\leq \liminf_{T \to +\infty} \frac{1}{T} \Phi_{c}(\pi(\varphi_{T} \theta), \pi(\theta)) = 0.
\]

Now suppose that \( \mu \in \mathcal{M}(L) \) is minimizing. By corollary 3-6.5, there is a set \( A \) of total \( \mu \)-measure such that if \( \theta \in A \) then there is a sequence \( T_{n} \to +\infty \) such that \( d(\theta, \varphi_{T_{n}} \theta) \to 0 \) and

\[
\lim_{n} \int_{0}^{T_{n}} L(\varphi_{t} \theta) + c \, dt = 0.
\]

Then

\[
0 \leq d_{c}(\pi \theta, \pi \varphi_{1} \theta) = \lim_{n} \left[ \Phi_{c}(\pi \theta, \pi \varphi_{1} \theta) + \Phi_{c}(\pi \varphi_{1} \theta, \pi \varphi_{T_{n}} \theta) \right] \\
\leq \lim_{n} \int_{0}^{T_{n}} L(\varphi_{t} \theta) + c \, dt = 0.
\]

This implies that \( \theta \) is static. Since \( \hat{\Sigma}(L) \) is closed and \( A \) is dense in \( \text{supp}(\mu) \) then \( \text{supp}(\mu) \subseteq \hat{\Sigma}(L) \).

\[ \square \]

It follows from theorem 2-4.1 that

3-6.2. Corollary.

*If \( M \) is compact then \( \hat{\Sigma}(L) \neq \emptyset \).*

Combining theorem 3-6.1 with corollary 3-5.3, we get
3-6.3. Corollary. If \( \mu \) is a minimizing measure then it is supported in the energy level \( E(\text{supp}(\mu)) = c(L) \).

3-6.4. Lemma. Let \((X, \mathcal{B}, \nu)\) be a probability space, \( f \) an ergodic measure preserving map and \( F : X \to \mathbb{R} \) an integrable function. Given \( A \in \mathcal{B} \) with \( \nu(A) > 0 \) denote by \( \hat{A} \) the set of point \( p \in A \) such that for all \( \varepsilon > 0 \) there exists an integer \( N > 0 \) such that \( f^N(p) \in A \) and

\[
\left| \sum_{j=0}^{N-1} F(f^j(x)) - N \int F \, d\nu \right| < \varepsilon.
\]

Then \( \nu(\hat{A}) = \nu(A) \).

Proof: Without loss of generality we can assume that \( \int F \, d\nu = 0 \). For \( p \in X \) denote

\[
S_N(p) := \sum_{n=0}^{N-1} F(f^n(p)).
\]

Let

\[
A(\varepsilon) := \{ p \in A \mid \exists N > 0 \text{ such that } f^N(p) \in A \text{ and } |S_N F(p)| < \varepsilon \}. \]

It is enough to prove that \( \nu(A(\varepsilon)) = \nu(a) \), because \( \hat{A} = \cap_n A(\frac{1}{n}) \). Let \( \hat{X} \) be the set of points for which the Birkhoff's theorem holds for \( F \) and the characteristic functions of \( A \) and of \( A(\varepsilon) \). Take \( x \in A \cap \hat{X} \) and let \( N_1 < N_2 < \cdots \) be the integers for which \( f^{N_i}(x) \in A \). Define \( \delta(k) \) by

\[
N_k \delta(k) = |S_{N_k} F(x)|.
\]

Since \( x \in \hat{X} \) we have that \( \lim_{k \to +\infty} \delta(k) = 0 \). Set

\[
c_k := S_{N_k} F(x),
\]

\[
S(k) := \{ 1 \leq j \leq k - 1 \mid \forall \ell > j \ |c_\ell - c_j| \geq \varepsilon \}.
\]

Then

\[
\varepsilon \#S(k) \leq \delta(k) N_k.
\]
Moreover if $j \notin S(k)$, then there is an $\ell > j$ such that $|c_\ell - c_j| \leq \varepsilon$, so that

$$|S_{N_\ell - N_j} F(f^{N_j}(x))| = |c_\ell - c_j| \leq \varepsilon.$$ 

Hence

$$j \notin S(k) \implies f^{N_j}(x) \in A(\varepsilon).$$

This implies that

$$\frac{1}{N_k} \# \{ 0 \leq j < N_k \mid f^j(x) \in A - A(\varepsilon) \} \leq \frac{1}{N_k} \# S(k) \leq \frac{\delta(k)}{\varepsilon}.$$ 

Since $\delta(k) \to 0$ when $k \to +\infty$ and $x \in \hat{X}$, we obtain

$$\nu(A - A(\varepsilon)) = \lim_{k \to +\infty} \frac{1}{N_k} \# \{ 0 \leq j < N_k \mid f^j(x) \in A - A(\varepsilon) \} = 0,$$

concluding the proof of the lemma.

\[\square\]

3-6.5. Corollary. If besides the hypothesis of lemma 3-6.4, the space $X$ is compact and metric, and $\mathcal{B}$ is the Borel $\sigma$-algebra, then for a.e. $x \in X$ the following property holds: for all $\varepsilon > 0$ there exists $N > 0$ such that $d(f^N(x), x) < \varepsilon$ and

$$\left| \sum_{j=0}^{N-1} F(f^j(x)) - N \int F \, d\nu \right| < \varepsilon$$

Proof: Given $\varepsilon > 0$ let $\{V_n(\varepsilon)\}$ be a countable basis of neighbourhoods with diameter $< \varepsilon$ and let $\hat{V}_n$ be associated to $V_n$ as in lemma 3-6.4. Then the full measure subset $\bigcap_{m \geq n} \hat{V}_n(\frac{1}{m})$ satisfies the required property.

\[\square\]
3-7 The Peierls barrier.

For $T > 0$ and $x, y \in M$ define

$$h_T(x, y) = \Phi_c(x, y; T) := \inf_{\gamma \in C_T(x,y)} A_{L+c}(\gamma).$$

So that the curves which realize $h_T(x, y)$ are the Tonelli minimizers on $C_T(x,y)$. Define the Peierls barrier as

$$h(x, y) := \lim_{T \to +\infty} \inf h_T(x, y).$$

The difference between the action potential and the Peierls barrier is that in the Peierls barrier the curves must be defined on large time intervals. Clearly

$$h(x, y) \geq \Phi_c(x, y).$$

3-7.1. Proposition.

If $h : M \times M \to \mathbb{R}$ is finite, then

1. $h$ is Lipschitz.

2. $\forall x, y \in M$, $h(x, x) \geq \Phi_c(x, y)$, in particular $h(x, x) \geq 0$, $\forall x \in M$.

3. $h(x, z) \leq h(x, y) + h(y, z)$, $\forall x, y, z \in M$.

4. $h(x, y) \leq \Phi_c(x, p) + h(p, q) + \Phi_c(q, y)$, $\forall x, y, p, q \in M$.

5. $h(x, x) = 0 \iff x \in \pi(\Sigma) = \mathcal{P}$.

6. If $\Sigma \neq \emptyset$, $h(x, y) \leq \inf_{p \in \pi(\Sigma)} \Phi_c(x, p) + \Phi_c(p, y)$.

Proof: Item 2 is trivial. Observe that for all $S, T > 0$ and $y \in M$,

$$h_{T+S}(x, z) \leq h_T(x, y) + h_S(y, z).$$

Taking $\lim \inf_{T \to +\infty}$ we get that

$$h(x, z) \leq h(x, y) + h_S(y, z), \quad \text{for all } S > 0.$$
Taking $$\liminf_{S \to +\infty}$$, we obtain item 3.

1. Taking the infimum on $$S > 0$$, we get that

$$h(x, z) \leq h(x, y) + \Phi_c(y, z) \quad \forall x, y, z \in M.$$  

$$\leq h(x, y) + A d_M(y, z),$$

where $$A$$ is a Lipschitz constant for $$\Phi_c$$. Changing the roles of $$x, y, z$$, we obtain that $$h$$ is Lipschitz.

4. Observe that

$$\inf_{S > T} h_S(x, y) \leq \Phi_c(x, p) + h_T(p, q) + \Phi_c(q, x).$$

Taking $$\liminf_{T \to +\infty}$$ we get item 4.

5. We first prove that if $$p \in \mathcal{P} = \pi(\hat{\Sigma})$$, then $$h(p, p) = 0$$. Take $$v \in \hat{\Sigma}$$ such that $$\pi(v) = p$$ and $$y \in \pi(\omega\text{-limit}(v))$$. Let $$\gamma(t) := \pi \varphi_t(v)$$ and choose $$t_n \uparrow +\infty$$ such that $$\gamma(t_n) \to y$$. Then

$$0 \leq h(p, p) \leq h(p, y) + \Phi_c(y, p)$$

$$\leq \lim_{n} A_{L+c}(\gamma|_{[0,t_n]}) + \Phi_c(y, p)$$

$$\leq \lim_{n} -\Phi_c(\gamma(t_n), p) + \Phi_c(y, p) = 0.$$  

Conversely, if $$h(x, x) = 0$$, then there exists a Tonelli minimizer $$\gamma_n \in C(x, x; T_n)$$ with $$T_n \to +\infty$$ and $$A_{L+c}(\gamma_n) \not\to 0$$. By lemma 3-2.3, $$|\dot{\gamma}|$$ is uniformly bounded. Let $$v$$ be an accumulation point of $$\dot{\gamma}_n(0)$$ and $$\eta(t) := \pi \varphi_t(v)$$. Then if $$\dot{\gamma}_n(0) \overset{k}{\to} v$$, for any $$s > 0$$ we have that

$$0 \leq \Phi_c(x, \pi \varphi_s v) + \Phi_c(\pi \varphi_s v, x)$$

$$\leq A_{L+c}(\eta|_{[0,s]}) + \Phi_c(\pi \varphi_s v, x)$$

$$\leq \lim_{k} A_{L+c}(\gamma_n|_{[0,s]}) + A_{L+c}(\gamma_n|_{[s,T_n]})$$

$$= 0.$$  

Thus $$v \in \hat{\Sigma}.$$
6. Using items 4 and 5, we get that
\[ h(x, y) \leq \inf_{p \in \pi(\Sigma)} [\Phi_c(x, p) + 0 + \Phi_c(p, y)]. \]

\[ \square \]

3-7.2. Proposition. If \( M \) is compact, then
\[ h(x, y) = \inf_{p \in \pi(\Sigma)} [\Phi_c(x, p) + \Phi_c(p, y)]. \]

Proof:
From proposition 3-7.1.6 we have that
\[ h(x, y) \leq \inf_{p \in \pi(\Sigma)} [\Phi_c(x, p) + \Phi_c(p, y)]. \]

In particular \( h(x, y) < +\infty \) for all \( x, y \in M \). Now let \( \gamma_n \in C_{T_n}(x, y) \) with \( T_n \to +\infty \) and \( A_{L+c}(\gamma_n) \to h(x, y) < +\infty \). Then \( \frac{1}{n} A_{L+c}(\gamma_n) \to 0 \).

Let \( \mu \) be a weak limit of a subsequence of the measures \( \mu_{\gamma_n} \). Then \( \mu \) is minimizing. Let \( q \in \pi(\text{supp}(\mu)) \) and \( q_n \in \gamma_n([0, T_n]) \) be such that \( \lim_n q_n = q \). Then,
\[ \Phi_c(x, q) + \Phi_c(q, y) \leq \Phi_c(x, q_n) + \Phi_c(q_n, y) + 2A d(q_n, q) \leq A_{L+c}(\gamma_n) + 2A d(q_n, q). \]

Letting \( n \to \infty \), we get that
\[ \Phi_c(x, q) + \Phi_c(q, y) \leq h(x, y). \]

\[ \square \]
3.8 Graph Properties.

In this section we shall see that the projection \( \pi : \tilde{\Sigma} \rightarrow M \) is injective. We shall call \( P := \pi(\tilde{\Sigma}) \) the Peierls set. \(^3\) Thus the projection \( \pi|_{\tilde{\Sigma}} \) gives an identification \( P \approx \tilde{\Sigma} \).

For \( v \in TM \), write \( x_v(t) = \pi \varphi_t(v) \). Given \( \varepsilon > 0 \), let

\[ \Sigma^\varepsilon := \{ w \in TM \mid x_w : [0, \varepsilon) \rightarrow M \text{ or } x_w : (-\varepsilon, 0] \rightarrow M \text{ is semistatic} \} . \]

3.8.1. Theorem. (Mañé) \([32]\)

For all \( p \in \pi(\tilde{\Sigma}) \) there exists a unique \( \xi(p) \in T_p M \) such that \((p, \xi(p)) \in \Sigma^\varepsilon \), in particular \((p, \xi(p)) \in \tilde{\Sigma} \) and \( \tilde{\Sigma} = \text{graph}(\xi) \).

Moreover, the map \( \xi : \pi(\tilde{\Sigma}) \rightarrow \Sigma \) is Lipschitz.

The proofs of the injectivity of \( \pi \) in this book only need that the solutions of the Euler-Lagrange equation are differentiable\(^4\). The reader may provide those proofs as exercises. The proof of the Lipschitz condition need the following lemma, due to Mather. For the proof see [39] or Mañé [29].

3.8.2. Mather’s Crossing lemma. \([39]\)

Given \( A > 0 \) there exists \( K > 0 \) \( \varepsilon_1 > 0 \) and \( \delta > 0 \) with the following property: if \( |v_i| < A \), \((p_i, v_i) \in TM \), \( i = 1, 2 \) satisfy \( d(p_1, p_2) < \delta \) and \( d((p_1, v_1), (p_2, v_2)) \geq K^{-1}d(p_1, p_2) \) then, if \( a \in \mathbb{R} \) and \( x_i : \mathbb{R} \rightarrow M \), \( i = 1, 2 \), are the solutions of \( L \) with \( x_i(a) = p_i \), \( \dot{x}_i(p_i) = v_i \), there exist solutions \( \gamma_i : [a - \varepsilon, a + \varepsilon] \rightarrow M \) of \( L \) with \( 0 < \varepsilon < \varepsilon_1 \), satisfying

\[
\begin{align*}
\gamma_1(a - \varepsilon) &= x_1(a - \varepsilon), & \gamma_1(a + \varepsilon) &= x_2(a + \varepsilon), \\
\gamma_2(a - \varepsilon) &= x_2(a - \varepsilon), & \gamma_2(a + \varepsilon) &= x_1(a + \varepsilon), \\
S_L(x_1|_{[a-\varepsilon,a+\varepsilon]}) + S_L(x_2|_{[a-\varepsilon,a+\varepsilon]}) &> S_L(\gamma_1) + S_L(\gamma_2)
\end{align*}
\]

\(^3\)This name is justified by proposition 3.7.1(5).

\(^4\)and hence a non-differentiable curve can not be a Tonelli minimizer.
Proof of theorem 3-8.1:

We prove that if \((p, v) \in \tilde{\Sigma}, (q, w) \in \Sigma^e\), and \(d(p, q) < \delta\), then

\[
d_{TM}\left((p, v), (q, w)\right) < K d_M(p, q)\,.
\]

Observe that this implies the theorem. For simplicity, we only prove the case in which \(x_v|_{[-\varepsilon,0]}\) is semistatic. Suppose it is false. Then by lemma 3-8.2 there exist \(\alpha, \beta : [-\varepsilon, \varepsilon] \rightarrow M\) such that

\[
\alpha(-\varepsilon) = x_w(-\varepsilon) =: q_{-\varepsilon}, \quad \alpha(0) = p,
\]

\[
\beta(-\varepsilon) = x_v(-\varepsilon) =: p_{\varepsilon}, \quad \beta(0) = q,
\]

and

\[
S_L(\alpha) + S_L(\beta) < S_L(x_w|_{[-\varepsilon,0]}) + S_L(x_v|_{[-\varepsilon,0]}).
\]

So

\[
\Phi_c(q_{-\varepsilon}, p) + \Phi_c(p_{-\varepsilon}, q) < \Phi_c(q_{-\varepsilon}, q) + \Phi_c(p_{-\varepsilon}, p)
\]

\[
= \Phi_c(q_{-\varepsilon}, q) - \Phi_c(p, p_{-\varepsilon})
\]

Thus

\[
\Phi_c(q_{-\varepsilon}, q) \leq \Phi_c(q_{-\varepsilon}, p) + \Phi_c(p, p_{-\varepsilon}) + \Phi_c(p_{-\varepsilon}, q) < \Phi_c(q_{-\varepsilon}, q)
\]

which is a contradiction. \(\square\)
Using the graph property 3-8.1 we can define an equivalence relation on \( \hat{\Sigma} \) by

\[
u, v \in \hat{\Sigma}, \quad u \equiv v \iff d_c(\pi(u), \pi(v)) = 0.
\]

The equivalence classes are called static classes. The continuity of the pseudometric \( d_c \) implies that a static class is closed, and it is invariant by proposition 3-5.5.

For \( v \in TM \) denote by \( \omega(v) \) its \( \omega \)-limit. Let \( \Gamma \) be a static class, the set

\[
\Gamma^+ = \{ v \in \Sigma^+(L) \mid \omega(v) \subseteq \Gamma \}
\]

is called the (forward) basin of \( \Gamma \). Clearly \( \Gamma^+ \) is forward invariant. Let

\[
\Gamma_0^+ = \bigcup_{t > 0} \varphi_t(\Gamma)
\]

\[
= \bigcup_{\varepsilon > 0} \{ v \in TM \mid x_v|_{x_v|_{-\varepsilon, +\infty}^\ast} \text{ is semistatic} \}.
\]

The set \( \pi(\Gamma^+ \setminus \Gamma_0^+) \) is called the cut locus of \( \Gamma^+ \).

3-8.3. Theorem. (Mañé)[32]

For every static class \( \Gamma \), the projection \( \pi: \Gamma_0^+ \to M \) is injective with Lipschitz inverse.

The projection \( \pi: \Gamma^+ \to M \) may not be surjective. But when \( M \) is compact for generic lagrangians \( \pi(\Gamma^+) = \pi(\Gamma^-) = M \) because there is only one static class (cf. theorem 5-0.1.(B)).

Proof: We prove that for \( K \) as in lemma 3-8.2, if \( v, w \in \Gamma_0^+ \) then

\[
d_{TM}(v, w) \leq K d_M(\pi(v), \pi(w)). \tag{3.12}
\]

Suppose it is false. Then there are \( v, w \in \Gamma_0^+ \) such that inequality (3.12) does not hold. Let \( \varepsilon > 0 \) be such that \( x_v|_{-\varepsilon, +\infty}^\ast \) and \( x_w|_{-\varepsilon, +\infty}^\ast \) are semistatic. By lemma 3-8.2, there exist \( \alpha \in \mathcal{C}_2(x_v(-\varepsilon), x_w(\varepsilon)) \) and \( \beta \in \mathcal{C}_2(x_w(-\varepsilon), x_v(\varepsilon)) \) such that

\[
A_{L+c}(\alpha) + A_{L+c}(\beta) + \delta < A_{L+c}(x_v|_{-\varepsilon, \varepsilon}) + A_{L+c}(x_w|_{-\varepsilon, \varepsilon}),
\]
for some \( \delta > 0 \). Let \( p, q \in \pi(\Gamma) \) and \( s_n t_n \to +\infty \) be such that \( x_v(s_n) \to +\infty \) and \( x_v(s_n) \to p, x_w(t_n) \to q \). Then

\[
\Phi_c(x_v(-\varepsilon), x_w(t_n)) + \Phi_c(x_w(-\varepsilon), x_v(s_n)) + \delta \\
\leq A_{L+c}(\alpha * x_w|_{\varepsilon, t_n}) + A_{L+c}(\beta * x_v|_{\varepsilon, s_n}) + \delta \\
< A_{L+c}(x_v|_{-\varepsilon, s_n}) + A_{L+c}(x_w|_{-\varepsilon, t_n}) \\
= \Phi_c(x_v(-\varepsilon), x_v(s_n)) + \Phi_c(x_w(-\varepsilon), x_w(t_n)).
\]

Letting \( n \to \infty \) and adding \( d_c(p, q) = 0 \), we have that

\[
\Phi_c(x_v(-\varepsilon), p) + \Phi_c(x_w(-\varepsilon), q) \\
\leq \Phi_c(x_v(-\varepsilon), q) + \Phi_c(x_w(-\varepsilon), p) + \Phi_c(q, p) + \Phi_c(p, q) \\
< \Phi_c(x_v(-\varepsilon), p) + \Phi_c(x_w(-\varepsilon), q).
\]

\( \Box \)
3-9 Coboundary Property.

The coboundary property was first presented by R. Mañé in [31] and further developed in [32] and by A. Fathi.

3-9.1 Theorem. (Mañé) [32]

If \( c = c(L) \), then \( (L + c)|_\Sigma \) is a Lipschitz coboundary. More precisely, taking any \( p \in M \) and defining \( G : \hat{\Sigma} \to \mathbb{R} \) by

\[
G(w) = \Phi_c(p, \pi(w)),
\]

then

\[
(L + c)|_\Sigma = \frac{dG}{df},
\]

where

\[
\frac{dG}{d\varphi}(w) := \lim_{h \to 0} \frac{1}{h} \left[ G(\varphi_h(w)) - G(w) \right].
\]

Proof: Let \( w \in \hat{\Sigma} \) and define \( F_w(v) := \Phi_c(\pi(w), \pi(v)) \). We have that

\[
\frac{dF_w}{d\varphi} \bigg|_w = \lim_{h \to 0} \frac{1}{h} \left[ F_w(\varphi_h w) - F_w(w) \right]
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left[ \Phi_c(\pi w, \pi \varphi_h w) - \Phi_c(\pi w, \pi w) \right]
\]

\[
= \lim_{h \to 0} \frac{1}{h} S_{L+c} (x_w|_{[0,h]})
\]

\[
= \lim_{h \to 0} \frac{1}{h} \int_{0}^{h} \left[ L(x_w(s), \dot{x}_w(s)) + c \right] ds
\]

\[
= L(w) + c.
\]

We claim that for any \( p \in M \) and any \( w \in \hat{\Sigma}, h \in \mathbb{R} \),

\[
G(\varphi_h w) = \Phi_c(p, \pi(\varphi_h w)) = \Phi_c(p, \pi(w)) + \Phi_c(\pi(w), \pi(\varphi_h w))
\]

\[
G(\varphi_h w) = \Phi_c(p, \pi(w)) + F_w(\varphi_h(w)).
\]

(3.13)
This is enough to prove the theorem because then

\[ \frac{dG}{d\varphi}_w = \left. \frac{d}{dh} F_h(\varphi_h w) \right|_{h=0} = \left. \frac{F_w}{d\varphi} \right|_w = L(w) + c, \]

and \( G \) is Lipschitz by proposition 2-1.1.

We now prove (3.13). Let \( q := \pi(w), \; x := \pi(\varphi_h w) \). We have to prove that

\[ \Phi_c(p, x) = \Phi_c(p, q) + \Phi_c(q, x). \] \hspace{1cm} (3.14)

Since the points \( q \) and \( x \) can be joined by the static curve \( x_w|_{[0,h]} \), then

\[ \Phi_c(x, q) = -\Phi_c(q, x). \]

Using twice the triangle inequality for \( \Phi_c \) we get that

\[ \Phi_c(p, q) \leq \Phi_c(p, x) + \Phi_c(x, q) = \Phi_c(p, x) - \Phi_c(q, x) \leq \Phi_c(p, q). \]

This implies (3.14). \qed
3.10 Covering Properties.

3.10.1. Theorem. \( \pi(\Sigma^+(L)) = M. \)

Proof: First suppose that \( \hat{\Sigma}(L) \neq \varnothing. \) Take \( p \in \pi(\hat{\Sigma}). \) Given \( x \in M \setminus \pi(\hat{\Sigma}), \) take a Tonelli minimizer \( \gamma_n \in C_{T_n}(x, p) \) such that

\[
A_{L+c}(\gamma_n) < \Phi_c(x, p) + \frac{1}{n}.
\]

By the apriori bounds 3.2.3, \( |\dot{\gamma}_n| < A \) and \( T_n > \frac{1}{A} d(x, p). \) Let \( v = \lim_k \dot{\gamma}_{n_k}(0) \) be an accumulation point of \( \langle \dot{\gamma}_n(0) \rangle. \) Let \( \eta(t) := \pi \varphi_t(v). \)

Then, if \( 0 < s < \lim \inf_k T_{n_k}, \) we have that

\[
A_{L+c}(|\eta|_{[0,s]}) = \lim_k A_{L+c}(|\gamma_n|_{[0,s]})
\]

\[
\leq \lim_k \left[ \Phi_c(\gamma_{n_k}(0), \gamma_{n_k}(s)) + \frac{1}{n_k} \right]
\]

\[
= \Phi_c(\eta(0), \eta(s)).
\]

Then \( \eta \) is semistatic on \( [0, S], \) where \( S = \lim \inf_k T_{n_k}. \) If \( S < +\infty \) then \( \eta(S) = \lim_k \gamma_{n_k}(T_k) = p. \) Since \( x \notin \pi(\hat{\Sigma}), \) this contradicts the graph property 3.8.1; hence \( S = +\infty. \) Thus \( \eta|_{[0, +\infty]} \) is semistatic and \( v \in \Sigma^+. \)

If \( \hat{\Sigma} = \varnothing, \) then by corollary 3.6.2, \( M \) is non-compact. Let \( x \in M \) and \( \langle y_n \rangle \subseteq M \) such that \( d_M(x, y_n) \to +\infty. \) Let \( \gamma_n \in C_{T_n}(x, y_n) \) be a Tonelli minimizer such that

\[
A_{L+c}(\gamma_n) < \Phi_c(x, y_n) + \frac{1}{n}.
\]

Then by lemma 3.2.3, \( |\dot{\gamma}_n| < A, \) and hence \( T_n \to +\infty. \) The rest of the proof is similar to the case above. \( \square \)
3-11 Recurrence Properties.

Let $\Lambda$ be the set of static classes. Define a reflexive partial order $\preceq$ in $\Lambda$ by

(a) $\preceq$ is reflexive.

(b) $\preceq$ is transitive.

(c) If there is $v \in \Sigma$ with the $\alpha$-limit set $\alpha(v) \subseteq \Lambda_i$ and $\omega$-limit set $\omega(v) \subseteq \Lambda_j$, then $\Lambda_i \preceq \Lambda_j$.

3-11.1. Theorem.

Suppose that $M$ is compact and the number of static classes is finite. Then given $\Lambda_i$ and $\Lambda_j$ in $\Lambda$, we have that $\Lambda_i \preceq \Lambda_j$.

![Diagram](image)

FIG. 2: CONNECTING ORBITS BETWEEN STATIC CLASSES.

The three closed curves represent the static classes and the other curves represent semistatic orbits connecting them.

Theorem 3-11.1 could be restated by saying that if the cardinality of $\Lambda$ is finite, then given two static classes $\Lambda_i$ and $\Lambda_j$ there exist classes $\Lambda_i = \Lambda_1, \ldots, \Lambda_n = \Lambda_j$ and semistatic vectors $v_1, \ldots, v_{n-1} \in \Sigma$ such that for all $1 \leq k \leq n - 1$ we have that $\alpha(v_k) \subseteq \Lambda_k$ and $\omega(v_k) \subseteq \Lambda_{k+1}$. In other words, between two static classes there exists a chain of static classes connected by heteroclinic semistatic orbits (cf. figure 2).
A proof of the following theorem can be found in [6]

3-11.2. Theorem. If $M$ is compact, then

1. $\Sigma(L)$ is chain transitive.
2. $\widehat{\Sigma}(L)$ is chain recurrent.

Now we proceed to prove theorem 3-11.1. Assume for the rest of this section that $M$ is compact.

3-11.3. Proposition.

If $v \in \Sigma$ is semistatic, then $\alpha(v) \subset \widehat{\Sigma}(L)$ and $\omega(v) \subset \widehat{\Sigma}(L)$. Moreover $\alpha(v)$ and $\omega(v)$ are each contained in a static class.

**Proof:** We prove only that $\omega(v) \subset \widehat{\Sigma}$. Let $\gamma(t) = \pi \varphi_t(v)$. Suppose that $t_n \to +\infty$ and $\gamma(t_n) \to w \in TM$. Let $\eta(t) = \pi \varphi_t(w)$. Since $\gamma$ and $\eta$ are solutions the Euler-Lagrange equation, then $\gamma|_{[t_n-s,t_n+s]} \to \eta|_{[-s,s]}$. Then

$$A_{L+c}(\eta|_{[-s,s]}) + \Phi_c(\eta(s), \eta(-s)) =$$

$$= \lim_n \left\{ A_{L+c}(\gamma|_{[t_n-s,t_n+s]}) + \lim_m A_{L+c}(\gamma|_{[t_m-s,t_m+s]}) \right\}$$

$$= \lim_n \lim_m A_{L+c}(\gamma|_{[t_n-s,t_m-s]})$$

$$= \lim_n \lim_m \Phi_c(\gamma(t_n-s), \gamma(t_m-s))$$

$$= \Phi_c(\eta(-s), \eta(-s)) = 0.$$

Thus $w \in \widehat{\Sigma}(L)$. Let $u \in \omega(v)$. We may assume that $\gamma(s_n) \to w$ with $t_n < s_n < t_{n+1}$. Then

$$d_c(\pi w, \pi u) = \Phi_c(\pi w, \pi u) + \Phi_c(\pi u, \pi w)$$

$$= \lim_n A_{L+c}(\gamma|_{[t_n,s_n]} + A_{L+c}(\gamma|_{[s_n,t_{n+1}]}$$

$$= \lim_n A_{L+c}(\gamma_n|_{[t_n,t_{n+1}]} = \Phi_c(\pi w, \pi w) = 0.$$
3-11.4. Proposition. Every static class is connected.

Proof: Let $\Lambda$ be a static class and suppose that it is not connected. Let $U_1, U_2$ be disjoint open sets such that $\Lambda \subseteq U_1 \cup U_2$ and $\Lambda \cap U_i \neq \emptyset$, $i = 1, 2$. Let $p_i \in \pi(U_i \cap \Lambda)$, $i = 1, 2$. Since $U_1$ and $U_2$ are disjoint sets, we can take a solution $x_{v_n} : [a_n, b_n] \to M$, $a_n < 0 < b_n$ of (E-L) such that $x_{v_n}(0) \notin \pi(U_1 \cup U_2)$, $x_{v_n}(a_n) = p_1$, $x_{v_n}(b_n) = p_2$ and

$$A_{L+c}(x_{v_n}) \leq \Phi_c(p_1, p_2) + \frac{1}{n}. \quad (3.15)$$

Let $u$ be a limit point of $v_n$, then $x_u : \mathbb{R} \to M$ is semistatic (see the proof of claim 2 item (a)). Then, for $a_n \leq s \leq t \leq b_n$,

$$d_c(p_1, p_2) \leq \Phi_c(p_1, x_{v_n}(s)) + \Phi_c(x_{v_n}(s), x_{v_n}(t)) + \Phi_c(x_{v_n}(t), p_2) + \Phi_c(p_2, p_1),$$

therefore

$$d_c(p_1, p_2) \leq \Phi_c(p_2, p_1) + \liminf_{n} [\Phi_c(p_1, x_{v_n}(s)) + \Phi_c(x_{v_n}(s), x_{v_n}(t)) + \Phi_c(x_{v_n}(t), p_2)]$$

$$\leq \Phi_c(p_2, p_1) + \liminf_{n} A_{L+c}(x_{v_n})$$

$$\leq d_c(p_1, p_2) = 0,$$

where in the last inequality we used (3.15). Hence

$$\Phi_c(p_1, x_u(s)) + \Phi_c(x_u(s), x_u(t)) + \Phi_c(x_u(t), p_2) + \Phi_c(p_2, p_1) = 0.$$

Combining the last equation with the triangle inequality we obtain

$$d_c(x_u(s), x_u(t)) \leq$$

$$\leq \Phi_c(x_u(s), x_u(t)) + [\Phi_c(x_u(t), p_2) + \Phi_c(p_2, p_1) + \Phi_c(p_1, x_u(s))] = 0.$$

So that $u \in \bar{\Sigma}$. Moreover, for $s = 0, t = 1$:

$$d_c(x_u(0), p_1) \leq$$

$$\leq \Phi_c(p_1, x_u(0)) + [\Phi_c(x_u(0), x_u(1)) + \Phi_c(x_u(1), p_2) + \Phi_c(p_2, p_1)] = 0.$$

Hence $x_u(0) \in \pi(\Lambda)$. On the other hand $x_u(0) \notin \pi(U_1 \cup U_2)$. This contradicts the fact that $\Lambda \subseteq U_1 \cup U_2$. \qed
Proof of theorem 3-11.1.

Given \( v \in TM \) denote by \( \alpha(v) \) and \( \omega(v) \) its \( \alpha \) and \( \omega \)-limits respectively. By proposition 3-11.4 the static classes are connected. Hence if we assume that there are only finitely many of them, the connected components of \( \tilde{\Sigma} \) are finite and must coincide with the static classes. For \( \varepsilon > 0 \), let \( \tilde{\Sigma}(\varepsilon) \) be the \( \varepsilon \)-neighborhood of \( \tilde{\Sigma} \), i.e.

\[
\tilde{\Sigma}(\varepsilon) := \{ v \in TM \mid d_{TM}(v, \tilde{\Sigma}) < \varepsilon \}.
\]

Fix \( \varepsilon > 0 \) small enough such that the connected components of \( \tilde{\Sigma}(\varepsilon) \) are the \( \varepsilon \)-neighborhoods of the static classes. So that for \( 0 < \delta < \varepsilon \), \( \tilde{\Sigma}(\delta) = \sum_{i=1}^{N(\varepsilon)} \Lambda_i(\delta) \), where \( \Lambda_i(\delta) \) are disjoint open sets containing exactly one static class and the number of components \( N(\varepsilon) \) is fixed for all \( 0 < \delta < \varepsilon \).

Now suppose that the theorem is false. Let \( \Lambda_i, \Lambda_k \in \Lambda \) be such that \( \Lambda_i \not\subseteq \Lambda_k \). Let

\[
A := \bigcup \{ \Lambda_j \mid \Lambda_i \not\subseteq \Lambda_j \}, \quad B := \bigcup \{ \Lambda_j \mid \Lambda_i \not\subseteq \Lambda_j \}.
\]

Given \( v \in \Sigma \) with \( \alpha(v) \subseteq A \) and \( 0 < \delta < \varepsilon \), define inductively

\[
s_1(v) := \inf \{ s \in \mathbb{R} \mid f_s(v) \not\in A(\varepsilon) \},
\]

\[
t_k(v) := \sup \{ t < s_k(v) \mid f_t(v) \in A(\delta) \},
\]

\[
T_k(v) := \inf \{ t > s_k(v) \mid f_t(v) \in A(\delta) \},
\]

\[
s_{k+1}(v) := \inf \{ s > T_k(v) \mid f_s(v) \not\in A(\varepsilon) \};
\]

\[
A_k = A_k(\delta) := \sup \{ |T_k(v) - t_k(v)| \mid v \in \Sigma, \alpha(v) \subseteq A \}.
\]
We split the rest of the proof of theorem 3-11.1 into the following claims:

Claim 1. \( A_k(\delta) < +\infty \) for all \( k = 1, 2, \ldots \) and all \( 0 < \delta < \varepsilon \).

Define
\[
\mathcal{M} := \{ v \mid v \in \Sigma, \alpha(v) \subseteq A \}.
\]

Claim 2.

(a) \( \overline{\mathcal{M}} \cap \mathcal{B} \neq \emptyset \).

(b) \( \lim k \sup A_k(\delta) = \sup_k A_k(\delta) = +\infty \).

Claim 3. There exist sequences \( v_n \in \Sigma \), \( 0 < s_n < t_n \) such that \( v_n \rightarrow u_1 \in A \), \( f_{s_n}(v_n) \rightarrow u_2 \notin A(\varepsilon) \), \( f_{t_n}(v_n) \rightarrow u_3 \in A \) and \( d_c(\pi u_1, \pi u_3) = 0 \).

We now use claim 3 to complete the proof of theorem 3-11.1. If \( u_1 \in \Lambda_j \subseteq A \), we shall prove that \( u_2 \in \Lambda_j \setminus A(\varepsilon) \), obtaining a contradiction and thus proving theorem 3-11.1. It is enough to show that \( d_c(\pi u_1, \pi u_2) = 0 \). Indeed

\[
d_c(\pi u_1, \pi u_2) = \Phi_c(\pi u_1, \pi u_2) + \Phi_c(\pi u_2, \pi u_1)
\leq \Phi_c(\pi u_1, \pi u_2) + \Phi_c(\pi u_2, \pi u_3) + \Phi_c(\pi u_3, \pi u_1)
\leq \lim_n \left[ \Phi_c(\pi v_n, \pi f_{s_n}(v_n)) + \Phi_c(\pi f_{s_n}(v_n), \pi f_{t_n}(v_n)) \right] + \Phi_c(\pi u_3, \pi u_1)
= \lim_n \Phi_c(\pi v_n, \pi f_{t_n}(v_n)) + \Phi_c(\pi u_3, \pi u_1)
= d_c(\pi u_1, \pi u_3) = 0,
\]

where the fourth equation holds because \( v_n \) is a semistatic vector.

\[\Box\]

Proof of claim 1:

Suppose that \( A_i < +\infty \) for \( i = 1, \ldots, k - 1 \) and \( A_k = +\infty \). The case \( k = 1 \) is similar. Then there exists \( v_n \in \Sigma \), with \( \alpha(v_n) \subseteq A \) and \( T_k(v_n) - t_k(v_n) \rightarrow +\infty \). We can assume that \( t_k(v_n) = 0 \) and that \( v_n \)
converges ($\Sigma$ is compact). Let $u = \lim_n v_n \in \partial A(\varepsilon)$. Then

$$m\{ t < 0 \mid f_t(v_n) \notin A(\varepsilon) \} \leq \sum_{i=1}^{k-1} A_i,$$

where $m$ is the Lebesgue measure on $\mathbb{R}$. This implies that

$$m\{ t < 0 \mid f_t(u) \notin A(\varepsilon) \} \leq \sum_{i=1}^{k-1} A_i$$

and hence $\alpha(u) \subset A$. Since $f_t(v_n) \notin A(\varepsilon)$ for $0 < t < T_k(v_n)$ and $T_k(v_n) \to +\infty$, then $f_t(u) \notin A(\varepsilon)$ for all $t > 0$ and hence $\omega(u) \subset B$. But then the orbit of $u$ contradicts the definition of $B$.

□

Proof of claim 2:

(a) Let $p \in \pi A$, $q \in \pi B$. For $n > 0$, let $x_{v_n} : [a_n, b_n] \to M$ be a solution of (E-L) such that $x_{v_n}(a_n) = p$, $x_{v_n}(b_n) = q$ and

$$A_{L+c}(x_{v_n}) \leq \Phi_c(p, q) + \frac{1}{n}.$$

This implies that

$$A_{L+c}(x_{v_n}|_{[s, t]}) \leq \Phi_c(x_{v_n}(s), x_{v_n}(t)) + \frac{1}{n} \quad (3.16)$$

for all $a_n \leq s \leq t \leq b_n$. We can assume that

$$\inf\{ s > a_n \mid x_{v_n}(s) \in B(\delta) \} = 0,$$

and that the sequence $v_n$ converges (cf. lemma 3-2.3). Let $u = \lim_n v_n \in \partial B(\delta)$. Taking limits in (3.16) we obtain that $x_u|_{[s, t]}$ is semistatic for all $\lim \inf_n a_n \leq s \leq t \leq \lim \sup_n b_n$.

Any limit point $w$ of $\dot{x}_{v_n}(a_n) = f_{a_n}(v_n)$ satisfies $\pi(w) = p \in \pi A$, and by the graph property (theorem 3-8.1), $w \in A$. Similarly, any limit point of $f_{b_n}(v_n)$ is in $B$. Since $A \cup B$ is invariant and $u \notin A \cup B$, then $\lim_n a_n = -\infty$, $\lim_n b_n = +\infty$. Hence $u \in \Sigma$. Since $f_t(v_n) \notin B(\delta)$ for all $a_n \leq t < 0$ and $a_n \to -\infty$, then $f_t(u) \notin B(\delta)$ for all $t < 0$. Hence
\( \alpha(u) \subseteq A \) and \( u \in B(\delta) \). Thus \( u \in M \cap B(\delta) \neq \emptyset \). Letting \( \delta \to 0 \), we obtain that \( M \cap B \neq \emptyset \).

(b) By claim 1 it is enough to show that \( \sup_k A_k(\delta) = +\infty \). If \( \sup_k A_k(\delta) < T \), then \( M \subseteq M(\delta, T) \), where

\[
M(\delta, T) = \{ v \in \Sigma \mid f_{[-T,T]}(v) \cap A(\delta) \neq \emptyset \}.
\]

Then \( M \cap B \subseteq M(\delta, T) \cap B = \emptyset \), because \( B \) is invariant and \( B \cap A(\delta) = \emptyset \). This contradicts item (a).

\[ \square \]

Proof of claim 3:

Given \( 0 < \delta < \varepsilon \), by claim 2(b) there exists \( k > N(\varepsilon) \) such that \( A_k(\delta) > 0 \). Hence there is \( v = v_\delta \in \Sigma \) with \( \alpha(v) \subseteq A \), such that the orbit of \( v \) leaves \( A(\varepsilon) \) and returns to \( A(\delta) \) at least \( k \) times. Since \( k > N(\varepsilon) \) there is one component \( \Lambda_j(\delta) \subseteq A(\delta) \) with two of these returns, i.e. there exist \( \tau_1(\delta) < s(\delta) < \tau_2(\delta) \) with \( f_{\tau_1}(v) \in \Lambda_j(\delta) \), \( f_s(v) \notin A(\varepsilon) \) and \( f_{\tau_2}(v) \in \Lambda_j(\delta) \). Now, there exists a sequence such that the repeated component \( \Lambda_j \subseteq \Lambda_j(\delta_n) \) is always the same. Let \( s_n := s(\delta_n) \), \( t_n := \tau_2(\delta_n) \) and choose a subsequence such that \( f_{\tau_1(\delta_n)}(v_n) \), \( f_{s_n}(v_n) \) and \( f_{t_n}(v_n) \) converge. Let \( u_1 = \lim_n f_{\tau_1(\delta_n)}(v_n) \in \cap_n \Lambda_j(\delta_n) = \Lambda_j \), \( u_3 = \lim_n f_{t_n}(v_n) \in \Lambda_j \) and \( u_2 = \lim_n f_{s_n}(v_n) \notin A(\varepsilon) \). Since \( u_1, u_3 \in \Lambda_j \), then \( d_c(\pi u_1, \pi u_3) = 0 \).

\[ \square \]
Chapter 4

Dynamics on prescribed energy levels.

4-1 The Hamilton-Jacobi equation.

Let $\omega$ be the canonical symplectic form on $T^*M$. A subspace $\lambda$ of $T_pT^*M$ is called isotropic if $\omega(X, Y) = 0$ for all $X, Y$ on $\lambda$. Since $\omega$ is nondegenerate, the isotropic subspaces have dimension $\leq n$, half of the dimension of $T^*M$. Isotropic spaces of dimension $n$ are called lagrangian subspaces. We say that a submanifold $W \subset T^*M$ is lagrangian if at each point $\theta \in W$, its tangent space $T_\theta W$ is a lagrangian subspace of $T_\theta T^*M$. In particular, $\dim W = \dim M = n$.

4-1.1. Theorem (Hamilton-Jacobi).

If the hamiltonian $H$ is constant on a lagrangian submanifold $N$, then $N$ is invariant under the hamiltonian flow.

Proof: We only have to show that the hamiltonian vectorfield $X$ is tangent to $N$. Since $H$ is constant on $N$, then $dH|_{TN} \equiv 0$. Since $\omega(X, \cdot) = dH$, then $\omega(X, \xi) = 0$ for all $\xi \in TN$. Since the tangent spaces to $N$ are lagrangian, they are maximal isotropic subspaces, therefore $X \in TN$. $\Box$
Some distinguished $n$-dimensional manifolds on $T^*M$ are the graph submanifolds, which are of the form

$$G_{\eta} = \{ (x, \eta_x) \mid x \in M \} \subset T^*M,$$

(4.1)

where $\eta_x$ is a 1-form on $M$. A lagrangian graph is a lagrangian graph submanifold. In fact,

4-1.2. Lemma. $G_{\eta}$ is a lagrangian graph if and only if the form $\eta$ is closed:

$$G_{\eta} \text{ is lagrangian } \iff d\eta \equiv 0$$

Proof: Choose local coordinates $q_1, \ldots, q_n$ of $M$. Then $\eta(q) = \sum_k p_k(q) dq_k$. A basis of the tangent space to the graph $G_{\eta}$ is given by $E_i = \left( \frac{\partial}{\partial q_i}, \sum_k \frac{\partial p_k}{\partial q_i}, \frac{\partial}{\partial p_k} \right)$. Applying $\omega = dp \wedge dq$,

$$\omega(E_i, E_j) = \frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i}.$$ 

Since

$$d\eta = \sum_{i<j} \left( \frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i} \right) dq_j \wedge dq_i,$$

then $\omega|_{TG_{\eta}} \equiv 0 \iff d\eta \equiv 0$.

Thus, we can associate a cohomology class $[\eta] \in H^1(M, \mathbb{R})$ to each lagrangian graph $G_{\eta}$. Lagrangian graphs with zero cohomology class are the graphs of the exact 1-forms: $G_{df}$, with $\eta = df$ and $f : M \to \mathbb{R}$ a smooth function. These are called exact lagrangian graphs.

The Hamilton-Jacobi equation for autonomous hamiltonians is

$$H(x, d_xu) = k, \quad u : M \to \mathbb{R}.$$  

(H-J)

Thus a smooth solution of the Hamilton-Jacobi equation corresponds to an exact invariant lagrangian graph.
4-2 Subsolutions of the Hamilton-Jacobi equation.

We say that a function \( u : M \to \mathbb{R} \) is a subsolution of the Hamilton-Jacobi equation if
\[
H(x, d_x u) \leq k,
\]
i.e. if \( d_x u \) is an exact lagrangian graph. We shall prove that for \( k > c(L) \) there is always a \( C^\infty \) subsolution of the Hamilton-Jacobi equation and for \( k < c(L) \) there are no differentiable subsolutions. Hence

4-2.1. Theorem. If \( M \) is any covering of the closed manifold \( N \), then
\[
c(L) = \inf_{f \in C^\infty(M, \mathbb{R})} \sup_{x \in M} H(x, d_x f)
= \inf \{ k \in \mathbb{R} : \text{there exists } f \in C^\infty(M, \mathbb{R}) \text{ such that } H(df) < k \},
\]
where \( H \) is the hamiltonian associated with \( L \).

Theorem 4-2.1 could be restated by saying that \( c(L) \) is the infimum of the values of \( k \in \mathbb{R} \) for which \( H^{-1}(-\infty, k) \) contains an exact lagrangian graph. This is a very geometric way of describing the critical value.

Theorem 4-2.1 will be an immediate consequence of lemma 4-2.2 and proposition 4-2.4 below.

4-2.2. Lemma.
If there exists a \( C^1 \) function \( f : M \to \mathbb{R} \) such that \( H(df) < k \), then \( k \geq c(L) \).

Proof: Recall that
\[
H(x, p) = \max_{v \in T_x M} \{ p(v) - L(x, v) \}.
\]
Since \( H(df) < k \) it follows that for all \( (x, v) \in TM \),
\[
d_x f(v) - L(x, v) < k.
\]
Therefore, if \( \gamma : [0, T] \to M \) is any absolutely continuous closed curve with \( T > 0 \), we have

\[
\int_0^T (L(\gamma, \dot{\gamma}) + k) \, dt = \int_0^T (L(\gamma, \dot{\gamma}) + k - d_\gamma f(\dot{\gamma})) \, dt > 0,
\]

and thus \( k \geq c(L) \). \( \square \)

### 4.2.3. Lemma.

Let \( k \geq c(L) \). If \( f : M \to \mathbb{R} \) is differentiable at \( x \in M \) and satisfies

\[
f(y) - f(x) \leq \Phi_k(x, y)
\]

for all \( y \) in a neighbourhood of \( x \), then \( H(x, d_x f) \leq k \).

**Proof:** Let \( \gamma(t) \) be a differentiable curve on \( M \) with \( (\gamma(0), \dot{\gamma}(0)) = (x, v) \). Then

\[
\limsup_{t \to 0^+} \frac{f(\gamma(t)) - f(x)}{t} \leq \liminf_{t \to 0^+} \frac{1}{t} \int_0^t [L(\gamma, \dot{\gamma}) + k] \, ds.
\]

Hence \( d_x f(v) \leq L(x, v) + k \) for all \( v \in T_x M \) and thus

\[
H(x, d_x f) = \max_{v \in T_x M} \{ d_x f(v) - L(x, v) \} \leq k.
\]

\( \square \)

### 4.2.4. Proposition.

For any \( k > c(L) \) there exists \( f \in C^\infty(M, \mathbb{R}) \) such that \( H(df) < k \).

**Proof:** We shall explain first how to prove the proposition in the case of \( M = \mathbb{R} \) and then we will lift the construction to an arbitrary covering \( M \).

Set \( c = c(L) \). Fix \( q \in M \) and let \( u(x) := \Phi_c(q, x) \). By the triangle inequality, we have that

\[
u(y) - u(x) \leq \Phi_c(x, y) \text{ for all } x, y \in M.
\]
By the previous lemma, \( H(d_x u) \leq c \) at any point \( x \in M \) where \( u(x) \) is differentiable.

We proceed to regularize \( u \). We can assume that \( M \subseteq \mathbb{R}^N \). Let \( U \) be a tubular neighbourhood of \( M \) in \( \mathbb{R}^N \), and \( \rho : U \to M \) a \( C^\infty \) projection along the normal bundle. Extend \( u(x) \) to \( U \) by \( \overline{u}(z) = u(\rho(z)) \). Then \( \overline{u}(z) \) is also Lipschitz.

Extend the lagrangian to \( U \) by

\[
\overline{L}(z, v) = L(\rho(z), d_z \rho(v)) + \frac{1}{2} |v - d_z \rho(v)|^2 .
\]

Then the corresponding hamiltonian satisfies \( \overline{H}(z, p \circ d_z \rho) = H(\rho(z), p) \) for \( p \in T^*_\rho(z) M \). At any point of differentiability of \( \overline{u} \), we have that \( d_z \overline{u} = d_{\rho(z)} u \circ d_z \rho \), and \( \overline{H}(d_z \overline{u}) = H(d_{\rho(z)} u) \leq c \).

Let \( \varepsilon > 0 \) be such that

(a) The \( 3\varepsilon \)-neighbourhood of \( M \) in \( \mathbb{R}^N \) is contained in \( U \).

(b) If \( x \in M \), \( (y, p) \in T^* \mathbb{R}^N = \mathbb{R}^{2N} \), \( \overline{H}(y, p) \leq c \) and \( d_{\mathbb{R}^N}(x, y) < \varepsilon \) then \( H(x, p) < k \).

Let \( \psi : \mathbb{R} \to \mathbb{R} \) be a \( C^\infty \) function such that \( \psi(x) \geq 0 \), \( \text{support}(\psi) \subseteq (-\varepsilon, \varepsilon) \) and \( \int_{\mathbb{R}^N} \psi(|x|) \, dx = 1 \). Let \( K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) be \( K(x, y) = \psi(|x - y|) \). Let \( N_\varepsilon \) be the \( \varepsilon \)-neighbourhood of \( M \) in \( \mathbb{R}^N \). Define \( f : N_\varepsilon \to \mathbb{R} \) by

\[
f(x) = \int_{\mathbb{R}^N} \overline{u}(y) \, K(x, y) \, dy .
\]

Then \( f \) is \( C^\infty \) on \( N_\varepsilon \).

Observe that \( \partial_x K(x, y) = -\partial_y K(x, y) \). Since \( \overline{u}(y) \) is Lipschitz, it is differentiable at Lebesgue almost every point of \( U \) (Rademacher's theorem, cf. [14]). Moreover it is weakly differentiable (cf. [14, Section 4.2.3]), that is, for any \( C^\infty \) function \( \varphi : U \to \mathbb{R} \) with compact support

\[
\int_{\mathbb{R}^N} (\varphi \, d\overline{u} + \overline{u} \, d\varphi) \, dx = 0 .
\]
Hence
\[- \int_{\mathbb{R}^N} \overline{u}(y) \, \partial_y K(x, y) \, dy = \int_{\mathbb{R}^N} K(x, y) \, d_y \overline{u} \, dy.\]

Now, since
\[d_x f = \int_{\mathbb{R}^N} \overline{u}(y) \, \partial_x K(x, y) \, dy,\]
we obtain
\[d_x f = \int_{\mathbb{R}^N} K(x, y) \, d_y \overline{u} \, dy.\]

From the choice of \(\varepsilon > 0\) we have that \(\overline{H}(x, d_y \overline{u}) < k\) for almost every \(y \in \text{supp} K(x, \cdot)\) and \(x \in M\). Since \(K(x, y) \, dy\) is a probability measure, by Jensen's inequality
\[H(d_x f) \leq \overline{H}(d_x f) \leq \int_{\mathbb{R}^N} \overline{H}(x, d_y \overline{u}) \, K(x, y) \, dy < k,\]
for all \(x \in M\).

Now, suppose that \(M\) is a covering of a compact manifold \(N\) with covering projection \(p\). Assume that \(N \subseteq \mathbb{R}^N\). Fix \(q \in M\) and set \(u(x) := \Phi_{c(L)}(q, x)\). We can regularize our function \(u\) similarly as we shall now explain. For \(\bar{x} \in M\) let \(x\) be the projection of \(\bar{x}\) to \(N\) and let \(\mu_x\) be the Borel probability measure on \(N\) defined by
\[\int_N \varphi \, d\mu_x = \int_{\mathbb{R}^N} (\varphi \circ p)(y) \, K(x, y) \, dy,\]
for any continuous function \(\varphi : N \to \mathbb{R}\). Then the support of \(\mu_x\) satisfies
\[\text{supp}(\mu_x) \subseteq \{y \in N : d_N(x, y) < \varepsilon\}.\]
Let \(\widehat{\mu}_{\bar{x}}\) be the Borel probability measure on \(M\) uniquely defined by the conditions: \(\text{supp}(\widehat{\mu}_{\bar{x}}) \subseteq \{\bar{y} \in M : d_M(\bar{x}, \bar{y}) < \varepsilon\}\) and \(p_* \widehat{\mu}_{\bar{x}} = \mu_x\). Then we have
\[\frac{d}{d\bar{x}} \int_M \varphi \, d\widehat{\mu}_{\bar{x}} = \int_M d\bar{y} \varphi \, d\widehat{\mu}_{\bar{x}}(\bar{y}),\]
for any weakly differentiable function $\varphi : M \to \mathbb{R}$. The same arguments as above show that

$$f(\bar{x}) = \int_M u(\bar{y}) \, d\mu_{\bar{x}}(\bar{y}),$$

satisfies $H(d^\perp \bar{x}) < k$. \hfill \Box

4-2.5. Corollary.

$$\alpha(q) = \inf_{[\omega] = q} \sup_{x \in N} H(x, \omega(x)).$$

**Proof:** Let us fix a closed one form $\omega_0$ such that $[\omega_0] = q$. By equality (2.29) we have that $\alpha(q) = c(\lambda - \omega_0)$. Hence, it suffices to show that

$$c(\lambda - \omega_0) = \inf_{[\omega] = q} \sup_{x \in N} H(x, \omega(x)). \quad (4.2)$$

It is straightforward to check that the hamiltonian associated with $\lambda - \omega_0$ is $H(x, p + \omega_0(x))$. Since all the closed one forms in the class $q$ are given by $\omega_0 + df$ where $f$ ranges among all smooth functions, equality (4.2) is now an immediate consequence of theorem 4-2.1. \hfill \Box

4-2.6. Corollary. If $k > c(L)$, then it is possible to see the dynamics of $\phi_t|_{E^{-1}(k)}$ as the reparametrization of the geodesic flow on the unit tangent bundle of an appropriately chosen Finsler metric on $M$.

**Proof:** If $k > c(L)$, then $H^{-1}(-\infty, k)$ contains an exact lagrangian graph. This means that there exists a smooth function $f : M \to \mathbb{R}$ such that $H(x, d_x f) < k$ for all $x \in M$. Therefore the new hamiltonian $H_{df}(x, p) \overset{\text{def}}{=} H(x, p + d_x f)$ is such that $H^{-1}_{df}(-\infty, k)$ contains the zero section of $T^*M$. Let $\varphi : T^*M \to T^*M$ be the map $\varphi(x, p) = (x, p + d_x f)$. Observe that the hamiltonian flow $\phi^*_t$ of $H$ and the hamiltonian flow $\psi_t$ of $H_{df}$ are related by $\psi_t \circ \varphi = \varphi \circ \phi^*_t$. Define now a new hamiltonian $G$ on $T^*M$ minus the zero section such that $G$ takes the value one on
$H^{-1}_{df}(k)$ and such that $G(x, \lambda p) = \lambda^2 G(x, p)$ for all positive $\lambda$. Since $G$ is positively homogeneous of degree two and convex in $p$, it follows that the Legendre transform $\mathcal{L}_G$ associated to $G$ is a diffeomorphism from $TM$ minus the zero section to $T^*M$ minus the zero section. Therefore the hamiltonian $G$ induces a Finsler metric on $M$ simply by taking $G \circ \mathcal{L}_G$.

Since by definition $G^{-1}(1) = H^{-1}_{df}(k)$ it follows that the hamiltonian flows of $G$ and $H^{-1}_{df}(k)$ coincide up to reparametrization on the energy level $G^{-1}(1) = H^{-1}_{df}(k)$ and therefore the Euler-Lagrange solutions of $L$ with energy $k$ are reparametrizations of unit speed geodesics of $G \circ \mathcal{L}_G$. $\square$

Given a Finsler metric $\sqrt{F}$ and an absolutely continuous curve $\gamma$ we can define its Finsler length as

$$l_F(\gamma) = \int \sqrt{F(\dot{\gamma})}$$

Observe that since the Finsler metric is homogeneous of degree one, the definition does not depend on the parametrization of the curve. Finally we define the Finsler distance as

$$D_F(x, y) = \inf\{l_F(\gamma)\}$$

where the infimum is taken over all absolutely continuous curves joining $x$ and $y$.

4-2.7. Corollary. [24] If $k > c(L)$, then there exists a Finsler metric $\sqrt{F}$ and a $C^\infty$ real valued function $f$ on $M$ such that $\Phi_k(x, y) = D_F(x, y) + f(y) - f(x)$. Moreover if $k > -\inf L$, then we can choose $f = 0$.

Proof: We begin with the last statement. Note that $L + k > 0$ if and only if $H(x, 0) < k$. Indeed

$$H(x, p) = \max_{v \in T_xM} (pv - L(x, v))$$
then

\[ H(x, 0) = \max_{v \in T_x M} (-L(x, v)) = -\min(L(x, v)). \]

So if \( k > -\inf L \) then \( H^{-1}(-\infty, k) \) contains the zero section of \( T^*M \).

Define as before a Hamiltonian \( G \) on \( T^*M \) minus the zero section such that \( G \) takes the value \( \mu \) on \( H^{-1}(k) \) and such that \( G(x, \lambda p) = \lambda^2 G(x, p) \) for all positive \( \lambda \). Let \( F \) be the associated Finsler metric on \( M \) by the Legendre transformation.

We claim that for an appropriate choice of \( \mu \) and if \( E(x, v) = k \) then

\[ \sqrt{F(x, v)} = L + k. \]

From proposition 3-5.1, for \( k > c(L) \) and for any \( x, y \in M \) there exists \( \gamma \) such that \( A_{L+k}(\gamma) = \Phi_k(x, y) \). Moreover \( \gamma \) is a solution of the Euler-Lagrange equation and has energy \( k \). Also, if \( k > c(L) \), every curve can be reparametrized to have energy \( k \) and the Finsler length does not depend on the reparametrization. So in the definitions of \( D_F \) and \( \Phi_k \) we may restrict ourselves to curves with energy \( k \) and theorem 4-2.7 follows in this case.

**Proof of the claim:** Since \( G \) is homogeneous of degree 2 it follows from Euler’s formula that \( F \) and \( G \) take the same value at Legendre related points.

Define \( \lambda(x, p) \) such that \( H(x, \frac{p}{\lambda}) = k \), then \( G(x, p) = \mu \lambda^2(x, p) \).

We have that

\[
\frac{\partial H}{\partial p}(x, \frac{p}{\lambda}) \lambda^{-1} - \frac{\partial H}{\partial p}(x, \frac{p}{\lambda}) \cdot p \lambda^{-2} \frac{\partial \lambda}{\partial p} = 0 \quad (4.3)
\]

and

\[
\frac{\partial G}{\partial p} = 2 \mu \lambda \frac{\partial \lambda}{\partial p}.
\]

Then using (4.3) multiplied by \( \lambda^2 \) we get

\[
\frac{\partial H}{\partial p}(x, \frac{p}{\lambda}) \cdot p \frac{\partial G}{\partial p} = 2 G(x, p) \frac{\partial H}{\partial p}(x, \frac{p}{\lambda}). \quad (4.4)
\]
Suppose now that \( E(x, v) = k \) and define \( P(x, v) = \frac{\partial L}{\partial v} \) then by definition we have

\[
\begin{align*}
\lambda(x, P(x, v)) &= 1, \quad (4.5) \\
G(x, P(x, v)) &= \mu, \quad (4.6) \\
\frac{\partial H}{\partial p}(x, P(x, v)) &= v, \quad (4.7)
\end{align*}
\]

and so

\[
\frac{\partial H}{\partial p}(x, P(x, v)) \cdot P(x, v) = v \frac{\partial L}{\partial v} = L + k \quad \text{(4.8)}
\]

\[
> 0. \quad \text{(4.10)}
\]

Hence from (4.4) we have

\[
\frac{\partial G}{\partial p}(x, P(x, v)) = \frac{2v}{v \cdot P(x, v)}. \quad \text{(4.9)}
\]

Then since \( \frac{\partial G}{\partial p} \) is homogeneous of degree one and from (4.8) \( v \cdot P(x, v) \) is positive we obtain

\[
\frac{\partial G}{\partial p}(x, [\frac{1}{2}v \cdot P(x, v)] P(x, v)) = v.
\]

So \( v \) is related to \( \frac{1}{2}v \cdot P(x, v)P(x, v) \) with respect to the Legendre transform of \( F \). Hence

\[
F(x, v) = G(x, \frac{1}{2}v \cdot P(x, v)P(x, v))
\]

\[
= \frac{(v \cdot P(x, v))^2}{4} G(x, P(x, v))
\]

\[
= \frac{(v \cdot P(x, v))^2}{4} \mu.
\]

So if \( \mu = 4 \)

\[
\sqrt{(F(x, v))} = v \cdot \frac{\partial L}{\partial v}
\]
Let now \( k > c(L) \), then by proposition 4-2.4 there exists a \( C^\infty \) real valued function on \( M \) \( f \), such that \( H(x, df_x) < k \). Define as before \( H_{df}(x, p) = H(x, p + df_x) \). The Legendre transformation \( L_{df} \) of \( H_{df} \) is

\[
L_{df}(x, v) = \max_{p \in T^*_x M} (pv - H_{df}(x, p))
= \max_{p \in T^*_x M} (pv - H(x, p + df_x))
= \max_{p \in T^*_x M} ((p - df_x)v - H(x, p))
= L(x, v) - df_x v.
\]

It turns out that

\[
E(L_{df}) = E(L),
\]
\[
c(L_{df}) = c(L),
\]
\[
\Phi_k(L_{df})(x, y) = \Phi_k(L)(x, y) - f(y) + f(x).
\]

So as the zero section is contained in \( H_{df}^{-1}(-\infty, k) \), \( L_{df} + k \) is positive and there is a Finsler metric such that

\[
\Phi_k(L_{df})(x, y) = D_F(x, y).
\]

So

\[
\Phi_k(L)(x, y) = D_F(x, y) + f(y) - f(x).
\]

\[
\square
\]
4-3 Anosov energy levels.

An Anosov energy level is a regular energy level on which the flow $\phi_t$ is an Anosov flow.

4-3.1. Theorem. If the energy level $E^{-1}(k)$ is Anosov, then

$$k > c_u(\lambda).$$

Proof: Suppose that the energy level $k$ is Anosov and set $\Sigma \overset{\text{def}}{=} \mathbb{H}^{-1}(k)$. Let $\pi : T^*N \rightarrow N$ denote the canonical projection. G.P. Paternain and M. Paternain proved in [49] that $\Sigma$ must project onto the whole manifold $N$ and that the weak stable foliation $\mathcal{W}^s$ of $\phi^*_t$ is transverse to the fibers of the fibration by $(n - 1)$-spheres given by

$$\pi|_\Sigma : \Sigma \rightarrow N.$$

Let $\tilde{N}$ be the universal covering of $N$. Let $\tilde{\Sigma}$ denote the energy level $k$ of the lifted hamiltonian $H$. We also have a fibration by $(n - 1)$-spheres

$$\tilde{\pi}|_{\tilde{\Sigma}} : \tilde{\Sigma} \rightarrow \tilde{N}.$$

Let $\tilde{\mathcal{W}}^s$ be the lifted foliation which is in turn a weak stable foliation for the hamiltonian flow of $H$ restricted to $\tilde{\Sigma}$. The foliation $\tilde{\mathcal{W}}^s$ is also transverse to the fibration $\tilde{\pi}|_{\tilde{\Sigma}} : \tilde{\Sigma} \rightarrow \tilde{N}$. Since the fibers are compact a result of Ehresman (cf. [3]) implies that for every $(x, p) \in \tilde{\Sigma}$ the map

$$\tilde{\pi}|_{\tilde{\mathcal{W}}^s(x, p)} : \tilde{\mathcal{W}}^s(x, p) \rightarrow \tilde{N},$$

is a covering map. Since $\tilde{N}$ is simply connected, $\tilde{\pi}|_{\tilde{\mathcal{W}}^s(x, p)}$ is in fact a diffeomorphism and $\tilde{\mathcal{W}}^s(x, p)$ is simply connected. Consequently, $\tilde{\mathcal{W}}^s(x, p)$ intersects each fiber of the fibration $\tilde{\pi}|_{\tilde{\Sigma}} : \tilde{\Sigma} \rightarrow \tilde{N}$ at just one point. In other words, each leaf $\tilde{\mathcal{W}}^s(x, p)$ is the graph of a one form. On the other hand it is well known that the weak stable leaves of an Anosov
energy level are lagrangian submanifolds. Since any closed one form in the universal covering must be exact, it follows that each leaf $\tilde{W}^s(x, p)$ is an exact lagrangian graph. The theorem now follows from lemma 4-2.2 and the fact that by structural stability there exists $\varepsilon > 0$ such that for all $k' \in (k - \varepsilon, k + \varepsilon)$ the energy level $k'$ is Anosov.  

For $e \in \mathbb{R}$, let $\mathcal{A}_e$ be the set of $\phi \in C^\infty(M)$ such that the flow of $H + \phi$ is Anosov in $(H + \phi)^{-1}(e)$ and let $\mathcal{B}_e$ be the set of $\phi \in C^\infty(M)$ such that $(H + \phi)^{-1}(e)$ contains no conjugate points. As is well known $\mathcal{A}_e$ is open in $C^k$ topology and $\mathcal{B}_e$ is closed. On the other hand G. and M. Paternain [47] have shown that $\mathcal{A}_e$ is contained in $\mathcal{B}_e$. It is proved in [9] the following

4-3.2. **Theorem.** The interior of $\mathcal{B}_e$ in the $C^2$ topology is $\mathcal{A}_e$.

This theorem is an extension to the Hamiltonian setting of a result of R. O. Ruggiero for the geodesic flow [55]. Theorems 4-3.2 and 4-3.1 have as corollary:

4-3.3. **Corollary.** Given a convex superlinear lagrangian $L$, $k < c_u(L)$ and $\varepsilon > 0$ there exists a smooth function $\psi : N \to \mathbb{R}$ with $|\psi|_{C^2} < \varepsilon$ and such that the energy level $k$ of $L + \psi$ possesses conjugate points.

**Proof:** Suppose now that there exists $\varepsilon > 0$ such that for every $\psi$ with $|\psi|_{C^2} < \varepsilon$, the energy level $k$ of $\lambda + \psi$ has no conjugate points. The main result in [9] says that in this case the energy level $k$ of $\lambda$ must be Anosov thus contradicting theorem 4-3.1.  

4-3.4. **Proposition.** If $k$ is a regular value of the energy such that $k < e$, then the energy level $k$ has conjugate points.

**Proof:** If an orbit does not have conjugate points then there exist along it two subbundles called the **Green subbundles**. They have the following properties: they are invariant, lagrangian and they have dimension $n = \dim N$. Moreover, they are contained in the same energy level as the orbit and they do not intersect the vertical subbundle (cf. [7]). If $k$ is
a regular value of the energy with \( k < e \), then \( \pi(\mathcal{E}^{-1}(k)) \) is a manifold with boundary and at the boundary the vertical subspace is completely contained in the energy level. Therefore the orbits that begin at the boundary must have conjugate points, because at the boundary two \( n \)-dimensional subspaces contained in the energy level (which is \( (2n - 1) \)-dimensional) must intersect.

\[ \square \]

4-4 Weak KAM Solutions.

Given a continuous function \( u : M \to \mathbb{R} \), we shall write \( u < L + c \) whenever \( u(x) - u(y) \leq \Phi_c(y, x) \) for all \( x, y \in M \). Let us define the sets

\[
\Gamma_0^+(u) := \{ v \in \Sigma^+ | u(x_v(t)) - u(x_v(0)) = \Phi_c(x_v(0), x_v(t)) \ \forall t > 0 \},
\]

\[
\Gamma^+(u) := \bigcup_{t > 0} \phi_t(\Gamma^+(u)),
\]

where \( \phi_t \) is the Euler-Lagrange flow on \( TM \).

We shall say, following Fathi, that a continuous function \( u : M \to \mathbb{R} \) is a weak KAM solution if \( u \) satisfies the following three conditions:

1. \( u \) is Lipschitz;

2. \( u < L + c; \)

3. \( \pi(\Gamma_0^+(u)) = M. \)

It is important to point out that using the action potentials it is quite simple to show the existence of a function \( u \) that satisfies only properties 1 and 2 above. Take any point \( p \in M \) and set \( u(x) = \Phi_c(p, x) \). Elementary properties of the action potential (cf. chapter 3) show that \( u \) satisfies 1 and 2. This is used in the proof of theorem 4-2.1 which is based on a convolution argument that smoothes out a function \( u \) that satisfies 1 and 2.

Fathi shows in [16] that weak KAM solutions exist assuming that \( M \) is compact. His proof is based on applying the Banach fixed point
theorem to a certain semigroup of operators defined on the space of continuous functions on $M$ divided by the constant functions and, as presented, it cannot be applied when $M$ is not compact. In our next theorem we show the existence of weak KAM solutions for $M$ an arbitrary covering of a compact manifold. We hope that this will clarify the relationship between Fathi’s approach and Mañé’s.

Given a semistatic vector $w \in \Sigma^+$, let $\gamma(t) = x_w(t)$ and define $u : M \to \mathbb{R}$ by

$$u(x) = \sup_{t>0} [\Phi_c(\gamma(0), \gamma(t)) - \Phi_c(x, \gamma(t))]. \quad (4.11)$$

The function $u$ thus defined clearly resembles the Busemann functions from Riemannian geometry. In fact, the supremum in (4.11) is a limit as we shall see in Section 5, thus if $\omega$-limit $\omega(w) \neq \emptyset$, then $u(x) = u(p) - \Phi_c(x, p)$ for all $x \in M$ and any $p$ in $\pi(\omega(w))$.

4-4.1. Theorem. The function $u(x)$ in (4.11) is a weak KAM solution.

Hence when $M$ is compact, since $\omega(w) \neq \emptyset$ for any semistatic vector $w$, the function $u(x) = -\Phi_c(x, p)$, where $p$ is any point in the static class of $\pi(\omega(w))$, is a weak KAM solution.

Recall the set

$$\Sigma^+ := \{ v \in TM \mid x_v : [0, +\infty) \to M \text{ is semistatic} \}$$

For the proof of theorem 4-4.1 we need the following

4-4.2. Lemma. $\Sigma^+ \neq \emptyset$.

Proof: If $M$ is compact then there exists a minimizing measure $\mu$ then it follows by corollary 3-6.2
Assume that $M$ is not compact. Then there is a sequence $\{q_n\} \subset M$ such that $d_M(q_0, q_n) \to +\infty$. Let $x_n : [0, T_n] \to M$ be a Tonelli minimizer such that $x_n(0) = q_0$, $x_n(T_n) = q_n$ and

$$A_{L+c}(x_n) \leq \Phi_c(q_0, q_n) + \frac{1}{n}. \quad (4.12)$$

Since for any $x : \mathbb{R} \to M$, the function $\delta(t) = A_{L+c}(x|_{[0, t]}) - \Phi_c(x(0), x(t))$ is non-decreasing, inequality (4.12) implies that

$$A_{L+c}(x_n|_{[s, t]}) \leq \Phi_c(x_n(s), x_n(t)) + \frac{1}{n} \quad (4.13)$$

for all $0 \leq s \leq t \leq T_n$.

By lemma 3-2.1, $|\dot{x}_n(t)| < A$ for all $n$ large enough, $0 \leq t \leq T_n$. Let $v_n = \dot{x}_n(0)$ and $v$ an density point of $\{v_n\}$. We can assume that $v_n \to v$.

Since $d_M(q_0, q_n) \to +\infty$, then $T_n \to +\infty$. Since $x_n|_{[0, t]} \xrightarrow{C^1} x_v|_{[0, t]}$ for all $t > 0$, from (4.13) we obtain that $x_v : [0, +\infty) \to M$ is semistatic. \quad \Box

**Proof of theorem 4-4.1**

Our candidate for a weak KAM solution is defined as follows. Given a semistatic vector $w \in \Sigma^+$, let $\gamma(t) = x_w(t)$ and define $u : M \to \mathbb{R}$ by

$$u(x) = \sup_{t > 0} \{\Phi_c(\gamma(0), \gamma(t)) - \Phi_c(x, \gamma(t))\}.$$

By the triangle inequality, for $x \in M$ and $t > 0$,

$$\Phi_c(\gamma(0), \gamma(t)) - \Phi_c(x, \gamma(t)) \leq \Phi_c(\gamma(0), x) + \Phi_c(x, \gamma(t)) - \Phi_c(x, \gamma(t)) = \Phi_c(\gamma(0), x).$$

Hence $u(x) \leq \Phi_c(\gamma(0), x) < +\infty$. Moreover, the function $\delta(t) := \Phi_c(\gamma(0), \gamma(t)) - \Phi_c(x, \gamma(t))$ is increasing in $t$ because if $0 \leq s \leq t$, then

$$\delta(t) - \delta(s) = \Phi_c(\gamma(0), \gamma(t)) - \Phi_c(\gamma(0), \gamma(s)) - \Phi_c(x, \gamma(t)) + \Phi_c(x, \gamma(s))$$

$$= \Phi_c(\gamma(0), \gamma(s)) + \Phi_c(\gamma(s), \gamma(t)) - \Phi_c(x, \gamma(t)) - \Phi_c(x, \gamma(s))$$

$$\geq 0.$$
In the last inequality we used the triangle inequality for the triangle \((x, \gamma(s), \gamma(t))\). Hence the supremum in the definition of \(u\) is a limit. For \(x, y \in M\), we have that

\[
\begin{align*}
\Phi_c(y, \gamma(t)) - \Phi_c(x, \gamma(t)) & = \lim_{t \to +\infty} \left[ \Phi_c(\gamma(0), \gamma(t)) - \Phi_c(x, \gamma(t)) - \Phi_c(\gamma(0), \gamma(t)) + \Phi_c(y, \gamma(t)) \right] \\
& = \lim_{t \to +\infty} \left[ \Phi_c(y, \gamma(t)) - \Phi_c(x, \gamma(t)) \right] \\
& \leq \lim_{t \to +\infty} \left[ \Phi_c(y, x) + \Phi_c(x, \gamma(t)) - \Phi_c(x, \gamma(t)) \right] \\
& \leq \Phi_c(y, x).
\end{align*}
\]

Hence \(u \prec L + c\). This property implies that

\[
|u(x) - u(y)| \leq \max\{ |\Phi_c(x, y)|, |\Phi_c(y, x)| \},
\]

and hence \(u(x)\) is Lipschitz, with the same Lipschitz constant as \(\Phi_c\).

We show now that \(M \setminus \pi(\tilde{\Sigma}) \subseteq \pi(\Gamma_0^+(u))\), where \(\Gamma_0^+(u)\) was defined in the introduction. Let \(x \in M \setminus \pi(\tilde{\Sigma})\) and let \(x_{v_n} : [0, T_n] \to M\) be a Tonelli minimizer such that \(x_{v_n}(0) = x\), \(x_{v_n}(T_n) = \gamma(n)\) and

\[
\Phi_c(x, \gamma(n)) \leq A_{L+c} \left( x_{v_n}|_{[0, T_n]} \right) \leq \Phi_c(x, \gamma(n)) + \frac{1}{n}.
\]

The same argument as in inequality (4.13) shows that

\[
A_{L+c} \left( x_{v_n}|_{[s, T_n]} \right) - \frac{1}{n} \leq \Phi_c(x_{v_n}(s), \gamma(n)) \leq A_{L+c} \left( x_{v_n}|_{[s, T_n]} \right),
\]

for all \(0 \leq s \leq T_n\), and then

\[
|\Phi_c(x, \gamma(n)) - \Phi_c(x_{v_n}(t), \gamma(n)) - A_{L+c} \left( x_{v_n}|_{[0, t]} \right)| \leq \frac{1}{n}.
\]

By lemma 3-2.1, \(|\dot{x}_{v_n}(t)| < A\) for all \(n\) large enough and \(0 \leq t \leq T_n\). We prove below that \(T_n \to +\infty\), then the same arguments as in lemma 4-4.2 show that any limit point of \(\{v_n\}\) is in \(\Sigma^+\) so we may assume that \(v_n \to v \in \Sigma^+\). Using the triangle inequality we get

\[
\Phi_c(x_v(t), \gamma(n)) - \Phi_c(x_v(t), x_{v_n}(t)) \leq \Phi_c(x_{v_n}(t), \gamma(n)) \leq \Phi_c(x_v(t), \gamma(n)) + \Phi_c(x_{v_n}(t), x_v(t))
\]

(4.15)
and then
\[ |\Phi_c(x, v, t), \gamma(n)) - \Phi_c(x, v, t), \gamma(n)) | \leq K d_M(x, v, t), x, v(t)), \]
where \( K \) is the Lipschitz constant of \( \Phi_c \). Combining (4.16) and (4.15) we obtain
\[ |\Phi_c(x, \gamma(n)) - \Phi_c(x, v, t), \gamma(n)) - A_{L+c}(x, v, |[0, t]) | \leq \]
\[ \leq \frac{1}{n} + K d_M(x, v, t), x, v(t)). \]

Therefore
\[ u(x, v, t) - u(x) = \lim_{n \to +\infty} [\Phi_c(x, \gamma(n)) - \Phi_c(x, v, t), \gamma(n))] = \lim_{n \to +\infty} A_{L+c}(x, v, |[0, t])] = A_{L+c}(x, v, |[0, t]) = \Phi_c(x, x, v(t)), \]

because \( x, v, |[0, t] \xrightarrow{C^1} x, v, |[0, t] \) and \( x, v \) is semistatic.

Now we prove that \( \lim_n T_n = +\infty \). Suppose that this is not the case. Then there exists a subsequence that we still denote by \( \{T_n\} \) such that \( \lim_n T_n = T_0 < +\infty \). Hence the speed \( |\dot{x}, v, | \) is uniformly bounded in \([0, T_0]\) and therefore we can assume that \( \{v, n\} \) converges to a vector \( v \), \( \lim \dot{\gamma}(n) = (p, w_1) \in \hat{\Sigma}, x, v, |[0, T_0] \xrightarrow{C^1} x, v, |[0, T_0], \) and that \( x, v, |[0, T_0] \) is semistatic. Note that \( \dot{x}, v, (T_0) \) has the form \((p, w_2)\). Since \( x, v, |[0, T_0] \) is semistatic, then the graph property in theorem 3-8.1 implies that \( w_1 = w_2 \). Since \( \hat{\Sigma} \) is invariant, then \( x \in \pi(\hat{\Sigma}) \). This contradicts the hypothesis \( x \in M \setminus \pi(\hat{\Sigma}) \).

Now let \((x, v) \in \hat{\Sigma} \) and \( t > 0 \). Let \( p = x, v, (t) \) and \( y \in M \). Since \( d_c(x, p) = 0 \), then
\[ \Phi_c(x, y) = \Phi_c(x, p) + \Phi_c(p, x) + \Phi_c(x, y) \geq \Phi_c(x, p) + \Phi_c(p, y) \geq \Phi_c(x, y). \]
Hence $\Phi_c(x, y) = \Phi_c(x, p) + \Phi_c(p, y)$. For $y = \gamma(s)$ (and $p = x_v(t)$), we have that

$$u(x_v(t)) - u(x) = \lim_{s \to +\infty} [\Phi_c(x, \gamma(s)) - \Phi_c(p, \gamma(s))] = \Phi_c(x, x_v(t))$$

$$= A_{L+c} (x_v|_{[0,t]}) .$$
Chapter 5

Generic Lagrangians.

In [27], Mañé introduced the concept of generic property of a lagrangian $L$. A property $P$ is said to be generic for the lagrangian $L$ if there exists a generic set $\mathcal{O}$ (in the Baire sense) on the set $C^\infty(M, \mathbb{R})$ such that if $\psi$ is in $\mathcal{O}$ then $L + \psi$ has the property $P$. One of Mañé's objectives was to show that Mather's theory of minimizing measures becomes much more accurate and stronger if we restrict ourselves to generic lagrangians. The main purpose of this chapter is to proof the following

5-0.1. Theorem. For every lagrangian $L$ there exists a generic set $\mathcal{O} \subseteq C^\infty(M, \mathbb{R})$ such that

(A) If $\psi$ is in $\mathcal{O}$ then $L + \psi$ has a unique minimizing measure, $\mu$ and this measure is uniquely ergodic.

(B) Moreover $\text{supp}(\mu) = \hat{\Sigma}(L + \psi) = \Sigma(L + \psi)$.

(C) When $\mu$ is supported on a periodic orbit or a fixed point, this orbit (point) $\Gamma$ is hyperbolic and its stable and unstable manifolds if intersect they do it transversally.

On Mañé [28], it is conjectured that there exists a generic set $\mathcal{O}$ such that this unique minimizing measure is supported on a periodic orbit or an equilibrium point.
The first statement of this was proved by Mañé in [27]. The second statement is proved in [11] and the third one in [7]. We will prove only the first part and give only the ideas of the proofs of the last part.

5-1 Generic Lagrangians.

Proof of (A)

Given a potential $\psi$ on $C^\infty(M, \mathbb{R})$ define

$$m(\psi) = \min_{\nu \in \overline{C}} \int L + \psi d\nu,$$

$$M(\psi) = \{ \nu \in \overline{C} : \int L + \psi d\nu = m(\psi) \}$$

Where $\overline{C}$ is the set of holonomic measures. For $\epsilon > 0$ let

$$O_\epsilon = \{ \psi : \text{diam } M(\psi) < \epsilon \}$$

This set is open, in fact if $\nu_n$ is in $M(\psi_n)$ then

$$\int L + \psi d\nu_n \leq m(\psi) + 2\|\psi - \psi_n\|_{C^0}$$

(5.1)

So by theorem 2-3.2 in chapter 2 if $\psi_n \to \psi$ then the sequence $\nu_n$ is precompact and the limit is in $M(\psi)$. From this follows that $O_\epsilon$ is open.

It remains to prove that it is also dense.

Given a compact convex set $K_0$ on $\overline{C}$ and potential $\psi$ on $C^\infty(M, \mathbb{R})$ define

$$m_0(\psi) = \min_{\nu \in K_0} \int \psi d\nu,$$

$$M_0(\psi) = \{ \nu \in K_0 : \int \psi d\nu = m_0(\psi) \}$$
5-1.1. Lemma. Let $K_0$ as before then if $\mu$ is an extremal point of $K_0$, for all $\epsilon > 0$ there exists $\psi$ on $C^\infty(M, \mathbb{R})$ such that

\[ \text{diam } M_0(\psi) < \epsilon \]
\[ d(\mu, M_0(\psi)) < \epsilon \]

Proof

Denote by $D$ the diagonal of $K_0 \times K_0$ for each pair $(\mu, \nu)$ in $K_0 \times K_0 - D$ take a potential $\psi_{(\mu, \nu)}$ such that $\int \psi_{(\mu, \nu)} d\mu \neq \int \psi_{(\mu, \nu)} d\nu$, then there is a neighbourhood $U(\mu, \nu)$ contained on $K_0 \times K_0$ such that $\int \psi_{(\mu, \nu)} d\mu' \neq \int \psi_{(\mu, \nu)} d\nu'$ for every $(\mu', \nu')$ in $U(\mu, \nu)$.

Take a covering $\{U(\mu_n, \nu_n)\}$ of $K_0 \times K_0 - D$ and set $\psi_n = \psi_{(\mu_n, \nu_n)}$ then if $(\mu, \nu)$ in $K_0 \times K_0 - D$ there exist $n$ such that

\[ \int \psi_n d\mu \neq \int \psi_n d\nu \quad (5.2) \]

Define $T_n : \overline{C} \to \mathbb{R}^n$ as

\[ T_n(\mu) = (\int \psi_1 d\mu, ..., \int \psi_n d\mu) \]

Using (5.2) and the compactness of $K_0 \times K_0$ it is easy to see that given $\epsilon$ there exist $\delta > 0$ and $n > 0$ such that

\[ S \subset \mathbb{R}^n, \text{diam } S < \delta \Rightarrow \text{diam } T^{-1}S < \epsilon \quad (5.3) \]

Let $B = T_n(k_0)$ then $B$ is a compact convex set, let $f : \mathbb{R}^n \to \mathbb{R}$ a linear function such that its minimum restricted to $B$ is attained in only one point $p$.

Define $\psi = \sum_i \lambda_i \psi_i$ where $f = \sum_i \lambda_i p_i$, then

\[ f \circ T_n = \sum_i \lambda_i \int \psi_i \]
and so
\[ M_0(\psi) = T_n^{-1}(p) \]
Then by (5.3) we get
\[ \text{diam } M_0(\psi) < \varepsilon \]

The following lemma proves the density of \( C_\varepsilon \) and hence the first part of (A).

5-1.2. Lemma. If \( \psi \) is on \( C^\infty(M, \mathbb{R}) \) and \( \mu \) is an extremal point of \( M(\psi) \) then for every neighbourhood \( U \) of \( \psi \) and every \( \varepsilon > 0 \) there exists \( \psi_1 \) on \( U \) such that
\[ \text{diam } M(\psi_1) = \varepsilon \]

Proof

For \( K_0 = M(\psi) \) applying the previous lemma, we can find given \( \varepsilon \) a \( \psi_1 \) such that \( \int \psi_1 d\nu \) attains its minimum, say \( m_1 \) for all measures \( \nu \) on \( K_0 = M(\psi) \) on a set \( S = M_0(\psi_1) \) such that \( d(\mu, S) < \varepsilon \). Set

\[
\begin{align*}
m_0 &= m(\psi) \\
f_0(\nu) &= \int L + \psi - m_0 d\nu \\
f_1(\nu) &= \int \psi_1 - m_1 d\nu
\end{align*}
\]

Then
\[ f_1(\nu) = f_0(\nu) \text{ if } \nu \in S \quad (5.4) \]
\[ f_1(\nu) \geq 0 \text{ if } \nu \in M(\psi) \quad (5.5) \]
\[ \nu \in M(\psi), f_1(\nu) = 0 \Rightarrow \nu \in S \quad (5.6) \]

\[ \nu \in \overline{C}, f_0(\nu) = 0 \Rightarrow \nu \in M(\psi) \quad (5.7) \]

For \( \lambda > 0 \) define

\[ f_\lambda = f_0 + \lambda f_1 \]

and set

\[ m(\lambda) = \min_{\nu \in \overline{C}} f_\lambda(\nu), \]

\[ M(\lambda) = \{ \nu \in \overline{C} : f_\lambda(\nu) = m(\lambda) \}. \]

We claim that

\[ \lim_{\lambda \to 0} \text{diam} (M(\lambda), \{ \mu \}) \leq \epsilon \quad (5.8) \]

This proves the lemma since

\[ M(\lambda) = M(\psi + \lambda \psi_1) \]

Proof of the claim

Suppose otherwise that there exist \( \lambda_n \to 0 \) and \( \mu_{\lambda_n}, \nu_{\lambda_n} \) on \( M(\lambda_n) = M(\psi + \lambda_n \psi_1) \) such that \( d(\mu_{\lambda_n}, \nu_{\lambda_n}) > \epsilon \). Then by (5.1) \( \{ \mu_{\lambda_n} \} \) and \( \{ \nu_{\lambda_n} \} \) are precompact and as in the proof of the open property we may assume that \( \mu_{\lambda_n} \to \mu \in M(\psi) \) and \( \nu_{\lambda_n} \to \nu \in M(\psi) \). Naturally \( d(\mu, \nu) \geq \epsilon \).

Now because of (5.4) we have that \( m(\lambda) \leq 0 \)

\[ 0 \geq m(\lambda_n) = f_0(\mu_{\lambda_n}) + \lambda f_1(\mu_{\lambda_n}) \geq \lambda_n f_1(\mu_{\lambda_n}) \]
So \( f_1(\mu_\lambda_n) \leq 0 \) and hence \( f_1(\mu) \leq 0 \), since \( \mu \in M(\psi) \) by (5.5) \( f_1(\mu) = 0 \) and then by (5.6) \( \mu \) is in \( S \). Similarly \( \nu \) is in \( S \). This is a contradiction with the fact that the diameter of \( S \) is less than \( \epsilon \).

The fact that \( \mu \) is uniquely ergodic follows from the fact that ergodic components of a minimizing measure are also minimizing. And the proof of (A) is complete.

It is worth to remark that the proof presented here is a particular case of Mañe's original [31] more general setting:

Let \( E, F \) be real convex spaces, \( K \) contained on \( F \) a metrizable convex subset and \( \phi : E \to F', L : F \to \mathbb{R} \) linear maps satisfying

- (a) The map \( E \times F \to \mathbb{R} \) defined by \( (w, x) \mapsto \phi(w)(x) \) is continuous.

- (b) For any \( x \neq y \) in \( K \) there exists \( w \) in \( E \) such that \( \phi(w)(x) \neq \phi(w)(y) \).

- (c) For all \( w \) in \( E \) and \( c \) in \( \mathbb{R} \) the set

\[
\{ x \in K : L(x) + \phi(w)(x) \leq c \}
\]

is compact.

Denote by

\[
m(w) = \min_{x \in K} L(x) + \phi(w)(x)
\]

which exists by (c). And

\[
M(w) = \{ x \in K : L(x) + \phi(w)(x) = m(w) \}
\]

5-1.3. Proposition. If \( E \) is a Frechet space then there exists a residual set \( \mathcal{O} \) contained on \( E \) such that if \( w \) is on \( E \) then \( M(w) \) has only one element.

The reader can verify that with the following choices, we get the desired result.
• (1) Let \( E \) be the Banach space \( C^\infty(M, \mathbb{R}) \)

• (2) As in section 2-3 let \( C_0^\ell \) be the set of continuous functions \( f \) on \( TM \) such that \( \sup \frac{f(x,v)}{1+|v|} \leq \infty \), and \( \overline{C} \) the set of holonomic probabilities. Let \( K \) be \( \overline{C} \) and \( F \) be the subspace of \( (C_0^\ell)^* \) spanned by \( \overline{C} \).

• (3) Finally let \( L : F \rightarrow \mathbb{R} \) is the linear map such that if \( \mu \) is in \( \overline{C} \) then \( L(\mu) = \int Ld\mu \); and for \( \psi \) in \( C^\infty(M, \mathbb{R}) \) \( \phi(\psi) \) is the restriction to \( F \) of the linear map on \( (C_0^\ell)^* \) such that \( w \mapsto \langle w, \psi \rangle \).

This general setting has some other applications see theorems A, C D in [28] and also [10]

Proof of (C)

Let \( \mathcal{O} \) be the residual given by (A). Let \( \mathcal{A} \) be the subset of \( \mathcal{O} \) of potentials \( \psi \) for which the measure on \( \mathcal{M}(L+\psi) \) is supported on a periodic orbit. Let \( \mathcal{B} := \mathcal{O} \setminus \mathcal{A} \) and let \( \mathcal{A}_1 \) be the subset of \( \mathcal{A} \) on which the minimizing periodic orbit is hyperbolic. We prove that \( \mathcal{A}_1 \) is relatively open on \( \mathcal{A} \). For, let \( \psi \in \mathcal{A}_1 \) and

\[
\mathcal{M}(L+\psi) = \{ \mu_\gamma \}
\]

where \( \mu_\gamma \) is the invariant probability measure supported on the hyperbolic periodic orbit \( \gamma \) for the flow of \( L+\psi \). We claim that if \( \phi_k \in \mathcal{A} \), \( \phi_k \rightarrow \psi \) and \( \mathcal{M}(L+\phi_k) = \{ \mu_{\eta_k} \} \), then \( \eta_k \rightarrow \gamma \). Indeed, since \( L \) is super-linear, the velocities in the support of the minimizing measures \( \mu_k := \mu_{\eta_k} \) are bounded (cf. corollary 2-4 and inequality 1.9), and hence there exists a subsequence \( \mu_k \rightarrow \nu \) converging weakly* to a some invariant measure \( \nu \) for \( L+\psi \). Then if \( \nu \neq \mu_\gamma \),

\[
\lim_k S_{L+\phi_k}(\mu_k) = S_{L+\psi}(\nu) > S_{L+\psi}(\mu_\gamma) .
\] (5.9)

Thus if \( \delta_k \) is the analytic continuation of the hyperbolic periodic orbit \( \gamma \) to the flow of \( L+\phi_k \) in the original energy level \( c(L+\psi) \), since \( \lim_k S_{L+\psi_k}(\mu_{\delta_k}) = S_{L+\psi}(\mu_\gamma) \), for \( k \) large we have that,

\[
S_{L+\phi_k}(\mu_{\delta_k}) < S_{L+\phi_k}(\mu_{\eta_k}) ,
\]
which contradicts the choice of \( \eta_k \). Therefore \( \nu = \mu_\gamma \). For energy levels \( h \) near to \( c(L + \psi) \) and potentials \( \phi \) near to \( \psi \), there exist hyperbolic periodic orbits \( \gamma_{\phi,h} \) which are the continuation of \( \gamma \). Now, on a small neighbourhood of a hyperbolic orbit there exists a unique invariant measure supported on it, and it is in fact supported in the periodic orbit. Thus, since \( \eta_k \rightarrow \gamma \), then \( \eta_k \) is hyperbolic. Hence \( \phi_k \in \mathcal{A}_1 \) and \( \mathcal{A}_1 \) contains a neighbourhood of \( \phi \) in \( \mathcal{A} \).

Let \( \mathcal{U} \) be an open subset of \( C^\infty(M,\mathbb{R}) \) such that \( \mathcal{A}_1 = \mathcal{U} \cap \mathcal{A} \). We shall prove below that \( \mathcal{A}_1 \) is dense in \( \mathcal{A} \). This implies that \( \mathcal{A}_1 \cup \mathcal{B} \) is generic. For, let \( \psi := \text{int} (C^\infty(M,\mathbb{R}) \setminus \mathcal{U}) \), then \( \mathcal{U} \cup \psi \) is open and dense in \( C^\infty(M,\mathbb{R}) \). Moreover, \( \psi \cap \mathcal{A} = \emptyset \) because \( \mathcal{A} \subseteq \overline{\mathcal{A}_1} \subseteq \overline{\mathcal{U}} \) and \( \psi \cap \mathcal{A} \subseteq \mathcal{A} \setminus \overline{\mathcal{U}} = \emptyset \). Since \( \mathcal{O} = \mathcal{A} \cup \mathcal{B} \) is generic and

\[
(U \cup \psi) \cap (A \cup B) = (U \cap A) \cup ((U \cup \psi) \cap B) \\
\subseteq \mathcal{A}_1 \cup \mathcal{B},
\]

then \( \mathcal{A}_1 \cup \mathcal{B} \) is generic.

The perturbation to achieve hyperbolicity in a fixed point is easy. Is very much as the mechanic case: \( L = \frac{1}{2} < v, v >_x -U(x) \). The reader can verify that if \( x_0 = \max U \) then the Dirac measure supported on the point \( (x_0,0) \) is minimizing. And it is well known that this critical point is hyperbolic if and only if the maximum has non-degenerate quadratic form.

In fact, from the Euler-Lagrange equation (E-L) we get that \( L_x(x_0,0) = 0 \). Differentiating the energy function (1.7) we see that \( (x_0,0) \) is a singularity of the energy level \( c(L) \). Moreover, the minimizing property of \( \mu \) implies that \( x_0 \) is a minimum of the function \( x \mapsto L_{xx}(x,0) \). In particular, \( L_{xx}(x_0,0) \) is positive semidefinite in linear coordinates in \( T_{x_0}M \). And it is hyperbolic if and only if it is positive definite. So to achieve hyperbolicity we must just add a small quadratic form.

The perturbation needed in the case of a periodic orbit the same spirit; Because of the graph property the projection of the orbit \( \Gamma, \pi(\Gamma) \)
is a simple closed curve. We add a $C^\infty$-small non negative potential $\psi$, which is zero if and only if $x$ is on $\pi(\Gamma)$ that is nondegenerate in the transversal direction. It follows that $\Gamma$ is also a minimizing solution of the perturbed lagrangian $L + \psi$.

To prove that it actually is hyperbolic is much more difficult. The reason is that the linearization of the flow (the Jacobi equation) is along the periodic orbit and hence non autonomous as in the case of a singularity.

To explain the idea of the proof we need some definitions. Let $H$ be the associated hamiltonian by the Legendre transformation on $T^*M$ and $\psi$ its flow. Denote by $\pi : T^*M \to M$ be the canonical projection and define the vertical subspace on $\theta \in T^*M$ by $\psi(\theta) = \ker(d\pi)$. Two points $\theta_1, \theta_2 \in T^*M$ are said conjugate if $\theta_2 = \psi_\tau(\theta_1)$ for some $\tau \neq 0$ and $d\psi_\tau(\psi(\theta_1)) \cap \psi(\theta_2) \neq \{0\}$.

A basic property of orbits without conjugate points is given by the following

5-1.4. Proposition. Suppose that the orbit of $\theta \in T^*M$ does not contain conjugate points and $H(\theta) = e$ is a regular value of $H$. Then there exist two $\varphi$-invariant lagrangian subbundles $E, F \subset T(T^*M)$ along the orbit of $\theta$ given by

$$E(\theta) = \lim_{t \to +\infty} d\psi_{-t} \left( \psi \left( \psi_t(\theta) \right) \right),$$

$$F(\theta) = \lim_{t \to +\infty} d\psi_t \left( \psi \left( \psi_{-t}(\theta) \right) \right).$$

Moreover, $E(\theta) \cup F(\theta) \subset T_\theta \Sigma$, $E(\theta) \cap \psi(\theta) = F(\theta) \cap \psi(\theta) = \{0\}$, $\langle X(\theta) \rangle \subseteq E(\theta) \cap F(\theta)$ and $\dim E(\theta) = \dim F(\theta) = \dim M$, where $X(\theta) = (H_p, -H_q)$ is the hamiltonian vectorfield and $\Sigma = H^{-1}\{e\}$.

These bundles where constructed for disconjugate geodesics of riemannian metrics by Green [22] and of Finsler metrics by Foulon [19]. In the general case where constructed in [7]
We will only sketch the proof.

Fix a riemannian metric on $M$ and the corresponding induced metric on $T^*M$. Then $T_\theta T^*M$ splits as a direct sum of two lagrangian subspaces: the vertical subspace $\psi(\theta) = \ker (d\pi(\theta))$ and the horizontal subspace $H(\theta)$ given by the kernel of the connection map. Using the isomorphism $K: T_\theta T^*M \rightarrow T_{\pi(\theta)}M \times T_{\pi(\theta)}^*M$, $\xi \mapsto (d\pi(\theta) \xi, \nabla_{\theta}(\pi \xi))$, we can identify $H(\theta) \approx T_{\pi(\theta)}M \times \{0\}$ and $\psi(\theta) \approx \{0\} \times T_{\pi(\theta)}^*M \approx T_{\pi(\theta)}M$. If we choose local coordinates along $t \mapsto \pi \psi_i(\theta)$ such that $t \mapsto \frac{\partial}{\partial q_i}(\pi \psi_i(\theta))$ are parallel vector fields, then this identification becomes $\xi \mapsto (dq(\xi), dp(\xi))$. Let $E \subset T_\theta T^*M$ be an $n$-dimensional subspace such that $E \cap \psi(\theta) = \{0\}$. Then $E$ is a graph of some linear map $S: H(\theta) \rightarrow \psi(\theta)$. It can be checked that $E$ is lagrangian if and only if in symplectic coordinates $S$ is symmetric.

Take $\theta \in T^*M$ and $\xi = (h, v) \in T_\theta T^*M = H(\theta) \oplus \psi(\theta) \approx T_{\pi(\theta)}M \oplus T_{\pi(\theta)}^*M$. Consider a variation

$$\alpha_s(t) = (q_s(t), p_s(t))$$

such that for each $s \in ]-\varepsilon, \varepsilon[\), $\alpha_s$ is a solution of the hamiltonian $H$ such that $\alpha_0(0) = \theta$ and $\frac{d}{ds}\alpha_s(0)|_{s=0} = \xi$.

Writing $d\psi_t(\xi) = (h(t), v(t))$, we obtain the hamiltonian Jacobi equations

$$\dot{h} = H_{pq} h + H_{pp} v,$$
$$\dot{v} = -H_{qq} h - H_{qp} v,$$

(5.10)

where the covariant derivatives are evaluated along $\pi(\alpha_0(t))$, and $H_{pq}$, $H_{qp}$, $H_{pp}$ and $H_{qq}$ are linear operators on $T_{\pi(\theta)}M$, that in local coordinates coincide with the matrices of partial derivatives $\left(\frac{\partial^2 H}{\partial q_i \partial q_j}\right)$, $\left(\frac{\partial^2 H}{\partial q_i \partial p_j}\right)$, $\left(\frac{\partial^2 H}{\partial p_i \partial q_j}\right)$ and $\left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right)$. Moreover, since the hamiltonian $H$ is convex, then $H_{pp}$ is positive definite.

We derive now the Ricatti equation. Let $E$ be a lagrangian subspace of $T_\theta T^*M$. Suppose that for $t$ in some interval $]-\varepsilon, \varepsilon[$ we have that
Then we can write \( d\psi_t(E) = \text{graph } S(t) \),
where \( S(t): H(\psi_t \theta) \to \psi(\psi_t \theta) \) is a symmetric map. That is, if \( \xi \in E \) then
\[
d\psi_t(\xi) = (h(t), S(t) \ h(t)).
\]
Using equation (5.10) we have that
\[
\dot{\mathcal{S}} h + S(H_{pq} h + H_{pp} S h) = -H_{qq} h - H_{qp} S h.
\]
Since this holds for all \( h \in H(\psi_t(\theta)) \) we obtain the Ricatti equation:
\[
\dot{\mathcal{S}} + S H_{pp} S + S H_{pq} + H_{qp} S + H_{qq} = 0. \tag{5.11}
\]
Let \( K_c(\theta): H(\theta) \to \psi(\theta) \) be the symmetric linear map such that
\[
\text{graph}(K_c(\theta)) = d\psi_c(\psi(\psi_c(\theta))).
\]
Define a partial order on the symmetric isomorphisms of \( T_{\pi(\theta)} M \) by writing \( A \succ B \) if \( A - B \) is positive definite.

The following proposition based essentially on the convexity of \( H \) proves 5-1.4.

5-1.5. Proposition. For all \( \varepsilon > 0 \),

(a) If \( d > c > 0 \) then \( K_{-\varepsilon} \succ K_d \succ K_c \).

(b) If \( d < c < 0 \) then \( K_\varepsilon \prec K_d \prec K_c \).

(c) \( \lim_{d \to +\infty} K_d = \mathcal{S}, \lim_{d \to -\infty} K_d = \mathcal{U} \).

(d) \( \mathcal{S} \preceq \mathcal{U} \).

(e) The graph of \( \mathcal{S} \) is the stable green bundle \( \mathcal{E} \) and the graph of \( \mathcal{U} \) is the unstable green bundle \( \mathcal{F} \).

An example of the relationship between the transversality of the Green subspaces and hyperbolicity appears in the following
5.1.6. Proposition. Let $\Gamma$ be a periodic orbit of $\psi_t$ without conjugate points. Then $\Gamma$ is hyperbolic (on its energy level) if and only if $\mathbb{E}(\theta) \cap \mathbb{F}(\theta) = \langle X(\theta) \rangle$ for some $\theta \in \Gamma$, where $\langle X(\theta) \rangle$ is the 1-dimensional subspace generated by the hamiltonian vectorfield $X(\theta)$. In this case $\mathbb{E}$ and $\mathbb{F}$ are its stable and unstable subspaces.

This proposition follows ideas of Eberlein [13] and Freire, [20].

It is known that minimizing orbits do not have conjugate points. So by proposition 5.1.4 and 5.1.6 to prove the density of hyperbolicity it is enough to perturb to make the Green bundles transverse. This is done using two formulas for the index. One in the lagrangian setting and another one in the hamiltonian setting.

Let $\Omega_T$ be the set of continuous piecewise $C^2$ vectorfields $\xi$ along $\gamma_{[0,T]}$. Define the index form on $\Omega_T$ by

$$ I(\xi, \eta) = \int_0^T \left( \dot{\xi} L_{uv} \dot{\eta} + \dot{\xi} L_{ux} \eta + \xi L_{uv} \dot{\eta} + \xi L_{ux} \eta \right) dt, \quad (5.12) $$

which is the second variation of the action functional for variations $f(s,t)$ with $\frac{\partial f}{\partial s} \in \Omega_T$. For general results on this form see Duistermaat [12].

From this formula it is easy to compare the index of the original and the perturbed lagrangian along the same solution $\Gamma$.

Finally we use the following transformation of the index form. It is taken from Hartman [23] and originally due to Clebsch [5] see also [7]. Let $\theta \in T^*M$ and suppose that the orbit of $\theta$, $\psi_t(\theta)$, $0 \leq t \leq T$ does not have conjugate points. Let $E \subset T_\theta T^*M$ be a lagrangian subspace such that $d\psi_t(E) \cap \psi(\psi_t(\theta)) = \{0\}$ for all $0 \leq t \leq T$.

Let $E(t) := d\psi_t(E)$ and let $H(t)$, $\psi(t)$ be a matrix solution of the hamiltonian Jacobi equation (5.10) such that $\det H(t) \neq 0$ and $E(t) = \text{Image}(H(t), \psi(t)) \subset T_{\psi_t(\theta)}(T^*M)$ is a lagrangian subspace. For $\xi = H\zeta \in \Omega_T$, $\eta = H\rho \in \Omega_T$, we obtain (see [7])

$$ I(\xi, \eta) = \int_0^T (H\zeta')^* (H_{pp})^{-1} (H\rho') dt + (H\zeta)^*(V\rho)|_0^T. \quad (5.13) $$
Define \( N(\theta) := \{ w \in T_0 M \mid \langle w, \dot{\gamma} \rangle = 0 \} \). Then \( N(\theta) \) is the subspace of \( T_0 M \) generated by the vectors \( \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \). Let \( v_0 \in N(\theta) \), \( |v_0| = 1 \) and let \( \xi^T(t) := \tilde{Z}_T(t) v_o \). Denote by \( \tilde{I}_T \) and \( I_T \) be the index forms on \([0, T]\) for \( \tilde{L} \) and \( L \) respectively. Using the solution \( (\tilde{Z}_T, \tilde{V}_T) \) on formula (5.13), we obtain that

\[
\tilde{I}_T(\xi^T, \xi^T) = -(\tilde{Z}_T(0) v_o)^* (\tilde{V}_T(0) v_o) = -v_o^* \tilde{K}_T(0) v_o. \tag{5.14}
\]

Moreover, in the coordinates \((x_1, \ldots, x_n; \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})\) on \( TU \) we have that

\[
\tilde{I}_T(\xi^T, \xi^T) = \int_0^T (\xi^T \tilde{L}_{vv} \dot{\xi}^T + 2 \dot{\xi}^T \tilde{L}_{xx} \xi^T + \xi^T \tilde{L}_{xx} \xi^T) \, dt
\]

\[= \int_0^T (\xi^T L_{vv} \dot{\xi}^T + 2 \dot{\xi}^T L_{xx} \xi^T + \xi^T L_{xx} \xi^T) \, dt \tag{5.15}\]

\[+ \int_0^T \varepsilon \sum_{i=2}^n |\xi_i^T|^2 \, dt.
\]

We have that \( \tilde{Z}_T(0) = I \) and for all \( t > 0 \), \( \lim_{t \to \infty} \tilde{Z}_T(t) = \tilde{h}(t) \) with \( \tilde{h}(t) \) the solution of the Jacobi equation for \( \tilde{H} \) corresponding to the stable Green bundle. Writing \( \pi_N(\xi) = (\xi_2, \xi_3, \ldots, \xi_n) \) then \( |\pi_N \tilde{h}(0) v_o| = |v_o| = 1 \) because \( v_0 \in N(\theta) \). Hence there exists \( \lambda > 0 \) and \( T_0 > 0 \) such that \( |\pi_N \tilde{Z}_T(t) v_o| > \frac{1}{2} \) for all \( 0 \leq t \leq \lambda \) and \( T > T_0 \). Therefore

\[
\tilde{I}_T(\xi^T, \xi^T) \geq I_T(\xi^T, \xi^T) + \varepsilon \lambda \tag{5.16}
\]

Let \( (h(t), v(t)) = d\psi_t o (d\pi|_{E(\theta)})^{-1} \) be the solution of the Jacobi equation for \( H \) corresponding to the stable Green subspace \( E \) and let \( S(\psi_t(\theta)) = v(t) h(t)^{-1} \) be the corresponding solution of the Riccati equation, with graph \( [S(\psi_t(\theta)) = E(\psi_t(\theta))] \). Using formula (5.13), and writing
\[ \xi^T(t) = h(t) \xi(t), \]
we have that

\[ I_T(\xi^T, \xi^T) = \int_0^T (h(\dot{\xi}) H_{pp}^{-1}(h(\dot{\xi}) dt + 0 - (h(0) \xi(0))^*(v(0) \xi(0)) \]

\[ I_T(\xi^T, \xi^T) \geq -v_o^* \mathcal{S}(\theta) v_o. \quad (5.17) \]

From (5.14), (5.16) and (5.17), we get that

\[ v_o^* \mathcal{S}(\theta) v_o \geq v_o^* \tilde{K}_T v_o + \frac{\epsilon \lambda}{4}. \]

From proposition 5-1.5, we have that \( \lim_{T \to +\infty} \tilde{K}_T(0) = \tilde{\mathcal{S}}(\theta) \), where \( \text{graph}(\tilde{\mathcal{S}}(\theta)) = \tilde{\mathcal{E}}(\theta) \), the stable Green bundle for \( \tilde{H} \). Therefore

\[ v_o^* \tilde{\mathcal{S}}(\theta) v_o \geq v_o^* \tilde{\mathcal{S}}(\theta) v_o + \frac{\epsilon \lambda}{4}. \quad (5.18) \]

Similarly, for the unstable Green bundles we obtain that

\[ v_o^* \mathcal{U}(\theta) v_o + \lambda_2 \leq v_o^* \tilde{\mathcal{U}}(\theta) v_o \quad \text{for } v_o \in N(\theta), |v_o| = 1. \quad (5.19) \]

for some \( \lambda_2 > 0 \) independent of \( v_o \).

From proposition 5-1.5 we have that \( \mathcal{U}(\theta) \succeq \mathcal{S}(\theta) \). From (5.18) and (5.19) we get that \( \tilde{\mathcal{U}}|N \succeq \mathcal{U}|N \succeq \mathcal{S}|N \succeq \tilde{\mathcal{S}}|N \). Since \( \tilde{\mathcal{E}}(\theta) = \text{graph}(\tilde{\mathcal{S}}(\theta)) \) and \( \tilde{\mathcal{F}}(\theta) = \text{graph}(\tilde{\mathcal{U}}(\theta)) \), we get that \( \tilde{\mathcal{E}}(\theta) \cap \tilde{\mathcal{F}}(\theta) \subseteq \{\tilde{\mathcal{X}}(\theta)\} \). Then proposition B shows that \( \Gamma \) is a hyperbolic periodic orbit for \( L + \phi \).

This proves that \( A_1 \) is dense in \( A \).

Let \( A_2 \) be the subset of \( A_1 \) of potentials \( \psi \) for which the minimizing hyperbolic periodic orbit \( \Gamma \) has transversal intersections. The proof that \( A_2 \) is open and dense in \( A_1 \) is similar to the previous proof, see [7].

### 5-2 Homoclinic Orbits.

Assume in this section that \( \hat{\Sigma} \) contains only one static class. By theorem 5-0.1.(A), this is true for generic lagrangians. By proposition 3-11.4,
the static classes are always connected, thus if we assume that there is only one static class, $\hat{\Sigma}$ must be connected.

Given $\varepsilon > 0$, let $U_\varepsilon$ be the $\varepsilon$-neighbourhood of $\pi(\hat{\Sigma})$. Since $\hat{\Sigma}$ is connected, the open set $U_\varepsilon$ is connected for $\varepsilon$ sufficiently small. Let $H_1(M, U_\varepsilon, \mathbb{R})$ denote the first relative singular homology group of the pair $(M, U_\varepsilon)$ with real coefficients.

We shall say that an orbit of $L$ is homoclinic to a closed invariant set $K \subset TM$ if its $\alpha$ and $\omega$-limit sets are contained in $K$.

Observe that to each homoclinic orbit $x : \mathbb{R} \to M$ to the set of static orbits $\hat{\Sigma}$ we can associate a homology class in $H_1(M, U_\varepsilon, \mathbb{R})$. Indeed, since there exists $t_0 > 0$ such that for all $t$ with $|t| \geq t_0$, $x(t) \in U_\varepsilon$, the class of $x|_{[-t_0, t_0]}$ defines an element in $H_1(M, U_\varepsilon, \mathbb{R})$. Let us denote by $\mathcal{H}$ the subset of $H_1(M, U_\varepsilon, \mathbb{R})$ given by all the classes corresponding to homoclinic orbits to $\hat{\Sigma}$.

5-2.1. Theorem. Suppose that $\hat{\Sigma}$ contains only one static class. Then for any $\varepsilon$ sufficiently small the set $\mathcal{H}$ generates over $\mathbb{R}$ the relative homology $H_1(M, U_\varepsilon, \mathbb{R})$. In particular, there exist at least $\dim H_1(M, U_\varepsilon, \mathbb{R})$ homoclinic orbits to the set of static orbits $\hat{\Sigma}$.

Let $U_\varepsilon$ be an $\varepsilon$-neighbourhood of $\text{supp}(\mu)$. From theorems 5-2.1 and 5-0.1 we obtain:

5-2.2. Corollary. Given a lagrangian $L$ there exists a generic set $\mathcal{O} \subset C^\infty(M, \mathbb{R})$ such that if $\psi \in \mathcal{O}$ the lagrangian $L + \psi$ has a unique minimizing measure $\mu$ in $\mathcal{M}^0(L + \psi)$ and this measure is uniquely ergodic. For any $\varepsilon$ sufficiently small the set $\mathcal{H}$ of homoclinic orbits to $\text{supp}(\mu)$ generates over $\mathbb{R}$ the relative homology $H_1(M, U_\varepsilon, \mathbb{R})$. In particular, there exist at least $\dim H_1(M, U_\varepsilon, \mathbb{R})$ homoclinic orbits to $\text{supp}(\mu)$.

To prove theorem 5-2.1 we consider finite coverings $M_0$ of $M$ whose group of deck transformations is given by the quotient of the torsion free part of $H_1(M, U_\varepsilon, \mathbb{Z})$ by a finite index subgroup. Using that the lifted lagrangian $L_0$ has the same critical value as $L$, we conclude that the
number of static classes of \( L_0 \) must be finite. Hence we can apply theorem 3-11.1 to \( L_0 \) to deduce that the group generated by the homoclinic orbits to the set of static orbits of \( L \) coincides with \( H_1(M, U_\varepsilon, \mathbb{R}) \).

We note that the homoclinic orbits that we obtain in theorem 5-2.1 and corollary 5-2.2 have energy \( c \) but they are not semistatic orbits of \( L \). However, they are semistatic for lifts of \( L \) to suitable finite covers.

Combining corollary 5-2.2, theorem 5-0.1 and lemma 5-2.4, we obtain

**5-2.3. Corollary.** Let \( M \) be a closed manifold with first Betti number \( \geq 2 \). Given a lagrangian \( L \) there exists a generic set \( \mathcal{O} \subset C^\infty(M, \mathbb{R}) \) such that if \( \psi \in \mathcal{O} \) the lagrangian \( L + \psi \) has a unique minimizing measure in \( M^0(L + \psi) \) and this measure is uniquely ergodic. When this measure is supported on a periodic orbit, this orbit is hyperbolic and the stable and unstable manifolds have transverse homoclinic intersections.

**5-2.4. Lemma.** Let \( M \) be a closed manifold with first Betti number \( b_1(M, \mathbb{R}) \geq 2 \). Then if \( A \subset M \) is a closed submanifold diffeomorphic to \( S^1 \) and \( U_\varepsilon \) denotes the \( \varepsilon \) neighborhood of \( A \), we have that \( H_1(M, U_\varepsilon, \mathbb{R}) \) is non zero for all \( \varepsilon \) sufficiently small.

**Proof:** Since \( A \) is diffeomorphic to a circle, the singular homology of the pair \( (M, U_\varepsilon) \) coincides with the singular homology of the pair \( (M, A) \) and therefore the vector space \( H_1(M, U_\varepsilon, \mathbb{R}) \) must have dimension \( \geq b_1(M, \mathbb{R}) - 1 \geq 1 \).

For the proof of theorem 5-2.1 we shall need the following lemma:

**5-2.5. Lemma.** Let \( p : M_1 \to M_2 \) be a covering such that \( c(L_1) = c(L_2) \). Then any lift of a semistatic curve of \( L_2 \) is a semistatic curve of \( L_1 \). Also the projection of a static curve of \( L_1 \) is a static curve of \( L_2 \). If in addition, \( p \) is a finite covering, then any lift of a static curve of \( L_2 \) is a static curve of \( L_1 \).
Proof: Observe first that for any \( k \in \mathbb{R} \) we have that
\[
\Phi^1_k(x, y) \geq \Phi^2_k(px, py),
\]
for all \( x \) and \( y \) in \( M_1 \). Hence if we write \( c = c(L_1) = c(L_2) \) we have
\[
\Phi^1_c(x, y) \geq \Phi^2_c(px, py), \tag{5.20}
\]
for all \( x \) and \( y \) in \( M_1 \).

Suppose now that \( x_2 : \mathbb{R} \rightarrow M_2 \) is a semistatic curve of \( L_2 \) and let \( x_1 : \mathbb{R} \rightarrow M_1 \) be any lift of \( x_2 \) to \( M_1 \). Using (5.20) and the fact that \( x_2 \) is semistatic we have for \( s \leq t \),
\[
\Phi^1_c(x_1(s), x_1(t)) \leq A_{L_1+c}(x_1|[s,t]) = A_{L_2+c}(x_2|[s,t]) = \Phi^2_c(x_2(s), x_2(t)) \leq \Phi^1_c(x_1(s), x_1(t)).
\]
Hence \( x_1 \) is semistatic for \( L_1 \).

Suppose now that \( x_1 : \mathbb{R} \rightarrow M_1 \) is a static curve of \( L_1 \) and let \( x_2 : \mathbb{R} \rightarrow M_2 \) be \( p \circ x_1 \). Using (5.20) and the fact that \( x_1 \) is static we have for \( s \leq t \),
\[
-\Phi^1_c(x_1(t), x_1(s)) = \Phi^1_c(x_1(s), x_1(t)) = A_{L_1+c}(x_1|[s,t]) = A_{L_2+c}(x_2|[s,t])
\geq \Phi^2_c(x_2(s), x_2(t)) \geq -\Phi^2_c(x_2(t), x_2(s)) \geq -\Phi^1_c(x_1(t), x_1(s)).
\]
Hence \( x_2 \) is static for \( L_2 \).

Suppose now that \( p \) is a finite covering and let \( x_2 : \mathbb{R} \rightarrow M_2 \) be a static curve of \( L_2 \). Let \( x_1 : \mathbb{R} \rightarrow M \) be any lift of \( x_2 \) to \( M_1 \). Since \( x_2 \) is static, given \( s \leq t \) and \( \varepsilon > 0 \), there exists a curve \( \alpha : [0, T] \rightarrow M_2 \) with \( \alpha(0) = x_2(t), \alpha(T) = x_2(s) \) such that
\[
A_{L_2+c}(x_2|[s,t]) + A_{L_2+c}(\alpha) \leq \varepsilon.
\]
Since \( p \) is a finite covering, there exists a positive integer \( n \), bounded from above by the number of sheets of the covering, such that the \( n \)-th
iterate of $x_2|_{[s,t]} * \alpha$ lifts to $M_1$ as a closed curve. Hence, there exists a curve $\beta$ joining $x_1(t)$ to $x_1(s)$ such that

$$A_{L_1+c}(x_1|_{[s,t]}) + A_{L_1+c}(\beta) \leq n\varepsilon,$$

and thus $x_1$ is static for $L_1$. \qed

**Proof of theorem 5.2.1:**

Let $U \overset{\text{def}}{=} U_\varepsilon$ denote the $\varepsilon$-neighborhood of $\pi(\hat{\Sigma}(L))$, where $\hat{\Sigma}(L)$ is the set of static vectors of $L$. Since we are assuming that $\hat{\Sigma}(L)$ contains only one static class, the set $U$ is also connected for small $\varepsilon$. Let $i : U \to M$ be the inclusion map. The vector space $H_1(M, U, \mathbb{R})$ is isomorphic to the quotient of $H_1(M, \mathbb{R})$ by $i_*(H_1(U, \mathbb{R}))$.

Let $H$ denote the torsion free part of $H_1(M, \mathbb{Z})$ and let $K$ denote the torsion free part of $i_* (H_1(U, \mathbb{Z}))$. Let us write $G \overset{\text{def}}{=} H/K = \mathbb{Z} \oplus k \cdot \mathbb{Z}$ where $k = \dim H_1(M, U, \mathbb{R})$. Let $J$ be a finite index subgroup of $G$. There is a surjective homomorphism $j : G \to G/J$ given by the projection.

If we take the Hurewicz map

$$\pi_1(M) \mapsto H_1(M, \mathbb{Z}),$$

and we compose it with the projections $H_1(M, \mathbb{Z}) \mapsto H$, $H \mapsto G$ and $j : G \to G/J$, we obtain a surjective homomorphism

$$\pi_1(M) \mapsto G/J,$$

whose kernel will be the fundamental group of a finite covering $M_0$ of $M$ with covering projection map $p : M_0 \to M$ and group of deck transformations given by the finite abelian group $G/J$.

Since $J$ is a subgroup of $G = H/K$, $G/J$ acts transitively and freely on the set of connected components of $p^{-1}(U)$ which coincides with the set of connected components of $p^{-1}(\pi(\hat{\Sigma}(L)))$. Therefore we have

**5.2.6. Lemma.** There is a one to one correspondence between elements in $G/J$ and connected components of $p^{-1}(\pi(\hat{\Sigma}(L)))$. 
Observe that to each homoclinic orbit $x : \mathbb{R} \to M$ to $\hat{\Sigma}(L)$ we can associate a homology class in $H/K$. Indeed, since there exists $t_0 > 0$ such that for all $t$ with $|t| \geq t_0$, $x(t) \in U$, the class of $x|_{[-t_0,t_0]}$ defines an element in $H_1(M, U, \mathbb{Z})$. Let us denote by $\mathcal{H}$ the subset of $H/K$ given by all the classes corresponding to homoclinic orbits to $\hat{\Sigma}(L)$.

5-2.7. Lemma. For any $J$ as above, the image of $\langle \mathcal{H} \rangle$ under $j$ is precisely $G/J$.

Proof: Let $L_0$ denote the lift of the lagrangian $L$ to $M_0$. Observe first that by proposition 2-6.2, $c(L) = c(L_0)$ and therefore by lemma 5-2.5 we have

$$\pi_0(\hat{\Sigma}(L_0)) = p^{-1}(\pi(\hat{\Sigma}(L))), \quad (5.21)$$

where $\pi_0 : TM_0 \to M_0$ is the canonical projection of the tangent bundle $TM_0$ to $M_0$. 

FIG. 1: Creating homoclinic connections with finite coverings.
Let us prove now that $L_0$ satisfies the hypothesis of theorem 3-11.1, that is, the number of static classes of $L_0$ is finite. In fact, we shall show that the projection to $M_0$ of a static class of $L_0$ coincides with a connected component of $p^{-1}(\pi(\hat{\Sigma}(L)))$. Using (5.21) and proposition 3-11.4 we see that the projection of a static class of $L_0$ to $M_0$ must be contained in a single connected component of $p^{-1}(\pi(\hat{\Sigma}(L)))$. Hence, it suffices to show that if $x$ and $y$ belong to a connected component of $p^{-1}(\pi(\hat{\Sigma}(L)))$ then $d^0_c(x, y) = 0$. Since we are assuming that $\hat{\Sigma}(L)$ contains only one static class we have that $d_c(px, py) = 0$. Since $p : M_0 \to M$ is a finite covering there are lifts $x_1$ of $px$ and $y_1$ of $py$ such that $d^0_c(x_1, y_1) = 0$. Since static classes are connected $x_1$ and $y_1$ must belong to the same connected component of $p^{-1}(\pi(\hat{\Sigma}(L)))$ and thus there is a covering transformation taking $x_1$ into $x$ and $y_1$ into $y$ which implies that $d^0_c(x, y) = 0$ as desired.

Now theorem 3-11.1 and (5.21) imply that every covering transformation in $G/J$ can be written as the composition of covering transformations that arise from elements in $\mathcal{H}$, that is, $j(\langle \mathcal{H} \rangle) = G/J$. $\square$

We shall need the following algebraic lemma.

**5-2.8. Lemma.** Let $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Given a finite index subgroup $J \subset G$ let us denote by $j : G \to G/J$ the projection homomorphism.

Let $A$ be a subgroup of $G$. If $A$ has the property that for all $J$ as above $j(A) = G/J$, then $A = G$.

**Proof:** The hypothesis readily implies that

\[
A/A \cap J \text{ is isomorphic to } G/J \quad (5.22)
\]

- If the rank of $A$ is strictly less than the rank of $G$, one can easily construct a subgroup $J \subset G$ with finite index such that $A \subset J$ and $G/J \neq \{0\}$. But this contradicts (5.22) because $A/A \cap J = \{0\}$.

- If the rank of $A$ equals the rank of $G$, then $A$ has finite index in $G$ and by (5.22) $G/A = \{0\}$ and thus $G = A$. 

Observe now that any set \( H \) of a free abelian group \( G \) of rank \( k \) such that the group generated by \( H \) is \( G \) must have at least \( k \) elements. Therefore if we combine lemma 5-2.7 and lemma 5-2.8 with \( \langle H \rangle = A \) we deduce that the set \( H \) of classes corresponding to homoclinic orbits generates \( G \) and must have at least \( k \) elements thus concluding the proof of theorem 5-2.1.
Appendix.

A Absolutely continuous functions.

A.1. Definition. A function $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous, if $\forall \epsilon > 0 \ \exists \delta > 0$ such that

$$\sum_{i=1}^{N} |t_i - s_i| < \delta \implies \sum_{i=1}^{N} |f(t_i) - f(s_i)| < \epsilon,$$

whenever $]s_1, t_1[, \ldots, ]s_N, t_N[$ are disjoint intervals in $[a, b]$.

A.2. Proposition.

The function $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous if and only if

(i) The derivative $f'(t)$ exists for a.e. $t \in [a, b]$.

(ii) $f' \in L^1([a, b])$.

(iii) $f(t) = f(a) + \int_a^t f'(s) \, ds$.

Proof: Define

$$\mu([s, t]) := f(t) - f(s).$$

We claim that $\mu$ defines a finite signed Borel measure on $[a, b]$. Indeed, let $\mathcal{A}$ be the algebra of finite unions of intervals. The function $\mu$ can be extended to a $\sigma$-additive function on $\mathcal{A}$. Moreover, if $B$ is a Borel set
and \( \{A_n\}_{n \in \mathbb{N}} \subset A \) is a family with \( A_n \downarrow B \), then \( \mu(B) := \lim_n \mu(A_n) \) exists because \( \mu(B_n \setminus B_m) \to 0 \) when \( n, m \to +\infty \).

Observe that the absolute continuity of \( f \) implies that \( \mu \ll m \), where \( m \) is the Lebesgue measure. Let \( g = \frac{d\mu}{dm} \) be the Radon-Nikodym derivative. Then \( g \in L^1 \) and

\[
f(t) - f(a) = \mu([a, t]) = \int_a^t g(s) \, ds.
\]

By the Lebesgue differentiation theorem

\[
\lim_{h \to 0} \frac{f(t + h) - f(t)}{h} = \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} g = g(t) \quad \text{for a.e.} \ t \in [a, b].
\]

Conversely, suppose that (i)-(iii) hold. Using (ii), let \( \mu(A) = \int_A f' \, dm \). Then by (iii),

\[
\mu([s, t]) = f(t) - f(s) \quad \text{for} \ s, t \in [a, b].
\]

Then \( \mu \ll m \) implies\(^1\) that \( f \) is absolutely continuous. \( \square \)

The Lebesgue differentiation theorem gives the following characterization.

\textbf{A.3. Corollary.}

The function \( f : [a, b] \to \mathbb{R} \) is absolutely continuous if and only if there exists \( g \in L^1([a, b]) \) such that \( f(t) = f(a) + \int_a^t g'(s) \, ds \).

\(^1\)If \( \mu \) is finite, then \( \mu \ll m \) is equivalent, using the Borel-Cantelli lemma, to

\[
\forall \varepsilon > 0 \ \exists \delta > 0, \ m(A) < \delta \implies |\mu|(A) < \varepsilon.
\]
B Convex functions.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said convex if

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

for all $0 \leq \lambda \leq 1$ and $x, y \in \mathbb{R}^n$. Equivalently, if the set $\{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid r \geq f(x)\}$ is convex.

For $x_0 \in \mathbb{R}^n$ the elements of the set

$$\partial f(x_0) := \{ p \in \mathbb{R}^n^* \mid f(x) \geq p(x - x_0) + f(x_0) \}.$$ 

are called **subderivatives** of $f$ at $x_0$, and the planes

$$\{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid r = p(x - x_0) + f(x_0)\}$$

are called **supporting hyperplanes** for $f$ at $x_0$. The functional $p \in \mathbb{R}^n^*$ is called the **slope** of the hyperplane.

For the proof of the following proposition see Rockafellar [54].

B.1. Proposition.

(a) $\partial f(x) \neq \emptyset$ for every $x \in \text{Dom}(f)$.

(b) A finite convex function is continuous and Lebesgue almost everywhere differentiable.

(c) If $\partial f(x) = \{p\}$ then $f$ is differentiable at $x$ and $f'(x) = p$. 
C The Frenshel and Legendre Transforms.

Given a convex function \( f : \mathbb{R}^n \to \mathbb{R} \) the Frenshel Transform (or the convex dual of \( f \)) is the function \( f^* : (\mathbb{R}^n)^* \to \mathbb{R} \cup \{+\infty\} \) defined by

\[
    f^*(p) = \max_{x \in \mathbb{R}^n} \left[ p x - f(x) \right]
\]

(C.1)

The function \( f \) admits a supporting hyperplane with slope \( p \in \mathbb{R}^n^* \) if and only if \( f^*(p) \neq +\infty \). If \( f \) is superlinear, then \( f^* \) is finite on all \( \mathbb{R}^n \).

C.1. Proposition.

1. If \( f \) is convex then \( f^* \) is convex.

2. If \( f \) and \( f^* \) are superlinear then \( f^{**} = f \).

3. \( f \) is superlinear if and only if \( f^* \) is bounded on balls, more explicitly,

\[
    f(x) \geq A |x| - B(A), \quad \forall x \in \mathbb{R}^n \iff f^*(p) \leq B(|p|), \quad \forall p \in \mathbb{R}^n^*
\]

4. If \( f \) is superlinear, the maximum C.1 is attained at some point \( x \in \mathbb{R}^n \).

**Proof:**

1. Given \( 0 \leq \lambda \leq 1 \) and \( p_1, p_2 \in \mathbb{R}^n^* \) we have that

\[
    f^*(\lambda p_1 + (1 - \lambda) p_2) = \max_{x \in \mathbb{R}^n} \left[ (\lambda p_1 + (1 - \lambda) p_2) x - f(x) \right]
    \leq \lambda \max_{x \in \mathbb{R}^n} [p_1 x - f(x)] + (1 - \lambda) \max_{x \in \mathbb{R}^n} [p_2 x - f(x)]
    = \lambda f^*(p_1) + (1 - \lambda) f^*(p_2).
\]

2. From (C.1) we get that

\[
    f(x) \geq p x - f^*(p) \quad \text{for all } x \in \mathbb{R}^n, \ p \in (\mathbb{R}^n)^*
\]
Hence,

\[ f(x) \geq \sup_{p \in \mathbb{R}^n} [p x - f^*(p)] = f^{**}(x). \]

Let \( p_x \in \partial f(x) \neq \emptyset. \) Then

\[ f(y) \geq f(x) + p_x (y - x), \quad \forall y \in \mathbb{R}^n. \]

Hence

\[ f^*(p_x) = \max_{y \in \mathbb{R}^n} [p_x y - f(y)] = p_x x - f(x). \]

And

\[ f(x) = p_x x - f^*(p_x) \leq \max_{p \in \mathbb{R}^n} [p x - f(x)] = f^{**}(x). \]

3. We have that

\[
\begin{align*}
    f^*_x(p) &= \max_{v \in \mathbb{R}^n} [p v - f_x(v)] \\
    &\leq \max_{v \in \mathbb{R}^n} [p v - |p| v] + B(|p|) \\
    &= B(|p|).
\end{align*}
\]

Conversely, suppose that \( f^*_x(p) \leq B(|p|) \). Given \( A \in \mathbb{R} \) and \( x \in \mathbb{R}^n \) there exists \( p_x \in \mathbb{R}^n \) such that \( |p_x| = A \) and \( p_x v = |p_x| |x| = A |x| \). Then

\[ f(x) = \max_{p \in \mathbb{R}^n} [p x - f^*(p)] \]

\[ \geq p_x v - B(|p_x|) = A |x| - B(A). \]

4. By item 3, \( f^* \) is finite. Let \( p \in \mathbb{R}^n \). If \( b > 0 \) is such that \( f(x) > (|p| + 1) |x| - b \), then

\[ p x - f(x) < b - |x| < f^*(p) - 1 \quad \text{for} \quad |x| > b + 1 - f^*(p). \]

Hence

\[ f^*(p) = \max_{|x| \leq b+1-f^*(p)} [p x - f(x)], \]
and the maximum is attained at some interior point $x_p$ in the closed ball $|x| \leq b + 1 - f^*(p)$.

C.2. Corollary.

If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and superlinear then so is $f^* : \mathbb{R}^{n^*} \to \mathbb{R}$. In this case $f^{**} = f$.

Observe that in this case we have

$$f^*(0) = - \min_{x \in \mathbb{R}^n} f(x) \quad \text{and} \quad f(0) = - \min_{p \in \mathbb{R}^{n^*}} f^*(p).$$

If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and superlinear we define the Legendre Transform $\mathcal{L} : \mathbb{R}^n \to 2^{\mathbb{R}^{n^*}}$ of $f$, by

$$\mathcal{L}(x) = \{ p \in \mathbb{R}^{n^*} \mid px = f(x) + f^*(p) \}, \quad (C.2)$$

C.3. Proposition. If $f : \mathbb{R}^n \to \mathbb{R}$ is $C^2$ and there is $a > 0$, such that

$$y \cdot f''(x) \cdot y \geq a |y|^2 \quad \text{for all} \ x, y \in \mathbb{R}^n,$$

then the Legendre transform $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^{n^*}$ is a $C^1$ diffeomorphism given by $\mathcal{L}(x) = df_x f$.

Proof: The function $f$ is convex and it is superlinear because

$$f(x) = f(0) + \int_0^1 f'(sx) \, ds$$

$$= f(0) + \int_0^1 \int_0^1 sx f''(tsx) x \, dt \, ds$$

$$\geq f(0) + \frac{1}{2} a |x|^2.$$
From (C.1) we get that
\[ p x \leq f(x) + f^*(p) \quad \text{for all } x \in \mathbb{R}^n, \ p \in \mathbb{R}^{n*}. \]  
(C.3)

By proposition C.2, \( f^* \) is superlinear. Then item 4 in proposition C.2 implies that
\[ \mathcal{L}(x) = \arg \max_{p \in \mathbb{R}^{n*}} \{ p x - f^*(p) \} \neq \emptyset. \]

Moreover, from (C.3), if \( p \in \mathcal{L}(x) \) then \( x = \arg \max_{x \in \mathbb{R}^n} \{ p x - f(x) \} \).
Thus \( p = d_x f = \mathcal{L}(x) \). This proves that \( \mathcal{L} \) differentiable and single valued. Moreover since \( d_x \mathcal{L} = f''(x) \) is non-singular, then \( \mathcal{L} \) is a local \( C^1 \) diffeomorphism.

Since
\[ (y - x) \cdot [d_y f - d_x f] = \int_0^1 f''(sx + (1 - s)y) \ ds > 0, \]
then \( x \mapsto d_x f = \mathcal{L}(x) \) is injective. We now prove that \( \mathcal{L} \) is surjective.
By item 4 in proposition C.1 the maximum
\[ f^*(p) = \max_{x \in \mathbb{R}^n} \{ p x - f(x) \} \]
is attained at some \( x_p \in \mathbb{R}^n \). Then \( p \in \mathcal{L}(x_p) \). \( \Box \)
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Index

$A_L(\mu)$, 41  
$C(x, y)$, 21  
$C(x, y; T)$, 35  
$C^0_{\ell}$, 26  
$C_T(x, y)$, 11, 51  
$M_\ell$, 26  
$S^+(\gamma)$, 35  
$[h]_{Lip}$, 37  
$C(M)$, 28, 41  
$\Phi_k(x, y)$, 21  
$\Phi_k(x, y; T)$, 58  
$\Sigma^+(L)$, 66  
$\Sigma^-(L)$, 66  
$\alpha(\omega) = \beta^*(\omega)$, 47  
$\beta(h)$, 47  
$A$, 66  
$H$, 37  
$M$, 48  
$M(L)$, 41  
$M(h)$, 47  
$\tilde{\Sigma} = \Sigma(L)$, 66  
$P = \pi(\Sigma(L))$, 66, 71  
$\tilde{\Sigma}(L)$, 66  
$\mu_\gamma$, 28, 41  
$\partial f(x)$, 133  
$\zeta$, 81  
$\rho(\mu)$, 44  
$c(L)$, 21, 41, 47  
$c_0(L)$, 48, 49  
$c_a(L)$, 49  
$c_u(L)$, 42, 50  
$d_1(\gamma_1, \gamma_2)$, 35, 52  
$d_k(x, y)$, 22  
$e_0$, 12  
$f'(x)$, 133  
$h(x, y)$, 71  
$h_T(x, y) = \Phi_c(x, y; T)$, 71

A

abelian cover, 49  
real –, 50  
absolutely continuous  
functions, 131–132  
metric, 35, 52  
absolutely equicontinuous, 52  
action  
of a measure, 41  
potential, 21  
of finite time, 58  
action potential  
Lipschitz property, 22  
algebraic function, 47
Anosov energy level, 100
asymptotic cycle, 44
Aubry set, 66
Aubry-Mather theory, 44–48

B
barrier
  Peierls', 71–73
beta function, 47
bound
  a priori – for speed, 58, 59
  for the energy, 13
  for the Peierls barrier, 71
  lower – for a lagrangian, 21
boundedness, 8

C
chain
  recurrent, 82
  transitive, 82
class
  static, 76
Cle's theorem, 54
closed form, 8, 44, 90
coboundary
  Lipschitz, 78
  properties, 78–79
condition
  boundedness, 8
  convexity, 7
  superlinearity, 7
configuration space, 45
continuity
  of the action potential, 22
  of the critical value, 25
  of the homology function $\rho$, 44
convex
dual, 47, 134
function, 133–137
convexity, 7
  of the $\alpha$ function, 47
  of the $\beta$ function, 47
covering, 42, 49–50, 91
  abelian, 49
  finite, 49, 123, 125, 126
  properties, 80
  real abelian, 50
  universal, 50, 100
critical value, 21, 22, 41, 47
  of a covering, 49
  of the abelian cover $(c_a)$, 49
  of the universal cover, 42, 50
  strict $(c_0)$, 48, 49
curve
  semistatic, 65
  energy of, 66
  static, 65
cut locus, 76

D
deck transformation, 123
defformation retract, 45
distance
  absolutely continuous, 35, 52
dual
  convex –, 47, 134
E
energy
function, 12–13
kinetic, 16
level
Anosov, 100
compactness, 13
of a semistatic curve, 66
potential –, 16
equicontinuous
absolutely, 52
Euler-Lagrange
equation, 8
magnetic lagrangian, 18
mechanic lagrangian, 16
riemannian lagrangian, 16
flow, 8
reversible, 17
example
c(L) > c₀(L) > e₀, 50
c₀(L) > c_u(L), 50
F
finite
– time potential, 58
covering, 123, 125
finite covering, 126
Finsler
lagrangian, 16
metric, 16
flow
embedded, 20
Euler-Lagrange, 8
Finsler geodesic –, 16, 95
reversible, 17
riemannian geodesic, 16
twisted geodesic, 18
form
canonical symplectic –, 14
closed, 8, 44, 90
Liouville's 1-form, 14
Frenshel transform, 14, 134
G
generic point, 45
geodesic flow
Finsler, 95
of a Finsler metric, 16
riemannian, 16
twisted –, 18
gradient, 17
graph
lagrangian, 90
properties, 74–77
submanifold, 90
growth
linear, 26
superlinear, 7, 21, 47, 134
H
Hamilton-Jacobi
theorem, 89
hamiltonian
riemannian, 16
holonomic measure, 28
homoclinic orbit
homology of –, 123
to a static class, 123
homology
  of a homoclinic orbit, 123
  of a measure, 44
Hurewicz homomorphism, 49, 50, 126
hyperplane
  supporting, 133, 134

I
invariant measures $M(L)$, 41
isotropic subspace, 89

K
kinetic energy, 16

L
lagrangian
  Finsler, 16
  flow, 8
    reversible, 17
  graph, 90
    cohomology class of, 90
    exact, 90
  lower bound, 21
  magnetic, 17
  mechanic, 16
  natural, 16
  riemannian, 16
  submanifold, 89
  subspace, 89
  symmetric, 17, 50
Legendre transform, 14, 136
Liouville’s 1-form, 14
Lipschitz
coboundary, 78
graph, 74
inverse, 76
property
  action potential’s, 22
  finite-time potential’s, 63
  Peierls barrier’s, 71
  weak KAM solutions’, 102
Rademacher’s theorem, 93
smallest – constant, 37
local
  static curve, 66
lower bound
  for a lagrangian, 21

M
Mañé
  set, 66
magnetic lagrangian, 17
  E-L equation, 18
Mather
  beta function, 47
  crossing lemma, 74
  minimizing lemma, 48
  set, 48, 66
  theory, 44–48
measure
  holonomic, 44
  Mather minimizing, 48
mechanic lagrangian, 16
  E-L equation, 16
metric
  absolutely continuous, 35
  absolutely continuous –, 52
INDEX

Finsler, 16

N
natural lagrangian, 16

P
partial order, 81
Peierls
  barrier, 71–73
  Lipschitz property, 71
set \( \mathcal{P} \), 66, 71, 74
  chain recurrence, 82
  non-emptyness, 68
phase space, 45
plane
  supporting, 133, 134
point
  generic, 45
potential
  action \(-\), 21
  energy, 16
probability
  holonomic, 44
  invariant, 41
properties
  coboundary \(-\), 78–79
  covering \(-\), 80
  graph \(-\), 74–77
  recurrence \(-\), 81–87

R
Rademacher's theorem, 93
real abelian cover, 50
recurrence
  properties, 81–87
  reversible flow, 17
  riemannian
    hamiltonian, 16
    lagrangian, 16
  riemannian lagrangian
    E-L equation, 16
  rotation of a measure, 44

S
Schwartzman, 44
semicontinuous, 47
semistatic curve, 65
  \( \alpha \) and \( \omega \)-limits, 82
  energy of, 66
  existence, 103
  lift of, 124
set
  Aubry, 66
  Mañé, 66
  Mather's, 48, 66
  Peierls, 66, 71, 74
    chain recurrence, 82
  slope of a hyperplane, 133, 134
static
  class, 76, 82
  curve
    local, 66
  orbit, 65
  energy of, 66
  existence, 68
  lift of, 124
  strict critical value \((c_0)\), 48, 49
subderivative, 133
submanifold
graph, 90
lagrangian, 89
subsolution of the Hamilton-Jacobi equation, 91

subspace
isotropic, 89
lagrangian, 89
superlinear, 134
superlinearity, 7, 21
of the $\alpha$ function, 47
of the $\beta$ function, 47
supporting hyperplane, 133, 134

surjectivity
of the homology function $\rho$, 46
of the projection of an energy level, 12

symmetric lagrangian, 17, 50
symplectic form, 14
canonical, 89

T

theorem
Hamilton-Jacobi, 89
Tonelli's, 51

Tonelli
minimizer, 51
theorem, 51

Tonelli minimizer, 35
torsion, 50

transform
Frenshel, 14, 134
Legendre, 14, 136

triangle inequality
for the action potential, 22
for the Peierls barrier, 71
twisted geodesic flow, 18