Delaunay type hypersurfaces in cohomogeneity one manifolds
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Delaunay 1841: a rotationally symmetric surface in $\mathbb{R}^3$ has CMC iff its profile curve is a *roulette* of a conic section.

- **Delaunay surfaces:** spheres, unduloids, nodoids, catenoids and cylinders.
- Similar constructions of rotationally invariant CMC hypersurfaces in $\mathbb{H}^n$, $\mathbb{R}^n$, $S^n$
Embedded CMC tori in $S^3$

- CMC Clifford tori in $S^3$: for each $0 < t < \pi/2$,

$$T^2_t := \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \right.$$

$$\left| x_1 \right|^2 + \left| x_2 \right|^2 = \cos^2 t, \left| x_3 \right|^2 + \left| x_4 \right|^2 = \sin^2 t \right\},$$

- $T^2_t$ are orbits of isometric $S^1 \times S^1$-action
- Singular orbits: geodesics $S^1$ at distance $\pi/2$; limits of $T^2_t$ as $t \to 0$ and $t \to \pi/2$
- Other rotationally symmetric CMC tori: *bifurcating families* of CMC tori of *unduloid type* (classified by Hynd, Park, McCuan 2009 and Perdomo 2010)
- Full classification (announced by Andrews, Li 2012): all embedded CMC tori in $S^3$ are rotationally symmetric (settles conjecture of Pinkall, Sterling 1989)
- Totally analogous bifurcation theory in higher dimensions: $S^m \times S^k \hookrightarrow S^{m+k+1}$, but classification is wide open

Assume that $p \in M$ is a nondegenerate critical point of the scalar curvature on $(M, g)$. Then, a neighborhood of $p$ is foliated by constant mean curvature topological spheres $\Sigma(\rho)$, for $\rho \in ]0, \rho_0[$.

Mahmoudi, Mazzeo, Pacard: GAFA 2006

For $r > 0$ small, geodesic $r$-tubes around a nondegenerate minimal submanifold $N^k \subset M^m$ ($k \leq m - 2$) can be deformed to CMC hypersurfaces with $H = \frac{m-1-k}{r(m-1)}$, except for a sequence $r_n \to 0$ of resonant radii.
Delaunay-type hypersurfaces:

- bifurcating branches of CMC hypersurfaces issuing from a natural 1-parameter family of symmetric CMC embeddings (orbits of isometric actions);
- partially preserve the symmetries of the natural branch;
- bifurcating branches condense onto a minimal submanifold (of higher codimension).

Natural ambient: Manifolds foliated by CMC hypersurfaces, with many symmetries, and condensing on minimal submanifolds.
Cohomogeneity one manifolds

- $(M, g)$ compact Riemannian manifold
- $G$ Lie group acting by isometries on $M$

**cohomogeneity one**: $\dim(M/G) = 1$

$$M/G = \begin{cases} [-1, 1] & \iff \text{two non-principal orbits} \\ S^1 & \iff \text{all orbits are principal} \end{cases}$$

$\gamma : [-1, 1] \to M$ horizontal geodesic, section $\implies$ polar action

- $H := G_{\gamma(t)}$ principal isotropy, $t \in ]-1, 1[$
- $K_{\pm} := G_{\gamma(\pm 1)}$ singular isotropies
- $H \subset \{K_-, K_+\} \subset G$

**Note**: $M$ simply connected $\implies$ non-principal orbits are *singular*. 
Geometry of cohomogeneity one manifolds

singular orbit $G/K_-$
isolated $\implies$ minimal

principal orbits $G/H$:
CMC hypersurfaces

$singular orbit G/K_+$
isolated $\implies$ minimal

$M/G = [-1, 1]$
Geometry of cohomogeneity one manifolds – 2

Tubular neighborhood of singular orbit

- $K_{\pm} \circ D_{\pm}$ slice representation
- $D(G/K_{\pm}) := G \times_{K_{\pm}} D_{\pm}$ Fiber bundle with fiber $D_{\pm}$ associated to $K_{\pm}$-principal bundle $K_{\pm} \to G \to G/K_{\pm}$
- $M \cong D(G/K_{-}) \cup_{G/H} D(G/K_{+})$ is obtained by gluing the two tubular neighborhoods along a principal orbit $G/H$. 

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Delaunay hypersurfaces in cohomogeneity one
Group diagram:

- $S_{\pm} = \partial D_{\pm}$ normal sphere to $G/K_{\pm}$
- $S_{\pm} = K_{\pm}/H$
- $K_{\pm} \circlearrowleft S_{\pm}$ transitive action

$M$ is determined by data

$$H \subset \{K_{-}, K_{+}\} \subset G$$

with $K_{\pm}/H$ diffeomorphic to spheres.
Collapse of singular orbits

- \( x_t : G/H \hookrightarrow M \) family of principal orbits, \( t \in ]-1, 1[ \)
- \( S \cong G/K_+ \) singular orbit at \( t = +1 \)
- \( x_t(G/H) \) is a geodesic tube around \( S \)
- \( x_t(G/H) \) is the total space of a homogeneous fibration:

\[
K/H \twoheadrightarrow x_t(G/H) \twoheadrightarrow S \cong G/K
\]

- As \( t \to 1 \), \( x_t(G/H) \) converges to \( S \) in the Hausdorff metric, i.e., the fibers (normal spheres) collapse to a point:

\[
x_t(G/H) \text{ condeses on } S \text{ as } t \to 1
\]

- \( \lim_{t \to 1} H_t = +\infty \), however, \( S \) is minimal!

Discuss minimality of limit submanifold.
Adapted metric

$G$-invariant metric on a $G$-manifold of cohomogeneity one:

$$g = g_t + dt^2, \quad t \in ]-1, 1[ $$

$g_t$ is a $G$-invariant metric on $x_t(G/H)$, with some conditions as $t \to \pm 1$. (Back-Hsiang 1987, ..., Mendes 2012 for polar actions)

**Definition**

$g$ is *adapted* near $S_{\pm}$ if the projection $(G/H, g_t) \xrightarrow{\pi} (G/K_{\pm}, \check{g}_{\pm1})$ is a Riemannian submersion for $t$ near $\pm 1$ (up to a factor $\alpha(t) \to 1$ as $t \to \pm 1$), i.e.:

$$\pi^* (\check{g}_{\pm1}) = \alpha(t) g_t$$
Existence of adapted metrics

Lie algebras: \[ h \subset k \subset g \]

Choose complements

\[ g = k + m, \quad [k, m] \subset m \]
\[ \ell = h + p, \quad [h, p] \subset p \]
\[ n := m + p \]

Then, \( g = dt^2 + g_t \) is adapted iff \( g_t \) is of the form on \( n \):

\[
g_t(\cdot, \cdot) = \alpha(t) A(\cdot, \cdot) + B_t(\cdot, \cdot),
\]

- \( A \): \( K \)-invariant inn. prod. on \( m \) coming from \( S_{\pm} = G/K_{\pm} \)
- \( B_t \): any \( H \)-invariant inn. prod. on \( p \).

Using a bi-invariant metric on \( G \) one proves easily:

**Proposition**

Every cohomogeneity one \( G \)-manifold \( M \) with \( M/G = [-1, 1] \) admits a metric that is adapted near both of its singular orbits.
A criterion in non-negative curvature

Criterion

Let $M$ be a cohomogeneity one manifold with an invariant metric $g$ of nonnegative sectional curvature. If $(M, g)$ has a totally geodesic principal orbit $N$, then the metric $g$ is adapted near both singular orbits (with $\alpha_{\pm} \equiv 1$).

Proof.

Assume $N$ disconnects $M$ (general case follows).

- $N \subset M$ totally geodesic & $\sec \geq 0 \implies \text{dist}(\cdot, N)$ concave.
- Each component $C_{\pm}$ of $M \setminus N$ is a loc. convex subset of $M$.
- $S_{\pm} = \{\text{points at maximal distance from } N\}$ soul of $C_{\pm}$
- By Perelman, the Sharafutdinov retraction onto the soul (projection from each principal orbit $G/H$ onto $S_{\pm}$) is a Riemannian submersion.
Main result

Theorem

*M* cohomogeneity one *G*-manifold, *H* principal isotropy, singular orbit \( S = G/K \). Assume:

- *S* is not a fixed point
- metric adapted near *S*
- either of the two normality assumptions (N1) or (N2) below.

Then, there are infinitely many bifurcating branches of CMC embeddings of \( G/H \) in *M* issuing from principal orbits arbitrarily close to *S*. Such embeddings are *K*-invariant, but not *G*-invariant.

(N1) *K* normal in *G*
(N2) *H* normal in *K*, and *K*-invariant metric \( g_t \) on \( G/H \) w.r. to a modified action.
(N1) implies:
(P) $K$-orbits (inside principal orbits) coincide with the fibers $(gK)H$ of homogeneous fibration:

$$K/H \longrightarrow G/H \longrightarrow G/K.$$ 

Under (N2), consider a different action:

$$K \times G/H \ni (k, gH) \longmapsto gk^{-1}H \in G/H.$$ 

Extends to a smooth isometric action of $K$ on regular part $M_0 = M \setminus \{S_{\pm}\}$ and (P) holds

(P) yields:
- Eigenvalues of the Jacobi operator for the $K$-symmetric CMC variational problem come from basic eigenvalues of the total space of the fibration $G/H \longrightarrow G/K$.

(Besson, Bordoni, 1991)
Some consequences of the normality assumptions

- (N1) or (N2) implies $S$ totally geodesic (fixed point set of $K$)
- (N1) implies that $K$-action is *fixed-point homogeneous*
- (N2) implies $\text{codim}(S) = 2, 4$
On the normality assumption (N2)

$H$ normal in $K$, $K/H = \text{sphere} \implies K/H \cong S^1$ or $K/H \cong S^3$.
Conversely:

**Proposition**

Let $K$ be a connected group and $H \subset K$ be a compact subgroup such that $K/H \cong S^1$. Then, $H$ is normal in $K$.

**Proof.**

- $H$ compact $\implies \exists$ $K$-invariant metric on $K/H \cong S^1$
- all Riemannian metrics on $S^1$ are round $\implies$ $K$-action given by a homomorphism $\varphi: K \to O(2)$
- $K$ connected $\implies \varphi(K) \subset SO(2)$
- $SO(2)$ acts freely on $S^1 \implies H = \text{stabilizer} = \text{Ker}(\varphi)$. 

Ex. 1: Delaunay-type spheres $S^{2n+1}$ in $\mathbb{C}P^{n+1}$

- $(M, g) = (\mathbb{C}P^{n+1}, g_{FS})$, $g_{FS}$ Fubini-Study metric

  ![Diagram]

- Singular orbits: $S_- = \{p\}$, $S_+ = \text{Cut}(p) \cong \mathbb{C}P^n$
- Principal orbits: $S^{2n+1}_t = (U(n+1)/U(n), g_t)$, $t \in ]0, \pi/2[$, geodesic spheres of radius $t$ centered at $p$, metrically Berger spheres
- $K/H \rightarrow G/H \rightarrow G/K$ is Hopf fibration $tS^1 \rightarrow S^{2n+1}_t \rightarrow \mathbb{C}P^n$
- $g_{FS}$ is adapted near $S_+$, $\alpha(t) = \sin^2 t$
- (N2) is satisfied: $U(n) \triangleleft U(n)U(1)$, $U(n)U(1)/U(n) \cong S^1$
Example 2: Delaunay-type spheres $S^{4n+3}$ in $\mathbb{H}P^{n+1}$

- $(M, g) = (\mathbb{H}P^{n+1}, g_{FS})$, $g_{FS}$ Fubini-Study metric

$$
\begin{align*}
\text{Sp}(n+1) & \quad \text{Sp}(n+1) \\
\text{Sp}(n) & \quad \text{Sp}(n)\text{Sp}(1)
\end{align*}
$$

- Singular orbits: $S^- = \{p\}$, $S^+ = \text{Cut}(p) \cong \mathbb{H}P^n$
- Principal orbits: $S^{4n+3}_t = (\text{Sp}(n+1)/\text{Sp}(n), g_t)$, $t \in ]0, \pi/2[$, geodesic spheres of radius $t$ centered at $p$, metrically Berger spheres
- $K/H \to G/H \to G/K$ is Hopf fibration $tS^3 \to S^{4n+3}_t \to \mathbb{H}P^n$
- $g_{FS}$ is adapted near $S^+$, $\alpha(t) = \sin^2 t$
- (N2) is satisfied: $\text{Sp}(n) \triangleleft \text{Sp}(n)\text{Sp}(1)$, $\text{Sp}(n)\text{Sp}(1)/\text{Sp}(n) \cong S^3$
Ex. 3: Other Delaunay-type hypersurfaces in CROSS

Grove, Wilking, Ziller JDG 2008: full description of cohom 1 actions on CROSS

### Essential cohom 1 actions on CROSS with (N2) with $H \triangleleft K_-$

<table>
<thead>
<tr>
<th>$M$</th>
<th>$G$</th>
<th>$K_-$</th>
<th>$K_+$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^{2k+3}$</td>
<td>$\text{SO}(2)\text{SO}(k + 2)$</td>
<td>$\Delta \text{SO}(2)\text{SO}(k)$</td>
<td>$\mathbb{Z}_2 \cdot \text{SO}(k + 1)$</td>
<td>$\mathbb{Z}_2 \cdot \text{SO}(k)$</td>
</tr>
<tr>
<td>$S^{15}$</td>
<td>$\text{SO}(2)\text{Spin}(7)$</td>
<td>$\Delta \text{SO}(2)\text{SU}(3)$</td>
<td>$\mathbb{Z}_2 \cdot \text{Spin}(6)$</td>
<td>$\mathbb{Z}_2 \cdot \text{SU}(3)$</td>
</tr>
<tr>
<td>$S^{13}$</td>
<td>$\text{SO}(2) \cdot G_2$</td>
<td>$\Delta \text{SO}(2)\text{SU}(2)$</td>
<td>$\mathbb{Z}_2 \cdot \text{SU}(3)$</td>
<td>$\mathbb{Z}_2 \cdot \text{SU}(2)$</td>
</tr>
<tr>
<td>$S^7$</td>
<td>$\text{SO}(4)$</td>
<td>$\text{SO}(1)\text{SO}(1)$</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
<td></td>
</tr>
<tr>
<td>$S^4$</td>
<td>$\text{SO}(3)$</td>
<td>$\text{SO}(1)\text{SO}(1)$</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
<td></td>
</tr>
<tr>
<td>$CP^{k+1}$</td>
<td>$\text{SO}(k + 2)$</td>
<td>$\text{SO}(2)\text{SO}(k)$</td>
<td>$\mathbb{Z}_2 \cdot \text{SU}(3)$</td>
<td>$\mathbb{Z}_2 \cdot \text{SU}(2)$</td>
</tr>
<tr>
<td>$CP^6$</td>
<td>$G_2$</td>
<td>$U(2)$</td>
<td>$\mathbb{Z}_2 \cdot \text{Spin}(6)$</td>
<td>$\mathbb{Z}_2 \cdot \text{SU}(3)$</td>
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</tr>
</tbody>
</table>
Ex. 4: Delaunay hypersurfaces in Kervaire spheres

\[ M_d^{2n-1} \subset \mathbb{C}^{n+1} \] defined by:
\[
\begin{cases}
  z_0^d + z_1^2 + \cdots + z_n^2 = 0, \\
  \|z_0\|^2 + \|z_1\|^2 + \cdots + \|z_n\|^2 = 1
\end{cases}
\]

\( n \) odd, \( d \) odd \( \Rightarrow M_d^{2n-1} \) homeom. to \( S^{2n-1} \);
\( 2n - 1 \equiv 1 \mod 8 \) \( \Rightarrow M_d^{2n-1} = \Sigma^{2n-1} \) exotic (Kervaire) spheres

Cohom 1 action (\( n = 3 \): Calabi, \( n \geq 3 \): Hsiang-Hsiang, 1967):

\[
\begin{array}{ccc}
\text{SO}(2) \times \text{SO}(n) & \xrightarrow{\Delta} & \mathbb{Z}_2 \times \text{SO}(n-2) \\
\downarrow & & \downarrow \\
\text{SO}(2) \times \text{SO}(n-2) & \xrightarrow{\text{SO}(2) \times \text{SO}(n-2)} & \text{SO}(n-2) \times \text{SO}(n-2)
\end{array}
\]

\( \triangleleft \)

\( \text{Z}_2 \times \text{SO}(n-2) \)

\( \text{N2} \) is satisfied:
\[ \mathbb{Z}_2 \times \text{SO}(n-2) \triangleleft \text{SO}(2) \times \text{SO}(n-2) \]
Constructions

Extensions:

- $M$ cohom 1 mfld, diagram $H \subset \{K_-, K_+\} \subset G$
- $G \hookrightarrow \tilde{G}$ extension of $G$
- Get cohom 1 bundle $\tilde{M}$ with $\tilde{G}$-action,
  $M \to \tilde{M} \to \tilde{G}/G$
- $M$ has (N2) $\Rightarrow \tilde{M}$ has (N2)

Products:

- $(H, K_+)$ pair of Lie groups with $K_+/H = S^n$
- $K_- := H \times S^1$ (or $K_- := H \times S^3$)
- $G$ any Lie group containing $K_\pm$
- E.g., $G = K_+ \times S^1$ (or $K_+ \times S^3$), $M = S^{n+2}$ sphere, principal orbits are $G/H = S^n \times S^1$ (or $S^n \times S^3$), singular orbits are $S_- = S^n$ and $S_+ = S^1$ (or $S^3$)
- (N2) is trivially satisfied
Ingredients of Proof

- Variational bifurcation theory: $t$-spectral flow of Jacobi operators

\[ J_t(\psi) = \Delta_{g_t}\psi - (\text{Ric}(\bar{\nu}) + \|S_t\|^2)\psi, \quad \psi : G/H \to \mathbb{R} \]

- Space of (unparameterized) $K$-invariant embeddings
  \[ x : G/H \to M \]

- Area functional with volume constraint & Palais’ symmetric criticality principle

- Eigenvalues of the Jacobi operators related to eigenvalues of Laplacian of a collapsing homogeneous fibration
Delaunay and Yamabe

Delaunay CMC problem \iff Yamabe problem in homogeneous fibration

Orbits of isometric actions are:
- CMC embeddings
- solutions of the Yamabe problem (constant scalar curvature)

**Fact.** Jacobi operators of the area functional and of the Yamabe functional are both Schrödinger operators with potential given by curvatures.

