Manifolds without conjugate points and their fundamental groups

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Main object of study: Closed Riemannian manifolds with no conjugate points (NCP)

- If M is NCP, $p \in M$ then $exp\colon T_pM \to M$ is the universal cover.
- If $\sec(M) \leq 0$ (NPC) then M is NCP.
- There are examples of NCP metrics that have some positive curvature but they are obtained by perturbing NPC metrics.

Question

Let M^n be closed NCP. Does it admit a NPC metric?

Little progress on this so weaker question

Question

Which properties of fundamental groups of NPC manifolds hold for NCP manifolds?

Theorem (Croke and Schroeder, 86)

Let M be closed NCP. Then

- *If $A \leq \pi_1(M)$ is abelian then the embedding $A \to \pi_1(M)$ is quasi-isometric.*
- *If $A \leq \pi_1(M)$ is solvable then A is virtually abelian.*
- *had to assume that the metric is real analytic (removed by Kleiner, Lebedeva using a different proof).*

Theorem (Ivanov-K, 2012)

Let $\bar{M}$ be a closed manifold that admits a C^∞ Riemannian metric without conjugate points. Then for every nontrivial element $\gamma \in \pi_1(\bar{M})$, its centralizer $Z(\gamma) < \pi_1(\bar{M})$ virtually splits over γ. This means that there exists a finite index subgroup $G < Z(\gamma)$ which is isomorphic to a direct product $\mathbb{Z} \times G'$ so that γ corresponds to the generator of the $\mathbb{Z}$ factor.

Example

Let S_g be a closed surface of genus $g > 1$. Then $T^1 S_g$ does not admit a NCP metric.

Displacement functions

Fix a nontrivial element $\gamma \in \Gamma$. The *displacement function* $d_\gamma \colon M \to \mathbb{R}_+$ is defined by

$$d_\gamma(x) = d(x, \gamma x), \qquad x \in M.$$

A complete geodesic $c \colon \mathbb{R} \to M$ is called an *axis* of γ if γ translates c forward along itself, i.e., there is a constant $L > 0$ such that $\gamma c(t) = c(t + L)$ for all $t \in \mathbb{R}$.

Displacement functions

Lemma

1. *The function* d_γ *assumes a positive minimum,* $\min d_\gamma$*. The set of points* $x \in M$ *where* $d_\gamma(x) = \min d_\gamma$*, is equal to* A_γ*.*
2. *The isometry* γ *translates all its axes by the same amount, namely* $\min d_\gamma$*. That is, if* c *is an axis of* γ *then* $\gamma(c(t)) = c(t + \min d_\gamma)$ *for all* $t \in \mathbb{R}$*.*
3. $\min d_{\gamma^m} = m \cdot \min d_\gamma$ *for every integer* $m \geq 1$*.*
4. A_γ *is equal to the set of critical points of* d_γ*. In particular* d_γ *has no critical points outside its minimum set.*
5. A_γ *is connected.*

Busemann functions

Definition

The Busemann function of a (minimizing) geodesic $c\colon \mathbb{R} \to M$ is a function $b_c\colon M \to \mathbb{R}$ defined by

$$b_c(x) = \lim_{t \to +\infty} d(x, c(t)) - t.$$

- $b_c(x) \leq d(x, c(t)) - t$.
- if $\alpha\colon M \to M$ is an isometry, then $b_c(x) = b_{\alpha c}(\alpha x)$ for all $x \in M$.
- if $\sigma(t) = c(t + L)$ where L is a constant, then $b_\sigma(x) = b_c(x) + L$.
- If c is an axis of γ then $b_c(\gamma x) = b_c(x) - \min d_\gamma$ for all $x \in M$.
- If c_1 is another axis of γ then b_c decays at unit rate along c_1, that is $b_c(c_1(t + t_1)) = b_c(c_1(t)) - t_1$ for all $t, t_1 \in \mathbb{R}$.

For a geodesic c in M, denote by b_c^- the Busemann function of the reverse geodesic $t \mapsto c(-t)$ and let $b_c^0 = b_c + b_c^-$.

- $b_c^0(x) \geq 0$ for any x.

Lemma

Let c be an axis of γ. Then $b_c^0 = 0$ on A_γ.

Proposition

Let $\gamma \in \Gamma \setminus \{e\}$, and let c and c_1 be axes of γ. Then $b_c - b_{c_1}$ is constant on M.

Splitting of centralizers

Lemma

Let $\alpha \in Z(\gamma)$ and c be an axis of γ. Then $b_c(\alpha x) - b_c(x)$ does not depend on $x \in M$.

Corollary

There exists a homomorphism $h\colon Z(\gamma) \to \mathbb{R}$ such that $h(\gamma) \neq 0$.

Corollary

$\pi(\gamma)$ is non-torsion (i.e. has infinite order) in $H_1(Z(\gamma))$.

Corollary

$Z(\gamma)$ virtually splits over γ

Open problems

Question

Let M^n be closed NCP. Is $\pi_1(M)$ semi-hyperbolic?

Question

Let $S_{g,1}$ be a surface of genus $g > 1$ with boundary S^1. Let $M_1^3 = M_2^3 = S_{g,1} \times S^1$. Let $M = M_1 \cup_\phi M_2$ where $\phi\colon T^2 \to T^2$ is a diffeomorphism given by the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Does M admit a NCP metric?