On the Finiteness of Attractors for Maps of the Interval Allowing Discontinuities

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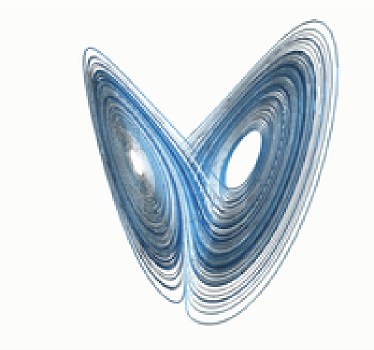
PART I

On The Structure of Lorenz Maps: Topological Attractors and Spectral Decomposition Theorem

Lorenz (63): truncating Navier-Stokes (weather forecasting modeling)

$$\begin{aligned}\dot{x} &= -10x + 10y \\ \dot{y} &= 28x - y - xz \\ \dot{z} &= -\frac{8}{3}z + xy\end{aligned} \tag{1}$$



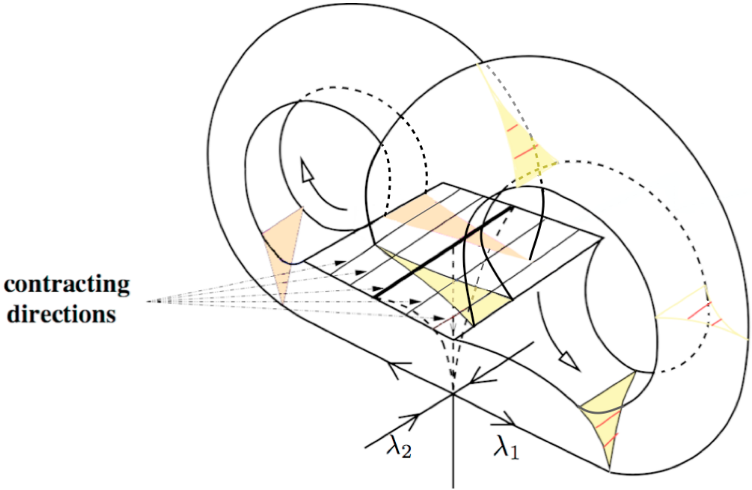


Figura : Guckenheimer/Williams (79): Geometric Model

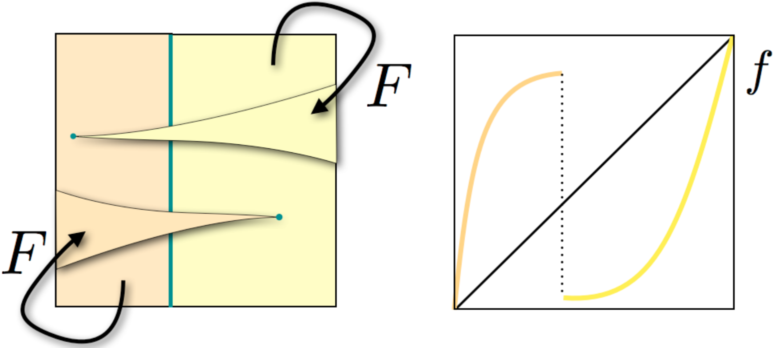


Figura : Poincaré and induced one-dimensional map
contracting case: (Arneodo/Coullet/Tresser (81)) $\lambda_3 + \lambda_1 < 0$

Definition (Basin of Attraction)

The basin of attraction of a set Λ is the set of points whose future orbits accumulate in Λ: $\beta(\Lambda) := \{x; \omega_f(x) \subset \Lambda\}$

Definition (Topological attractor)

a **transitive** compact set Λ is a *topological attractor* if $\beta(\Lambda)$ is relevant in the topological sense, that is, $\beta(\Lambda)$ is residual in an open set.

Definition (Metrical attractor)

a **transitive** compact set Λ is a *metrical attractor* if $\beta(\Lambda)$ is relevant in the metrical sense, that is, $\mathrm{Leb}\left(\beta(\Lambda)\right) > 0$.

A C^3 map f has **negative Schwarzian derivative** if $Sf(x) < 0$ $\forall\, x$ such that $Df(x) \neq 0$, where

$$Sf(x) = \frac{D^3 f(x)}{Df(x)} - \frac{3}{2}\left(\frac{D^2 f(x)}{Df(x)}\right)^2$$

[Guckenheimer (79)] f unimodal, $Sf < 0$, non-flat.

Uniqueness of Topological Attractors

- *Attracting Periodic Orbit*
- *Solenoid*
- *Cycle of Intervals*

(uses Milnor-Thurston Theorem).

[Blokh-Lyubich (89,91)] f unimodal, $Sf < 0$, non-flat.

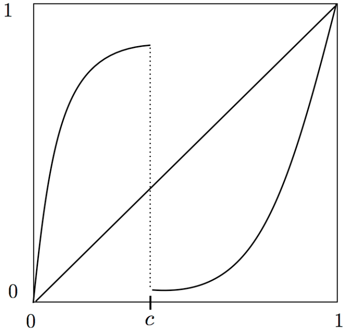
Uniqueness of Metrical Attractors:

- *Attracting Periodic Orbit*
- *Solenoid*
- *Cycle of Intervals*
- *Cantor Set K contained in the cycle of transitive intervals, containing the critical point c and s.t.* $K = \omega(c)$

- Milnor:examples where the metrical attractor contains the topological one and vice-versa.
- S-unimodal maps: metrical $\subseteq$ topological.
- quasi-quadratic maps (f with $Sf < 0$ and non-degenerate critical point: $|f''(c)| \neq 0$, Lyubich,94): topological of Guckenheimer coincide with the metrical ones obtained in B-L (case 4 doesn't occur).
- General case: NO (1996, Bruin, Keller, Nowicki, van Strien) unimodal maps with metrical attractor distinct to the topological one (*Wild attractors* do exist).

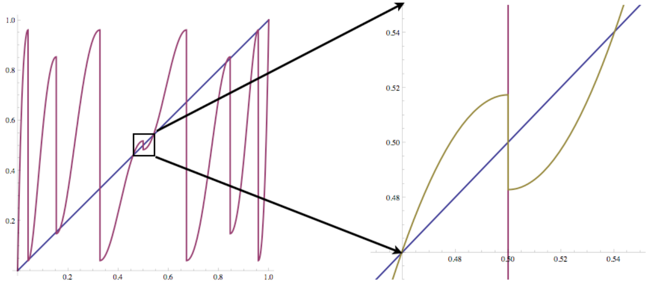
Definition (Lorenz Maps)

A contracting Lorenz Map is C^2 map $f : [0,1] \setminus \{c\} \to [0,1]$, with $f(0) = 0$, $f(1) = 1$, $f' > 0$ and $\lim_{x \to c} f'(x) = 0$.



renormalization interval: $(a, b) \ni c$, first return conjugated to Lorenz:

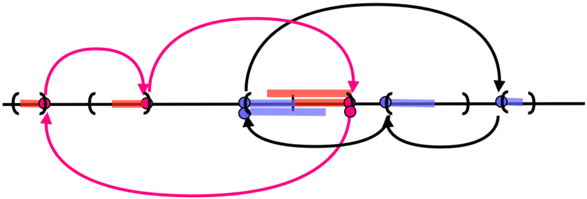
$$f^{\text{period}(a)}([a, c)) \subset [a, b] \supset f^{\text{period}(b)}((c, b]).$$



J RENORMALIZ. INTERV. $\to$ RENORMALIZATION CYCLE $U_J \subset K_J$ NICE TRAPPING REGION

$$U_J = \Big(\cup_{i=0}^{\text{period}(a)} f^i((a,c)) \Big) \cup \Big(\cup_{i=0}^{\text{period}(b)} f^i((c,b)) \Big).$$

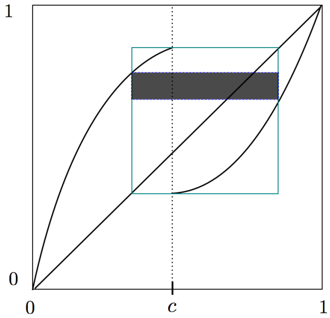
Each connected component of it is contained in a pullback of J. K_J is the *nice trapping region*, the union of these pre-images of J.



Theorem:

- If f infinitely renormalizable $J_n \supsetneq J_{n+1}$ then $\bigcap_n J_n = \{c\}$.
- *Solenoid*: $\Lambda \subset \bigcap_{n=0}^{\infty} K_{J_n}$
- Λ is a Cantor set, minimal (every orbit is dense), $\omega(x) = \Lambda$, $\forall x \in \Lambda$. In particular, Λ contains the critical point, whose forward orbit is dense in Λ.

In the case of a finite number of renormalizations, one possibility is that in the last renormalization we see a gap map. In this case, the dynamics restricted to (v_0, v_1) is semi-conjugated to a rotation.



- If it is semi-conjugated to a RATIONAL ROTATION, Λ is one or a pair of periodic attracting orbits.
- If it is semi-conjugated to an IRRATIONAL ROTATION, Λ is a minimal set, "Cherry attractor".
  In this case, $\omega(x) = \Lambda$, $\forall x \in \Lambda$ and $c \in \Lambda$. And similarly to the circle maps:
    - Λ is CONJUGATED to a irrational rotation, and it is an interval;
    - or is SEMI-CONJUGATED and Λ is a Cantor set.

f contracting Lorenz, $Sf < 0$.

- Fact 1: f has ONE or TWO periodic attracting orbits (follows from Singer's Theorem)
- Fact 2:

$$\text{(f has periodic attracting orbit)}$$
$$\Downarrow$$
$$(\exists U \text{ open and dense } Leb(U) = 1$$
$$\text{s.t. } \forall x \in U,$$
$$x \text{ is in the basin of a periodic attracting orbit)}$$

Theorem (—,Pinheiro):

$f \in C^3$ non-flat contracting Lorenz, $Sf < 0$

$\exists!$ topological attractor Λ, one of the following:

- *Attracting Periodic Orbit(s):*
  - Λ *is ONE Attracting Periodic Orbit* OR,
  - $\Lambda = \Lambda_1 \cup \Lambda_2$, *TWO Attracting Periodic Orbits*. (*)
- *Cherry attractor.*
- *Solenoidal attractor.*
- *Chaotic attractor:*
  - *Finite union of intervals* OR,
  - *Cantor set* (only if there is a wandering interval).

$\omega_f(x) = \Lambda$ residually in $[0,1]$, except
$\omega_f(x) = \Lambda_1$ or $\omega_f(x) = \Lambda_2$ residually in $[0,1]$ in (*).

OBSERVE THAT:
IN THE LORENZ CONTEXT WE DONT HAVE
MILNOR-THURSTON THEOREM AS A TOOL.
We don't have available two main tools that Guckenheimer had: Milnor-Thurston and, from this, the non-existence of non-wandering intervals!!!

Theorem(—) [Spectral Decomposition for Contracting Lorenz Maps] If $f \in C^3$, non-flat, $Sf < 0$, then $\exists n_f \in \mathbb{N} \cup \{\infty\}$, compact sets $\{\Omega_n\}_{0 \leq n \leq n_f}$ s.t.:

1. $\Omega(f) = \bigcup_{0 \leq j \leq n_f} \Omega_j$
2. $f(\Omega_n) = \Omega_n$
   - (i) $\Omega_0 \subset \{0, 1\}$ or $\Omega_0 = [0, 1]$ (if $n_f = 0$)
     - $\Omega_n \cap \Omega_m = \emptyset$, $\forall\, 0 \leq n \neq m < n_f$.
     - $\Omega_{n_{f-1}} \cap \Omega_{n_f} = \emptyset$ or $\Omega_{n_{f-1}} \cap \Omega_{n_f} = 1$ or 2 periodic orbits $\iff$ c_- or c_+ preperiodic
   - (ii) $f|_{\Omega_j}$ is topologically transitive $\forall j$, $0 < j < n_f$.
   - (iii) Ω_{n_f} is transitive or union of a pair of attracting periodic orbits.
3. For each $0 < j < n_f$ there is a decomposition of Ω_j into closed sets
$$\Omega_j = X_{0,j} \cup X_{1,j} \cup \cdots \cup X_{\ell_j,j}$$
such that $\#(X_{a,j} \cap X_{b,j}) \leq 1$ when $a \neq b$, the first return map to $X_{0,j}$ is topologically exact (in particular, topologically mixing) and, for each $i \in \{1, \cdots, \ell_j\}$, there is $1 \leq s_{i,j} \leq \ell_j$ such that $f^{s_{i,j}}(X_{i,j}) \subset X_{0,j}$.

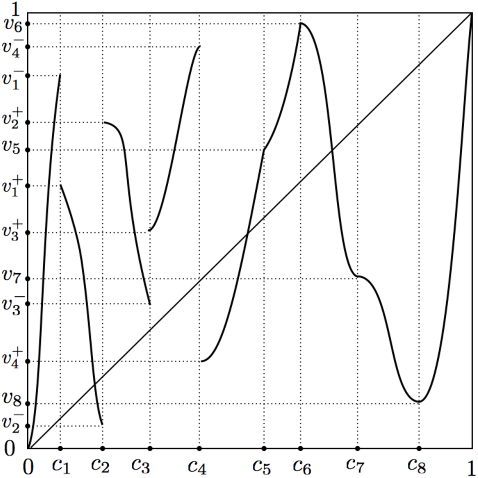
PART II

Finiteness of Attractors

(joint work with Palis and Pinheiro)

Definition (generalized S-multimodal maps)

A generalized S-multimodal map is a C^3 local diffeomorphism $f : [0,1] \setminus \mathcal{C}_f \to [0,1]$ with negative Schwarzian derivative with $f(\{0,1\}) \subset \{0,1\}$, $\mathcal{C}_f \subset (0,1)$ and $\#\mathcal{C}_f < \infty$. The set $\mathcal{C}_f$ is called the critical set of f.

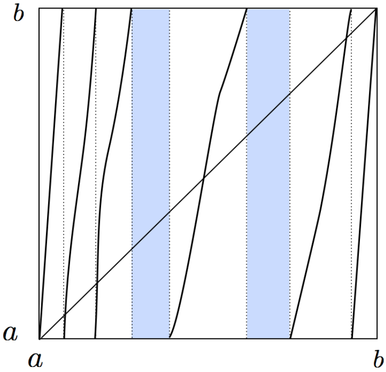


Lemma

Let $V \subset (a,b)$ be an open set, $\mathcal{P}$ be the set of c.c. of V and $F : V \to (a,b)$ be s.t.:

1. $F(P) = (a,b)$ *diffeomorphically* $\forall P \in \mathcal{P}$;
2. $\text{diameter}(\mathcal{P}_n(x)) \stackrel{n\to\infty}{\longrightarrow} 0$, $\forall x \in \bigcap_{j\geq 0} F^{-j}(V)$, *where* $\mathcal{P}_n(x)$ *is the connected component of* $\bigcap_{j=0}^{n} F^{-j}(V)$ *that contains* x;
3. $\exists K > 0$ *such that* $\left|\frac{DF^n(p)}{DF^n(q)}\right| \leq K$, $\forall n$, $\forall p, q \in \mathcal{P}_n(x)$, $\forall x \in \bigcap_{j\geq 0} F^{-j}(x)$

Then, either $Leb(\bigcap_{j\geq 0} F^{-j}(V)) = 0$ *or* $|b-a|$.
Furthermore, if $Leb(\bigcap_{j\geq 0} F^{-j}(V)) = |b-a|$ *then* $\omega_F(x) = [a,b]$ *for a.e.* $x \in [a,b]$.



FORMER LEMMA SAYS THESE HOLES CANNOT OCCUR IF $Leb(\bigcap_{j\geq 0} F^{-j}(V)) > 0$

From now on f will be a generalized S-multimodal map with critical set $\mathcal{C}_f$ and critical values $\mathcal{V}_f$.

Let $\mathbb{B}$ be the union of the basins of all periodic attractors.

Lemma (Critical Dichotomy)

Let $I \subset [0,1]$ be an interval such that

$$\mathcal{O}_f^+(\mathcal{V}_f) \cap I = \emptyset.$$

and $SF < 0$, where $F : I^* \to I$ is the first return map to I by f. If $Leb(I \setminus \mathbb{B}) > 0$, then either

1. *$\omega_f(x) \cap I = \emptyset$ for Lebesgue almost every $x \in I \setminus \mathbb{B}$ or*
2. *$\omega_f(x) \supset I$ for Lebesgue almost every $x \in I$.*

The set of generators of an attractor

If $\Lambda \subset [0,1]$ is an attractor define

$$G_f(\Lambda) := \{x\,;\,\omega_f(x) = \Lambda\} \subset \beta_f(\Lambda).$$

If Λ is a finite attractor, $G_f(\Lambda) = \beta_f(\Lambda)$.

Let

$$\mathbb{G} = \Bigg(\bigcup_{\Lambda \text{ is a finite attractors}} G_f(\Lambda) \Bigg) \cup \Bigg(\bigcup_{\Lambda \text{ is a cycle of intervals}} G_f(\Lambda) \Bigg).$$

Proposition

$$Leb\Big(\mathbb{G} \cup \beta_f\big(\overline{\mathcal{O}_f^+(\mathcal{V}_f)} \big) \Big) = 1.$$

As for every $v \in \mathcal{V}_f \cap \omega(x) \neq \emptyset$ the orbit of v and its closure remains in $\omega_f(x)$, we have

$$\bigcup_{v \in \mathcal{V}_f \cap \omega(x)} \overline{\mathcal{O}_f^+(v)} \subset \omega_f(x) \subset \bigcup_{v \in \mathcal{V}_f} \overline{\mathcal{O}_f^+(v)}$$

Proposition

Let f as before and having periodic points accumulating by both sides in every critical point. Let $\omega(x) \subset \overline{\mathcal{O}^+(v_1)} \cup \cdots \cup \overline{\mathcal{O}^+(v_s)}$. *If* $v_1 \notin \omega(x)$ *then* $\omega(x) \subset \overline{\mathcal{O}^+(v_2)} \cup \cdots \cup \overline{\mathcal{O}^+(v_s)}$.

Re-stating the proposition:

$$\omega_f(x) \subset \bigcup_{v \in \mathcal{V}_f \cap \omega(x)} \overline{\mathcal{O}_f^+(v)}$$

Theorem (—, Palis, Pinheiro)

For Lebesgue almost every $x \notin \mathbb{G}$ we have that

$$\omega_f(x) = \bigcup_{v \in \mathcal{V}_f \cap \omega(x)} \overline{\mathcal{O}_f^+(v)}$$

or we can re-write it as

Theorem (Labor Day's Theorem)

There is a finite collection $\{A_1, \cdots, A_s\}$ *of compact transitive positively invariant sets, with* $s \leq 2^{\#\mathcal{V}_f} - 1$ *such that*

$$Leb\big(G_f(A_1) \cup \cdots \cup G_f(A_s)\big) = 1.$$

Furthermore,

1. $\exists U \subset [0,1] \setminus C_f$ *with* $Leb(U) = 1$ *s.t. if* $x \in U$ *then* $\omega_f(x) = A_j$ *for some* $1 \leq j \leq s$*;*
2. *either* A_j *is a finite attractor or a cycle of intervals or* A_j *is a Cantor set with* $A_j = \overline{\mathcal{O}_f^+(v_1)} \cup \cdots \cup \overline{\mathcal{O}_f^+(v_{k_j})}$ *and* $\{v_1, \cdots, v_{k_j}\} \subset \mathcal{V}_f$*.*

Remarks:

- Observe that the previous results guarantees that there are at most $2^{\#\mathcal{V}_f} - 1$ proving the finitude of attractors in this case
- This estimate can be sharpened, and in the case of one single critical point we have **Theorem:** There is only one single attractor.
- We aim the C^2 case. While we know it can occur de Melo example of infinite many periodic attractors without the non-flat condition, we expect to have the result valid in the complementary set of these infinite basins, this complementary set going to a finite number of attractors.

Thanks!