Prices and Asymptotics for Discrete Variance Swaps

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Draft: July 21, 2012

Abstract

We study the fair strike of a discrete variance swap for a general time-homogeneous stochastic volatility model. In the special cases of Heston and Hull-White stochastic volatility models we give simple explicit expressions (improving Broadie and Jain (2008a) for the Heston case). We give conditions on parameters under which the fair strike of a discrete variance swap is higher or lower than that of the continuous variance swap. Interest rates and correlation between underlying price and its volatility are key elements in this analysis. We derive asymptotics for the discrete variance swaps and compare our results with those of Broadie and Jain (2008a), Jarrow et al. (2012) and Keller-Ressel (2011).

Key-words: Discrete Variance swap, Heston model, Hull-White model.

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1 Introduction

A variance swap is a derivative contract which pays at a fixed maturity Tthe difference between a given level (fixed leg) and a realized level of variance over the swap's life (floating leg). Nowadays, variance swaps on stock indices are broadly used and highly liquid. Less standardized variance swaps could be linked to other types of underlying assets such as currencies or commodities. They can be useful to hedge volatility risk exposure or to take positions on future realized volatility. For example, Carr and Lee (2007) price options on realized variance and realized volatility by using variance swaps as pricing and hedging instruments. See Carr and Lee (2009) for an history of volatility derivatives. As noted by Jarrow et al. (2012), most academic studies¹ focus on continuously sampled variance and volatility swaps. However existing volatility derivatives tend to be based on the realized variance computed from the discretely sampled log asset price and continuously sampled derivatives prices may only be used as approximations. As pointed out in Sepp (2012), some care is needed to replace the discrete realized variance by the continuous quadratic variation. By standard probability arguments, the discretely sampled realized variance converges to the quadratic variation of the log stock process in probability. However, this does not guarantee that it converges in expectation. Jarrow et al. (2012) provide sufficient conditions such that the convergence in expectation happens when the stock is modeled by a general semi-martingale, and concrete examples where this convergence fails.

In this paper we study discretely sampled volatility derivatives in a general time-homogeneous model for stochastic volatility. For discretely sampled volatility derivatives, it is difficult to use the elegant and model-free

¹See, for example, Howison, Rafailidis and Rasmussen (2004), Windcliff, Forsyth and Vetzal (2006), Benth, Groth and Kufakunesu (2007) and Broadie and Jain (2008b).

approach of Neuberger (1994) and Dupire (1993), who independently proved that the fair strike for a continuously sampled variance swap on any underlying price process with continuous path is simply two units of the forward price of the log contract. Building on these results, Carr and Madan (1998) published an explicit expression to obtain this forward price from option prices (by synthesizing a forward contract with vanilla options). This work was recently extended by Carr, Lee and Wu (2012) to the case when the underlying asset price is driven by a time-changed Lévy process (thus extending the Dupire-Neuberger theory to the case when there are jumps in the path of the underlying asset price). In this paper, we adopt a parametric approach which allows us to derive explicit closed-form expressions and asymptotic behaviors with respect to key parameters such as the maturity of the contract, the risk-free rate, the sampling frequency or the correlation between the underlying asset and its volatility. This is in line with the work of Broadie and Jain (2008a) in which the Heston model and the Merton jump diffusion model are considered. See also Itkin and Carr (2010) who study discretely sampled volatility derivatives in the 3/2 stochastic volatility model.

Our main contributions are as follows. We give an expression of the fair strike of the discretely sampled variance swap and derive its sensitivity to interest rates in a general time-homogeneous stochastic volatility model. In the case of the (correlated) Heston (1993) model and (correlated) Hull-White (1987) model, we obtain simple explicit closed-form formulas for the respective fair strikes of continuously and discretely sampled variance swaps. In the Heston model, our formula simplifies the results of Broadie and Jain (2008a) and is easy to analyze. Consequently, we are able to give asymptotic behaviors with respect to key parameters of the model and to the sampling frequency. In particular, we provide explicit conditions under which the discretely sampled variance swap is less valuable than the continuously sampled variance swap although the contrary is commonly observed in the literature (see Bülher (2006) for example). Thus the "convex-order conjecture" formulated by Keller-Ressel (2011) may not hold for stochastic volatility models with correlation. We discuss practical implications and illustrate the risk to underestimate or overestimate prices of discretely sampled volatility derivatives when using a model for the corresponding continuously sampled ones with numerical examples.

The paper is organized as follows. Section 2 deals with the general timehomogeneous stochastic volatility model. Section 3 and 4 derive formulas for the fair strike of a discrete variance swap in the Heston and Hull-White model. Section 5 contains asymptotics for the Heston and Hull-White model and discusses the "convex-order conjecture". Section 6 contains a numerical analysis.

2 Pricing of Variance Swaps in a General Stochastic Volatility Model

In this section, we consider the problem of pricing discrete variance swap under the following general time-homogeneous stochastic volatility model (M) where the stock price and its volatility can possibly be correlated. We assume a constant² risk-free rate $r \ge 0$ and that under a risk-neutral probability measure Q,

$$(M) \quad \begin{cases} \frac{dS_t}{S_t} = rdt + m(V_t)dW_t^{(1)} \\ dV_t = \mu(V_t)dt + \sigma(V_t)dW_t^{(2)} \end{cases}$$
(1)

where $W^{(1)}$ and $W^{(2)}$ are standard correlated Brownian motions with $\mathbb{E}[dW_t^{(1)}dW_t^{(2)}] = \rho dt$. Denote by $J \subset \mathbb{R}$ the state space of the volatility process V. We assume that $\mu, \sigma : J \to \mathbb{R}$ are Borel functions satisfying the following Engelbert-Schmidt conditions, $\forall x \in J, \sigma(x) \neq 0, \frac{1}{\sigma^2(x)}, \frac{\mu(x)}{\sigma^2(x)}, \frac{m^2(x)}{\sigma^2(x)} \in L^1_{loc}(J)$. $L^1_{loc}(J)$ denotes the class of locally integrable functions. Under the above conditions, the SDE (1) for V has a unique (in law) weak solution that possibly exits its state space J (see Theorem 5.15, p341, Karatzas and Shreve (1991)). Denote the exit time of V by ζ , then $Q\left(\int_0^t m^2(V_s)ds < \infty\right) = 1$ on $\{t < \zeta\}, t \in [0, +\infty)$. We also assume that $\forall x \in J, \frac{m(x)}{\sigma(x)}$ is differentiable at all $x \in J$.

In particular, this general model includes the Heston, Hull-White, 3/2and Stein-Stein models as special cases. In what follows, we study discretely and continuously sampled variance swaps with maturity T. In a variance swap, one counterparty agrees to pay at a fixed maturity T a notional amount times the difference between a fixed level and a realized level

 $^{^2\}mathrm{The}$ impact of stochastic interest rates on variance swaps is studied by Hörfelt and Torné (2010).

of variance over the swap's life. If it is continuously sampled, the realized variance corresponds to the quadratic variation of the underlying log price. When it is discretely sampled, it is the sum of the squared increments of the log price. We define their respective "fair" strikes as follows.

Definition 2.1. The fair strike of the "discrete variance swap" associated with the partition $0 = t_0 < t_1 < ... < t_n = T$ of the time interval [0,T] is defined as

$$K_d^M(n) := \frac{1}{T} \sum_{i=0}^{n-1} \mathbb{E}\left[\left(\ln \frac{S_{t_{i+1}}}{S_{t_i}} \right)^2 \right],$$
 (2)

where the underlying asset price S follows the general stochastic volatility model (1) and where the exponent M refers to the model.

Definition 2.2. The fair strike of the "continuous variance swap" is defined as

$$K_c^M := \frac{1}{T} \mathbb{E}\left[\int_0^T m^2(V_s) ds\right],\tag{3}$$

where S follows the dynamics in the general stochastic volatility model (1). In popular stochastic volatility models, $m(v) = \sqrt{v}$, so that $K_c^M = \frac{1}{T} \mathbb{E} \left[\int_0^T V_s ds \right]$.

The derivation of the fair strike of a discrete variance swap in the general stochastic volatility model (1) is based on the following proposition.

Proposition 2.1. Under the dynamics (1) for the stochastic volatility model, assuming the conditions given in Section 2 on the functions,

$$\mathbb{E}\left[\left(\ln\frac{S_{t+\Delta}}{S_t}\right)^2\right] = r^2\Delta^2 - r\Delta\int_t^{t+\Delta}\mathbb{E}\left[m^2(V_s)\right]ds + \frac{1}{4}\mathbb{E}\left[\left(\int_t^{t+\Delta}m^2(V_s)ds\right)^2\right] \\ + (1-\rho^2)\int_t^{t+\Delta}\mathbb{E}\left[m^2(V_s)\right]ds + \rho^2\mathbb{E}\left[\left(f(V_{t+\Delta}) - f(V_t)\right)^2\right] \\ + \rho^2\mathbb{E}\left[\left(\int_t^{t+\Delta}h(V_s)ds\right)^2\right] + \rho\mathbb{E}\left[\int_t^{t+\Delta}h(V_s)ds\int_t^{t+\Delta}m^2(V_s)ds\right] \\ - \rho\mathbb{E}\left[\left(f(V_{t+\Delta}) - f(V_t)\right)\int_t^{t+\Delta}(2\rho h(V_s) + m^2(V_s))ds\right], \quad (4)$$

where

$$f(v) = \int_{\cdot}^{v} \frac{m(z)}{\sigma(z)} dz \quad and \quad h(v) = \mu(v)f'(v) + \frac{1}{2}\sigma^{2}(v)f''(v).$$
(5)

Proof. See Appendix A.

Proposition 2.1 gives the key equation in the analysis of discrete variance swaps. From now on, for simplicity, we consider the equi-distant sampling scheme in (2). Under this scheme, $t_i = iT/n$ and $\Delta = t_{i+1} - t_i = T/n$. From (4) it is clear that the price of a discrete variance swap only depends on the risk-free rate r up to the second order, as there is no higher order terms of r. Interestingly, the second order coefficient of this expansion is model-independent whereas the first order coefficient is directly related to the strike of the corresponding continuously-sampled variance swap. This appears clearly in the following proposition.

Proposition 2.2 (Sensitivity to r). Assume a constant sampling period $\frac{T}{n}$ and denote respectively by K_c^M and K_d^M the fair strikes of the variance swap when it is continuously or discretely sampled in a given model M. The fair strike of the discrete variance swap can be expressed as

$$K_{d}^{M}(n) = b^{M}(n) - \frac{T}{n}K_{c}^{M}r + \frac{T}{n}r^{2},$$
(6)

where $b^M(n)$ does not depend on r. Its sensitivity to the risk-free rate r is equal to

$$\frac{dK_d^M(n)}{dr} = \frac{T}{n}(2r - K_c^M) \tag{7}$$

so that the minimum of K_d^M as a function of r is attained when the risk-free rate takes the value r^* given by

$$r^* = \frac{K_c^M}{2}.\tag{8}$$

Proof. The proof of Proposition 2.2 is immediate: it follows from the definition of the discretely sampled variance swap given in (2). The expansion (6) is obtained by summing (4) for the expected squares of log returns obtained in Proposition 2.1 and by noting that the term $b^M(n)$ is model-dependent but does not depend on r. The sensitivity to r in (7) is obtained by differentiating with respect to r and its minimum r^* is obvious given the sign of the derivative (7).

The next proposition deals with the special case when the risk-free rate r and the correlation coefficient ρ are both equal to 0.

Proposition 2.3 (Fair strike when r = 0% and $\rho = 0$). In the special case when the constant risk-free rate is 0, and the underlying stock price is not correlated to its volatility, we observe that

$$K_d^M(n) \ge K_c^M.$$

Proof. Using Proposition 2.1 when r = 0% and $\rho = 0$, we obtain

$$\mathbb{E}\left[\left(\ln\frac{S_{t+\Delta}}{S_t}\right)^2\right] = \frac{1}{4}\mathbb{E}\left[\left(\int_t^{t+\Delta} m^2(V_s)ds\right)^2\right] + \int_t^{t+\Delta}\mathbb{E}\left[m^2(V_s)\right]ds.$$
 (9)

We then add up the expectations of the squares of the log increments (as in (2)) and find that the fair strike of the discrete variance swap is always larger than the fair strike of the continuous variance swap given in (3). \Box

Proposition 2.3 has already appeared in the literature in the special models. See for example Corollary 6.2 of Carr, Lee and Wu (2012) where this result is proved in the more general setting of time-changed Lévy processes with independent time changes. However we will see in the remainder of this paper that Proposition 2.3 may not hold under more general assumptions.

3 Fair Strike of the Discrete Variance Swap in the Heston model

Assume that we work under the Heston stochastic volatility model with the following dynamics

$$(H) \quad \begin{cases} dS_t = rS_t dt + \sqrt{V_t}S_t dW_t^{(1)}, \\ dV_t = \kappa(\theta - V_t)dt + \gamma\sqrt{V_t}dW_t^{(2)} \end{cases}$$
(10)

where $\mathbb{E}\left[dW_t^{(1)}dW_t^{(2)}\right] = \rho dt$. It is a special case of the general model (1) where we choose

$$m(x) = \sqrt{x}, \ \mu(x) = \kappa(\theta - x), \ \sigma(x) = \gamma\sqrt{x}.$$
 (11)

Using (24) in Lemma A.1 in the Appendix with $f(v) = \frac{v}{\gamma}$ and $h(v) = \frac{\kappa}{\gamma}(\theta - v)$, the stock price is

$$S_t = S_0 e^{rt - \frac{1}{2}\xi_t + (V_t - V_0 - \kappa\theta t + \kappa\xi_t)\frac{\rho}{\gamma} + \sqrt{1 - \rho^2} \int_0^t \sqrt{V_s} dW_s^{(3)}$$
(12)

where $\xi_t = \int_0^t V_s ds$ and $W_t^{(3)}$ is such that $dW_t^{(1)} = \rho dW_t^{(2)} + \sqrt{1 - \rho^2} dW_t^{(3)}$.

Using Proposition 2.1 for the general stochastic volatility model, we then derive a closed-form expression for the fair strike of a discrete variance swap and compare it with the fair strike of the continuous variance swap.

Proposition 3.1 (Fair Strikes in the Heston Model). In the Heston stochastic volatility model (10), the fair strike (2) of the discrete variance swap is

$$\begin{split} K_d^H(n) &= \frac{a^2 T}{n} + \frac{b^2}{T} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \mathbb{E}[V_s V_u] ds du + \left(\frac{2ab}{n} + \frac{1-\rho^2}{T}\right) \int_0^T \mathbb{E}[V_s] ds \\ &+ \frac{\rho^2}{T\gamma^2} \sum_{i=0}^{n-1} \mathbb{E}[(V_{t_{i+1}} - V_{t_i})^2] + \frac{2\rho a}{n\gamma} (\mathbb{E}[V_T] - \mathbb{E}[V_0]) \\ &+ \frac{2\rho b}{\gamma T} \sum_{i=0}^{n-1} \left(\mathbb{E}\left[V_{t_{i+1}} \int_{t_i}^{t_{i+1}} V_s ds\right] - \mathbb{E}\left[V_{t_i} \int_{t_i}^{t_{i+1}} V_s ds\right]\right), \end{split}$$

where $a = r - \frac{\rho \kappa \theta}{\gamma}$ and $b = \frac{\rho \kappa}{\gamma} - \frac{1}{2}$. This expression can be computed explicitly as a function of the model parameters as follows

$$K_{d}^{H}(n) = \frac{1}{8n\kappa^{3}T} \left\{ n \left(\gamma^{2} \left(\theta - 2V_{0} \right) + 2\kappa \left(V_{0} - \theta \right)^{2} \right) \left(e^{-2\kappa T} - 1 \right) \frac{1 - e^{\frac{\kappa T}{n}}}{1 + e^{\frac{\kappa T}{n}}} \right. \\ \left. + 2\kappa T \left(\kappa^{2}T \left(\theta - 2r \right)^{2} + n\theta \left(4\kappa^{2} - 4\rho\kappa\gamma + \gamma^{2} \right) \right) \right. \\ \left. + 4 \left(V_{0} - \theta \right) \left(n \left(2\kappa^{2} + \gamma^{2} - 2\rho\kappa\gamma \right) + \kappa^{2}T \left(\theta - 2r \right) \right) \left(1 - e^{-\kappa T} \right) \right. \\ \left. - 2n^{2}\theta\gamma \left(\gamma - 4\rho\kappa \right) \left(1 - e^{-\frac{\kappa T}{n}} \right) + 4 \left(V_{0} - \theta \right) \kappa T\gamma \left(\gamma - 2\rho\kappa \right) \frac{1 - e^{-\kappa T}}{1 - e^{\frac{\kappa T}{n}}} \right\}.$$

$$(13)$$

The fair strike of the continuous variance swap is

$$K_c^H = \frac{1}{T} \mathbb{E}\left[\int_0^T V_s ds\right] = \theta + (1 - e^{-\kappa T}) \frac{V_0 - \theta}{\kappa T}.$$
 (14)

Proof. See Appendix B for the proof of the fair strike of the discrete variance swap. The formula for the fair strike of a continuous variance swap is already well-known and can be found for example in Broadie and Jain (2008a), formula (4.3) p. 772. \Box

Proposition 3.1 gives an explicit formula for the fair strike of a discrete variance swap as a function of the model parameters. This simplifies the expression obtained by Broadie and Jain (2008a) in equations (A-29) and (A-30) p. 793, where several sums from 0 to n are involved and can actually be computed explicitly as shown by the expression (13) above. We verified that our formula agrees with numerical examples presented in Table 5 (column 'SV') on page 782 of Broadie and Jain (2008a).

Contrary to what is stated in the introduction of the paper by Zhu and Lian (2011), the techniques of Broadie and Jain (2008a) can easily be extended to other types of payoffs. The following propositions gives explicit expressions for the volatility derivatives considered by Zhu and Lian (2011).

Proposition 3.2. For the following set of dates $t_i = \frac{iT}{n}$ with i = 0, 1...n, denote $\Delta = T/n$, and assume $\alpha = 2\kappa\theta/\gamma^2 - 1 \ge 0$. Then the fair price of a discrete variance swap with payoff $\frac{1}{T}\sum_{i=0}^{n-1} \left(\frac{S_{t_{i+1}}-S_{t_i}}{S_{t_i}}\right)^2$ is equal to

$$K_d^{zl}(n) = \frac{1}{T} \sum_{i=0}^{n-1} E\left[\left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}\right)^2\right] = \frac{1}{T} \left(a_0 + \sum_{i=1}^{n-1} a_i\right) + \frac{n - 2ne^{r\Delta}}{T}$$

where we define $a_i = E\left[\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right]$, for i = 0, 1, ..., n-1. Then we have that $a_0 = \frac{e^{2r\Delta}}{S_0^2}M(2,\Delta)$ and for i = 1, 2, ..., n-1,

$$a_{i} = e^{2r\Delta + \frac{\kappa\theta}{\gamma^{2}} \left((\kappa - 2\gamma\rho - d(2))\Delta - 2\ln\left(\frac{1 - g(2)e^{-d(2)\Delta}}{1 - g(2)}\right) \right) + V_{0}\frac{\eta(t_{i})q(2)}{\eta(t_{i}) - q(2)}e^{-\kappa t_{i}} \left(\frac{\eta(t_{i})}{\eta(t_{i}) - q(2)}\right)^{\alpha + 1}$$

with $M(u,t) = E[e^{uX_t}],$

$$M(u,t) = S_0^u e^{\frac{\kappa\theta}{\gamma^2} \left((\kappa - \gamma\rho u - d(u))t - 2\ln\left(\frac{1 - g(u)e^{-d(u)t}}{1 - g(u)}\right) \right)} e^{V_0 \frac{\kappa - \gamma\rho u - d(u)}{\gamma^2} \frac{1 - e^{-d(u)t}}{1 - g(u)e^{-d(u)t}}}$$

using the following auxiliary functions

$$d(u) = \sqrt{(\kappa - \gamma \rho u)^2 + \gamma^2 (u - u^2)}; \qquad g(u) = \frac{\kappa - \gamma \rho u - d(u)}{\kappa - \gamma \rho u + d(u)};$$
$$q(u) = \frac{\kappa - \gamma \rho u - d(u)}{\gamma^2} \frac{1 - e^{-d(u)\Delta}}{1 - g(u)e^{-d(u)\Delta}}; \qquad \eta(u) = \frac{2\kappa}{\gamma^2} \left(1 - e^{-\kappa u}\right)^{-1}.$$

Recall that M(u,t) is defined for $u < \eta(t)$. For example $M(2,\Delta)$ is well defined when $\gamma^2 T/n < 1$.

Proof. See Appendix C.

The formula in the above Proposition 3.2 is consistent with the one obtained by Zhu and Lian (2011). In particular, we are able to partially reproduce the numerical results presented in Table 3.1 p. 246 of Zhu and Lian (2011) using their set of parameters: $\kappa = 11.35$, $\theta = 0.022$, $\gamma = 0.618$, $\rho = -0.64$, $V_0 = 0.04$, r = 0.1, T = 1 and $S_0 = 1$ (all numbers match except the case when n = 4 we get 263.2 instead of 267.6).

Proposition 3.2 gives a formula for pricing the variance swap with payoff $\frac{1}{T}\sum_{i=0}^{n-1} \left(\frac{S_{t_{i+1}}-S_{t_i}}{S_{t_i}}\right)^2$ but it is straightforward to extend its proof to the following payoff $\frac{1}{T}\sum_{i=0}^{n-1} \left(\frac{S_{t_{i+1}}-S_{t_i}}{S_{t_i}}\right)^k$ with an arbitrary integer power k.

4 Fair Strike of the Discrete Variance Swap in the Hull-White model

The correlated Hull-White stochastic volatility model is as follows

$$(HW) \quad \begin{cases} \frac{dS_t}{S_t} = rdt + \sqrt{V_t}dW_t^{(1)} \\ dV_t = \mu V_t dt + \sigma V_t dW_t^{(2)} \end{cases}$$

where $\mathbb{E}[dW_t^{(1)}dW_t^{(2)}] = \rho dt$. Referring to equation (1), we have

$$m(x) = \sqrt{x}, \ \mu(x) = \mu x, \ \sigma(x) = \sigma x.$$
(15)

so it is straightforward to determine $f(v) = \frac{2}{\sigma}\sqrt{v}$, $h(v) = \left(\frac{\mu}{\sigma} - \frac{\sigma}{4}\right)\sqrt{v}$, and apply (24) in Lemma A.1 in the Appendix to obtain

$$S_{T} = S_{0} \exp\left\{rT - \frac{1}{2} \int_{0}^{T} V_{t} dt + \frac{2\rho}{\sigma} (\sqrt{V_{T}} - \sqrt{V_{0}}) -\rho\left(\frac{\mu}{\sigma} - \frac{\sigma}{4}\right) \int_{0}^{T} \sqrt{V_{t}} dt + \sqrt{1 - \rho^{2}} \int_{0}^{T} \sqrt{V_{t}} dW_{t}^{(3)}\right\}.$$
 (16)

Proposition 4.1 (Fair Strikes in the Hull-White Model). In the Hull-White stochastic volatility model (15), the fair strike (2) of the discrete variance swap is

$$K_d^{HW}(n) = \frac{1}{T} \sum_{i=0}^{n-1} \mathbb{E}\left[\left(\ln \frac{S_{t_{i+1}}}{S_{t_i}} \right)^2 \right]$$

which can be computed explicitly as a function of the model parameters as follows

$$\begin{split} K_d^{HW}(n) &= \frac{r^2 T}{n} + \left(1 - \frac{rT}{n}\right) K_c^{HW} - \frac{V_0^2 \left(e^{\left(2\,\mu + \sigma^2\right)T} - 1\right) \left(e^{\frac{\mu T}{n}} - 1\right)}{2T\mu(\mu + \sigma^2) \left(e^{\frac{\left(2\,\mu + \sigma^2\right)T}{n}} - 1\right)} + \frac{V_0^2 \left(e^{\left(2\,\mu + \sigma^2\right)T} - 1\right)}{2T(2\mu + \sigma^2)(\mu + \sigma^2)} \\ &+ \frac{8\rho \left(e^{\frac{3(4\mu + \sigma^2)T}{8}} - 1\right) V_0^{3/2} \sigma(e^{\frac{\mu T}{n}} - 1)}{\mu T \left(4\,\mu + 3\,\sigma^2\right) \left(e^{\frac{3(4\mu + \sigma^2)T}{8n}} - 1\right)} - \frac{64\rho \left(e^{\frac{3(4\mu + \sigma^2)T}{8}} - 1\right) V_0^{3/2} \sigma}{3T(4\mu + \sigma^2) \left(4\,\mu + 3\,\sigma^2\right)}. \end{split}$$

$$(17)$$

The fair strike of the continuous variance swap is

$$K_{c}^{HW} = \frac{1}{T} \mathbb{E}\left[\int_{0}^{T} V_{s} ds\right] = \frac{V_{0}}{T\mu} (e^{\mu T} - 1).$$
(18)

Proof. The proof can be found in Appendix D.

5 Asymptotics

In the general stochastic volatility model, we give an expansion in the riskfree rate parameter r (Proposition 2.2). This section presents asymptotics for the discrete variance swaps in the Heston and Hull-White model based on the explicit expressions derived in the previous sections.

The expansions as a function of the number n of sampling periods are given in Propositions 5.1 and 5.3 (respectively for the Heston model and the Hull-White model). They are consistent with Proposition 4.2 of Broadie and Jain (2008a) in which it is proved that $K_d^H(n) = K_c^H + \mathcal{O}\left(\frac{1}{n}\right)$, but are more precise in that at least the first leading term is given explicitly. See also Theorem 3.8 of Jarrow et al. (2012) in a more general context.

Expansions as a function of the maturity T (for small maturities) are also given in order to complement results of Keller-Ressel and Muhle-Karbe (2012) (see for example Corollary 2.7 which gives qualitative properties of this discretization gap³ when the maturity $T \rightarrow 0$).

5.1 Heston Model

We first expand the fair strike of the discrete variance swap with respect to the number of sampling periods n.

Proposition 5.1 (Expansion of the fair strike of the discrete variance swap w.r.t n). The asymptotic behavior of the fair strike of a discrete variance swap in the Heston model is given by

$$K_d^H(n) = K_c^H + \frac{a}{n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$
(19)

where

$$a = r^2 T - rTK_c^H + \left(\frac{\gamma(\theta - V_0)}{2\kappa}(1 - e^{-\kappa T}) - \frac{\theta\gamma T}{2}\right)\rho + \left(\frac{\theta^2}{4} + \frac{\theta\gamma^2}{8\kappa}\right)T + c_1$$

with

$$c_{1} = \frac{\left[\gamma^{2}\theta - 2\kappa(V_{0} - \theta)^{2}\right]\left(e^{-2T\kappa} - 1\right) + 2(V_{0} - \theta)(e^{-T\kappa} - 1)\left[\gamma^{2}(e^{-T\kappa} - 1) - 4\kappa\theta\right]}{16\kappa^{2}}$$

Proof. This proposition is a straightforward expansion from (13) in Proposition 3.1. \Box

We know from Proposition 2.2 that $K_d^H(n) = b^H(n) + \frac{T}{n}r(r - K_c^H)$. It is thus clear that *a* contains all the terms in the risk-free rate *r* and thus that

³See Definition 2.6 of Keller-Ressel and Muhle-Karbe (2012).

all the higher terms in the expansion (19) with respect to n are independent of the risk-free rate.

Remark 5.1. The first term in the expansion (19), a, is a linear function of ρ . Observe that the coefficient in front of ρ , $\frac{\gamma(\theta-V_0)}{2\kappa}(1-e^{-\kappa T})-\frac{\theta\gamma T}{2}$ is negative⁴, so that a is always a decreasing function of ρ . We have that

$$a \ge 0 \quad \Leftrightarrow \quad \rho \leqslant \rho_0^H$$

where

$$\rho_0^H = \frac{r^2 T - rTK_c^H + \left(\frac{\theta^2}{4} + \frac{\theta\gamma^2}{8\kappa}\right)T + c_1}{-\left(\frac{\gamma(\theta - V_0)}{2\kappa}(1 - e^{-\kappa T}) - \frac{\theta\gamma T}{2}\right)}.$$
(20)

Proposition 5.2 (Expansion of the fair strike of the discrete variance swap for small maturity). In the Heston model, an expansion of $K_d^H(n)$ when $T \to 0$ is calculated as

$$K_d^H(n) = V_0 + b_1^H T + b_2^H T^2 + \mathcal{O}\left(T^3\right)$$
(21)

where

$$b_1^H = \frac{\kappa(\theta - V_0)}{2} + \frac{1}{4n} \left((V_0 - 2r)^2 - 2\gamma V_0 \rho \right)$$

$$b_2^H = \frac{\kappa^2(V_0 - \theta)}{6} + \frac{(V_0 - \theta)\kappa(\gamma \rho + 2r - V_0) + \frac{\gamma^2 V_0}{2}}{4n} + \frac{\gamma \rho \kappa(V_0 + \theta) - \frac{\gamma^2 V_0}{2}}{12n^2}$$

Note also that

$$K_{c}^{H} = V_{0} + \frac{\kappa}{2} \left(\theta - V_{0}\right) T + \frac{\kappa^{2}}{6} \left(V_{0} - \theta\right) T^{2} + \mathcal{O}\left(T^{3}\right)$$

and we have that

$$K_d^H(n) - K_c^H = \frac{1}{4n} \left((V_0 - 2r)^2 - 2\gamma V_0 \rho \right) T + \mathcal{O}(T^2).$$

Proof. This proposition is a straightforward expansion from (13) in Proposition 3.1. \Box

Proposition 5.2 is consistent with Corollary 2.7 [b] of Keller-Ressel and

⁴This can be easily seen from the fact that for all x > 0, $(\theta - V_0)(1 - e^{-x}) - \theta x \leq \theta(1 - e^{-x} - x) < 0$ which is especially true for $x = \kappa T > 0$.

Muhle-Karbe (2012) (where it is clear that the limit when T goes to 0 of $K_d(n) - K_c$ is 0).

Notice that in the case $\rho \leq 0$, in the Heston model, $K_d^H(n)$ is nonnegative and decreasing in n as the maturity T goes to 0. However this property cannot be generalized to all correlation levels as it depends on the sign of $(V_0 - 2r)^2 - 2\gamma V_0 \rho$.

5.2 Hull-White Model

Proposition 5.3 (Expansion of the fair strike of the discrete variance swap with n). In the Hull-White model, the expansion of $K_d^{HW}(n)$ is given by

$$K_d^{HW}(n) = K_c^{HW} + \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)$$
(22)

where

$$\begin{aligned} a_1 &= r^2 T - rTK_c^{HW} + \frac{V_0^2}{4} \frac{e^{(2\mu + \sigma^2)T} - 1}{2\mu + \sigma^2} - \frac{4\rho\sigma V_0^{\frac{3}{2}}}{3} \frac{e^{\frac{3}{8}(4\mu + \sigma^2)T} - 1}{4\mu + \sigma^2} \\ a_2 &= -\frac{V_0^2 \sigma^2 T}{24} \frac{e^{(2\mu + \sigma^2)T} - 1}{2\mu + \sigma^2} - \frac{\rho V_0^{\frac{3}{2}} \sigma T(4\mu - 3\sigma^2)}{36} \frac{e^{\frac{3}{8}(4\mu + \sigma^2)T} - 1}{4\mu + \sigma^2} \\ a_3 &= -\frac{\mu T^2 V_0^2(\mu + \sigma^2)}{48} \frac{e^{(2\mu + \sigma^2)T} - 1}{2\mu + \sigma^2} + \frac{\mu T^2 \rho \sigma V_0^{\frac{3}{2}}(4\mu + 3\sigma^2)}{72} \frac{e^{\frac{3}{8}(4\mu + \sigma^2)T} - 1}{4\mu + \sigma^2} \end{aligned}$$

Proof. This proposition is a straightforward expansion from (17) in Proposition 4.1.

Observe that $K_d^{HW}(n) = b^{HW}(n) - \frac{K_c^{HW}T}{n}r + \frac{T}{n}r^2$ where $b^{HW}(n) = K_d^{HW}(r=0) > K_c^{HW}$ is independent of r. Comparing with the general expression in Proposition 2.2, it is clear that a_1 contains all the terms depending on r. Therefore none of the higher coefficients a_2 , a_3 , ... depends on r.

If we neglect higher order terms in the expansion (22), we observe that the position of the discrete variance swap with respect to the continuous variance swap is driven by the sign of a_1 and we have the following observation. **Remark 5.2.** The first term in the expansion (22), a_1 , is a linear function of ρ .

$$a_1 \ge 0 \quad \Leftrightarrow \quad \rho \leqslant \rho_0^{HW}$$

where

$$\rho_0^{HW} = \frac{3(4\mu + \sigma^2) \left(r^2 T - rT K_c^{HW} + \frac{V_0^2}{4} \frac{e^{(2\mu + \sigma^2)T} - 1}{2\mu + \sigma^2} \right)}{4\sigma V_0^{\frac{3}{2}} (e^{\frac{3}{8}(4\mu + \sigma^2)T} - 1)} > 0.$$

 ρ_0^{HW} can take values strictly higher than 1 as it appears clearly in the right panel of Figure 4. In this latter case, the fair strike of the discrete variance swap is higher than the fair strike of the continuous variance swap for all levels of correlation and for sufficiently high values of n. As noted in (8), the minimum value of $K_d^{HW}(n)$ as a function of r is obtained when $r = r^* = \frac{K_c^{HW}}{2}$. After replacing r by r^* in the expression of ρ_0^{HW} , ρ_0^{HW} can easily be shown to be positive⁵.

Proposition 5.4 (Expansion of the fair strike of the discrete variance swap for small maturity). In the Hull-White model, $K_d^{HW}(n)$ can be expanded when $T \to 0$ as

$$K_d^{HW}(n) = V_0 + b_1^{HW}T + b_2^{HW}T^2 + \mathcal{O}\left(T^3\right)$$
(23)

where

$$\begin{split} b_1^{HW} &= \frac{V_0 \, \mu}{2} + \frac{1}{4n} \left((V_0 - 2r)^2 - 2\rho V_0^{3/2} \sigma \right) \\ b_2^{HW} &= \frac{V_0 \mu^2}{6} + \frac{V_0}{4n} \left(\frac{\sigma^2 V_0}{2} - \frac{3\rho \, V_0^{1/2} \sigma (\sigma^2 + 4\mu)}{8} + \mu (V_0 - 2r) \right) \\ &+ \frac{V_0^{3/2} \sigma \left(\rho (3\sigma^2 - 4\mu) - 4\sigma \sqrt{V_0} \right)}{96n^2} \end{split}$$

Note also

$$K_{c}^{HW} = V_{0} + \frac{V_{0}\mu}{2}T + \frac{V_{0}\mu^{2}}{6}T^{2} + \mathcal{O}\left(T^{3}\right)$$

⁵It reduces to study the sign of $\frac{e^{(2\mu+\sigma^2)T}-1}{(2\mu+\sigma^2)T} - \frac{(e^{\mu T}-1)^2}{\mu^2 T^2}$. It is an increasing function of σ so that it is larger than $\frac{e^{2\mu T}-1}{2\mu T} - \frac{(e^{\mu T}-1)^2}{\mu^2 T^2}$ which is always positive as its minimum is 0 obtained when $\mu T = 0$.

and in particular, we have that

$$K_d^{HW}(n) - K_c^{HW} = \frac{1}{4n} \left((V_0 - 2r)^2 - 2\rho V_0^{3/2} \sigma \right) T + \mathcal{O}(T^2).$$

Proof. This proposition is a straightforward expansion from (17) in Proposition 4.1.

Note that we have similar patterns as in the Heston model case in Proposition 5.2.

5.3 Discussion on the convex-order conjecture

As motivated in Keller-Ressel (2011) it is of interest to study the systematic bias for fixed n and T when using the quadratic variation to approximate the realized variance. Bülher (2006) and Keller-Ressel and Muhle-Karbe (2012) show numerical evidence of this bias (see also Section 6 for further evidence in the Heston and Hull-White models). Keller-Ressel (2011) proposes the following "convex-order conjecture", $\mathbb{E}[f(RV(X,\mathcal{P}))] \ge \mathbb{E}[f([X,X]_T)]$ where f is convex, \mathcal{P} refers to the partition of [0,T] in n + 1 division points and $X = \log(S_T/S_0)$. $RV(X,\mathcal{P})$ is the discrete realized variance $(\sum_{i=1}^{n} (\log(S_{t_i}/S_{t_{i-1}}))^2)$ and $[X,X]_T$ is the continuous quadratic variation $(\int_0^T m^2(V_s) ds$ in our setting).

When f(x) = x and the correlation can be positive, the conjecture is violated, see for example Figure 1 to 4 where $K_d^M(n)$ can be below K_c^M . When $\rho = 0$ the process has conditionally independent increments and satisfies other assumptions in Keller-Ressel (2011), it ensures that $K_d^M(n) \ge K_c^M$ which is consistent with his results.

6 Numerics

This last section illustrates the previous propositions obtained in the Heston and Hull-White models through numerical examples, and provides interesting comparisons.

6.1 Parameter choice

Given parameters for the Heston model, we then choose the parameters in the Hull-White model so that the continuous strikes match. Precisely, we obtain μ by solving numerically

$$K_c^H = K_c^{HW}$$

and find σ such that the variances of V_T in the respective Heston and Hull-White models match. From (30) and (31), the variance for V_T for the Heston model is given by

$$Var^{H}(V_{T}) = \frac{\gamma^{2}}{2\kappa} (\theta + 2e^{-\kappa T}(V_{0} - \theta) + e^{-2\kappa T}(\theta - 2V_{0})).$$

The variance for V_T for the Hull-White model can be computed using (37),

$$Var^{HW}(V_T) = V_0^2 e^{2\mu T} (e^{\sigma^2 T} - 1).$$

					(matched)				
					Heston			Hull-White	
	Т	r	V_0	ρ	γ	θ	κ	μ	σ
Set 1	1	3.19%	0.010201	-0.7	0.31	0.019	6.21	1.003	0.42
Set 2	5	5%	0.09	-0.3	1	0.09	2	2.9×10^{-9}	0.52

Table 1: Parameters sets

The parameters for the Heston model are taken from reasonable parameter sets. Precisely the first set of parameters is similar to the one used by Broadie and Jain (2008a). The second set corresponds to Table 2 in Broadie and Kaya (2006). The values for the parameters of the Hull-White model are obtained consistently using the procedure described above.

6.2 Sensitivity to the number of sampling periods

Figure 1 displays cases when the fair strike of the discrete variance swap $K_d^M(n)$ may be smaller than the fair strike of the continuous variance swap K_c^M . The first graph obtained in the Heston model (the model M is denoted by the exponent H for Heston) shows that K_d^H is first higher than K_c^H ,

crosses this level and stays below K_c^H until it converges to the value K_c^H as $n \to \infty$. It means that options on discrete realized variance may be overvalued when the continuous quadratic variation is used to approximate the discrete realized variance. Note that this unusual pattern happens when $\rho = 0.7$, which is usual for example in foreign exchange markets.

Insert Figure 1

Figure 2 highlights another type of convergence showing the complexity of the behaviour of the discrete variance swap with respect to the continuous variance swap.

Insert Figure 2

Figure 3 illustrates the asymptotic expansion with respect to the number n of discretization steps (Proposition 5.1 in the Heston model and Proposition 5.3 in the Hull White model). It shows that the first term of this expansion is already highly informative as it appears clearly to fit very well for small values of n in both models.

Insert Figure 3

Figure 3 displays on the same graphs the discrete fair strike $K_d(n)$ and the first term of the expansion formula $K_c^H + \frac{a}{n}$ for the Heston model and $K_c^{HW} + \frac{a_1}{n}$ for the Hull-White model (see Propositions 5.1 and 5.3 for the exact expressions of a and a_1).

Insert Figure 4

Figure 4 illustrates that the discrete fair strike (for a daily monitoring) can be lower than the continuously sampled strike as $K_d^M - K_c^M$ may be negative for high values of the correlation coefficient both in the Heston and Hull-White model. In Remark 5.1 and 5.2, it is noted that the first term in the asymptotic expansion with respect to n is linear in ρ . From Figure 3 it is clear that the first term has an important explanatory power. This justifies the linear behaviour observed in Figure 4 of the difference between discrete and continuous fair strikes with respect to ρ . Computations of ρ_0^H

and ρ_0^{HW} for each of the risk-free rate levels r = 0%, r = 3.2% and r = 6% confirm that it is always positive when r = 0% (otherwise it would contradict Proposition 2.3) and that it can be higher than 1 which ensures that for n sufficiently high, the discrete fair variance swap rate is always higher than the continuous fair strike.

Recall that the minimums of ρ_0^H and of ρ_0^{HW} are obtained at $r_H^* = \frac{K_c^H}{2}$ and $r_{HW}^* = \frac{K_c^{HW}}{2}$ (see (8)).

Insert Figure 5

Figure 5 shows that as the time to maturity T goes to 0, the discrete fair strike is converging to the continuous fair strike at a quadratic rate. This is consistent with Proposition 5.2 and Proposition 5.4.

Insert Figure 6

Figure 6 shows that the discrepancy between the discrete fair strike and the continuous fair strike is exacerbated by the volatility of the underlying variance process. In particular the continuous fair strike K_c^H is independent of γ . For each γ we compute the corresponding σ for the Hull-White model such that the variances match as described in Section 6.1. We observe then similar patterns in the Heston and Hull-White models.

7 Conclusions

This paper presents explicit expressions for fair strikes of discretely sampled or continuously sampled derivatives in the Heston and the Hull-White models. They are consistent with the literature, more explicit (as there is no sums involved in the discrete fair strikes), and easier to use. Asymptotics are new and consistent with theoretical results obtained in the literature.

A Proof of Proposition 2.1

Using Itō's lemma and Cholesky decomposition, (1) becomes

$$d(\log(S_t)) = \left(r - \frac{1}{2}m^2(V_t)\right)dt + \rho m(V_t)dW_t^{(2)} + \sqrt{1 - \rho^2}m(V_t)dW_t^{(3)}$$
$$dV_t = \mu(V_t)dt + \sigma(V_t)dW_t^{(2)}$$

where $W_t^{(2)}$ and $W_t^{(3)}$ are two standard independent Brownian motions.

Proposition 2.1 is then a direct application of the following lemma (see Lemma 3.1 of Bernard and Cui (2011) for its proof).

Lemma A.1. Under the model given in (1), we have

$$S_T = S_0 \exp\left\{rT - \frac{1}{2} \int_0^T m^2(V_t) dt + \rho(f(V_T) - f(V_0)) -\rho \int_0^T h(V_t) dt + \sqrt{1 - \rho^2} \int_0^T m(V_t) dW_t^{(3)}\right\}, \quad (24)$$

where $f(v) = \int_0^v \frac{m(z)}{\sigma(z)} dz$ and $h(v) = \mu(v) f'(v) + \frac{1}{2} \sigma^2(v) f''(v)$.

Now from equation (24) in Lemma A.1, we compute the following key element in the fair strike of the discrete variance swap. Assume the time interval is $[t, t + \Delta]$, then we have

$$\ln\left(\frac{S_{t+\Delta}}{S_t}\right) = r\Delta - \frac{1}{2}\int_t^{t+\Delta} m^2(V_s)ds + \rho\left(f(V_{t+\Delta}) - f(V_t) - \int_t^{t+\Delta} h(V_s)ds\right) + \sqrt{1-\rho^2}\int_t^{t+\Delta} m(V_s)dW_s^{(3)}.$$

Then we can compute

$$\mathbb{E}\left[\left(\ln\frac{S_{t+\Delta}}{S_t}\right)^2\right] = r^2\Delta^2 + \frac{1}{4}\mathbb{E}\left[\left(\int_t^{t+\Delta}m^2(V_s)ds\right)^2\right] - r\Delta\mathbb{E}\left[\int_t^{t+\Delta}m^2(V_s)ds\right] + \mathbb{E}\left[A^2\right] \\ + \mathbb{E}\left[\left(2r\Delta - \int_t^{t+\Delta}m^2(V_s)ds\right)A\right] + (1-\rho^2)\mathbb{E}\left[\int_t^{t+\Delta}m^2(V_s)ds\right],$$
(25)

where $A = \rho \left(f(V_{t+\Delta}) - f(V_t) - \int_t^{t+\Delta} h(V_s) ds \right)$, and $A^2 = \rho^2 \left(\left(f(V_{t+\Delta}) - f(V_t) \right)^2 + \left(\int_t^{t+\Delta} h(V_s) ds \right)^2 - 2(f(V_{t+\Delta}) - f(V_t)) \int_t^{t+\Delta} h(V_s) ds \right)$ Using the above expressions for A and A^2 in (25), we obtain

$$\mathbb{E}\left[\left(\ln\frac{S_{t+\Delta}}{S_t}\right)^2\right] = r^2\Delta^2 + \frac{1}{4}\mathbb{E}\left[\left(\int_t^{t+\Delta}m^2(V_s)ds\right)^2\right] + (1-\rho^2 - r\Delta)\mathbb{E}\left[\int_t^{t+\Delta}m^2(V_s)ds\right] \\
+ \rho^2\mathbb{E}[\left((f(V_{t+\Delta}) - f(V_t))^2\right] + \rho^2\mathbb{E}\left[\left(\int_t^{t+\Delta}h(V_s)ds\right)^2\right] + 2r\rho\Delta\mathbb{E}[(f(V_{t+\Delta}) - f(V_t))] \\
- \mathbb{E}\left[\left(f(V_{t+\Delta}) - f(V_t)\right)\int_t^{t+\Delta}(2\rho^2h(V_s) + \rho m^2(V_s))ds\right] - 2r\rho\Delta\mathbb{E}\left[\int_t^{t+\Delta}h(V_s)ds\right] \\
+ \rho\mathbb{E}\left[\left(\int_t^{t+\Delta}h(V_s)ds\right)\left(\int_t^{t+\Delta}m^2(V_s)ds\right)\right].$$
(26)

By Itō's lemma, we have that f defined in Lemma A.1 verifies $df(V_t) = h(V_t)dt + m(V_t)dW_t^{(2)}$. Integrating the above SDE from t to $t + \Delta$, we have

$$f(V_{t+\Delta}) - f(V_t) = \int_t^{t+\Delta} h(V_s) ds + \int_t^{t+\Delta} m(V_s) dW_s^{(2)}.$$

Thus,

$$\mathbb{E}\left[f(V_{t+\Delta}) - f(V_t)\right] - \mathbb{E}\left[\int_t^{t+\Delta} h(V_s)ds\right] = \mathbb{E}\left[\int_t^{t+\Delta} m(V_s)dW_s^{(2)}\right] = 0.$$
(27)

Rearrange (26) and use (27) to simplify the terms, which ends the proof of Proposition 2.1. $\hfill \Box$

B Proof of Proposition 3.1

Proof. To prove Proposition 3.1, we first prove the following lemma.

Lemma B.1. In the Heston stochastic volatility model (10),

$$\mathbb{E}\left[\left(\ln\frac{S_{t+\Delta}}{S_t}\right)^2\right] = b^2 \mathbb{E}\left[\left(\int_t^{t+\Delta} V_s ds\right)^2\right] + (1 - \rho^2 + 2ab\Delta) \mathbb{E}\left[\int_t^{t+\Delta} V_s ds\right] \\ + a^2 \Delta^2 + \frac{\rho^2}{\gamma^2} \mathbb{E}\left[(V_{t+\Delta} - V_t)^2\right] + \frac{2\rho a}{\gamma} \Delta \left(\mathbb{E}[V_{t+\Delta}] - \mathbb{E}[V_t]\right) \\ + \frac{2\rho b}{\gamma} \left(\mathbb{E}\left[V_{t+\Delta} \int_t^{t+\Delta} V_s ds\right] - \mathbb{E}\left[V_t \int_t^{t+\Delta} V_s ds\right]\right)$$

where $a = r - \frac{\rho \kappa \theta}{\gamma}$ and $b = \frac{\rho \kappa}{\gamma} - \frac{1}{2}$.

Proof. We apply Proposition 2.1 to the Heston stochastic volatility model. Using the expressions of f and h given in (5) together with the functions corresponding to the Heston model in (11), we have that $f(x) = \frac{x}{\gamma}$ and $h(x) = \frac{\kappa \theta - \kappa x}{\gamma}$ and we obtain an expression of the expected square of the log return of the underlying stock. After some simplifications, we have

$$\mathbb{E}\left[\left(\ln\frac{S_{t+\Delta}}{S_t}\right)^2\right] = \left(r - \frac{\rho\kappa\theta}{\gamma}\right)^2 \Delta^2 + \left(\frac{\rho\kappa}{\gamma} - \frac{1}{2}\right)^2 \mathbb{E}\left[\left(\int_t^{t+\Delta} V_s ds\right)^2\right] + \frac{\rho^2}{\gamma^2} \mathbb{E}[\left((V_{t+\Delta} - V_t)^2\right] + \frac{2\rho\Delta(r\gamma - \rho\kappa\theta)}{\gamma^2} \mathbb{E}[\left(V_{t+\Delta} - V_t\right)] + \frac{\rho(2\rho\kappa - \gamma)}{\gamma^2} \mathbb{E}\left[\left(V_{t+\Delta} - V_t\right)\int_t^{t+\Delta} V_s ds\right] + \left(1 - \rho^2 + \Delta\left(-r + \frac{2r\rho\kappa}{\gamma} + \frac{\rho\kappa\theta}{\gamma} - \frac{2\theta\rho^2\kappa^2}{\gamma^2}\right)\right) \mathbb{E}\left[\int_t^{t+\Delta} V_s ds\right]. \quad (28)$$

Now given the notations that $a = r - \rho \kappa \theta / \gamma$ and $b = \rho \kappa / \gamma - 1/2$, we can rewrite the above formula in the form appearing in Lemma B.1.

Using Lemma B.1, we can sum the different quantities to compute the fair strike (2) of the discrete variance swap. Note that $\Delta = T/n$. Replace t by t_i in the above equation and then we have $t_i + \Delta = t_{i+1}$ and

$$\mathbb{E}\left[\left(\ln\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right] = b^2 \mathbb{E}\left[\left(\int_{t_i}^{t_{i+1}} V_s ds\right)^2\right] + \left(\frac{2abT}{n} + 1 - \rho^2\right) \mathbb{E}\left[\int_{t_i}^{t_{i+1}} V_s ds\right] + \frac{a^2 T^2}{n^2} + \frac{\rho^2}{\gamma^2} \mathbb{E}\left[\left(V_{t_{i+1}} - V_{t_i}\right)^2\right] + \frac{2\rho aT}{n\gamma} (\mathbb{E}[V_{t_{i+1}}] - \mathbb{E}[V_{t_i}]) + \frac{2\rho b}{\gamma} \left(\mathbb{E}\left[V_{t_{i+1}}\int_{t_i}^{t_{i+1}} V_s ds\right] - \mathbb{E}\left[V_{t_i}\int_{t_i}^{t_{i+1}} V_s ds\right]\right).$$

From the definition in (2), we sum the above quantities and we obtain

$$K_{d}^{M} = \frac{1}{T} \sum_{i=0}^{n-1} \mathbb{E} \left[\left(\ln \frac{S_{t_{i+1}}}{S_{t_{i}}} \right)^{2} \right] \\ = \frac{1}{T} \left(\frac{a^{2}T^{2}}{n} + b^{2} \sum_{i=0}^{n-1} \int_{t_{i}}^{t_{i+1}} \int_{t_{i}}^{t_{i+1}} \mathbb{E}[V_{s}V_{u}] ds du + \left(\frac{2abT}{n} + 1 - \rho^{2} \right) \int_{0}^{T} \mathbb{E}[V_{s}] ds \\ + \frac{\rho^{2}}{\gamma^{2}} \sum_{i=0}^{n-1} \mathbb{E}[(V_{t_{i+1}} - V_{t_{i}})^{2}] + \frac{2\rho aT}{n\gamma} (\mathbb{E}[V_{T}] - \mathbb{E}[V_{0}]) \\ + \frac{2\rho b}{\gamma} \sum_{i=0}^{n-1} \left(\mathbb{E} \left[V_{t_{i+1}} \int_{t_{i}}^{t_{i+1}} V_{s} ds \right] - \mathbb{E} \left[V_{t_{i}} \int_{t_{i}}^{t_{i+1}} V_{s} ds \right] \right) \right).$$
(29)

Furthermore, for all $t \ge 0$,

$$\mathbb{E}[V_t] = \theta + e^{-\kappa t} (V_0 - \theta).$$
(30)

and for all $0 < s \leq t$,

$$\mathbb{E}[V_t V_s] = \theta^2 + e^{-\kappa t} (V_0 - \theta) \left(\theta + \frac{\gamma^2}{\kappa}\right) + e^{-\kappa s} \theta (V_0 - \theta) + e^{-\kappa (t+s)} \left((\theta - V_0)^2 + \frac{\gamma^2}{2\kappa} (\theta - 2V_0)\right) + \frac{\gamma^2}{2\kappa} \theta e^{-\kappa (t-s)}.$$
 (31)

In particular this formula holds for t = s and gives $\mathbb{E}[V_t^2]$. These formulas already appear in Broadie and Jain (2008a) (formula (A-15)). To compute K_d^H , (31) is the only expression needed, it should then be integrated and summed in various way.

We have computed all terms in (29) with the help of Maple. It turns out that in the case of the Heston model, all terms can be computed explicitly and the final simplified expression for (29) does not require any sums or integrals. We finally obtain an explicit formula for K_d^H as a function of the parameters of the model.

C Proof of Proposition 3.2

Proof. Denote the log stock price without drift as $X_t = \ln S_t - rt$, and $X_0 = x_0$. Denote $V_0 = v_0$, $\Delta = T/n$. We have that $\mathbb{E}\left[\left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}\right)^2\right] = \mathbb{E}\left[\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right] + 1 - 2e^{r\Delta}$. Thus the goal is to calculate the second moment $\mathbb{E}\left[\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right]$, and note that it is closely linked to the moment generating

function of the log stock price X. Recall the following formulation of the moment generating function $M(u,t) = \mathbb{E}[e^{uX_t}]$ from Albrecher et al. (2007).

$$M(u,t) = S_0^u \exp\left\{\frac{\kappa\theta}{\gamma^2} \left((\kappa - \gamma\rho u - d(u))t - 2\ln\left(\frac{1 - g(u)e^{-d(u)t}}{1 - g(u)}\right) \right) \right\}$$
$$\times \exp\left\{ V_0 \frac{\kappa - \gamma\rho u - d(u)}{\gamma^2} \frac{1 - e^{-d(u)t}}{1 - g(u)e^{-d(u)t}} \right\}.$$
(32)

where the auxiliary functions are given by

$$d(u) = \sqrt{(\kappa - \gamma \rho u)^2 + \gamma^2 (u - u^2)};$$

$$g(u) = \frac{\kappa - \gamma \rho u - d(u)}{\kappa - \gamma \rho u + d(u)}.$$
(33)

We need to separate out the case of i = 0 and i = 1, ..., n - 1. For the first case, we have

$$\mathbb{E}\left[\left(\frac{S_{t_1}}{S_0}\right)^2\right] = \frac{1}{S_0^2} \mathbb{E}\left[e^{2\ln S_{t_1}}\right] = \frac{e^{2rt_1}}{S_0^2} M(2, t_1) = \frac{e^{2r\Delta}}{S_0^2} M(2, \Delta).$$
(34)

For the second case, with i = 1, 2, ..., n, we have

$$\mathbb{E}\left[\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right] = \mathbb{E}\left[e^{2\ln\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)}\right] = e^{2r\Delta}\mathbb{E}\left[\mathbb{E}\left[e^{2(X_{t_{i+1}}-X_{t_i})} \mid \mathcal{F}_{t_i}\right]\right]$$
$$= \exp\{2r\Delta + \frac{\kappa\theta}{\gamma^2}\left((\kappa - 2\gamma\rho - d(2))\Delta - 2\ln\frac{1 - g(2)e^{-d(2)\Delta}}{1 - g(2)}\right)\}$$
$$\times \mathbb{E}\left[\exp\left\{V_{t_i}\frac{\kappa - 2\gamma\rho - d(2)}{\gamma^2}\frac{1 - e^{-d(2)\Delta}}{1 - g(2)e^{-d(2)\Delta}}\right\}\right].$$
(35)

We first define $\alpha = 2\kappa\theta/\gamma^2 - 1 \ge 0$, and $\eta(t) = \frac{2\kappa}{\gamma^2}(1 - e^{-\kappa t})^{-1}$. Then from Theorem 3.1⁶ in Hurd and Kuznetsov (2008), we have

$$\mathbb{E}[e^{uV_T}] = \left(\frac{\eta(T)}{\eta(T) - u}\right)^{\alpha + 1} e^{v_0 \frac{\eta(T)u}{\eta(T) - u}e^{-\kappa T}}.$$
(36)

and the above holds for $u < \eta(T)$. As u = 2, a sufficient condition for $u < \eta(T)$ to hold is $\gamma^2 T < 1$ (since $2 < \eta(T)$ is equivalent to $1 - \frac{\kappa}{\gamma^2} < e^{-\kappa T}$).

⁶Note that in terms of our notations, the parameters in Hurd and Kuznetsov (2008) and our parameters have the correspondence $a = \kappa \theta, b = \kappa, c = \gamma$.

Combine equations (35) and (36), we finally have for i = 1, ..., n - 1,

$$\mathbb{E}\left[\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right] = e^{2r\Delta + \frac{\kappa\theta}{\gamma^2}\left((\kappa - 2\gamma\rho - d(2))\Delta - 2\ln\frac{1 - g(2)e^{-d(2)\Delta}}{1 - g(2)}\right)} e^{V_0 \frac{\eta(t_i)q(2)}{\eta(t_i) - q(2)}e^{-\kappa t_i}} \left(\frac{\eta(t_i)}{\eta(t_i) - q(2)}\right)^{\alpha + 1}$$

where $q(u) = \frac{\kappa - \gamma \rho u - d(u)}{\gamma^2} \frac{1 - e^{-d(u)\Delta}}{1 - g(u)e^{-d(u)\Delta}}$. Then the result follows by summing the above terms $a_i, i = 0, 1, ..., n - 1$.

D Proof of Proposition 4.1

Proof. For the Hull-White model, from the key equation in Proposition 2.1, we have

$$\mathbb{E}\left[\left(\ln\frac{S_{t+\Delta}}{S_t}\right)^2\right] = r^2\Delta^2 + \rho^2 q^2 \mathbb{E}\left[\left(\int_t^{t+\Delta}\sqrt{V_s}ds\right)^2\right] + \frac{4r\rho\Delta}{\sigma} \mathbb{E}\left[\left(\sqrt{V_{t+\Delta}} - \sqrt{V_t}\right)\right] + \frac{1}{4}\mathbb{E}\left[\left(\int_t^{t+\Delta}V_sds\right)^2\right] + (1-\rho^2 - r\Delta)\int_t^{t+\Delta}\mathbb{E}\left[V_s\right]ds + \frac{4\rho^2}{\sigma^2}\mathbb{E}\left[\left(\sqrt{V_{t+\Delta}} - \sqrt{V_t}\right)^2\right] - \frac{4\rho^2}{\sigma}q\mathbb{E}\left[\left(\sqrt{V_{t+\Delta}} - \sqrt{V_t}\right)\int_t^{t+\Delta}\sqrt{V_s}ds\right] - \frac{2\rho}{\sigma}\mathbb{E}\left[\left(\sqrt{V_{t+\Delta}} - \sqrt{V_t}\right)\int_t^{t+\Delta}V_sds\right] - 2r\rho\Delta q\int_t^{t+\Delta}\mathbb{E}\left[\sqrt{V_s}\right]ds + \rho q\mathbb{E}\left[\left(\int_t^{t+\Delta}\sqrt{V_s}ds\right)\left(\int_t^{t+\Delta}V_sds\right)\right].$$

where $q = \frac{\mu}{\sigma} - \frac{\sigma}{4}$. We then assume equi-distant sampling, i.e. $\Delta = T/n$, with $t_i = \frac{iT}{n}, i = 0, 1, 2, ..., n$ and sum these terms to compute $K_d^{HW}(n)$. We now compute a few expectations that are useful in the simplification of the fair discrete variance swap $K_d^{HW}(n)$. In the Hull-White model, the stochastic variance process V_t follows a Geometric Brownian motion. Thus we have $V_t = V_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t^{(2)}\right)$. Note that

$$\mathbb{E}\left[V_s^a\right] = V_0^a e^{a\mu s} e^{\frac{a^2 - a}{2}\sigma^2 s}.$$
(37)

which will be useful below for a = 1/2, a = 1 and a = 2.

$$\mathbb{E}\left[V_{s}\right] = V_{0}e^{\mu s}, \quad \mathbb{E}\left[\sqrt{V_{s}}\right] = \sqrt{V_{0}}e^{\frac{\mu}{2}s - \frac{1}{8}\sigma^{2}s} = \sqrt{V_{0}}e^{\frac{\sigma}{2}qs}, \quad \mathbb{E}\left[V_{s}^{2}\right] = V_{0}^{2}e^{2\mu s + \sigma^{2}s}$$

The fair strike for the continuous variance swap is straightforward and is equal to $\mathbb{E}\left[\int_{0}^{T} V_{s} ds\right] = \frac{V_{0}}{\mu} (e^{\mu T} - 1)$. Similarly,

$$\mathbb{E}\left[\int_0^T \sqrt{V_s} ds\right] = \int_0^T \sqrt{V_0} e^{\frac{\sigma}{2}qs} ds = \sqrt{V_0} \frac{2}{\sigma q} \left(e^{\frac{\sigma qT}{2}} - 1\right).$$
(38)

and for s < u, we have the following results

$$\mathbb{E}\left[V_s V_u\right] = V_0^2 \exp\left(\mu(u+s) + \sigma^2 s\right),\tag{39}$$

$$\mathbb{E}\left[\sqrt{V_s}\sqrt{V_u}\right] = V_0 \exp\left(\frac{\mu}{2}(u+s) - \frac{\sigma^2}{8}(u-s)\right),\tag{40}$$

$$\mathbb{E}\left[\sqrt{V_s} V_u\right] = V_0^{\frac{3}{2}} \exp\left(\mu\left(\frac{s}{2} + u\right) + \frac{3\sigma^2}{8}s\right),\tag{41}$$

$$\mathbb{E}\left[V_s\sqrt{V_u}\right] = V_0^{\frac{3}{2}} \exp\left(\mu\left(s+\frac{u}{2}\right) - \frac{\sigma^2}{8}u + \frac{\sigma^2}{2}s\right).$$
(42)

We further observe that

$$\mathbb{E}\left[\left(\int_{t}^{t+\Delta} V_{s} ds\right)^{2}\right] = 2\int_{t}^{t+\Delta} \int_{t}^{u} \mathbb{E}\left[V_{s} V_{u}\right] ds du,$$
$$\mathbb{E}\left[\left(\int_{t}^{t+\Delta} \sqrt{V_{s}} ds\right)^{2}\right] = 2\int_{t}^{t+\Delta} \int_{t}^{u} \mathbb{E}\left[\sqrt{V_{s}} \sqrt{V_{u}}\right] ds du,$$

and that

$$\mathbb{E}\left[\left(\int_{t}^{t+\Delta}\sqrt{V_{s}}ds\right)\left(\int_{t}^{t+\Delta}V_{u}du\right)\right] = \int_{t}^{t+\Delta}\int_{t}^{u}\mathbb{E}\left[\sqrt{V_{s}}V_{u}\right]dsdu + \int_{t}^{t+\Delta}\int_{u}^{t+\Delta}\mathbb{E}\left[\sqrt{V_{s}}V_{u}\right]dsdu.$$
(43)

Finally we use the two expressions (41) and (42) to replace $\mathbb{E}\left[\sqrt{V_s}V_u\right]$ in the above integrals in (43).

The fair discrete variance swap is computed as

$$K_d^{HW}(n) = \frac{1}{T} \sum_{i=0}^{n-1} \mathbb{E}\left[\left(\ln \frac{S_{t_{i+1}}}{S_{t_i}} \right)^2 \right].$$
 (44)

where

$$\begin{split} & \mathbb{E}\left[\left(\ln\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right] = (1-\rho^2 - \frac{rT}{n})\int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}} \mathbb{E}\left[V_s\right]ds + r^2\frac{T^2}{n^2} \\ & -\frac{2\rho}{\sigma}\mathbb{E}\left[\left(\sqrt{V_{\frac{(i+1)T}{n}}} - \sqrt{V_{\frac{iT}{n}}}\right)\int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}} V_sds\right] + \rho^2q^2\mathbb{E}\left[\left(\int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}} \sqrt{V_s}ds\right)^2\right] \\ & +\frac{4\rho^2}{\sigma^2}\mathbb{E}\left[\left(\sqrt{V_{\frac{(i+1)T}{n}}} - \sqrt{V_{\frac{iT}{n}}}\right)^2\right] - \frac{4\rho^2}{\sigma}q\mathbb{E}\left[\left(\sqrt{V_{\frac{(i+1)T}{n}}} - \sqrt{V_{\frac{iT}{n}}}\right)\int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}} \sqrt{V_s}ds\right] \\ & +\frac{1}{4}\mathbb{E}\left[\left(\int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}} V_sds\right)^2\right] + \rho q\mathbb{E}\left[\left(\int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}} \sqrt{V_s}ds\right)\left(\int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}} V_sds\right)\right]. \end{split}$$

with $q = \frac{\mu}{\sigma} - \frac{\sigma}{4}$. After some tedious calculations, the expression (44) can be simplified to the one appearing in Proposition 4.1.

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Parameters correspond to Set 1 in Table 1 except for ρ which can take three possible values $\rho = -0.7$, $\rho = 0$ or $\rho = 0.7$ and for T which is equal to T = 1 for the two upper graphs and T = 1/12 for the two lower graphs. When T = 1/12, the parameters for the Hull-White model are adjusted according to the procedure described in Section 6.1. In the case when T = 1/12, one has $\mu = 4.03$ and $\sigma = 1.78$.



Figure 2: Sensitivity to the number of sampling periods n and to ρ

Parameters are set to unusual values to show that any types of behaviors can be expected. $\rho = 0.6, r = 3.19\%, \theta = 0.019, \kappa = .1, V_0 = 0.8$ and γ takes three possible values: 0.5, 1.5 and 2.



Figure 3: Asymptotic Expansion $K_c^M + a/n$ with respect to the number of sampling periods n and to ρ

Parameters correspond to Set 2 in Table 1 except for ρ which can take three possible values $\rho = -0.3$, $\rho = 0$ or $\rho = 0.3$. The upper graphs correspond to large number of discretization steps whereas lower graphs have relatively small values of n.



Figure 4: Asymptotic Expansion with respect to the correlation coefficient ρ and the risk-free rate r

Parameters correspond to Set 1 in Table 1 except for r which can take three possible values r = 0%, r = 3.2% or r = 6%. Here n = 250 which corresponds to a daily monitoring as T = 1.



Figure 5: Discrete and Continuous Fair Strikes with respect to the Maturity T and to V_0

Parameters correspond to Set 2 in Table 1 except for T and V_0 . Also we choose a monthly monitoring to compute the discrete fair strike. When $\theta = V_0$, K_c^H is independent of the maturity T.



Figure 6: Discrete and Continuous Fair Strikes with respect to the parameter γ and to V_0

Parameters correspond to Set 2 in Table 1 except for γ and V_0 that are indicated on the graphs. A monthly monitoring is used to compute the discrete fair strike. The continuous fair strike K_c^H is independent of γ , so that it is easy to identify the different curves on the graph.