Pricing exotic options with an implied integrated variance

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Abstract

In this paper, we consider the pricing of exotic options using two option-implied densities: the density of the implied integrated variance and the risk-neutral price density. We present a Bayesian method to estimate both densities and an option-implied correlation coefficient between the stock price and volatility shocks. We discuss how the convexity adjustment can be avoided when pricing volatility derivatives and discuss the types of additional information that can be provided by the distribution of the implied integrated variance when pricing digital and barrier options. We then present a simple algorithm to estimate the implied integrated variance between two future moments in time and to price two pathdependent options, namely, compound and cliquet options.

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1 Introduction

One of the central problems in financial mathematics is the pricing and hedging of derivatives such as options. The accuracy of correct pricing and hedging is substantially dependent on how well the volatility of the underlying asset has been estimated. In this paper, we estimate two option-implied densities, namely the density of an implied integrated variance and a risk-neutral price density, and use these densities to price three types of exotic options: volatility options, semi-path-dependent options and path-dependent options. We adopt a Bayesian approach where, instead of an absolute truth, we search for evidence of the consistency or inconsistency of a given hypothesis.

The Black-Scholes implied volatility or variance and the model-free volatility or variance are common option-based forecasts of the volatility or variance. The model-free volatility, presented by Britten-Jones and Neuberger (2000), is defined as the risk-neutral expected realized volatility implied by options and can be computed as a weighted average of a continuum of European options with different strike prices and one maturity. The model-free volatility is independent of strike prices and can be computed between two future moments in time. The performance of the implied volatility and the model-free volatility in forecasting the future volatility is considered by Muzzioli (2010), where additional literature on the topic is also presented.

The implied integrated variance can be seen as a stochastic extension of the Black-Scholes implied variance. The distribution of the implied integrated variance can be estimated from option prices using either a general option payoff formula or the Hull-White formula (1987). Similar to the model-free variance, the distribution of the implied integrated variance is also independent of strike prices and can be computed between two future moments in time. The main difference between the implied integrated variance and the model-free variance is that we consider a distribution of the variance, rather than a fixed value, in the former. We can extend the Hull-White approach of pricing European options to pricing some exotic options and computing hedge ratios using an option implied integrated variance.

The inverse problem of implied integrated variance is considered by Friz and Gatheral (2005) and, later, by Kaila (2008), assuming an uncorrelated volatility. In this paper, we will simultaneously estimate an implied integrated variance and an implied correlation coefficient by applying a method presented by Kaila (2012). An estimate of the risk-neutral price density is obtained directly from these estimates. The inverse problem of an implied integrated variance is ill-posed, as explained by Friz and Gatheral (2005). In their paper, the authors coped with the ill-posedness by first regularizing the ill-posed problem and then using a Bayesian lognormal prior. Instead of regularizing, Kaila (2012) recasts the deterministic ill-posed inverse problem of an implied integrated variance in the form of a statistical inference of the distribution of the unknowns and adopts a Bayesian approach. In addition to providing an estimate of the unknowns, this approach provides information on the reliability of the estimates. Following Friz and Gatheral (2005), the distribution of the integrated variance is assumed to be lognormal, such that estimating the entire distribution reduces to estimating the mean and variance of the distribution. A statistical approach to estimating an option-implied volatility is also adopted by Cont and Ben Hamida (2005), who solved an unregularized, ill-posed problem of implied volatility with evolutionary optimization and Monte Carlo sampling.

The main contribution of this paper is that we propose pricing formulas for several exotic options assuming a correlated stochastic volatility, using both the option payoff formula and the correlated Hull-White formula. A statistical approach takes into account the effect of noisy data. The use of distributions instead of fixed values in pricing provides additional information on the possible pricing errors we are making. The problem of missing data is often encountered when estimating model-free volatilities or risk-neutral price densities, but it can be avoided to some extent when the distribution of the integrated variance is already reflected in a few option prices with different strike prices. Estimates of the implied integrated variance can be used when pricing volatility derivatives, assuming either a correlated or uncorrelated volatility. The convexity adjustment (related to volatility swaps and calls, for example) is avoided by using a distribution of the implied integrated variance instead of the corresponding expectation when pricing these derivatives. Finally, two different path-dependent options are priced using the distributions of the implied integrated variance and the risk-neutral price densities between two future moments in time.

This paper is organized as follows: Section 2 solves the inverse problem of an implied integrated variance and an implied correlation coefficient and provides the corresponding estimates of the risk-neutral price density. Section 3 addresses volatility derivatives, while digital options and barrier options are priced in Section 4. In Section 5, we consider compound options and cliquet options, and we offer a conclusion in Section 6.

2 Implied integrated variance

We assume a continuous-time economy with a finite trading interval $[0, T]$. Let $(\Omega, \mathcal{F}, P)$ be a probability space, equipped with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$. We denote a risk-neutral measure by P^*, equivalent to the natural measure P. The stock price process $(X_t)_{0\leq t\leq T}$ and the volatility process $(\sigma_t)_{0\leq t\leq T}$ are modeled as the following diffusion processes:

$$\begin{aligned} dX_t &= rX_t dt + \sigma_t X_t(\sqrt{1-\rho^2}dW_t^* + \rho dZ_t^*), \\ d\sigma_t &= \theta\sigma_t dt + \nu\sigma_t dZ_t^*, \end{aligned} \tag{1}$$

where r is the riskless rate of return, and θ and ν are the drift of the volatility and the volatility of the volatility, respectively. The P^*-Brownian motions W_t^* and Z_t^* are independent and $\int_0^T \sigma_s^2 ds < \infty$. We denote by $\rho = [-1, 1]$ the constant correlation coefficient between the shocks of the stock price process and the volatility process. When $\rho \neq 0$, we say that the volatility is correlated and when $\rho = 0$, we say that the volatility is uncorrelated.

We define the integrated variance, denoted by $\bar{\sigma}_{tT}^2$, as

$$\bar{\sigma}_{tT}^2 = \frac{1}{T-t}\int_t^T \sigma_s^2 ds, \quad 0 \leq t < T,$$

conditional on the filtration $\mathcal{F}_t$. The integrated variance is approximated in the literature by the realized variance. We call the square root of the integrated variance, $s_{tT} = \sqrt{\bar{\sigma}_{tT}^2}$, the integrated volatility.

We begin with a variation of the approach presented by Romano and Touzi (1997) and Willard (1997), apply the two-dimensional Itô formula to $\log(X_t)$ and integrate from t to

T, $0 \leq t < T$, obtaining the following:

$$\log\left(\frac{X_T}{X_t}\right) = \rho\phi_{tT} + \big((r - \frac{1}{2}\bar{\sigma}^2_{\rho,tT})(T-t) + \sqrt{1-\rho^2}\int_t^T \sigma_s dW_s^*\big), \quad \text{where} \tag{2}$$

$$\phi_{tT} = -\frac{1}{2}\rho\bar{\sigma}^2_{tT}(T-t) + \int_t^T \sigma_s dZ_s^* \quad \text{and} \quad \bar{\sigma}^2_{\rho,tT} = (1-\rho^2)\bar{\sigma}^2_{tT}. \tag{3}$$

Rather than ϕ_{tT}, Romano and Touzi (1997) and Willard (1997) apply $\xi_{tT} = \exp(\rho\phi_{tT})$ and show how X_T/X_t is lognormally distributed conditional on X_t, $\bar{\sigma}^2_{tT}$ and ξ_{tT}. We now have two lognormal distributions under the measure P^*:

1. Conditional on X_t, $\bar{\sigma}^2_{tT}$ and ϕ_{tT}, the return X_T/X_t is lognormally distributed:

$$\frac{X_T}{X_t} \sim l\mathcal{N}(\rho\phi_{tT} + \big(r - \frac{1}{2}\bar{\sigma}^2_{\rho,tT}\big)(T-t), \bar{\sigma}^2_{\rho,tT}(T-t)).$$

2. Conditional on X_t, σ_t and $\bar{\sigma}^2_{tT}$, ϕ_{tT} is shifted lognormally distributed:

$$\phi_{tT} \sim l\mathcal{N}\big(\log\big(\frac{\sigma_t}{\nu}\big) + (\theta - \frac{1}{2}\nu^2)(T-t), \nu^2(T-t)\big) - \frac{\sigma_t}{\nu}e^{\theta(T-t)} - \frac{1}{2}\rho\bar{\sigma}^2_{tT}(T-t), \tag{4}$$

   where θ and ν can be estimated from the distribution of $\bar{\sigma}^2_{tT}$. The proof is given in the Appendix and in the paper by Kaila (2012).

As a result, given X_t, the distribution $\pi(\bar{\sigma}^2_{tT})$ and ρ, we can estimate the conditional distribution $\pi(X_T \mid X_t)$.

Now consider a European call on the stock, with price $U_t = U_t(x; K, T; \sigma_t^2, r)$ given by the following equation:

$$U_t = e^{-r(T-t)}\mathbb{E}^*\{(X_T - K)^+ \mid x = X_t, \sigma_t^2, \mathcal{F}_t\} = e^{-r(T-t)}\mathbb{E}^*\{h(X_T) \mid x, \sigma_t^2, \mathcal{F}_t\}, \tag{5}$$

where K is the strike price and T is the maturity. We denote by $\mathbb{E}^*$ the expectation with respect to the measure P^*.

Hull and White (1987) showed that when the volatility is stochastic and uncorrelated, the prices of European options are given as expectations of the corresponding Black-Scholes prices (1973) with respect to the integrated variance. Renault and Touzi (1996) explained how the Hull-White formula produces a symmetric volatility smile when the implied volatilities are plotted as a function of the log-moneyness $\log(K/X_t e^{r(T-t)})$. The asymmetric smiles that are observed in the market can therefore be explained by a correlation between the stock price process and the volatility process; as a result, the assumption of an uncorrelated volatility is unjustified in practice.

Romano and Touzi (1997) and Willard (1997) extended the Hull-White formula to options with correlated volatility. The correlated Hull-White call price is given by $U_t^{\mathrm{HW},\rho}(x;\sigma_t^2) = U_t^{\mathrm{HW},\rho}(x;K,T;\sigma_t^2,r) = \mathbb{E}^*\big\{\mathbb{E}^*\big\{U_t^{\mathrm{BS}}(\xi_{tT}(\bar{\sigma}_{tT}^2,\rho)x;\bar{\sigma}_{\rho,tT}^2,r) \mid \bar{\sigma}_{tT}^2,\mathcal{F}_t\big\} \mid \sigma_t^2,\mathcal{F}_t\big\}$, where the Black-Scholes call price $U_t^{\mathrm{BS}}(x;\sigma^2) = U_t^{\mathrm{BS}}(x;K,T;\sigma^2,r)$ is

$$U_t^{\mathrm{BS}}(x;\sigma^2) = x\Phi(d_+) - e^{-r(T-t)}K\Phi(d_-), \quad d_\pm = \frac{\log(x/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}}, \tag{6}$$

with $\Phi(z) = 1/\sqrt{2\pi}\int_{-\infty}^{z} e^{-y^2/2}dy$. We substitute $\rho\phi_{tT} = \log(\xi_{tT})$ into the correlated Hull-White formula so that

$$U_t^{\mathrm{HW},\rho}(x;\sigma_t^2) = \mathbb{E}^*\big\{\mathbb{E}^*\{U_t^{\mathrm{BS}}(e^{(\rho\phi_{tT}(\bar{\sigma}_{tT}^2,\rho))}x;\bar{\sigma}_{\rho,tT}^2,r) \mid \bar{\sigma}_{tT}^2,\mathcal{F}_t\} \mid \sigma_t^2,\mathcal{F}_t\big\}. \tag{7}$$

Two common inverse problems are to estimate the risk-neutral price density, i.e., the density of the stock price X_T implied by option market prices u_t^{obs} under a risk-neutral measure P^* defined by the markets, and to estimate the Black-Scholes implied volatility

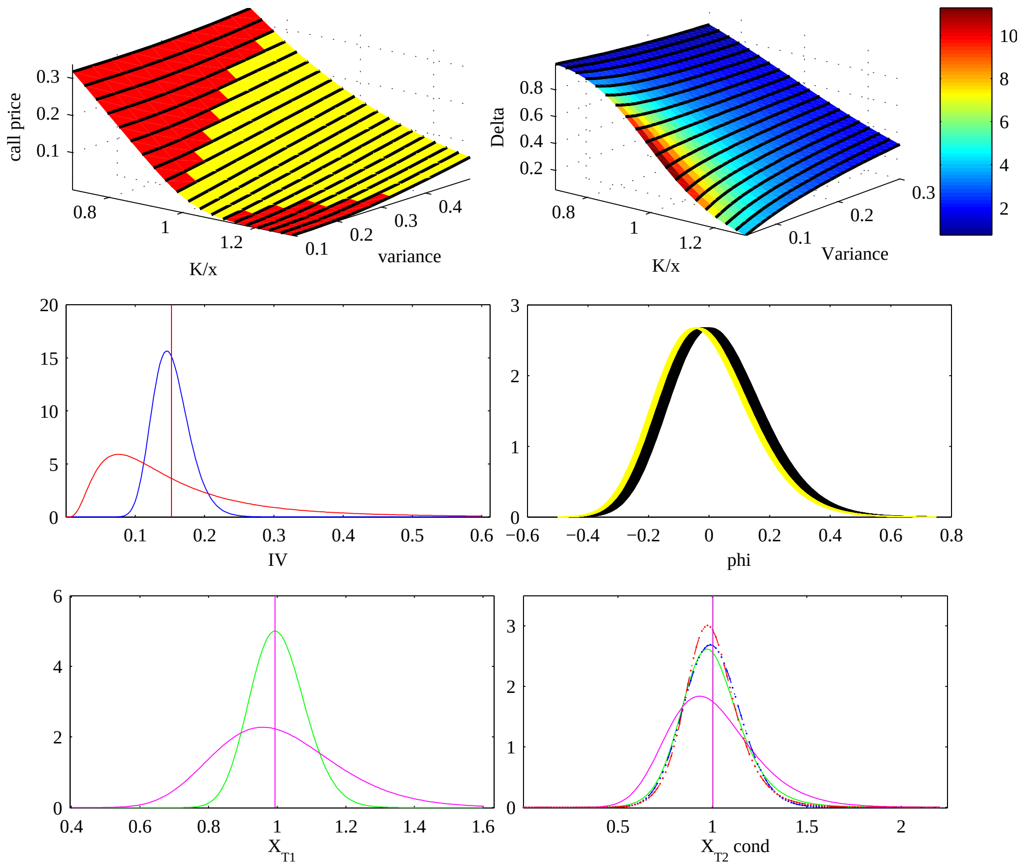

Figure 1: Due to the convexity of the Black-Scholes formula (6) with respect to the volatility and the underlying and the convexity of the payoff pricing formula (5) with respect to the underlying, different distributions of $\bar{\sigma}^2_{tT}$ and ϕ_{tT} result in different option prices for the same strike prices and maturities. As a result, assuming that option market prices coincide with the prices given by the Hull-White formula (7) or the payoff pricing formula, option market prices should provide information on the distributions of $\bar{\sigma}^2_{tT}$ and X_T. In the first row and in the left panel, a surface indicating the convexity (yellow) and concavity (red) of the Black-Scholes formula with respect to σ^2 is plotted. In the right panel, we plot the surface of the Black-Scholes delta. The convexity of the Black-Scholes formula with respect to the underlying is indicated with different colors. In the second row, we plot in the left panel two randomly chosen distributions of the integrated variance with the same expectation (vertical line), namely, $\bar{\sigma}^2_{tT,1} \sim l\mathcal{N}(.15, .03)$ (blue) and $\bar{\sigma}^2_{tT,2} \sim l\mathcal{N}(.125, .5)$ (red). In the right panel, the conditional distributions $\pi(\phi_{tT} \mid \bar{\sigma}^2_{tT})$ computed with different values of ρ, specifically $\rho_1 = -.85$ (black) and $\rho_2 = .4$ (yellow), are shown, assuming that $\bar{\sigma}^2_{tT} \in [0, .6]$. In the bottom row, two randomly chosen densities of X_t with the same expectation are plotted in the left panel, and in the right panel, the corresponding conditional densities $\pi(X_T \mid X_t)$ are shown, computed with (2), $\bar{\sigma}^2_{tT,2}$ and $\rho = 0$. We also plot in the right panel the densities X_T computed with $\bar{\sigma}^2_{tT,2}$ and ρ_1 (blue) and ρ_2 (red), assuming that $X_t = 1$ ($T_1 = .5$ years, $r = 0$).

I. As a stochastic extension of the Black-Scholes implied variance, we define the distribution of the implied integrated variance and the implied correlation coefficient to be, respectively, the distribution of $\bar{\sigma}^2_{tT}$ and the value of ρ for which the correlated Hull-White option prices equal the true option prices U_t^{true}: $U_t^{\text{HW},\rho}(x; K, T; \sigma_t^2, r) = U_t^{\text{true}}(K, T)$, given options with strike prices K_i, $1 \leq i \leq L$, and a maturity T. In the inverse problem of the implied integrated variance, the parameter to estimate is a probability density function $\pi(\bar{\sigma}^2_{tT})$ implied by a series of option prices with different strike prices and the same maturity. Then, using the uncorrelated Hull-White formula, for example, we estimate $\pi(\bar{\sigma}^2_{tT})$ from

$$U_{t,i}^{\text{true}} = \mathbb{E}^*\{U_{t,i}^{\text{BS}}(x; K_i, T; \bar{\sigma}^2_{tT}, r) \mid \bar{\sigma}^2_{tT}, \mathcal{F}_t\} = \int_t^T U_{t,i}^{\text{BS}}(x; K_i, T; \bar{\sigma}^2_{tT}, r)\pi(\bar{\sigma}^2_{tT})d\bar{\sigma}^2_{tT},$$

where the subscript i refers to the strike price K_i, $1 \leq i \leq L$, and the integral is computed with respect to all possible realizations of the integrated variance. In practice, we will have to estimate the unknowns from the noisy option prices observed in the markets. Figure 1 illustrates why Hull-White option prices with different strike prices contain information on the distributions of $\bar{\sigma}^2_{tT}$ and ϕ_{tT}. It also illustrates how an initial distribution $\pi(X_t)$ affects the return (X_T/X_t) similarly to the distributions of ϕ_{tT} and ξ_{tT}.

The inverse problem of implied integrated variance was first considered by Friz and Gatheral (2005), who assumed an uncorrelated volatility. They showed how this problem is unique if a continuum of options with different strike prices is provided. However, the problem is ill-posed in the sense that small errors in the data may result in large errors in the estimates of the unknown. Friz and Gatheral (2005) applied least-squares optimization using traditional regularization techniques and general assumptions regarding the probability density and then use a lognormal prior distribution in the Bayesian sense.

Rather than using regularization techniques, we will adopt a Bayesian statistical approach. In this statistical approach, the entire deterministic inverse problem is recasted in the form of a statistical inference of the distribution of the unknown. We then model both the unknown $X \in \mathbb{R}^n$ and the observation $Y \in \mathbb{R}^m$, as well as the noise $\epsilon \in \mathbb{R}^k$, as random variables or vectors. Then, given the model function F, $Y = F(X) + \epsilon$, the unknown X is estimated from a realization $y \in \mathbb{R}^m$ of Y. In addition to an estimate of the unknown, this approach provides information on the reliability of the estimate. A statistical approach for the inverse problem of option-implied volatilities is presented by Cont and Ben Hamida (2005), where the unregularized ill-posed problem is solved using evolutionary optimization and Monte Carlo sampling. As an example, the local volatility surface implied by options with different strike prices and maturities is estimated.

In the Bayesian framework prior information on the unknown $x = X$, encoded in a prior distribution $P_{\text{prior}}(x)$, is combined with information on the realization y of Y, modeled as a likelihood function $P_{\text{lik}}(y \mid x)$. The solution of the Bayesian inverse problem is the posterior density, given by the Bayes formula: $P_{\text{post}}(x \mid y) \propto P_{\text{prior}}(x)P_{\text{lik}}(y \mid x)$, where $\propto$ means "up to a normalizing constant". The posterior density can be visualized by point estimates, such as Maximum-A-Posteriori (MAP) estimates and Conditional Mean (CM) estimates. A brief introduction to statistical Bayesian inverse problems is given in the Appendix, and they are discussed in more depth by Kaipio and Somersalo (2005). Jacquier, Polson and Rossi (1994) and (2005) apply a Bayesian approach when estimating the stochastic uncorrelated or correlated volatility process from option prices. A literature review on the more recent use of Bayesian methods in quantitative finance is provided by Jacquier and Polson (2010).

When estimating the implied integrated variance, we will not search for one probability density $\pi(\bar{\sigma}^2_{tT})$. Instead, given the option data and the noise level, we will try to estimate the most likely distribution, as well as the probability of other distributions. Based on prior knowledge, we know that the total probability must equal one. Following Friz and Gatheral (2005), we make a prior assumption that the distribution of $\bar{\sigma}^2_{tT}$ is lognormal: $\bar{\sigma}^2_{tT} \sim l\mathcal{N}(\mu, \varsigma)$, such that the problem of estimating the entire distribution reduces to estimating μ and ς, which are random variables in our statistical approach. The lognormality of the realized variance is reported by Andersen, Bollerslev, Diebold and Ebens (2001), Andersen, Bollerslev, Diebold and Labys (2001) and Zumbach, Dacorogna, Olsen and Olsen (1999). A shifted lognormal distribution of realized variance is assumed by Carr and Lee (2007) and a gamma distribution is assumed by Carr, Madan, Geman and Yor (2005).

Because options have different market prices for bids and offers and there is no unique price to be used when estimating the integrated variance, we will model option prices as normally distributed random processes $U_t^{\text{true}} \sim \mathcal{N}(\hat{U}_t, \mathrm{Var_t})$. The mean $\hat{U}_t$ is given as the average of the realized bid and offer prices, $\hat{U}_t = (u_t^{\text{bid}} + u_t^{\text{offer}})/2$ and the variance is $\mathrm{Var_t} = \mathrm{V}(\mathrm{u_t^{bid}} - \mathrm{u_t^{offer}})$, where V is a positive finite constant. We assume that the observed option price u_t^{obs} is a realization of the arbitrage-free true option price U_t^{true} and that each observed option price coincides with the correlated Hull-White price (7) or the corresponding option payoff price (5) up to a small normally distributed error $e_{t,i} \sim \mathcal{N}(0, \mathrm{Var_{t,i}})$, such that

$$u_{t,i}^{\text{obs}} = U_{t,i}^{\text{HW},\rho}(x; K_i, T; \sigma_t^2, r) + e_{t,i} \quad \text{or} \quad u_{t,i}^{\text{obs}} = \mathbb{E}^*\{h_i(X_T)\} + e_{t,i}, \tag{8}$$

where $1 \leq i \leq L$.

Next, we briefly present an algorithm to compute MAP estimates of $\bar{\sigma}^2_{tT} \sim l\mathcal{N}(\mu, \varsigma)$, assuming an uncorrelated volatility. A similar algorithm that can be used to solve the entire inverse problem with correlated volatility and unknown ρ using either the payoff formula or the correlated Hull-White formula is presented in detail in the Appendix. The latter algorithm is from the study by Kaila (2012).

We fix time t and write $u^{\text{obs}} = u_t^{\text{obs}}$, $U^{\text{BS}} = U_t^{\text{BS}}$, $U^{\text{true}} = U_t^{\text{true}}$. For the computation, we must first discretize all of the distributions and the pricing function. We assume that the distribution of the implied integrated variance $\bar{\sigma}^2 = \bar{\sigma}_{tT}^2$ contains positive values in the interval $[a, a + M_\sigma]$, divide this interval into n_σ points, $\bar{\sigma}_j^2$ and denote by $z \in \mathbb{R}^{n_\sigma}$ the discretized probability distribution of $\bar{\sigma}^2$, such that z_j is the probability of $\bar{\sigma}_j^2$, $1 \leq j \leq n_\sigma$. We assure that $\sum_{j=1}^{n_\sigma} z_j = 1$ with $z_j = z_j / \big(M_\sigma / n_\sigma \sum_{j=1}^{n_\sigma} z_j\big)$. We denote

$$A = \frac{M_\sigma}{n_\sigma} \sum_{j=1}^{n_\sigma} U^{\text{BS}}(x; K, T; \bar{\sigma}_j^2, r), \quad u^{\text{obs}} = Az + e.$$

According to the prior assumption, $z \sim l\mathcal{N}(\mu, \varsigma)$. The prior density $P_{\text{prior}}(z)$ consists of all possible lognormal distributions z, denoted by $z_{\text{log}}(\mu, \varsigma)$. We model the hyperprior of the pair $\theta = (\mu, \varsigma)$ as a uniform distribution: $P_{\text{hyper}}(\mu, \varsigma) = \mathcal{U}((\mu_{\min}, \mu_{\max}) \times (\varsigma_{\min}, \varsigma_{\max}))$.

Because we have assumed a normally distributed error e, the likelihood function is given by the Gaussian kernel:

$$P_{\text{lik}}(u^{\text{obs}} \mid z) \propto \exp\bigg(-\frac{1}{2}\big((u^{\text{obs}} - Az)^T \Gamma^{-1} (u^{\text{obs}} - Az)\big)\bigg),$$

where the covariance matrix $\Gamma = \operatorname{diag}(\text{Var}_1, \text{Var}_2, \ldots, \text{Var}_L)$. The posterior density $P_{\text{post}}(z \mid u^{\text{obs}}) \propto P_{\text{lik}}(u^{\text{obs}} \mid z) P_{\text{prior}}(z) P_{\text{hyper}}(\mu, \varsigma)$, which is the solution of the inverse problem, reduces under the lognormality assumption of z to

$$P_{\text{post}}(z \mid u^{\text{obs}}) \propto P_{\text{lik}}(u^{\text{obs}} \mid z_{\text{log}}(\mu, \varsigma)) P_{\text{hyper}}(\mu, \varsigma).$$

In our case, the Maximum-A-Posteriori (MAP) estimator $z_{\mathrm{MAP}}(\mu_{\mathrm{MAP}}, \theta_{\mathrm{MAP}})$, which is characterized by $z_{\mathrm{MAP}} = \arg\max_{z \in \mathbb{R}^{n_\sigma}} P_{\mathrm{post}}(z \mid u^{\mathrm{obs}})$, leads to the following optimization problem:

$$z_{\mathrm{MAP}}(\mu_{\mathrm{MAP}}, \varsigma_{\mathrm{MAP}}) = \arg\min_{z_{\log}(\mu,\varsigma))} \big((u^{\mathrm{obs}} - Az_{\log}(\mu,\varsigma))^T \Gamma^{-1} (u^{\mathrm{obs}} - Az_{\log}(\mu,\varsigma))\big), \quad (9)$$

where (μ, ς) is given by the hyperprior. A simple method to estimate z_{MAP} or $\theta_{\mathrm{MAP}} = (\mu_{\mathrm{MAP}}, \varsigma_{\mathrm{MAP}})$ is to draw a sample $S = \{\theta_1, \theta_2, \ldots, \theta_N\}$ of independent realizations $\theta_j \in \mathbb{R}^2$ from the hyperprior $P_{\mathrm{hyper}}(\mu, \varsigma)$ and compute the θ_j that minimizes the right-hand side of (9). However, this method can be computationally costly because the sample size N must be large enough such that S represents the uniform distributions of μ and ς. Alternatively, we can compute estimates with the grid search method, where we approximate the uniformly distributed continuous hyperprior $P_{\mathrm{hyper}}(\mu, \varsigma)$ by a discretized hyperprior. This discretized hyperprior is a grid consisting of n_μ evenly spaced values of μ within $[\mu_{\min}, \mu_{\max}]$ and n_ς values of ς within $[\varsigma_{\min}, \varsigma_{\max}]$. The norm on the right-hand side of (9) is evaluated at all of the grid points, and θ_{MAP} and z_{MAP} are given by the grid point (μ_i, ς_j), $1 \le i \le n_\mu$, $1 \le j \le n_\varsigma$ minimizing the norm. If we know ρ and have good hyperpriors, an estimate z_{MAP} is obtained in less than a minute (computed with 2.2 GHz Intel Core i7). We explain in the Appendix how the hyperpriors can be chosen.

We cannot expect the approximate solution to yield a smaller residual error than the measurement error e and should accept as suitable approximations of the unknowns all the estimates satisfying $\|u^{\mathrm{obs}} - Az\| \simeq \|e\| = \|u^{\mathrm{obs}} - U^{\mathrm{true}}\|$. We denote such distributions $z_{\log}$ by z_{err}, and those with the smallest and largest expectations by $z_{\min}$ and $z_{\max}$, respectively. This discrepancy principle, the effect of noisy data and the bid-ask spread in the volatility estimation have also been considered by Cont and Hamida (2005).

Theoretically, the distribution of the implied integrated variance is the same for all strike prices and is already reflected in a couple of option prices due to the convexity of the Black-Scholes formula and the payoff formula with respect to the underlying and the volatility, as illustrated in Figure 1. If the option data are very noisy or the investors in in-the-money options have different expectations of the volatility than those investing in at-the-money or out-of-the-money options, estimates of the implied integrated variance could vary according to the strike prices of the options. The accuracy of the out-of-the-money calls and in-the-money puts is essentially lower than that of the rest of the options.

In Figure 2, we illustrate the performance of the grid search method with the MAP estimates of $\bar{\sigma}^2_{tT}$ implied by S&P 100 call options maturing in August 2009. We plot z_{MAP} and z_{err}, as well as the pricing errors related to these estimates. As supports for the discrete probability distributions, we use $\bar{\sigma}^2_{tT} \in [1e^{-5}, 1]$, $n_\sigma = 500$ and $\phi_{tT} \in [-.5, .5]$, $n_\phi = 100$. The discrete hyperprior of the pair (μ, ς) consists of $n_\mu = 100$ evenly distributed points μ on the interval $[\log(.5I^2_{\mathrm{ATM}}), \log(2I^2_{\mathrm{ATM}})]$, where I_{ATM} is the implied at-the-money volatility, and $n_\varsigma = 6$ prior values $\varsigma_{\mathrm{prior}}$ that are unevenly chosen on the interval $[.005, .3]$. We have chosen the prior values of ς unevenly in order to minimize the required computation; the option prices are not sensitive to whether the variance of $\bar{\sigma}^2_{tT}$ is $\varsigma = .005$ or $\varsigma = .007$, for example. Additionally, based on prior information on the typical values of ρ implied by S&P 100 options in 2009, the discretized hyperprior of ρ is the set $[-.7, -.75, -.8, -.85, -.875, -.9, -.925]$. Because we do not have actual information on the bid-ask spreads, we assume that $e \sim \mathcal{N}(0, (.02u^{\mathrm{obs}})^2)$.

To close the section, let us consider the integrated variance between two future moments in time $0 \leq T_1 < T_2$:

$$\bar{\sigma}_{12}^2 = \bar{\sigma}_{T_1 T_2}^2 = \frac{1}{T_{12}} \int_{T_1}^{T_2} \sigma_s^2 ds, \quad T_{12} = T_2 - T_1.$$

Given two estimates of $\bar{\sigma}_{tT_i}^2 \sim l\mathcal{N}(\mu_i, \varsigma_i)$ implied by the options u_i with different strike prices and two maturities $T_i,\ i = 1, 2$, we can compute an estimate of $\bar{\sigma}_{12}^2$. For $\bar{\sigma}_{12}^2 \sim l\mathcal{N}(\mu_{12}, \varsigma_{12})$, the estimates μ_{12} and ς_{12} are obtained from μ_i and ς_i, $i = [1, 2]$ by

$$\begin{aligned}
\mu_{12} &= \big(\log(\frac{T_2}{T_{12}} e^{\mu_2 + .5\varsigma_2} - \frac{T_1}{T_{12}} e^{\mu_1 + .5\varsigma_1})\big) - \frac{1}{2}\varsigma_{12}, \\
\varsigma_{12} &= \log\left(1 + \frac{\frac{T_2}{T_{12}}(e^{\varsigma_2} - 1)e^{2\mu_2 + \varsigma_2} - \frac{T_1}{T_{12}}(e^{\varsigma_1} - 1)e^{2\mu_1 + \varsigma_1}}{(\frac{T_2}{T_{12}} e^{\mu_2 + .5\varsigma_2} - \frac{T_1}{T_{12}} e^{\mu_1 + .5\varsigma_1})^2}\right).
\end{aligned}$$

The corresponding proof is provided in the Appendix. We will need $\bar{\sigma}_{12}^2$ when pricing some path-dependent options in Section 5.

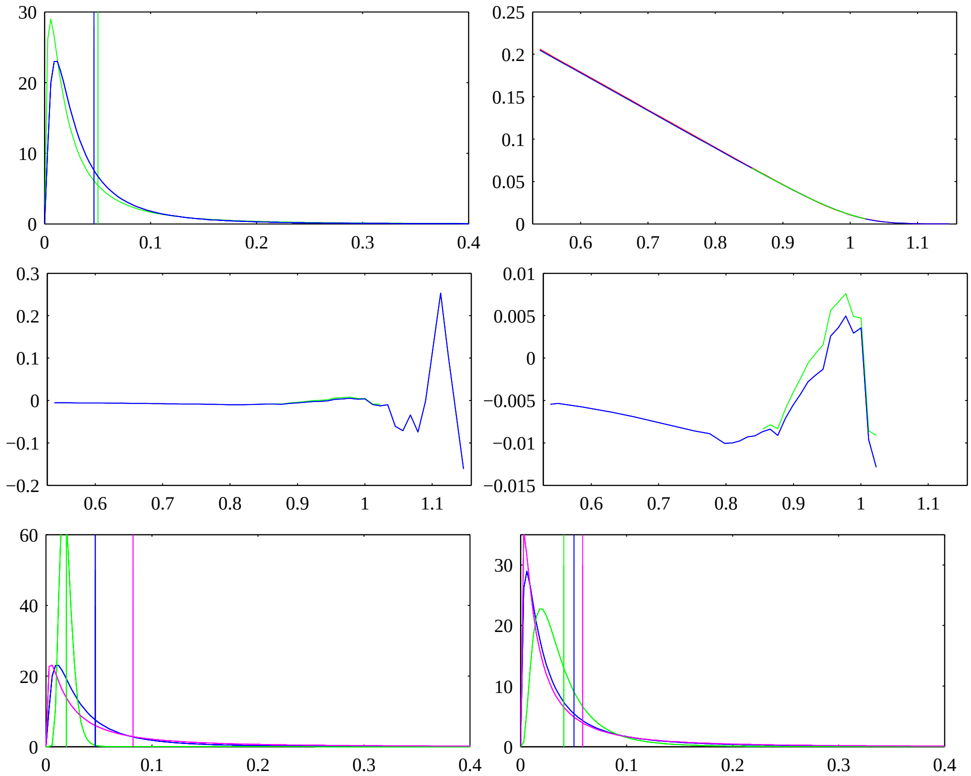

Figure 2: MAP estimates. In the first row, we plot in the left panel as distributions and expectations (vertical lines) the estimates $z_{\text{MAP,all}}$ computed from all $L = 54$ call options (blue) and $z_{\text{MAP,nATM}}$ from $L = 16$ near-at-the-money calls with moneyness $.85 \leq M \leq 1.03$ (green). In the right panel, the corresponding $L = 54$ true call prices u^{obs} (red) and the call prices u_{MAP} computed with the correlated Hull-White formula and the estimates z_{MAP} in the left panel are plotted using the same colors. In the second row, we plot in the left panel the relative errors between the estimated prices and the true prices, $(u^{\text{obs}} - u_{\text{MAP}})/u^{\text{obs}}$, computed with $z_{\text{MAP,all}}$ (blue) and $z_{\text{MAP,nATM}}$ (green). As the accuracy of the out-of-the money prices tends to be less than the accuracy of in-the-money and at-the-money prices, we plot in the right panel the same information on the relative errors but for the in-the-money and at-the-money options only. The MAP estimates of the correlation coefficient implied by all $L = 52$ options and $L = 16$ near-at-the-money options are $\rho_{\text{MAP,all}} = -.85$ and $\rho_{\text{MAP,nATM}} = -.8$, respectively. In the bottom row, we plot estimates $z_{\min}$ (green) and $z_{\max}$ (red) that should be accepted according to the discrepancy principle, as well as z_{MAP} (blue). In the left panel, we have used all $L = 54$ options and $\rho_{\text{MAP,all}}$ and in the right panel, we have used the $L = 16$ near-at-the-money options and $\rho_{\text{MAP,nATM}}$ (the estimates are computed from S&P 100 calls maturing in August 2009 and the assumed noise is $e \sim \mathcal{N}(0, (.02u^{\text{obs}})^2)$).

3 Volatility derivatives

Integrated and realized variances and volatilities are closely related to some volatility derivatives, such as variance and volatility swaps and options. These instruments provide the possibility to speculate on or hedge risks associated with the size of the movement of an underlying product, such as a stock index, an exchange rate or an interest rate. Variance and volatility swaps are forward contracts that pay the difference between the realized variance or volatility and an agreed fixed amount, the strike, at maturity. Variance and volatility calls and puts are similar to European calls and puts, with an underlying realized variance or volatility.

There are two main approaches in pricing volatility derivatives, based on either the underlying stock price process or the information provided by the market prices of European options. Option-implied volatility derivatives are discussed in detail by Carr and Lee (2008), and volatility derivatives more generally are discussed by Dupire (1992), Carr and Madan (1998), Derman et al. (1999a), (1999b) and, more recently, by Carr et al. (2005) and Carr and Sun (2008). The volatility derivatives markets are considered by Carr and Lee (209), who have also provided a literature review.

The payoff functions of a variance swap with strike price $K^{\mathrm{H}}_{\mathrm{var}}$, denoted by $H_{\mathrm{var}}(\bar{\sigma}^2_{tT})$, and of a variance call with price $K^{\mathrm{h}}_{\mathrm{var}}$, denoted by $h_{\mathrm{var}}(\bar{\sigma}^2_{tT})$, are given by

$$H_{\mathrm{var}}(\bar{\sigma}^2_{tT}) = \bar{\sigma}^2_{tT} - K^{\mathrm{H}}_{\mathrm{var}} \quad \text{and} \quad h_{\mathrm{var}}(\bar{\sigma}^2_{tT}) = (\bar{\sigma}^2_{tT} - K^{\mathrm{h}}_{\mathrm{var}})^+,$$

and the corresponding payoff functions of a volatility swap with a strike price $K^{\mathrm{H}}_{\mathrm{vol}}$ and a volatility call with a strike price $K^{\mathrm{h}}_{\mathrm{vol}}$, denoted by $H_{\mathrm{vol}}(\bar{s}_{tT})$ and $h_{\mathrm{vol}}(\bar{s}_{tT})$, are given by

$$H_{\mathrm{vol}}(\bar{s}_{tT}) = \bar{s}_{tT} - K^{\mathrm{H}}_{\mathrm{vol}} \quad \text{and} \quad h_{\mathrm{vol}}(\bar{s}_{tT}) = (\bar{s}_{tT} - K^{\mathrm{h}}_{\mathrm{vol}})^{+}.$$

In this paper, for simplicity, we will assume that the swap strikes $K^{\mathrm{H}}_{\mathrm{var}} = 0$ and $K^{\mathrm{H}}_{\mathrm{vol}} = 0$.

Based on the work of Neuberger (1994), Demeterfi, Derman, Kamal and Zou (1999) show how variance swaps can be replicated by holding a static position in European calls and puts with all strikes and trading dynamically on the underlying asset. The logarithm of stock price return is obtained by

$$\log\left(\frac{X_T}{X_t}\right) = \frac{X_T - X_t}{X_t} - \int_0^{X_t} \frac{(K - X_T)^{+}}{K^2} dK - \int_{X_t}^{\infty} \frac{(X_T - K)^{+}}{K^2} dK. \tag{10}$$

The integrated variance is replicated by a portfolio of calls and puts where each of the options is weighted by the squared strikes. Carr and Itkin (2010) note three types of errors related to the synthetizing of $\log(X_T/X_t)$ with (10): errors due to jumps in asset prices, extrapolation and interpolation errors due to a finite number of available option quotes instead of the continuum of options that is needed in the replication, and errors in computing the realized return variance.

Javahari, Wilmott and Haug (2004) value volatility swaps using the GARCH process. Gatheral (2006) proposes that the value of the variance swap can be expressed as a weighted average of Black-Scholes implied variances $I^2 = I^2(\log(X_t/K))$, with weights given by the log-moneyness, such that

$$\mathbb{E}^*\{I^2\} = \int_{\mathbb{R}} \Phi'(z) I^2(z) dz,$$

where $z = \log(X_t/K)$ and $\Phi'(y) = e^{y^2/2}/\sqrt{2\pi}$. Carr and Lee (2007) and (2008) construct a synthetic swap by trading a log contract of the underlying and bonds. Their approach does not make prior assumptions on the dynamics of the volatility and is robust with respect to correlated volatility. Carr and Itkin (2010); Carr, Geman, Madan and Yor (2010); and Carr, Lee and Wu (2012) propose an asymptotic method with the assumption that the underlying process is modeled by a Levy process with stochastic time change. Friz and Gatheral (2005) price volatility and variance swaps with the expectation of the implied integrated variance, assuming an uncorrelated volatility. Furthermore, based on the Black-Scholes formula, they derive a closed-form pricing formula for variance calls.

Our approach is an extension of that of Friz and Gatheral (2005). We assume a correlated volatility and use the distribution of the implied integrated variance to price volatility options. We show how the distribution provides information on the pricing error we may be making and explain how noisy option data should be taken into account when pricing volatility derivatives. We also estimate the distributions of variance and volatility swaps between two future moments in time. The prices of various volatility derivatives computed with implied integrated variances and volatilities are arbitrage-free prices. Unless the markets for these volatility derivatives are liquid, the corresponding market prices may differ substantially from the arbitrage-free prices. The error between the arbitrage-free prices and the market prices, as well as possible systemization in the error, is a subject of further research.

Variance swaps and calls can be perfectly replicated under the classical derivatives pricing theory and are to some extent straightforward to price and hedge. The same is not true for volatility swaps and calls. Hedging a volatility swap using variance swaps requires a dynamic position in the log contract replicating the variance swap. For example, according to (10), this position depends on a strip of vanilla options. Because some of these options will be very far out-of-the-money, the trading is performed at large bid-offer spreads and the re-balancing of the hedge becomes expensive.

Let us look at the problem of pricing volatility swaps and calls. Denote by $\hat{\sigma}^2_{tT}$ and $\hat{s}_{tT} = \mathbb{E}^*\{\bar{s}_{tT}\} = \mathbb{E}^*\{\sqrt{\bar{\sigma}^2_{tT}}\}$ the expectations of the integrated variance and volatility, respectively. As the expectation of the square root of a random variable is less than the square root of the expectation, we have

$$\hat{s}_{tT} = \mathbb{E}^*\{\sqrt{\bar{\sigma}^2_{tT}}\} \leq \sqrt{\mathbb{E}^*\{\bar{\sigma}^2_{tT}\}} = \sqrt{\hat{\sigma}^2_{tT}}, \tag{11}$$

The difference between the expectation of the integrated volatility and the square root of $\hat{\sigma}^2_{tT}$ is called the convexity adjustment. Sometimes, $\sqrt{\mathbb{E}^*\{\bar{\sigma}^2_{tT}\}}$ is called the model-free implied volatility. The VIX index approximates this volatility.

Brockhaus and Long (2000) suggest a volatility convexity correction that relates variance and volatility products. Heston and Nandi (2000) provide an analytical solution to price volatility swaps and options and demonstrate how to hedge different volatility derivatives by trading only on the underlying asset and a risk-free asset. Broadie and Jain (2008) derive partial differential equations to price volatility derivatives and to compute Greeks in order to hedge these derivatives, and Windcliff, Forsyth and Vetzal (2006) suggest improving the delta hedge of the volatility derivative by an additional gamma hedge. As explained by Friz and Gatheral (2005), under the prior assumption $\bar{\sigma}^2_{tT} \sim l\mathcal{N}(\mu, \varsigma)$, the integrated volatility $\bar{s}_{tT} \sim l\mathcal{N}(\mu/2, \varsigma/4)$ and the convexity adjustment is given by

$$\sqrt{\mathbb{E}^*\{\bar{\sigma}^2_{tT}\}} - \mathbb{E}^*\{\sqrt{\bar{\sigma}^2_{tT}}\} = (e^{.5\varsigma} - 1)\mathbb{E}^*\{\sqrt{\bar{\sigma}^2_{tT}}\}.$$

This formula provides a convexity correction for the VIX index. Carr and Lee (2008) add to (11) a lower bound $(\sqrt{2\pi}/X_t)\mathbb{E}^*\{(X_T - X_t)^+\}$ and an upper bound based on the log-return $\log(X_T/X_t)$ and show how to superreplicate the payoff of $\sqrt{\bar{\sigma}_{tT}^2}$.

We suggest that when pricing variance and volatility swaps, not only the expectations but the entire distributions of the integrated variances and volatilities are important. In particular, the volatility call is easily computed from the distribution of the corresponding swap. In Figure 3, we present examples of the MAP estimates of variance and volatility swaps, $H_{\text{var}}(\bar{\sigma}_{tT}^2)$ and $H_{\text{vol}}(\bar{s}_{tT})$, respectively, as well as the corresponding payoffs of variance and volatility calls $h_{\text{var}}(\bar{\sigma}_{tT}^2)$ and $h_{\text{vol}}(\bar{s}_{tT})$ as distributions and expectations, computed from S&P 100 ATM calls maturing in June 2009 with $\rho_{\text{MAP}} = -.85$, assuming that the error $e_1 \sim \mathcal{N}(0, (.02u^{\text{obs}})^2)$ and $K_{\text{var}} = .04$, $K_{\text{vol}} = \sqrt{.04} = .2$. We also indicate the discrete points $\bar{\sigma}_{tT,j}^2$ $1 \leq j \leq n_\sigma$ that are .67 standard deviations from $\hat{\sigma}_{tT}^2$. With the probability $P = .5$, the integrated variance is between these points. The distributions z_{MAP}, $z_{\min}$ and $z_{\max}$ in Figure 2 provide an idea of the effect of noisy data on the prices of variance and volatility derivatives. According to the discprepancy principle, both the estimates $z_{\min}$ and $z_{\max}$ should be considered as suitable estimates of $\bar{\sigma}_{tT}^2$.

To close the section, we plot the surfaces of the implied variance and the implied volatility between two future moments in time, estimated from S&P 100 call options with different strike prices and the maturities T_1 in March and T_2 in April 2010. The estimates were computed four times, every 15 days, starting 60 days before the maturity T_1. Information provided by these surfaces can be used when pricing variance and volatility swaps and options.

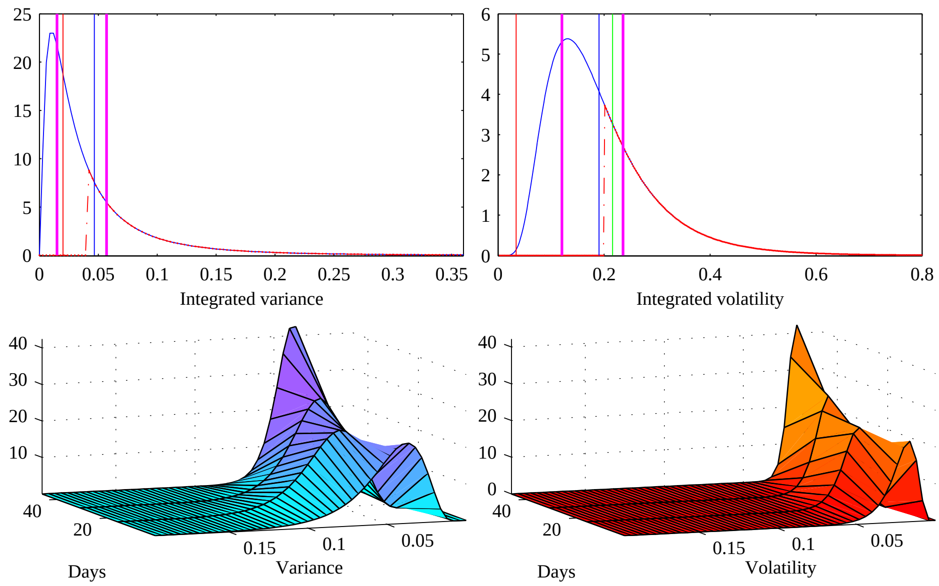

Figure 3: Variance and volatility derivatives. In the top row and in the left panel, estimates of a variance swap $\bar{\sigma}^2_{tT}$ (blue) and the payoff $h_{\text{var}}(\bar{\sigma}^2_{tT})$ of a variance call (red) are plotted as expectations (vertical lines) and distributions, based on S&P 100 calls maturing in June 2009 with $T = 1/10$ years. The right panel presents the corresponding data for volatility swaps and calls, as well as $\mathbb{E}\{\bar{\sigma}^2_{tT}\}$ (green). With the probability $P = .5$, the implied integrated variance and volatility are within the points indicated with magenta lines. The strike prices are $K_{\text{var}} = .04$ and $K_{\text{vol}} = \sqrt{.04} = .2$. In the bottom row, estimates of the implied integrated variance $\bar{\sigma}^2_{12}$ (left panel) and the corresponding implied integrated volatility $\bar{s}_{12} = sqrt\bar{\sigma}^2_{12}$ (right panel) between two future moments in time are plotted. Based on S&P 100 calls maturing in June and July 2009, we have estimated both distributions every 15 days, beginning 60 days before the first maturity in March.

4 Pricing digital options and barrier options

In this section, we use the risk-neutral price density and the implied integrated variance to price digital options and one type of semi-path-dependent options, namely, the barrier options. The pricing of both types of options when the volatility is stochastic has been considered by Fouque, Papanicolaou and Sircar (2000), where an asymptotic method exploiting volatility clustering is applied, and by Tahani (2005), where these options are priced with a Taylor expansion around two average volatilities and the Hull-White formula. Our contribution to the literature is to present how both types of options can be priced using the correlated Hull-White paradigm. We also show how both the distribution of the implied integrated variance and knowledge of the noise level in the option prices provide additional information on the size of the pricing error that we are potentially making.

The price $D_t(x;\sigma_t^2) = D_t(x;K,T,Q;\sigma_t^2,r)$ of a digital call is given by

$$\begin{aligned} D_t(x;\sigma_t^2) &= e^{-r(T-t)} Q\mathbb{E}^*\{(X_T > K) \mid x, \sigma_t^2, \mathcal{F}_t\} \\ &= e^{-r(T-t)} Q\mathbb{E}^*\{h_{\mathrm{D}}(X_T) \mid x, \sigma_t^2, \mathcal{F}_t\}, \end{aligned}$$

where Q is a positive constant. When the volatility is constant, the price of a digital call is given by $D_t^{\mathrm{BS}}(x;\sigma^2) = e^{-r(T-t)}Q\Phi(d_-(\sigma^2))$. We suggest that in the case of stochastic correlated volatility, the corresponding price would be

$$D_t^{\mathrm{HW},\rho}(x;\sigma_t^2) = e^{-r(T-t)} Q\mathbb{E}^*\{\Phi(d_-(\bar{\sigma}^2_{\rho,tT})) \mid \sigma_t^2, \mathcal{F}_t\},$$

applying the Hull-White paradigm. Figure 4 presents estimates of such digital options as functions of moneyness, computed from the S&P 100 calls that are maturing in November

2010.

Now consider the pricing of digital options using the risk-neutral price density. The distribution of $\bar{\sigma}^2_{tT}$ provides additional information on the reliability of the risk-neutral price density and the digital option prices. According to (2), the risk-neutral price density is the expectation of lognormal distributions $\pi(X_T \mid \bar{\sigma}^2_{tT})$ conditional on $\bar{\sigma}^2_{tT}$. In Figure 4, we plot such lognormal distributions, conditional on the discrete values $\bar{\sigma}^2_{tT,j}$, $1 \leq j \leq n_\sigma$ within one and two standard deviations from the expectation $\hat{\sigma}^2_t$. With the probabilities of $P_1 \approx .7$ and $P_2 \approx .95$, the risk-neutral price density is within the range of these distributions. We then plot the expected distribution of X_T and the digital option prices corresponding to each distribution.

Next, consider the pricing of barrier options. Barrier options are semi-path-dependent options whose payoff depends on whether the underlying asset price hits a specified value during the lifetime of the option. For example, a down-and-in option becomes a European call or put if the underlying stock price X_t reaches a predetermined level B at any time before the maturity T, and a down-and-out option is a European option that becomes worthless if X_t falls below a level B. The payoffs of a down-and-in call, h_t^{DI}, and of a down-and-out call, h_t^{DO}, are given by

$$
\begin{aligned}
h_t^{\mathrm{DI}} &= (x; K, B, T; \sigma^2, r) = \mathbb{E}^*\{(X_T - K)^+ 1_{[X_t > B \text{ for any } t \leq T](X_t)}\}, \\
h_t^{\mathrm{DO}} &= (x; K, B, T; \sigma^2, r) = \mathbb{E}^*\{(X_T - K)^+ 1_{[X_t < B \text{ for all } t \leq T](X_t)}\},
\end{aligned}
$$

where $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$, otherwise.

A common model for pricing and hedging barrier options, based on a reflection theorem in the Black-Scholes model, was initially considered by Carr, Ellis and Gupta (1998)

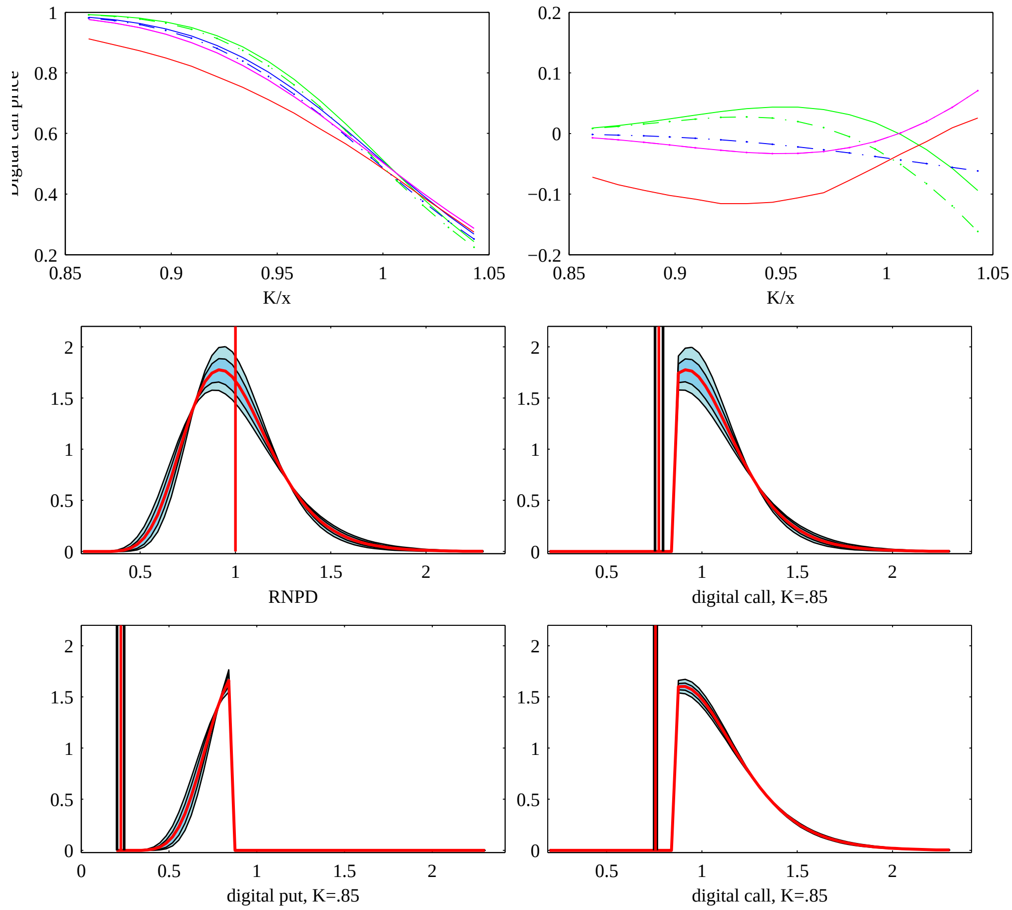

Figure 4: In the left panel in the top row, we plot the digital option prices D_{MAP} computed using the estimate of z_{MAP} (red), prices $D_{.14}$ and $D_{.86}$ computed using the discrete values $\bar{\sigma}^2_{tT,.14}$ and $\bar{\sigma}^2_{tT,.86}$ that are at 1.5 standard deviation from the expectation $\hat{\sigma}^2_{tT}$ (blue and green), and prices D_I computed using the implied variance I^2 (magenta). The prices computed with $z_{\min}$ and $z_{\max}$ are plotted with dashed lines. In the right panel, we present the corresponding relative differences between each price and D_{MAP}, $(D_{\text{MAP}} - D_I)/D_{\text{MAP}}$, for example (S&P 100 maturing in November 2008, $T = 1.5$ months, $Q = 1$). In the second row, we consider estimates of the risk-neutral price density computed with $\bar{\sigma}^2_{tT} \sim l\mathcal{N}(.15, .05)$ and $\rho = -.15$. We plot in red the expected distribution of X_T (left panel) and the corresponding digital option payoff $h_{\text{D}}(X_T)$ (right panel), both as distributions and expectations. We then plot in darker and lighter blue lognormal densities of $X_{T,j}$ conditional on each point of $\bar{\sigma}^2_{tT,j}$, $1 \leq j \leq n_\sigma$ that is within 1 or 2 standard deviations from $\hat{\sigma}^2_t$ (left), as well as the corresponding digital option payoffs as densities (right). With the probability $P \approx .95$, the payoff of the digital option price is within the two black lines. In the bottom row, we plot the same information for a put option (left panel) and a call option when $\bar{\sigma}^2_{tT} \sim l\mathcal{N}(.12, .01)$ and $\rho = -.45$ (right panel) ($Q = 1$, $T = .5$ years, $r = 0$).

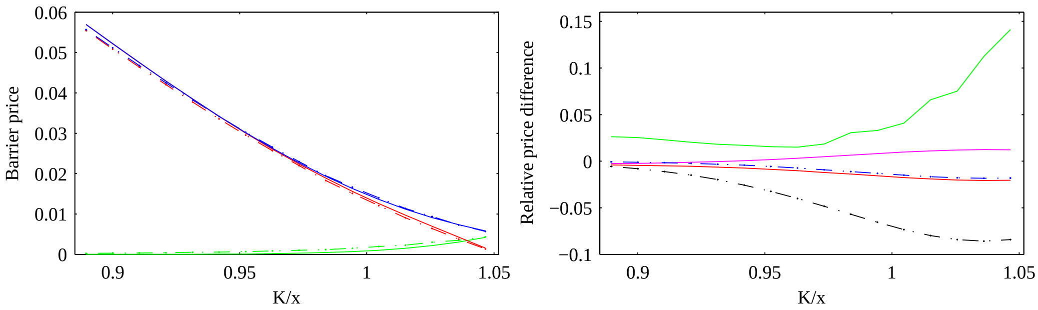

Figure 5: We plot in the left panel the prices of a down-and-in call (green), down-and-out call (red) and a vanilla call (blue), computed using z_{MAP} implied by S&P 100 calls maturing in November 2010 ($T \approx 2$ months, $B = .95K$). The corresponding prices computed with the implied variance I^2 are indicated with dashed lines. The relative differences between the down-and-out prices computed with z_{MAP} and the prices computed with $\bar{\sigma}^2_{tT,.14}$ (magenta), $\bar{\sigma}^2_{tT,.86}$ (red) and I^2 (green), $z_{\min}$ (black) and $z_{\max}$ (blue) are plotted in the right panel.

and Carr and Chou (1997a), (1997b). Barrier options with deterministic volatility are considered by Andersen, Andreasen and Eliezer (2002), Nalholm and Poulsen (2006), Poulsen (2006) and Carr and Nadtochiy (2011), who also provide a literature review of barrier options.

When the volatility is constant, barrier options can be priced based on the Black-Scholes formula. For example, the price of the down-and-in barrier option, $DI_t^{\mathrm{BS}}(x;\sigma^2) = DI_t^{\mathrm{BS}}(x;K,B,T;\sigma^2,r)$ with $B \le K$ is given by

$$DI_t^{\mathrm{BS}}(x;\sigma^2) = x\left(\frac{B}{x}\right)^{2\Psi}\Phi(y) - Ke^{-r(T-t)}\left(\frac{B}{x}\right)^{2\Psi-2}\Phi(y-\sigma\sqrt{(T-t)})$$

where

$$\Psi = \frac{r}{\sigma^2}+\frac{1}{2}, \quad y = \frac{\ln(B^2/(xK))}{\sigma\sqrt{(T-t)}} + \Psi\sigma\sqrt{(T-t)}.$$

and the price of the down-and-out option $DO_t^{\mathrm{BS}}(x;\sigma^2) = DO_t^{\mathrm{BS}}(x;K,B,T;\sigma^2,r)$ is $DO_t^{\mathrm{BS}}(x;\sigma^2) = U_t^{\mathrm{BS}}(x;\sigma^2) - DI_t^{\mathrm{BS}}(x;\sigma^2)$. When $B > K$, the price of a down-out barrier option is given by

$$
\begin{aligned}
DO_t^{\mathrm{BS}}(x;\sigma^2) &= x\Phi(x_1) - Ke^{-r(T-t)}\Phi(x_1 - \sigma\sqrt{(T-t)}) \\
&- x\left(\frac{B}{x}\right)^{2\Psi}\Phi(y_1) + Ke^{-r(T-t)}\left(\frac{B}{x}\right)^{2\Psi-2}\Phi(y_1 - \sigma\sqrt{(T-t)}),
\end{aligned}
$$

where

$$
x_1 = \frac{\log(x/B)}{\sigma\sqrt{(T-t)}} + \Psi\sigma\sqrt{(T-t)}, \quad y_1 = \frac{\log(B/x)}{\sigma\sqrt{(T-t)}} + \Psi\sigma\sqrt{(T-t)},
$$

and $DI_t^{\mathrm{BS}}(x;\sigma^2) = U_t^{\mathrm{BS}}(x;\sigma^2) - DO_t^{\mathrm{BS}}(x;\sigma^2)$.

We suggest that when the volatility is stochastic, the barrier option prices could be given as expectations of the Black-Sholes barrier option prices with respect to the integrated variance and ϕ_{tT}. For example, the prices of a down-and-in and a down-and-out option would be as follows:

$$
\begin{aligned}
DI_t^{\mathrm{HW},\rho}(x;\sigma_t^2) &= \mathbb{E}^*\{\mathbb{E}^*\{DI_t^{\mathrm{BS}}(e^{\rho\phi_{tT}}x;\bar{\sigma}^2_{\rho,tT}) \mid \bar{\sigma}^2_{tT}, \mathcal{F}_t\} \mid \sigma_t^2, \mathcal{F}_t\} \\
DO_t^{\mathrm{HW},\rho}(x;\sigma_t^2) &= \mathbb{E}^*\{\mathbb{E}^*\{DO_t^{\mathrm{BS}}(e^{\rho\phi_{tT}}x;\bar{\sigma}^2_{\rho,tT}) \mid \bar{\sigma}^2_{tT}, \mathcal{F}_t\} \mid \sigma_t^2, \mathcal{F}_t\}.
\end{aligned}
$$

We present a proof of these formulas with stochastic volatility in these Appendix, closely following the proof by Carr and Chou (1997b) associated with constant volatility. In particular, because the implied integrated variance is independent of the strike prices, this variance is quite robust with respect to the changes in the prices of the underlying during the lifetime of the option, compared with using the implied volatility corresponding to a certain strike price.

Figure 5 illustrates barrier option prices based on estimates z_{MAP}, $z_{\min}$ and $z_{\max}$ computed from S&P 100 calls maturing in November 2010. We also computed two prices with the discrete values $\bar{\sigma}^2_{tT,j}$ that are within 1.5 standard deviations from $\hat{\sigma}^2_{tT} = \mathbb{E}\{\bar{\sigma}^2_{tT}\}$, denoted by $\bar{\sigma}^2_{.14}$ and $\bar{\sigma}^2_{.86}$. Both the expectation and the shape of the distribution of $\bar{\sigma}^2_{tT}$ affect the prices of the barrier options.

5 Path-dependent options

In this section, we consider the pricing of two types of path-dependent options: compound options and cliquet options. Both types of options are sensitive to changes in volatility. When pricing these options, we will need estimates of $\bar{\sigma}^2_{12}$, the implied integrated variance between two future moments in time, namely, T_1 and T_2, $0 \leq T_1 < T_2$. Figure 6 presents such estimates and the corresponding risk-neutral price densities $\pi(X_{12}) = \pi(X_{T_2} \mid X_{T_1})$. First, for a fixed time t, we estimate these densities four months into the future based on the prices of options maturing in five consecutive months. We then take two options with consecutive maturities T_1 and T_2 and present how the estimates of $\bar{\sigma}^2_{12}$ and the corresponding density of X_{12} change as a function of time. We estimate both densities every 15 days, beginning 60 days before T_1.

Let us then price compound options, i.e., options-on-options, with two strike prices and two expiration dates. On the date of the first maturity T_1, the holder of a compound option has the right to buy or sell at price K_1 a new option with strike price K_2 and maturity T_2. Analytical pricing formulas have been proposed by Geske (1979), Hodges and Selby (1987) and Rubinstein (1992), provided that the volatility is constant, and Fouque and Han (2005) suggest a perturbation approximation when the volatility is stochastic.

We consider here the pricing of a put-on-call option; the pricing of a put-on-put, a call-on-call or a call-on-put is similar. To keep the notation simple, we assume that $\rho = 0$. Extension to correlated volatility is straight forward. Suppose that we have estimates of $\bar{\sigma}^2_{12}$ and X_{T_1}. Then, conditional on X_{T_1}, the price of the put on a call option is given by

$$C_{\text{PoC,COND}} = e^{-r(T_1-t)} \max(0, K_1 - \mathbb{E}^*\{U^{\text{BS}}_t(X_{T_1}; K_2, T_2 - T_1; \bar{\sigma}^2_{12}, r) \mid X_{T_1}, \mathcal{F}_t\}),$$

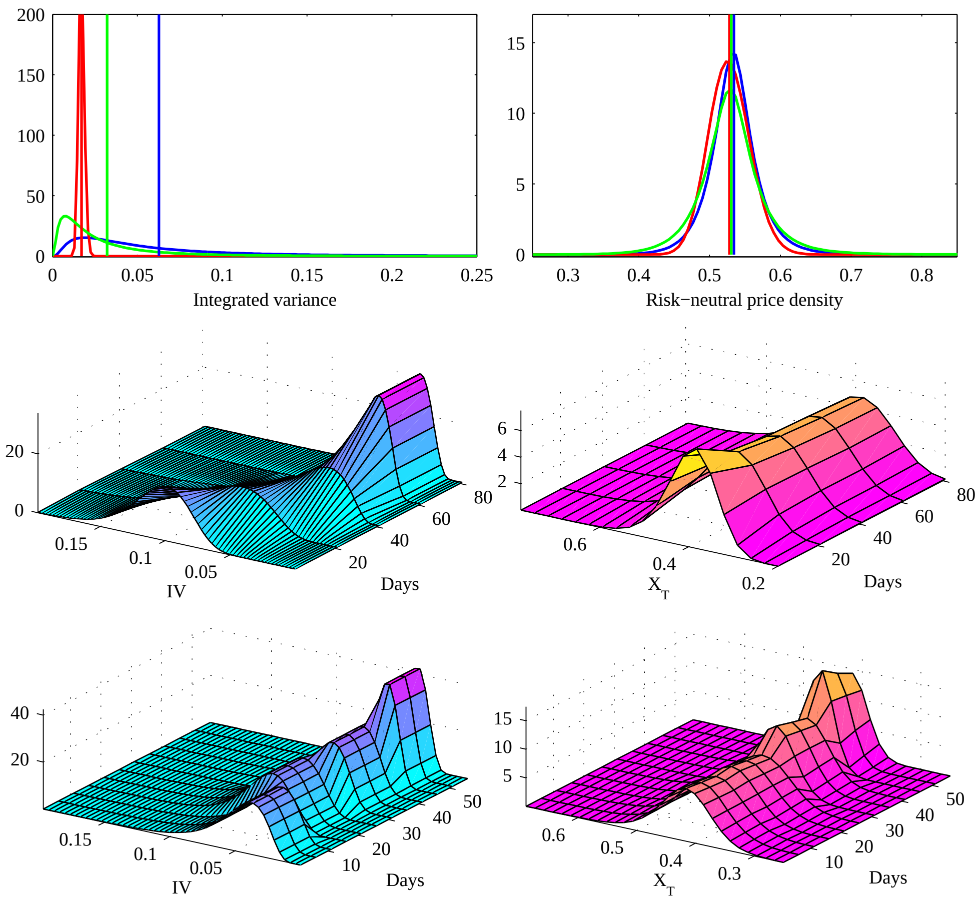

Figure 6: Estimates of $\bar{\sigma}^2_{tT}$ and X_T between two future moments in time. In the left panel in first row, we plot $\bar{\sigma}^2_{tT_1}$ (red), $\bar{\sigma}^2_{tT_2}$ (green), and $\bar{\sigma}^2_{12}$ (blue), as well as the corresponding expectations, estimated at t. The corresponding risk-neutral price densities are plotted in the right panel. In the second row, we estimate at time t the implied integrated variance and the risk-neutral price density during 90 future days. We separately estimate each time interval $[T_i, T_{i+1}]$, $1 \leq i \leq 4$, using options with the corresponding maturities. We plot in the left panel the surface of the MAP estimates of $\bar{\sigma}^2_m$, $m = [12, 23, 34, 45]$, and in the right panel, we plot the corresponding estimates of X_m (t_0 =October 14, T_1 =November 20, T_2 =December 18, T_3 =January 15, T_4 =February 19, T_5 =March 19, 2009). In the bottom row, we plot estimates of $\bar{\sigma}^2_{12}$ (left panel) and X_T (right panel) using options maturing in June and July 2009. The densities are estimated every 15 days, beginning on March 24, 2009.

and the price C_{PoC} is obtained by integrating $C_{\mathrm{PoC,COND}}$ with respect to X_{T_1}.

The chooser is an option where, at time T_1, the holder of the claim can decide to choose either a call or a put option, with maturity T_2 and strike price K_2. If we denote the payoffs of a call and a put conditional on X_{T_1} by $h_U = U_t^{\mathrm{BS}}(X_{T_1}; K_2, T_2 - T_1; \bar{\sigma}_{12}^2, r)$ and $h_P = P_t^{\mathrm{BS}}(X_{T_1}; K_2, T_2 - T_1; \bar{\sigma}_{12}^2, r)$, respectively, the price of a chooser put option, conditional on X_{T_1}, is then given by

$$C_{\mathrm{Ch,COND}} = e^{-r(T_1-t)}(K_1 - \max(h_U, h_P)),$$

and the price C_{Ch} is obtained by integrating $C_{\mathrm{Ch,COND}}$ with respect to X_{T_1}.

Cliquet options are exotic options consisting of a series of forward starting options whose terms are set on the reset dates. Haug and Haug (2001) present a method to price cliquet options based on a binomial tree developed by Cox, Ross and Rubinstein (1979), and Gatheral (2006) prices cliquet options with local volatility. Numerical pricing methods and various volatilities are investigated by Windcliff, Forsyth and Vetzal (2006) and den Iseger and Oldenkamp (2005).

We price a cliquet option with $\bar{\sigma}_{12}^2$ and X_{12}. We consider one particular cliquet, namely, a European-style reset put option with a single reset date, with payoff $CL(x; T_1, T_2)$ given by

$$CL(x; T_1, T_2) = \begin{cases} X_{T_1} - X_{T_2}, & \text{if } (X_{T_1} > x, \quad X_{T_2} \leq X_{T_1}), \\ x - X_{T_2}, & \text{if } (X_{T_1} \leq x, \quad X_{T_2} \leq x), \\ 0, & \text{otherways.} \end{cases} \tag{12}$$

A closed-form pricing formula for this type of cliquet option with constant volatility is given by Gray and Whaley (1999), where the expected terminal payoffs of the cliquet are divided into three possible outcomes and the probabilities for each outcome are computed.

Consider now the pricing of cliquet options using the implied integrated variance. Assume that we know the distributions of X_{T_1} and $\bar{\sigma}_{12}^2$ and denote the Black-Scholes put option price by P_t^{BS}. Conditional on X_{T_1} and $\bar{\sigma}_{12}^2$, the discounted payoff corresponding to $X_{T_1} - X_{T_2}$ in the first row in (7) is given as follows:

$$h_1 = e^{-r(T_1-t)}\mathbb{E}^*\{P_t^{\mathrm{BS}}(X_{T_1}; X_{T_1}, T_2 - T_1; \bar{\sigma}_{12}^2, r)\mathbf{1}_{X_{T_1}\geq x}(X_{T_1}) \mid x, X_{T_1}, \bar{\sigma}_{12}^2, \mathcal{F}_t\},$$

and the discounted payoff corresponding to $x - X_{T_2}$ is given by

$$h_2 = e^{-r(T_1-t)}\mathbb{E}^*\{P_t^{\mathrm{BS}}(X_{T_1}; x, T_2 - T_1; \bar{\sigma}_{12}^2, r)\mathbf{1}_{X_{T_1}<x}(X_{T_1}) \mid x, X_{T_1}, \bar{\sigma}_{12}^2, \mathcal{F}_t\}$$

The price of the reset put option is obtained by integrating h_1 and h_2 first with respect to X_{T_1} and then with respect to $\bar{\sigma}_{12}^2$. Figure 7 presents an example of reset put option prices, based on information provided by S&P 100 calls. We also illustrate the effect of noisy data and, based on the discrepancy principle considered in Section 2, we plot a minimum and maximum price for this cliquet option, computed using $z_{\min}$ and $z_{\max}$. Because we do not have information on the bid-ask spread of the options considered, in order to illustrate the phenomenon, we assume a high noise level of $e \sim \mathcal{N}(0, (.045u_t^{\mathrm{obs}})^2)$.

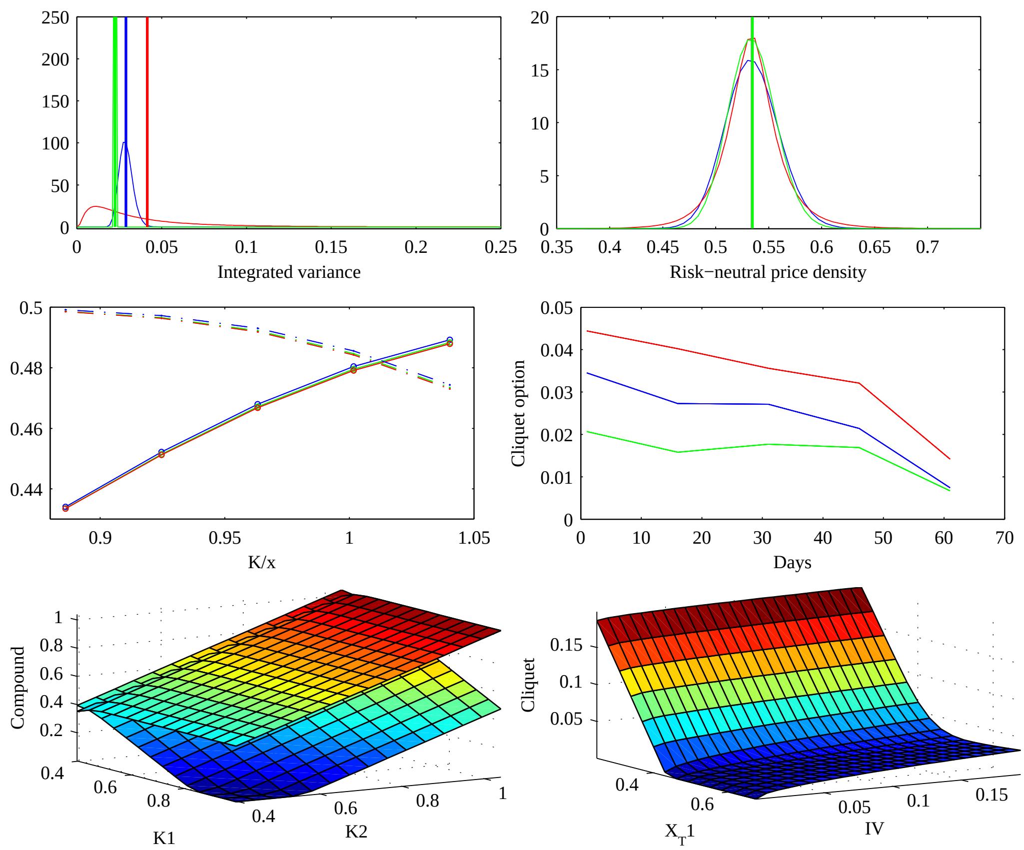

Figure 7: Compound and cliquet options. In the left panel of the first row, we plot the MAP estimate z_{MAP} of $\bar{\sigma}_{12}^2$ (blue), as well as the estimates $z_{\min}$ (green) and $z_{\max}$ (red) accepted by the discrepancy principle when $e \sim \mathcal{N}(0, (.045 u_t^{\text{obs}})^2)$. We estimated the distributions from S&P 100 call options maturing in June and July 2009 so that $T_1 = 60$ and $T_2 = 80$ days. The corresponding estimates of the risk-neutral price density are plotted in the right panel. In the second row, the prices of a put-on-call and a put-on-put option (dotted line) are plotted in the left panel and the prices of a cliquet option are plotted in the right panel. All of these options were computed using the densities of the first row and plotted with the corresponding colors. The effect of noisy data can particularly be observed in the estimates of cliquet prices. In the bottom row, we plot surfaces of compound put-on-call and put-on-put options as functions of strike prices (left panel), and we plot cliquet options as functions of the underlying and the integrated variance (right panel).

6 Concluding remarks

In this paper, we have considered the pricing of exotic option, namely, some volatility options, semi-path-dependent options and pathdependent options, with two option-implied densities: the density of the integrated variance and the risk-neutral price density. We have assumed a stochastic correlated volatility and proposed a Bayesian approach for the ill-posed inverse problem of an implied integrated variance and an implied correlation coefficient. We have presented an algorithm with which to calculate Maximum-A-Posteriori (MAP) estimates of the distribution and expectation of this variance from a series of European options with different strike prices and, possibly, different maturities. Based on the discrepancy principle, we have explained how noisy data affect these estimates.

Due to the convexity of the Black-Scholes formula with respect to the volatility and the underlying, the distribution of the integrated variance and the risk-neutral price density can be estimated from a couple of option prices with different strike prices, thus avoiding the problem of missing option price data often encountered when estimating the risk-neutral price density. Because the implied integrated variance is independent of strike prices, it is robust with respect to changes in the price of the underlying.

We have considered the pricing of variance and volatility swaps and options and showed how the convexity adjustment can be avoided by computing volatility swaps and options from the distribution of the integrated variance instead of the corresponding expectation. We then priced digital options and barrier options and showed what additional information can be obtained from the distribution of the implied integrated variance. Finally, we presented a simple algorithm with which to estimate the implied integrated variance between two future moments in time and used this variance to price compound and chooser options as well as cliquet options.

7 Appendix

proof of (4)**, page 6**

We write

$$\sigma_T = \sigma_t + \int_t^T \theta\sigma_s ds + \int_t^T \nu\sigma_s dZ_s^* = \sigma_t \exp((\theta - \frac{1}{2}\nu^2)(T-t) + \nu Z_{T-t}^*)$$

and denote $I_{tT} = \int_t^T \sigma_s dZ_s^*$. Now,

$$I_{tT} = \frac{1}{\nu}(\sigma_t \exp((\theta - \frac{1}{2}\nu^2)(T-t) + \nu Z_{T-t}^*) - \sigma_t - \int_t^T \theta\sigma_s ds), \tag{13}$$

and I_{tT} has a shifted lognormal distribution, conditional on $\int_t^T \sigma_s ds$:

$$(I_{tT} \mid \int_t^T \sigma_s ds) \sim l\mathcal{N}\big(\log\big(\frac{\sigma_t}{\nu}\big) + (\theta - \frac{1}{2}\nu^2)(T-t), \nu^2(T-t)\big) - \frac{\sigma_t}{\nu} - \frac{\theta}{\nu}\int_t^T \sigma_s ds.$$

Integrating with respect to $\int_t^T \sigma_s ds$ and noticing that $\mathbb{E}^*\{\theta \int_t^T \sigma_s ds\} = \sigma_t(e^{\theta(T-t)} - 1)$ gives

$$\phi_{tT} \sim l\mathcal{N}\big(\log\big(\frac{\sigma_t}{\nu}\big) + (\theta - \frac{1}{2}\nu^2)(T-t), \nu^2(T-t)\big) - \frac{\sigma_t}{\nu}e^{\theta(T-t)} - \frac{1}{2}\rho\bar{\sigma}_{tT}^2(T-t).$$

Now, we want to estimate ϕ_{tT} conditional on $\bar{\sigma}_{tT}^2$. We can expect to have information on the initial volatility σ_t at time t. Let us explain how the factors θ and ν in (13) are related to $\bar{\sigma}_{tT}^2$.

Assuming that $\bar{\sigma}_{tT}^2 \sim l\mathcal{N}(\mu, \varsigma)$, the expectation $\mathbb{E}^*\{\bar{\sigma}_{tT}^2\} = e^{(\mu+.5\varsigma)}$ and the variance $\mathrm{Var}^*(\bar{\sigma}_{\mathrm{tT}}^2) = (\mathrm{e}^{\varsigma}-1)\mathrm{e}^{(2\mu+\varsigma)}$. The pth moment of a lognormally distributed random variable $R \sim l\mathcal{N}(A, B^2)$ is given by $\mathbb{E}\{R^p\} = \exp(pA + .5p^2B^2)$. Denote $A = (\nu - .5\theta^2)(T-t)$

and $B = \theta\sqrt{T-t}$. The expectation and the variance of σ_T^2 are then given by

$$\begin{aligned} &\mathbb{E}^*\{\sigma_T^2\} = \sigma_t^2 e^{2A+2B^2} = \sigma_t^2 e^{(2\theta+\nu^2)(T-t)}, \quad \mathbb{E}^*\{\sigma_T^4\} = \sigma_t^4 e^{4A+8B^2} = \sigma_t^4 e^{(4\theta+6\nu^2)(T-t)}, \\ &\mathrm{var}(\sigma_\mathrm{T}^2) = \mathbb{E}\{\sigma_T^4\} - (\mathbb{E}\{\sigma_T^2\})^2. \end{aligned} \tag{14}$$

If $\mathbb{E}^*\{\sigma_t^2\}$ grew linearly as a function of time, we would have $\mathbb{E}^*\{\bar{\sigma}_{tT}^2\} - \sigma_t^2 = (\mathbb{E}^*\{\sigma_T^2\} - \sigma_t^2)/2$, such that

$$\mathbb{E}^*\{\sigma_T^2\} = 2\mathbb{E}^*\{\bar{\sigma}_{tT}^2\} - \sigma_t^2, \quad 0 \le t \le T. \tag{15}$$

In reality, $\mathbb{E}^*\{\sigma_t^2\}$ is convex as a function of time. It is, however, nearly linear when the volatility is low, and we will use this equality.

Similar to the expectation, the variance $\mathrm{Var}^*(\sigma_\mathrm{t}^2)$ is nearly linear as a function of time within the range of the typical values of $\bar{\sigma}_{tT}^2$. For a discrete sequence of variances $\mathrm{Var}^*(\sigma_{\mathrm{t_i}}^2)$, $1 \le i \le N$, $t_N = T$, we can approximate

$$\sum_{i=1}^N \mathrm{Var}^*(\sigma_{\mathrm{t_i}}^2) \approx \frac{1}{2}\mathrm{Var}^*(\sigma_{\mathrm{t_N}}^2).$$

Now, $\bar{\sigma}_{tT}^2(T-t) \approx \sum_{i=1}^N \sigma_{t_i}^2$ as $N \to \infty$ and

$$\mathrm{Var}^*\left(\sum_{\mathrm{i=1}}^{\mathrm{N}} \sigma_{\mathrm{t_i}}^2\right) = \sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{Var}^*(\sigma_{\mathrm{t_i}}^2) + 2\sum_{\mathrm{i,j:i<j}} \mathrm{cov}^*(\sigma_{\mathrm{t_i}}^2, \sigma_{\mathrm{j}}^2),$$

where the covariance cov^* is computed with respect to the measure P^*. This covariance term cannot be computed with information provided by an estimate of $\bar{\sigma}_{tT}^2$. Within the range of possible variances of the integrated variance, according to empirical experiment the following relationship appears to hold fairly well:

$$\mathrm{Var}^*(\sigma_\mathrm{T}^2) = 3\mathrm{Var}^*(\bar{\sigma}_{\mathrm{tT}}^2). \tag{16}$$

The drift θ and the volatility ν of the volatility process (1) can now be easily computed from (14), (15) and (16).

Statistical inverse problems, page 10

Consider a statistical inverse problem with an unknown hidden random variable $A \in \mathbb{R}^n$. An observed random variable $B \in \mathbb{R}^m$ is related to A by the mapping $\mathcal{C}$ such that $B = \mathcal{C}(A, E)$, with $\mathcal{C} : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$. The random variable $E \in \mathbb{R}^k$ accounts for the measurement noise and other poorly defined parameters. The goal of the inverse problem is to obtain information about A by measuring B.

In the Bayesian framework, randomness means a lack of information, and subjective beliefs about a random event are expressed in terms of probabilities. We combine prior information about the random variable A with information provided by an observed realization b of B. The conditional density $P_{\text{lik}}(b \mid a)$, called the likelihood function, describes the probability of the realization b of B given the realization a of A. The prior information is encoded in a prior density $P_{\text{prior}}(a)$. Applying the Bayes formula to these densities leads to the posterior density

$$P_{\text{post}}(a \mid b) \propto P_{\text{prior}}(a) P_{\text{lik}}(b \mid a),$$

which is the solution of the Bayesian inverse problem. Here $\propto$ means 'up to a normalizing constant".

Often, the posterior density is not available in closed form. Two common estimates that can be made to visualize the posterior density are the Maximum-A-Posteriori (MAP) estimate and the conditional mean (CM) estimate. The computation of the MAP estimate a_{MAP}, characterized by $a_{\text{MAP}} = \arg\max_{a \in \mathbb{R}^n} P_{\text{post}}(a \mid b)$, leads to an optimization problem, and the computation of the CM estimate a_{CM}, given by $a_{\text{CM}} = \mathbb{E}\{a \mid b\}$, results in Markov Chain Monte Carlo (MCMC) sampling and an integration problem.

Algorithm for computing a Maximum-A-Posteriori (MAP) estimate, page 12

We present an algorithm with which to estimate a discretized distribution of the integrated variance and use a discretized version of the option-pricing formula so that the algorithm can be used directly with Matlab or other computational software.

We fix time t and write $u^{\text{obs}} = u_t^{\text{obs}}$, $u^{\text{BS}} = u_t^{\text{BS}}$, $U^{\text{true}} = U_t^{\text{true}}$. We assume that the distribution of the implied integrated variance $\bar{\sigma}^2 = \bar{\sigma}_{tT}^2$ has positive values in the interval $[a, a+M_\sigma]$, divide this interval into n_σ points $\bar{\sigma}_j^2$ and denote the corresponding discretized distribution by $z \in \mathbb{R}^{n_\sigma}$, where z_j is the probability of $\bar{\sigma}_j^2$, $1 \leq j \leq n_\sigma$. To assure that the total probability equals one, we calibrate: $z_j = z_j/(M_\sigma/n_\sigma \sum_{j=1}^{n_\sigma} z_j)$.

In the same way, we define a discrete support $[b, b+M_X]$ for the risk-neutral price density $\pi(X_T)$, divide this interval into n_X points X_j and denote the corresponding discretized distribution by $g \in \mathbb{R}^{(n_X \times n_\sigma \times n_\phi)}$. Additionally, we define a discrete support $[c, c+M_\phi]$ for the distribution of $\phi = \phi_{tT}$, divide this interval into n_ϕ points ϕ_j and and denote the corresponding discretized distribution by $q \in \mathbb{R}^{(n_\sigma \times n_\phi)}$. We can choose the support of $\bar{\sigma}_{tT}^2$ very generally, for example $\bar{\sigma}_{tT}^2 \in [1e{-}5, 1]$. The implied volatility smile can be used when defining the hyper prior $P_{\text{hyper}}(\mu, \theta)$; at-the-money implied variance provides information on the expectation $\hat{\sigma}_{tT}^2$ and the slope of the smile provides information on the variance $\text{Var}^*(\bar{\sigma}_{\text{tT}}^2)$. The support of ϕ_{tT} usually has values in the range $[-.5, .5]$. According to empirical experiments, the number of support points $n_\sigma \in [100, 500]$ and $n_\phi \in [50, 200]$ seems to provide accurate estimates of the unknowns. The computation time of the MAP estimates can be essentially reduced if prior information on $\bar{\sigma}_{tT}^2$ and ρ from the previous moment of time is available.

According to the prior assumption, $z \sim l\mathcal{N}(\mu, \varsigma)$, such that the prior density $P_{\text{prior}}(z)$ consists of all the distributions z that are lognormal. We denote such distributions by $z_{\log} = z_{\log}(\mu, \varsigma)$. We define the hyperpriors of the pair (μ, ς) and ρ as uniform distributions: $P_{\text{hyper}}(\mu, \varsigma) = \mathcal{U}((\mu_{\min}, \mu_{\max}) \times (\varsigma_{\min}, \varsigma_{\max}))$, $P_{\text{hyper}}(\rho) = \mathcal{U}(\rho_{\min}, \rho_{\max})$. We denote $\theta = (\mu, \varsigma, \rho)$, where θ, as well as all the unknowns are random variables.

We approximate u^{obs} with a discretized version of the pricing formula. When applying the correlated Hull-White formula (7), we assume that $u_i^{\text{obs}} = U^{\text{HW},\rho}(x; K_i, T; \sigma_t^2, r) + e_i$, discretize this pricing formula and write:

$$
\begin{aligned}
&b_{i,j,k} = U_i^{\text{BS}}(xe^{\rho\phi_{j,k}}; K_i, T; \bar{\sigma}_j^2, r) \quad b_{i,j} = \frac{M_\phi}{n_\phi} \sum_{k=1}^{n_\phi} b_{i,j,k} q_{j,k} \\
&u_i^{\text{obs}} \approx \frac{M_\sigma}{n_\sigma} \sum_{j=1}^{n_\sigma} b_{i,j} z_j + e_i,
\end{aligned} \tag{17}
$$

where e_i denotes the uncertainty corresponding to the strike price K_i, $1 \leq i \leq L$. In matrix form, we write: $u^{\text{obs}} = Bz + e$.

As an alternative to the Hull-White formula, we can estimate the implied integrated variance $\bar{\sigma}_{tT}^2$and the implied correlation coefficient ρ from the payoff formula (5). In this case, we assume that $u_i^{\text{obs}} = \mathbb{E}\{h_i(X_T)\} + e_i$ and consider the discretized distribution g. We denote by $g_{l,j,k}$ the probability of the discrete point $X_{T,l}$ conditional on $\bar{\sigma}_j^2$ and ϕ_k and $(X_T \mid \sigma_j^2, \phi_{j,k}) \sim l\mathcal{N}(\rho\phi_{j,k} \log(x) + ((r - .5\bar{\sigma}_{\rho,j}^2)(T-t), \bar{\sigma}_{\rho,j}^2)(T-t))$. Then

$$
\begin{aligned}
g_{l,j} &= \frac{M_\phi}{N_\phi} \sum_{k=1}^{n_\phi} g_{l,j,k} q_{j,k}, \quad g_l = \frac{M_\sigma}{n_\sigma} \sum_{j=1}^{n_\sigma} g_{l,j} z_j \\
u_i^{\text{obs}} &\approx \frac{M_X}{n_X} \sum_{l=1}^{n_X} h_i(X_{T,l}) g_l + e_i.
\end{aligned}
$$

We write $u^{\text{obs}} = Cg + e$.

From this point forward, we will denote by A either B or C, depending on whether we use the Hull-White formula or the payoff formula, respectively. Because we have assumed a normally distributed error e, the likelihood function of u^{obs} is obtained from (17) using the Gaussian kernel,

$$P_{\text{lik}}(u^{\text{obs}} \mid z) \propto \exp\bigg(-\frac{1}{2}\big((u^{\text{obs}} - Az)^T \Gamma^{-1} (u^{\text{obs}} - Az)\big)\bigg),$$

where the covariance matrix $\Gamma = \text{diag}(\text{Var}_1, \text{Var}_2, \ldots, \text{Var}_L)$. The posterior density is given by $P_{\text{post}}(z \mid u^{\text{obs}}) \propto P_{\text{lik}}(u^{\text{obs}} \mid z) P_{\text{prior}}(z) P_{\text{hyper}}(\mu, \varsigma) P_{\text{hyper}}(\rho)$. Under the prior assumption of lognormality, the posterior density reduces to $P_{\text{post}}(z \mid u^{\text{obs}}) \propto P_{\text{lik}}(u^{\text{obs}} \mid z_{\log}(\mu, \varsigma) P_{\text{hyper}}(\mu, \varsigma) P_{\text{hyper}}(\rho)$. A MAP estimate $\theta_{\text{MAP}} = (\mu_{\text{MAP}}, \varsigma_{\text{MAP}}, \rho_{\text{MAP}})$ can be computed with

$$\theta_{\text{MAP}} = \arg\min_\theta \big((u^{\text{obs}} - Az_{\log}(\mu, \varsigma))^T \Gamma^{-1} (u^{\text{obs}} - Az_{\log}(\mu, \varsigma))\big) \tag{18}$$

by drawing a sample $S = (\theta_1, \theta_2, \ldots, \theta_N)$ from the hyperpriors $P_{\text{hyper}}(\mu, \varsigma)$ and $P_{\text{hyper}}(\rho)$ and computing which θ_i, $1 \le i \le N$, minimizes the right-hand side of (18). The computation of the MAP estimate can be costly because the sample size N should be so large that the sample S represents the uniformly distributed hyperpriors. We have explained in Section 2 how a MAP estimate can be very quickly and easily computed by approximating the continuous distributions of the hyperpriors with discrete distributions and solving the least-squares optimization problem with the grid method.

We can compute the MAP estimates for the unknown distributions z_{MAP} and g_{MAP} with θ_{MAP}. The MAP estimate of the discrete distribution of the implied integrated volatility is $y_{\mathrm{MAP}} \sim l\mathcal{N}(\mu_{\mathrm{MAP}}/2, \varsigma_{\mathrm{MAP}}/4)$. The reliability of the MAP estimates can be assessed with Markov Chain Monte Carlo (MCMC) techniques, for example, as explained by Kaila (2012). This approach provides conditional mean (CM) estimates and posterior distributions of the unknowns. Alternatively, we could apply a sequential Monte Carlo method, such as the evolutionary optimization method presented by Cont and Ben Hamida (2005).

Implied integrated variance between two future moments in time $0 \leq T_1 < T_2$, page 15

Let $\bar{\sigma}^2_{1,j}$, $1 \leq j \leq n_\sigma$, be a realization of $\bar{\sigma}^2_{tT_1}$. For lognormal distributions, conditioning on $T_1\bar{\sigma}^2_{1,j}$ affects the expectation by $T_1\bar{\sigma}^2_{1,j} + \mathbb{E}^*\{T_{12}\bar{\sigma}^2_{12}\} = \mathbb{E}^*\{T_{12}\bar{\sigma}^2_{12} + T_1\bar{\sigma}^2_{1,j}\}$ and the variance by $T_1\bar{\sigma}^2_{1,j} + \mathrm{Var}^*(T_{12}\bar{\sigma}^2_{12}) = \mathrm{Var}^*(T_{12}\bar{\sigma}^2_{12})$, where $T_{12} = T_2 - T_1$. Based on $\mathbb{E}^*\{T_2\bar{\sigma}^2_{tT_2}\} = \mathbb{E}^*\{\mathbb{E}^*\{T_2\bar{\sigma}^2_{tT_2} \mid T_1\bar{\sigma}^2_{tT_1}\}\} = \mathbb{E}^*\{\mathbb{E}^*\{T_{12}\bar{\sigma}^2_{12} + T_1\bar{\sigma}^2_{tT_1} \mid T_1\bar{\sigma}^2_{tT_1}\}\}$, we solve $\hat{\sigma}^2_{12} = \mathbb{E}^*\{\bar{\sigma}^2_{12}\} = T_2/T_{12}\hat{\sigma}^2_{tT_2} - T_1/T_{12}\hat{\sigma}^2_{tT_1}$.

According to the law of total variance, assuming that $T_1 = T_{12}$, we have $\mathrm{Var}^*(\bar{\sigma}^2_{tT_2}) = \mathrm{Var}^*(\mathbb{E}^*\{\bar{\sigma}^2_{tT_2} \mid \bar{\sigma}^2_{tT_1}\}) + \mathbb{E}^*\{\mathrm{Var}^*(\bar{\sigma}^2_{tT,2} \mid \bar{\sigma}^2_{tT_1})\} = \mathrm{Var}^*(\bar{\sigma}^2_{tT_1}) + \mathrm{Var}^*(\bar{\sigma}^2_{12})$. For any $T_1 + T_{12} = T_2$, this yields to $\mathrm{Var}^*(\bar{\sigma}^2_{12}) = T_2/T_{12}\mathrm{Var}^*(\bar{\sigma}^2_{tT_2}) - T_1/T_{12}\mathrm{Var}^*(\bar{\sigma}^2_{tT_1})$.

The estimates μ_{12} and ς_{12} are then obtained from μ_i and ς_i, $i = [1, 2]$, by

$$
\begin{aligned}
\mu_{12} &= \log(\mathbb{E}^*\{\bar{\sigma}^2_{12}\}) - \frac{1}{2}\varsigma_{12} \\
&= \log\left(\mathbb{E}^*\{\frac{T_2}{T_{12}}\bar{\sigma}^2_{tT_2}\} - \mathbb{E}^*\{\frac{T_1}{T_{12}}\bar{\sigma}^2_{tT_1}\}\right) - \frac{1}{2}\varsigma_{12} \\
&= \log(\frac{T_2}{T_{12}}e^{\mu_2+.5\varsigma_2} - \frac{T_2}{T_{12}}e^{\mu_1+.5\varsigma_1}) - \frac{1}{2}\varsigma_{12}, \\
\varsigma_{12} &= \log\left(1 + \frac{\mathrm{Var}^*(\bar{\sigma}^2_{12})}{(\mathbb{E}^*\{\bar{\sigma}^2_{12}\})^2}\right) \\
&= \log\left(1 + \frac{\frac{T_2}{T_{12}}\mathrm{Var}^*(\bar{\sigma}^2_{tT_2}) - \frac{T_1}{T_{12}}\mathrm{Var}^*(\bar{\sigma}^2_{tT_1})}{(\mathbb{E}^*\{\frac{T_2}{T_{12}}\bar{\sigma}^2_{tT_2}\} - \mathbb{E}^*\{\frac{T_1}{T_{12}}\bar{\sigma}^2_{tT_1}\})^2}\right) \\
&= \log\left(1 + \frac{\frac{T_2}{T_{12}}(e^{\varsigma_2} - 1)e^{2\mu_2+\varsigma_2} - \frac{T_1}{T_{12}}(e^{\varsigma_1} - 1)e^{2\mu_1+\varsigma_1}}{(\frac{T_2}{T_{12}}e^{\mu_2+.5\varsigma_2} - \frac{T_1}{T_{12}}e^{\mu_1+.5\varsigma_1})^2}\right).
\end{aligned}
$$

Barrier options, page 27

Carr and Chou (1997b) derived a closed form price for barrier option assuming a constant volatility. We follow their presentation closely but assume that the volatility is stochastic.

Lemma: Consider two portfolios P and Q of European options with maturity T and payoffs

$$
P(X_T) = \begin{cases} f(X_T) & \text{if } M \leq X_T \leq N, \\ 0 & \text{otherwise.} \end{cases}
$$

and

$$
Q(X_T) = \begin{cases} (\frac{X_T}{B})^\psi f(\frac{B^2}{X_T}) & \text{if } \frac{B^2}{N} \leq X_T \leq B^2 M, \\ 0 & \text{otherwise,} \end{cases}
$$

where the power $\psi = 1 - \frac{2r}{\bar{\sigma}^2_{tT}}$ and $B > 0$. Then, P and Q have the same value whenever the spot is B.

Proof: Assume that $\rho = 0$. Conditional on $\bar{\sigma}^2_{tT}$, when the spot equals B, the value of P

is

$$V_t^P(B) = e^{-r\tau} \int_M^N \frac{f(X_T)}{X_T\sqrt{2\pi\bar{\sigma}_{tT}^2\tau}} \exp\left[-\frac{(\ln(X_T/B) - (r - .5\bar{\sigma}_{tT}^2)\tau)^2}{2\bar{\sigma}_{tT}^2\tau}\right] dX_T,$$

where $\tau = T - t$, $0 \leq t < T$. Let $\hat{X} = B^2/X_T$. Then, $dX_T = -B^2/\hat{X}^2 d\hat{X}$ and conditional on $\bar{\sigma}_{tT}^2$:

$$\begin{aligned} V_t^P(B) &= e^{-r\tau} \int_{B^2/M}^{B^2/N} f\left(\frac{B^2}{\hat{X}}\right) \frac{1}{\hat{X}\sqrt{2\pi\bar{\sigma}_{tT}^2\tau}} \exp\left[-\frac{(\ln(B/\hat{X}) - (r - .5\bar{\sigma}_{tT}^2)\tau)^2}{2\bar{\sigma}_{tT}^2\tau}\right] d\hat{X} \\ &= e^{-r\tau} \int_{B^2/N}^{B^2/M} \left(\frac{\hat{X}}{B}\right)^{\psi} f\left(\frac{B^2}{\hat{X}}\right) \frac{1}{\hat{X}_T\sqrt{2\pi\bar{\sigma}_{tT}^2\tau}} \exp\left[-\frac{(\ln(\hat{X}/B) - (r - .5\bar{\sigma}_{tT}^2)\tau)^2}{2\bar{\sigma}_{tT}^2\tau}\right] d\hat{X}, \end{aligned}$$

Conditional on $\bar{\sigma}_{tT}^2$, $V_t^P(B)$ exactly matches the risk-neutral value of Q, namely $V_t^Q(B)$. This holds also when $V_t^P(B)$ and $V_t^Q(B)$ are integrated with respect to $\bar{\sigma}_{tT}^2$. Carr and Chou (1997b) explain how this Lemma is used to replicate the payoff of any single barrier option. For correlated volatility, we condition on $\bar{\sigma}_{tT}^2$ and ϕ_{tT} and proceed as above.

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