Half-Space Theorems in $\mathbb{H}^n \times \mathbb{R}^\ell$

G. Pacelli Bessa

Universidade Federal do Ceará - UFC

Encounters in Geometry
Cabo Frio, RJ.

The content of this talk is a joint work with
Jorge H. S. Lira and Adriano Medeiros

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- M. Anderson and L. Rodriguez proved the strong half-space in a complete oriented non-compact 3-dimensional Riemannian manifold N with Ricci curvature $Ric_N \geq 0$ and sectional curvature bounded $K_N \leq b$.

  They proved that any two complete properly immersed oriented minimal surfaces, intersect unless they are totally geodesic and parallel leaves in a local product structure.

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- The Strong Half-Space Theorem in this setting was settled independently by Bessa-Jorge-Oliveira [2001] and by Rosenberg [2001]. Two complete minimal surfaces with bounded curvature intersect unless they are parallel planes.
- Bessa-Jorge-Oliveira[2001] also proved the *Mixed Half-Space Theorem*. A complete properly minimal surface and a complete minimal surface with bounded curvature must intersect unless they are parallel planes.

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- This new theory started to be developed after a seminal paper by Rosenberg published in 2002.
- The classical theory of minimal/cmc surfaces in $\mathbb{R}^3$, in some sense, guides the research in this new theory, although the results are very sensitive to the geometry of N.
- In this spirit, Hauswirth, Rosenberg and Spruck [2008] proved a version of the Meeks' Half-Space Theorem in $\mathbb{H}^2 \times \mathbb{R}$.

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- The minimal surfaces of $\mathbb{R}^3$ correspond in $\mathbb{H}^2 \times \mathbb{R}$ to surfaces with constant mean curvature $H = \frac{1}{2}$.

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$$\sup |H| > \frac{m - \ell}{m}.$$

Here $p \in \Sigma$ and $\rho_\Sigma(x) = \mathrm{dist}_\Sigma(p, x)$.

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- For instance $G_1(t) = t^2 \log^2(1+t)$ for $t \gg 1$.

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- And $G_2 \leq 0$ or more generally G_2 satisfies the Kneser type criterion: $G_2^- = \max\{-G_2, 0\} \in L^1(\mathbb{R}^+)$ and $\int_t^{+\infty} G_2^-(s)\mathrm{d}s \leq \frac{1}{4t}$ on $(0,\infty)$.

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- In the best cases, $-G_2 \leq -c^2 < 0$.
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- The philosophy is that: the data $(p, \Sigma_0, G_2, h, h')$ yields a region $\Omega \subset N$ such that if $\varphi\colon M \to N$ is a properly immersed submanifold such that $\varphi(M) \subset \Omega$ then $\sup |H| \geq \inf_\Omega \frac{h'}{h}$.

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- Usually Ω is a "side" of a tubular neighborhood of Σ_0.
- The condition $-G_1(\rho(x)) \leq K_N^{rad}(x)$ plus properness implies that M is stochastically complete, a notion I am going to discuss right now.

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on M, for any/all $\lambda > 0$, vanish identically. [Grigor'yan-BAMS 1999]

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- $\int^{+\infty} \frac{t}{\log(\mathrm{vol} B(o,t))} = +\infty$ [Grigor'yan86]

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The weak maximum principle (at infinity) holds on a Riemannian manifold M if for every $u \in C^2(M)$, with $u^* := \sup_M u < +\infty$, there exists a sequence $\{x_k\} \subset M$ along which

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An important observation: the function u need to be C^2 only in the neighborhood of the points x_k.

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- However, there are stochastically complete minimal surfaces in a half-space of $\mathbb{R}^3$. For instance, the minimal surface of Jorge-Xavier between two parallel planes.

Curvature estimates

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- On the other hand

$$\begin{aligned} \triangle_M f &= g' \triangle_M (\rho \circ \varphi) + g'' |\nabla(\rho \circ \varphi)|^2 \\ &= g' \sum_{i=1}^{m} \mathrm{Hess}_N \rho(e_i, e_i) + \langle \mathrm{grad}\, \rho, m\mathbf{H} \rangle + g'' |\nabla(\rho \circ \varphi)|^2 \end{aligned}$$

- Where $e_i \in TM$.

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$$\begin{aligned}
\triangle_M f &\geq g' \sum_{i=1}^m \mathrm{Hess}_N \rho(e_i,e_i) - g'|m\mathbf{H}| + g''|\nabla(\rho\circ\varphi)|^2 \\
&= g'\frac{h'}{h}\sum_{i=1}^m\sum_{j=2}^n b_{ij}^2 + \sum_{i=1}^m a_i^2 g'' - g'|m\mathbf{H}| \qquad (4)\\
&= g'\frac{h'}{h}(m - \sum_{i=1}^m a_i^2) - g''\sum_{i=1}^m a_i^2 - g'|m\mathbf{H}| \qquad (5)\\
&= m(h' - h|\mathbf{H}|)
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Let $\varphi\colon M \to N \times \mathbb{R}^\ell$ be an stochastically complete submanifold so that $\varphi(M) \subset \mathscr{U} \times \mathbb{R}^\ell \subset (-\infty, T) \times \mathbb{R}^\ell$ and $\varphi(M) \cap \rho^{-1}(0,t_0) \times \mathbb{R}^\ell \neq \emptyset$. Then

$$\sup_M |\mathbf{H}| \geq \frac{m-\ell}{m} \inf_{t\in(0,t_0)} \frac{h'}{h}(t)\cdot$$

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- It turns out that $\mathscr{H}_1 = \Gamma_S(2)$ is the Hsiang-Salavessa's surface.

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Let $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be a complete surface with constant mean curvature $|H| \leq \frac{1}{2}$, Σ different from the Hsiang-Salavessa's Graph $\Gamma_S(2)$. Then Σ can not be properly immersed in the mean convex side of $\Gamma_S(2)$.

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- In $\mathbb{H}^n \times \mathbb{R}$ the Hsiang-Salavessa's graphs $\Gamma_S(n)$ are constant mean curvature $H = (n-1)/n$ discs with one vertical end. This disc separates $\mathbb{H}^n \times \mathbb{R}$ in two components. One side $\mathscr{H}$ is mean convex and it is foliated by spheres with a common point of tangency $o \in \mathbb{H}^n \times \{0\}$.

We have the following version of Earp-Nelli half-space theorem.

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Let Σ be a hypersurface properly immersed into the mean convex side $\mathscr{H}$ of $\Sigma(\frac{n-1}{n})$. Then $\sup_\Sigma |H| > \frac{m-1}{m}$.

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- Define $f = g \circ \rho \circ \varphi$, where $g = \int_0^t \sinh(s)ds$, $\rho = \mathrm{dist}_{\mathbb{H}^n}(\cdot, \Sigma_0)$ the signed distance. As before we can show that

$$\sup_\Sigma |\mathbf{H}| \geq \frac{m-1}{m} \frac{h'}{h}(d) > (m-1)/m$$

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Definition

A wedge $\mathscr{W}(p,v,a) \subset N \times \mathbb{R}^\ell$ determined by v and $a \in (0,1)$ is the set

$$\mathscr{W}(p,v,a) = \mathscr{C}_N(p,v,a) \times \mathbb{R}^\ell.$$

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We can prove the following mean curvature estimate.

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Let $\varphi\colon M \hookrightarrow N \times \mathbb{R}^\ell$ be a stochastically complete m-dimensional immersed submanifold M of the wedge $C_N(p,v,a) \times \mathbb{R}^\ell \subset N \times \mathbb{R}^n$, where $m \geq \ell + 1$, N is a Hadamard manifold with sectional curvature $K_N \leq b \leq 0$ and $C_N(p,v,a)$, $p \in N$, $v \in T_pN$, $a \in (0,1)$, is a non-degenerate cone in N. Then

$$\sup_M |H| > 0 \text{ if } b = 0 \text{ and } \sup_M |H| \geq \frac{(m-\ell)|b|}{m} \text{ if } b < 0.$$

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- The question is: What is the the Half-Space theorem to stochastically complete surfaces of $\mathbb{R}^3$. My guess is: Every geodesically and stochastically complete minimal surfaces intersect any catenoid.

Questions

- Complete proper minimal surfaces of $\mathbb{R}^3$ are stochastically complete.
- Complete minimal surfaces of $\mathbb{R}^3$ with bounded curvature are stochastically complete.
- The Jorge-Xavier minimal surface betwee two planes can be build so that it is stochastically complete.
- The question is: What is the the Half-Space theorem to stochastically complete surfaces of $\mathbb{R}^3$. My guess is: Every geodesically and stochastically complete minimal surfaces intersect any catenoid.
- What is known? L. Jorge, J. Lira and myself proved that a stochastically minimal surface can not be “inside”a catenoid with finite hight. This is better than a cone but not as satisfactory as I wished.
  THANK YOU ALL.