Optimal Portfolio Selection

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- This talk is joint work with **Phelim Boyle** (Wilfrid Laurier University, Waterloo, Canada), **Jit Seng Chen** (University of Waterloo) and with **Steven Vanduffel** (Vrije Universiteit Brussel (VUB), Belgium).

- **Outline:**

1. What is **cost-efficiency**? Illustration in the **binomial** model
2. Characterization of optimal investment strategies for an investor with **law-invariant preferences** and a **fixed investment horizon**
3. Illustration in the **Black and Scholes** model
4. How to use cost-efficiency to **optimize** your investment strategies? Or your hedging strategies?
5. More applications on how to "choose" a utility?
6. Extension to
    - A **multidimensional** market.
    - The case when investors have **state-dependent constraints**.

Traditional Approach to Portfolio Selection

Consider an investor with **increasing law-invariant** preferences and a **fixed** horizon. Denote by X_T the investor's final wealth.

- Optimize an increasing law-invariant objective function
  1. $\max_{\mathbf{X_T}} (\mathbf{E_P}[\mathbf{U}(\mathbf{X_T})])$ where U is increasing.
  2. Minimizing Value-at-Risk (a quantile of the cdf)
  3. Probability target maximizing: $\max_{\mathbf{X_T}} \mathbf{P}(\mathbf{X_T} > \mathbf{K})$ (quantile hedging)
  4. ...
- for a given **cost** (budget)

$$cost\ at\ 0 = E_Q[e^{-rT} X_T]$$

Find optimal strategy X_T^* $\Rightarrow$ Optimal cdf F of X_T^*

What is "cost-efficiency"?

Cost-efficiency is a criteria for evaluating payoffs independent of the agents' preferences.

Cost-Efficiency

A strategy (or a payoff) is **cost-efficient** if any other strategy that generates the same distribution under P costs at least as much.

This concept was originally proposed by Dybvig.

- Dybvig, P., 1988a. "Distributional Analysis of Portfolio Choice," *Journal of Business*, **61**(3), 369-393.
- Dybvig, P., 1988b. "Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market," *Review of Financial Studies*, **1**(1), 67-88.

Important observation

Consider an investor with

- Law-invariant preferences
- Increasing preferences
- A fixed investment horizon

It is clear that the optimal strategy must be **cost-efficient**.

Therefore optimal portfolios in the traditional settings discussed before are cost-efficient.

The rest of this section is about characterizing cost-efficient strategies.

Implications of this characterization

- **Characterization of optimal strategies**. This characterization can then be used to solve optimal portfolio problems by restricting the set of possible strategies.
- **Improving dynamic strategies** that are used with a fixed investment horizon T: CPPI, Stop-loss...
- **Improving hedging strategies** that are used with a fixed investment horizon T such as quantile hedging, probability maximization...
- **Improving European contracts** (retail investment products, EIAs, structured products): replacing path-dependent complex contracts by simpler contracts.

Main Assumptions

- Consider an arbitrage-free and complete market.
- Given a strategy with final payoff X_T at time T.
- There exists a unique probability measure Q, such that its price at 0 is

$$c(X_T) = \mathbb{E}_Q[e^{-rT} X_T]$$

Cost-efficient strategies

- Given a cdf F under the **physical measure** P.

The distributional price is defined as

$$PD(F) = \min_{\{Y \mid Y \sim F\}} c(Y) = \min_{\{Y \mid Y \sim F\}} \mathbb{E}_Q[e^{-rT}Y]$$

- The strategy with payoff X_T is **cost-efficient** if

$$PD(F) = c(X_T)$$

- Given a strategy with payoff X_T at time T. Its price at 0 is

$$P_X = E_Q[e^{-rT}X_T]$$

- F: distribution of the cash-flow at T of the strategy

The "loss of efficiency" or "efficiency cost" is equal to

$$P_X - PD(F)$$

A Simple Illustration

Let's illustrate what the "efficiency cost" is with a simple example. Consider :

- A market with 2 assets: a bond and a stock S.
- A discrete 2-period binomial model for the stock S.
- A strategy with payoff X_T at the end of the two periods.

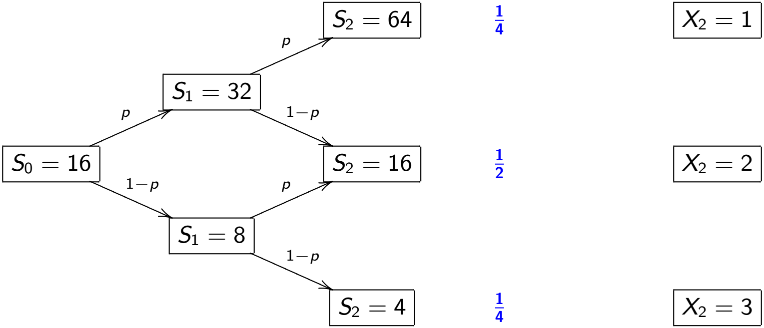
Example of

- $X_T \sim Y_T$ under P
- but with different prices

in a 2-period (arbitrage-free) binomial tree ($T = 2$).

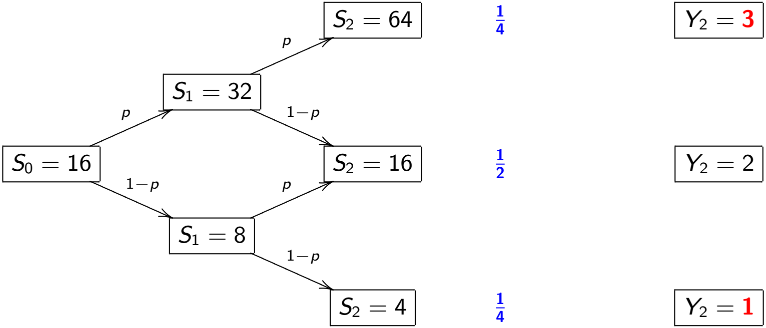
A simple illustration for X_2, a payoff at $T = 2$

Real-world probabilities: $p = \frac{1}{2}$



Y_2, a payoff at $T = 2$ distributed as X_2

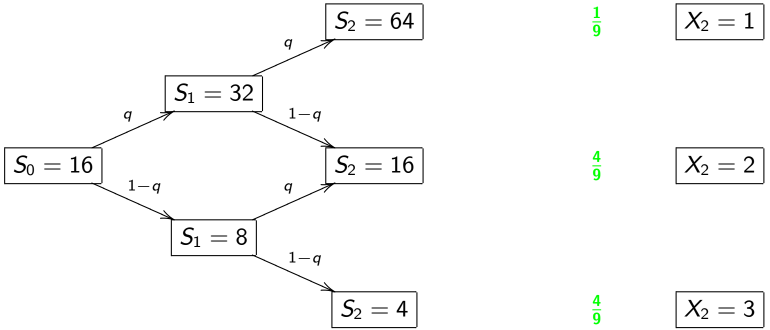
Real-world probabilities: $p = \frac{1}{2}$



X_2 and Y_2 have the same distribution under the physical measure

X_2, a payoff at $T = 2$

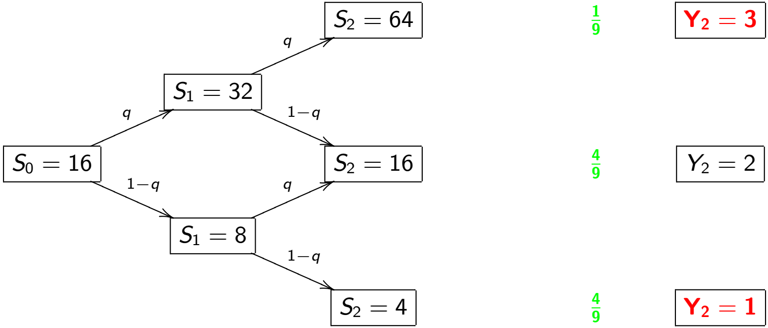
risk neutral probabilities: $q = \frac{1}{3}$.



$$c(X_2) = \text{Price of } X_2 = \left(\frac{1}{9} + \frac{4}{9}2 + \frac{4}{9}3\right) = \frac{21}{9}$$

Y_2, a payoff at $T = 2$

risk neutral probabilities: $q = \frac{1}{3}$.



$$c(Y_2) = \left(\frac{1}{9}3 + \frac{4}{9}2 + \frac{4}{9}1\right) = \frac{15}{9}$$

$$c(X_2) \;=\; \text{Price of } X_2 = \left(\frac{1}{9} + \frac{4}{9}2 + \frac{4}{9}3\right) = \frac{21}{9}$$

Characterization of Cost-Efficient Strategies

Assumptions: General setting

To characterize cost-efficiency, we need to introduce the "state-price process"

- Consider an arbitrage-free and complete market.
- Given a strategy with payoff X_T at time T. There exists a unique risk-neutral probability Q, such that its price at 0 is

$$c(X_T) = \mathbb{E}_Q[e^{-rT}X_T]$$

- P ("physical measure") and Q ("risk-neutral measure") are two equivalent probability measures:

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right)_T, \quad \mathbf{c(X_T)} = \mathbb{E}_Q[e^{-rT}X_T] = \mathbb{E}_{\mathbf{P}}[\xi_{\mathbf{T}}\mathbf{X_T}].$$

ξ_T is called "state-price process" and is also sometimes referred to as "deflator" or "pricing kernel".

The state-price process

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right)_T$$

plays a very important role in cost-efficiency.

We will show that a sufficient condition for a payoff X_T to be **cost-efficient** strategy is to move in the opposite direction with the state-price process ξ_T:

"When ξ_T increases, then X_T decreases".

We say that these variables are anti-monotonic.

Formally, we have the following definitions.

Sufficient Condition for Cost-efficiency

A random pair (X, Y) is anti-monotonic if

there exists a non-increasing relationship between them.

Theorem (Sufficient condition for cost-efficiency)

Any random payoff X_T with the property that (X_T, ξ_T) is **anti-monotonic** *is* **cost-efficient**.

Note the absence of additional assumptions on ξ_T (it holds in discrete and continuous markets) and on X_T (no assumption on non-negativity).

Idea of the proof (1/2)

Minimizing the price $c(X_T) = E[\xi_T X_T]$ when $X_T \sim F$ amounts to finding the dependence structure that **minimizes the correlation** between the strategy and the state-price process

$$\min_{X_T} \ \mathbb{E}\left[\xi_T X_T\right]$$
$$\text{subject to} \ \begin{cases} X_T \sim F \\ \xi_T \sim G \end{cases}$$

Recall that

$$\text{corr}(X_T, \xi_T) = \frac{\mathbb{E}[\xi_T X_T] - \mathbb{E}[\xi_T]\mathbb{E}[X_T]}{\text{std}(\xi_T)\,\text{std}(X_T)}.$$

Idea of the proof (2/2)

We can prove that when the distributions for both X_T and ξ_T are fixed, we have

$$(X_T, \xi_T) \text{ is anti-monotonic} \Rightarrow \text{corr}[X_T, \xi_T] \text{ is minimal.}$$

Minimizing the cost $E[\xi_T X_T] = c(X_T)$ of a strategy therefore amounts to minimizing the correlation between the strategy and the state-price process

Illustration of the "state-price process" in discrete model.

Consider the following payoff X_2 after 2 periods:

$$X_2 = \begin{cases} 1 & \text{if } S_2 = 64, \\ 2 & \text{if } S_2 = 16, \\ 3 & \text{if } S_2 = 4. \end{cases}$$

Recall that

$$\xi_T = e^{-rT} \left(\frac{dQ}{dP}\right)_T$$

and that

$$c(X_2) = E_P[\xi_2 X_2] = E_Q[e^{-2r} X_2]$$

$r = 0\%,\ p = \frac{1}{2},\ q = \frac{1}{3}.$

$$
\begin{tikzcd}
 & & \boxed{\xi_2 = e^{-2r}\frac{q^2}{p^2} = \frac{4}{9}} \\
 & \boxed{\xi_1 = e^{-r}\frac{q}{p}} \arrow[ur, "p"] \arrow[dr, "1-p"] & \\
\boxed{\xi_0 = 1} \arrow[ur, "p"] \arrow[dr, "1-p"'] & & \boxed{\xi_2 = e^{-2r}\frac{q(1-q)}{p(1-p)} = \frac{8}{9}} \\
 & \boxed{\xi_1 = e^{-r}\frac{1-q}{1-p}} \arrow[ur, "p"] \arrow[dr, "1-p"] & \\
 & & \boxed{\xi_2 = e^{-2r}\frac{(1-q)^2}{(1-p)^2} = \frac{16}{9}}
\end{tikzcd}
$$

$$
\begin{aligned}
c(X_2) &= E_P[\xi_2 X_2] = \left(\frac{1}{4}\,\frac{4}{9}\,1 + \frac{1}{2}\,\frac{8}{9}\,2 + \frac{1}{4}\,\frac{16}{9}\,3\right). \\
&= E_Q[e^{-2r} X_2] = \left(\frac{1}{9}\,1 + \frac{4}{9}\,2 + \frac{4}{9}\,3\right).
\end{aligned}
$$

Illustration of the "state-price process" in the Black-Scholes model.

Under the physical measure P,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^P$$

Then

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right) = a\left(\frac{S_T}{S_0}\right)^{-b}$$

where $a = e^{\frac{\theta}{\sigma}(\mu-\frac{\sigma^2}{2})t-(r+\frac{\theta^2}{2})t}$ and $b = \frac{\mu-r}{\sigma^2}$.

Explicit Representation for Cost-efficiency

Assume ξ_T is continuously distributed (for example a Black-Scholes market)

Theorem

The cheapest strategy that has cdf F is given explicitly by

$$X_T^\star = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right).$$

Note that $X_T^\star \sim F$ and $X_T^\star$ is a.s. **unique** *such that*

$$PD(F) = c(X_T^\star) = \mathbb{E}[\xi_T X_T^\star]$$

where $PD(F)$ is the distributional price

$$PD(F) = \min_{\{X_T \mid X_T \sim F\}} e^{-rT}\mathbb{E}_Q[X_T] = \min_{\{X_T \mid X_T \sim F\}} \mathbb{E}[\xi_T X_T]$$

and F^{-1} is defined as follows:

$$F^{-1}(y) = \min\left\{x \;/\; F(x) \geqslant y\right\}.$$

Copulas and Sklar's theorem

The joint cdf of a couple (ξ_T, X) can be decomposed into 3 elements

- The marginal cdf of ξ_T: G
- The marginal cdf of X_T: F
- A copula C

such that

$$P(\xi_T < \xi, X_T < x) = C(G(\xi), F(x))$$

Idea of the proof (1/3)

Solving this problem amounts to finding bounds on copulas!

$$\min_{X_T} \mathbb{E}\left[\xi_T X_T\right]$$
$$\text{subject to } \begin{cases} X_T \sim F \\ \xi_T \sim G \end{cases}$$

The distribution G is known and depends on the financial market. Let C denote a copula for (ξ_T, X).

$$\mathbb{E}[\xi_T X] = \int \int (1 - G(\xi) - F(x) + C(G(\xi), F(x)))dxd\xi, \quad (1)$$

Bounds for $\mathbb{E}[\xi_T X]$ are derived from bounds on the copula C.

Idea of the proof (2/3)

It is well-known that any copula verify

$$\max(u+v-1,0) \leqslant C(u,v) \leqslant \min(u,v)$$

(Fréchet-Hoeffding Bounds for copulas) where the lower bound is the "anti-monotonic copula" and the upper bound is the "monotonic copula".

Let U be uniformly distributed on $[0,1]$.

- The cdf of $(U, 1-U)$ is $P(U \leqslant u, 1-U \leqslant v) = \max(u+v-1,0)$ (**anti-monotonic** copula)
- the cdf of (U,U) is $P(u,v) = \min(u,v)$ (**monotonic** copula).

Idea of the proof (3/3)

Consider a strategy with payoff X_T distributed as F. Note that $U = F_\xi(\xi_T)$ is uniformly distributed over $(0,1)$.

Note that ξ_T and $X_T^* := F^{-1}(1 - G(\xi_T))$ are anti-monotonic and that $X_T^* \sim F$.

Note that ξ_T and $Z_T^* := F^{-1}(G(\xi_T))$ are comonotonic and that $Z_T^* \sim F$.
The cost of the strategy with payoff X_T is $c(X_T) = E[\xi_T X_T]$.

$$E[\xi_T F^{-1}(1 - G(\xi_T))] \leqslant c(X_T) \leqslant E[\xi_T F^{-1}(G(\xi_T))]$$

that is

$$E[\xi_T X_T^*] \leqslant c(X_T) \leqslant E[\xi_T Z_T^*].$$

Path-dependent payoffs are inefficient

Corollary

To be cost-efficient, the payoff of the derivative has to be of the following form:

$$X_T^{\star} = F^{-1}\left(1 - F_{\xi}\left(\xi_T\right)\right)$$

It becomes a European derivative written on S_T when the state-price process ξ_T can be expressed as a function of S_T. Thus path-dependent derivatives are in general not cost-efficient.

Corollary

Consider a derivative with a payoff X_T which could be written as

$$X_T = h(\xi_T)$$

Then X_T is cost efficient if and only if h is non-increasing.

Examples

in the Black-Scholes setting

to improve strategies

Black-Scholes Model

Under the physical measure P,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^P$$

Then

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right) = a\left(\frac{S_T}{S_0}\right)^{-b}$$

where $a = e^{\frac{\theta}{\sigma}(\mu - \frac{\sigma^2}{2})t - (r + \frac{\theta^2}{2})t}$ and $b = \frac{\mu - r}{\sigma^2}$.

Theorem (Cost-efficiency in Black-Scholes model)

To be cost-efficient, the contract has to be a European derivative written on S_T and non-decreasing w.r.t. S_T (when $\mu > r$). In this case,

$$\mathbf{X_T^\star = F^{-1}\left(F_{S_T}\left(S_T\right)\right)}$$

Implications

In a Black Scholes model (with 1 risky asset), optimal strategies for an investor with a **fixed horizon investment** and **law-invariant preferences** are always of the form

$$g(S_T)$$

with g non-decreasing.

Maximum price = Least efficient payoff

Theorem

Consider the following optimization problem:

$$\max_{\{X_T \mid X_T \sim F\}} c(X_T) = \max_{\{X_T \mid X_T \sim F\}} \mathbb{E}[\xi_T X_T]$$

Assume ξ_T is continuously distributed. *The unique strategy $Z_T^\star$ that generates the same distribution as F with the highest cost can be described as follows:*

$$Z_T^\star = F^{-1}\left(F_\xi\left(\xi_T\right)\right) = F^{-1}\left(1 - F_{S_T}\left(S_T\right)\right)$$

Geometric Asian contract in Black-Scholes model

Assume a strike K. The payoff of the Geometric Asian call is given by

$$X_T = \left(e^{\frac{1}{T}\int_0^T \ln(S_t)dt} - K\right)^+$$

which corresponds in the discrete case to $\left(\left(\prod_{k=1}^{n} S_{\frac{kT}{n}}\right)^{\frac{1}{n}} - K\right)^+$.

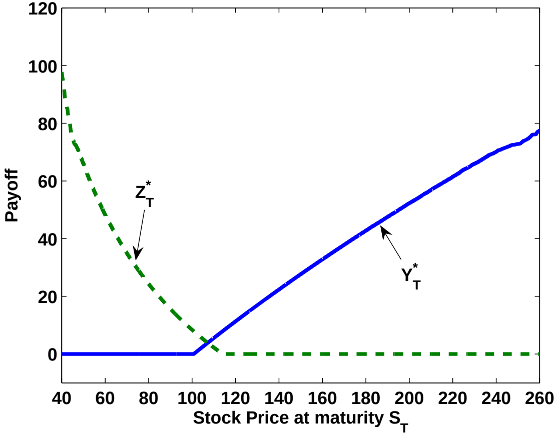
The efficient payoff that is distributed as the payoff X_T is a power call option

$$X_T^\star = d\left(S_T^{1/\sqrt{3}} - \frac{K}{d}\right)^+$$

where $d := S_0^{1-\frac{1}{\sqrt{3}}} e^{\left(\frac{1}{2}-\sqrt{\frac{1}{3}}\right)\left(\mu-\frac{\sigma^2}{2}\right)T}$.

Similar result in the discrete case.

Example: Discrete Geometric Option



With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100,\ T = 1$ year, $K = 100$.

$$C(X_T^\star) = 5.3 < Price(geometric\ Asian) = 5.5 < C(Z_T^\star) = 8.4.$$

Put option in Black-Scholes model

Assume a strike K. The payoff of the put is given by

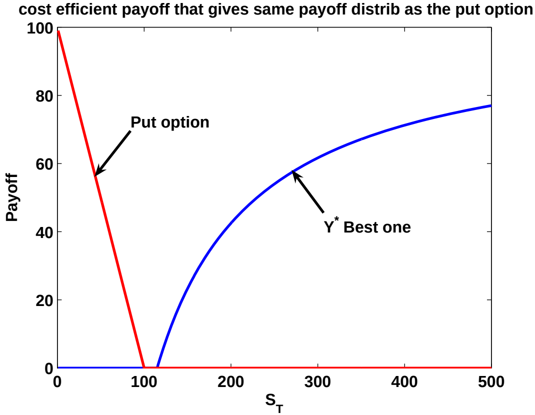
$$L_T = (K - S_T)^+ .$$

The payout that has the **lowest** cost and that has the same distribution as the put option payoff is given by

$$X_T^\star = F_L^{-1}\left(F_{S_T}\left(S_T\right)\right) = \left(K - \frac{S_0^2 e^{2\left(\mu - \frac{\sigma^2}{2}\right)T}}{S_T}\right)^+ .$$

This type of power option "dominates" the put option.

Cost-efficient payoff of a put



With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100,\ T = 1$ year, $K = 100$.
Distributional price of the put = 3.14
Price of the put = 5.57
Efficiency loss for the put = 5.57-3.14= 2.43

Up and Out Call option in Black and Scholes model

Assume a strike K and a barrier threshold $H > K$. Its payoff is given by

$$L_T = (S_T - K)^+ \mathbb{1}_{\max_{0 \leqslant t \leqslant T}\{S_t\} \leqslant H}$$

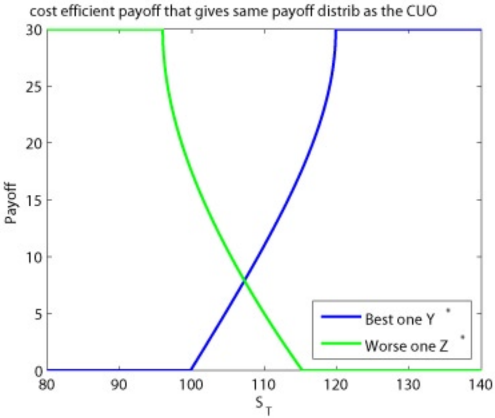
The payoff that has the **lowest** cost and is distributed such as the barrier up and out call option is given by

$$X_T^\star = F_L^{-1}\left(1 - F_\xi\left(\xi_T\right)\right)$$

The payoff that has the **highest** cost and is distributed such as the barrier up and out call option is given by

$$Z_T^\star = F_L^{-1}\left(F_\xi\left(\xi_T\right)\right)$$

Cost-efficient payoff of a Call up and out



With $\sigma = 20\%, \mu = 9\%, S_0 = 100,\ T = 1$ year, strike $K = 100,\ H = 130$

Distributional Price of the CUO $= 9.7374$

Price of CUO $= P_{cuo}$

Worse case $= 13.8204$

Efficiency loss for the CUO $= P_{cuo}$-9.7374

Analysis of CPPI strategy

In a CPPI strategy

- the investor chooses floor values F_t for his portfolio at future times t (here $F_t = Fe^{ct}$).
- at each time t the cushion C_t is

$$C_t = V_t - F_t$$

where V_t is the actual portfolio value and F_t is the floor value.
- The amount allocated to one or more risky assets is called the exposure E_t and is given by

$$E_t = \begin{cases} \ell C_t & \text{if } C_t > 0 \\ 0 & \text{if } C_t \leqslant 0 \end{cases}$$

where $\ell > 0$ is reflecting the degree of leverage.
- The remaining proportion $V_t - E_t$ will then be invested in the risk free account which is assumed to grow at a fixed rate $r > 0$.

In a one-dimensional Black-Scholes market with continuous trading and without borrowing restrictions a single-asset CPPI gives rise to a path-independent payoff increasing in the underlying asset, and thus is cost-efficient.

However, in reality trading occurs at discrete times and may be subject to borrowing constraints.

Assume no transactions costs and it is not allowed to go short. In particular this means that the exposure

$$E_t \leqslant V_t.$$

We will also assume that the rebalancing occurs on a monthly basis and that there are no transaction costs involved.

Methodology

The CPPI strategy gives rise to a payoff $H(T)$ at time T and we apply Monte Carlo methods to estimate the distribution function F of $H(T)$ in a straightforward way.

The cost-efficient payoff X_T^* is given by

$$X_T^* = F^{-1}\left(F_{S_T}(S_T)\right). \tag{2}$$

Finally, we apply Monte Carlo simulation under the risk neutral measure $\mathbb{Q}$ to derive the approximate price for X^* and estimate the inefficiency cost of the CPPI strategy.

Inefficiency of CPPI

Inefficiency cost of single-asset CPPI structure in percentage

T=1

$\ell \setminus \sigma$	**0.15**	**0.20**	**0.25**	**0.30**	**0.35**
4	0.012	0.020	0.038	0.065	0.108
7	0.057	0.122	0.215	0.318	0.423

T=5

$\ell \setminus \sigma$	**0.15**	**0.20**	**0.25**	**0.30**	**0.35**
4	0.219	0.347	0.523	0.749	1.018
7	0.709	1.211	1.650	1.988	2.221

where $c = r = 0.04$ and $\mu = 0.10$ and for various parameters for the multiplier ℓ, the volatility σ and the risk horizon T, we express the increase in cost when applying the CPPI strategy as compared to pursuing the path-independent alternative.

Some Applications of Cost-Efficiency

Applications

1. Equivalence between the Expected Utility Maximization setting and the Cost-Efficient strategy.
2. Solving well-known problems in a simpler way
3. Understanding implications of the choice of utility in terms of "distribution of terminal wealth"
4. A series of examples: Mean-variance optimization, Exponential utility, Power utility, Yaari's theory, Goal reaching (target probability maximization)...

Equivalence

For each expected utility maximizer, one may construct the optimal wealth $X_T^\star$ at the investment horizon T. It has a cdf F, and one can show that

$$X_T^\star = F^{-1}(1 - F_{\xi_T}(\xi_T)) \quad a.s.$$

The optimal wealth for an expected utility maximizer is characterized by its cdf.

Power utility (CRRA) & the LogNormal Distribution

$$U(x)=\frac{x^{1-\eta}}{1-\eta}.$$

The optimal wealth obtained with an initial budget x_0 is calculated as $[U']^{-1}(k\xi_T)$ where k is chosen to meet the budget constraint. After some straightforward calculations,

$$X_T^{\star}=x_0 e^{rT} e^{A}\left(\frac{S_T}{S_0}\right)^{\frac{\lambda}{\eta\sigma}}$$

where $\lambda=\frac{\mu-r}{\sigma}$ is the instantaneous Sharpe ratio for the risky asset S and where $A=-\frac{\lambda}{\eta\sigma}(\mu-\frac{\sigma^2}{2})T-\frac{\lambda^2 T}{2\eta^2}+\frac{\lambda^2 T}{\eta}$. $X_T^{\star}$ is obviously lognormal and its cdf F_{CRRA} can be written as

$$F_{CRRA}(y)=\Phi\left(\frac{\eta\ln\left(\frac{y}{x_0}\right)-\eta rT-\lambda^2 T+\frac{\lambda^2 T}{2\eta}}{\lambda\sqrt{T}}\right).$$

Exponential utility & the Normal Distribution

The exponential utility investor maximizes expected utility of final wealth where the utility function is given by

$$U(x) = -\exp(-\gamma x),$$

where γ is the constant absolute risk aversion parameter. The optimal wealth obtained with an initial budget x_0 is given by

$$X_T^\star = e^{rT}x_0 + \frac{T}{\gamma}\lambda^2 - \frac{\lambda}{\gamma\sigma}\left(\mu - \frac{\sigma^2}{2}\right)T + \frac{\lambda}{\gamma\sigma}\ln\left(\frac{S_T}{S_0}\right)$$

where $\lambda = \frac{\mu - r}{\sigma}$ is the instantaneous Sharpe ratio for the risky asset S. Its cdf F_{EXP} corresponds to the cdf of a normal distribution with mean $e^{rT}x_0 + \frac{T}{\gamma}\lambda^2$ and variance $\frac{\lambda^2 T}{\gamma^2}$

$$F_{EXP}(y) = \Phi\left(\frac{\gamma y - \gamma e^{rT}x_0 - T\lambda^2}{\lambda\sqrt{T}}\right)$$

Mean Variance Optimum

Let W_0 denote the initial wealth. The solution of the following *mean-variance* optimization problem

$$\max_{c(X_T)=W_0} \left(E\left(X_T\right) - \alpha \cdot \mathrm{Var}\left(X_T\right)\right) \tag{3}$$

is denoted by $X_T^\star$ and is given by

$$X_T^\star = a - b.\xi_T. \tag{4}$$

Here a and b are determined by

$$a = W_0 e^{rT} + \frac{1}{2\alpha}\left(\frac{Var(\xi_T)}{E^2(\xi_T)} + 1\right), \quad b = \frac{1}{2\alpha} e^{rT}.$$

Mean Variance Optimum in Black Scholes

Assume a Black-Scholes market, mean-variance efficient portfolios are Lognormal + constant.
The maximal sharpe ratio SR is then given as

$$SR = \sqrt{e^{\theta^2 T} - 1}$$

where $\theta = \frac{\mu - r}{\sigma}$.

Improving strategies

An alternative approach

Second-Order Stochastic Dominance

We will say that X_T dominates Y_T for the second-order stochastic dominance order

$$Y_T \prec_{ssd} X_T$$

if for all concave utility

$$E[U(Y_T)] \leqslant E[U(X_T)]$$

A Very Different Approach

Improving by conditioning!

Theorem

Any payoff X_T which cannot be expressed as a function of the state-price process ξ_T at time T is strictly dominated in the sense of second-order stochastic dominance by

$$H_T^{\star} = E\left[X_T \,|\, \sigma(\xi_T)\right] = g(\xi_T),$$

which is a function of ξ_T. Consequently in the Black and Scholes framework, any strictly path-dependent payoff is dominated by a path-independent payoff.

- Same cost.
- Different distribution.
- H_T^* is preferred by all risk-averse investors.

Example: the Lookback Option

Consider a lookback call option with strike K. The payoff on this option is given by

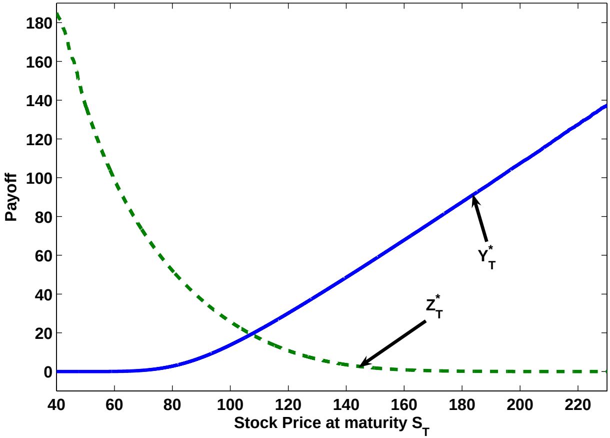
$$L_T = \left(\max_{0 \leqslant t \leqslant T} \{S_t\} - K \right)^+ .$$

The cost efficient payoff with the same distribution

$$Y_T^\star = F_L^{-1}\left(1 - F_\xi\left(\xi_T\right)\right).$$

The payoff that has the highest cost and has the same distribution as the payoff L_T is given by $Z_T^\star = F_L^{-1}\left(F_\xi\left(\xi_T\right)\right).$

Example: the Lookback Option

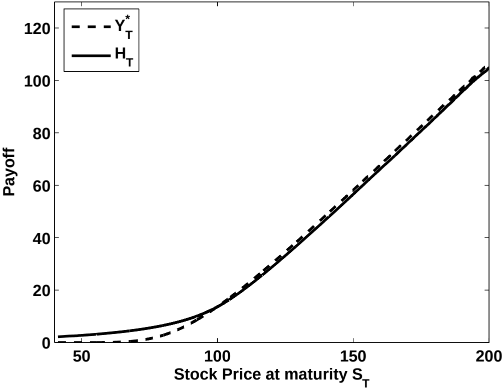


With $\sigma = 20\%, \mu = 9\%, r = 5\% S_0 = 100,\ T = 1$ year, $K = 100$.

Distributional Price of the lookback $= 18.85$

Price of the lookback call $= 19.17$

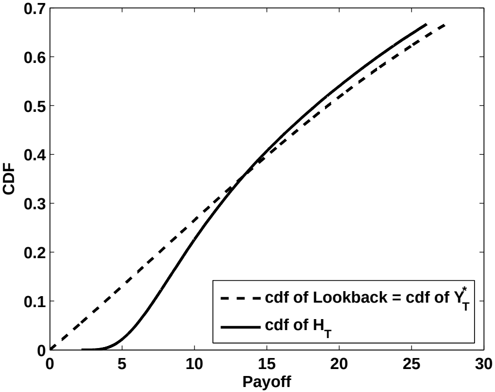
Example: the Lookback Option



With $\sigma = 20\%, \mu = 9\%, r = 5\% S_0 = 100,\ T = 1$ year, $K = 100$.

Comparison of the two payoffs

Example: the Lookback Option



With $\sigma = 20\%, \mu = 9\%, r = 5\% S_0 = 100,\ T = 1$ year, $K = 100$.

Comparison of the cdf of the two payoffs

Cost-Efficient Strategies in a Multidimensional Market

Multidimensional Case

Assume that

$$\frac{\mathrm{d}S^i(t)}{S^i(t)} = \mu_i \mathrm{d}t + \sigma_i \, \mathrm{d}B^i(t), \qquad i = 1, 2, \cdots, m.$$

where the m-dimensional vector $\mu^T = (\mu_1 \ \cdots \ \mu_m)$ is the drift with $\mu \neq r \, \mathbf{1}$, with $\mathbf{1}^T = (1 \ 1 \dots 1)$. Furthermore the processes $\left\{B^i(t), \ t \geqslant 0\right\}$ are correlated with ρ_{ij}:

$$\forall t, s \geqslant 0, \quad \rho_{ij} = \mathrm{Corr}\left(B^i(t), B^j(t+s)\right).$$

Furthermore, we also define a $(m \times m)$ positive definite matrix $\mathbf{\Sigma}$ as

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_m^2 \end{pmatrix},$$

where $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_i$. Note that $\sigma_{ij} = \sigma_{ji}$ and also that $\sigma_{ii} = \sigma_i^2$.

Multidimensional Case

It is then well-known that for $i = 1, \ldots, m$

$$S^i(t) = S^i(0) \exp(X^i(t)) \qquad t > 0,$$

where

$$X^i(t) = \left(\mu_i - \frac{1}{2}\sigma_i^2\right) t + \sigma_i \; B^i(t).$$

Also it is easily seen that the elements of the matrix $\mathbf{\Sigma}$ describe the covariances between the yearly logreturns $X^i(k) - X^i(k-1)$ of the m assets $(i = 1, 2, ..., m)$ and k is a positive integer.

Constant Mix Strategy

Recall that a strategy at time t is denoted by $\pi(t)^T = (\pi_1(t),\ \pi_2(t), \cdots,\ \pi_m(t))$ where $\pi_i(t)$ is the fraction of the wealth that is invested in risky asset i at time t. The residual, i.e. $1 - \sum_{i=1}^{n} \pi_i(t)$ is invested in the riskfree asset which grows at the constant continuously compounded interest rate r.

A constant portfolio $\pi(t) = \pi = (\pi_1,\ \pi_2, \cdots,\ \pi_m)^T$, where the fractions invested in the different assets remain constant over time is called a **constant mix strategy**.

It follows that under the $\mathbb{P}-$measure the dynamics of the price process $\{S^{\pi}(t),\ t \geqslant 0\}$ of the security that is constructed according to a non-zero vector π is given by

$$\frac{\mathrm{d}S^{\pi}(t)}{S(t)} = \left(\sum_{i=1}^{m} \pi_i\ (\mu_i - r) + r\right)\mathrm{d}t + \sum_{i=1}^{m} \pi_i\ \sigma_i \mathrm{d}B^i(t) \tag{5}$$

It is easy to see that it can be recast as

$$\frac{\mathrm{d}S^{\pi}(t)}{S(t)} = \mu(\pi)\,\mathrm{d}t + \sigma(\pi)\,\mathrm{d}B^{\pi}(t). \tag{6}$$

where $\{B^{\pi}(t),\ t \geqslant 0\}$ is a standard Brownian motion defined $B^{\pi}(t) = \frac{1}{\sqrt{\pi^T \cdot \mathbf{\Sigma} \cdot \pi}} \sum_{i=1}^{m} \pi_i\ \sigma_i\ B^i(t)$, with $\mu(\pi) = r + \pi^T \cdot (\mu - r\ \mathbf{1})$, and $\sigma^2(\pi) = \pi^T \cdot \mathbf{\Sigma} \cdot \pi$.

Theorem (State-price process for a Black-Scholes market)

The process $\{\xi(t),\ t \geqslant 0\}$ *with* $\xi(t) = e^{-rt} e^{-\frac{\theta_*^2}{2}t - \theta_* B^{\pi^*}(t)}$ *is the state-price process where* π^* *corresponds to the* **market portfolio**. *Then the state-price* $\xi(t)$ *can be written as an explicit function of the market portfolio price* $S^{\pi^*}(t)$:

$$\xi(t) = a \cdot \left(\frac{S^{\pi^*}(t)}{S^{\pi^*}(0)} \right)^{-b}, \tag{7}$$

where $a = \exp\left(\frac{\theta_*}{\sigma_*} \left(\mu_* - \frac{\sigma_*^2}{2} \right) t - \left(r + \frac{\theta_*^2}{2} \right) t \right)$ *and* $b = \frac{\theta_*}{\sigma_*}$.

Market Portfolio

Assume that $\mathbf{1}^T \cdot \mathbf{\Sigma}^{-1} \cdot (\mu - r\mathbf{1}) > 0$ and $\sigma > 0$. The unique solution of the following *mean-variance* optimization problem

$$\max_{\pi} \; \mu(\pi) \text{ subject to } \sigma(\pi) = \sigma, \tag{8}$$

is denoted by π^σ.Hence, for the appropriate choice of σ, there will be a unique mean-variance efficient portfolio π^σ fully invested in the risky assets ($\mathbf{1} \cdot \pi^\sigma = 1$). We call it the "market portfolio" and denote it by π^* :

$$\pi^* = \frac{\mathbf{\Sigma}^{-1} \cdot (\mu - r\mathbf{1})}{\mathbf{1}^T \cdot \mathbf{\Sigma}^{-1} \cdot (\mu - r\mathbf{1})},$$

where $\mu_* = r + \frac{(\mu - r\mathbf{1})^T \cdot \mathbf{\Sigma}^{-1} \cdot (\mu - r\mathbf{1})}{\mathbf{1}^T \cdot \mathbf{\Sigma}^{-1} \cdot (\mu - r\mathbf{1})}$, and $\sigma_*^2 = \frac{(\mu - r\mathbf{1})^T \cdot \mathbf{\Sigma}^{-1} \cdot (\mu - r\mathbf{1})}{(\mathbf{1}^T \cdot \mathbf{\Sigma}^{-1} \cdot (\mu - r\mathbf{1}))^2}$.

Cost-efficiency in Multidimensional model

Theorem (Given payoff $\Longrightarrow$ cheapest payoff with the same distribution)

Consider a payoff $H(T)$ with distribution function F (under P) which is assumed to be strictly increasing. Consider the payoff $Y^*(T)$ given as

$$Y^*(T) = F^{-1}\left(F_{S^{\pi^*}(T)}(S^{\pi^*}(T))\right).$$

Then the payoff $Y^*(T)$ will have F as its $\mathbb{P}-$distribution function. Further, it will hold that

$$c(Y^*(T)) \leqslant c(H(T)).$$

Finally, $Y^*(T)$ is the almost surely unique way to achieve the cheapest payoff with distribution F at time T.

Inefficiency of the Buy and Hold Strategy

The buy-and-hold strategy can be dominated by purchasing a series of power options with zero strike on the market portfolio. Consider

$$H(T) = W_0 \left(\sum_{i=1}^{m} \alpha_i e^{X^i(T)} + \left(1 - \sum_{i=1}^{m} \alpha_i\right) e^{rT} \right),$$

where $X^i(T) \sim N\left((\mu_i - \frac{\sigma_i^2}{2})T, \sigma_i^2 T\right)$ $(i = 1, 2, \ldots, m)$ are log-returns over the period $[0, T]$.

$H(T)$ is the payoff of a buy-and-hold strategy evaluated at the end of the investment horizon T, where at time $t = 0$, one invests α_i in the $i-$th risky asset and $1 - \sum_{i=1}^{m} \alpha_i$ in the riskless asset and where one does not trade afterwards.

Improving a buy and hold strategy

Since H_T is not a function of the market portfolio, it can be strictly improved

- using **cost-efficiency**.
  In general the distribution of $H(T)$ is not well-known but we could approximate the cost-efficient strategy by ways of Monte Carlo techniques. The dominating strategy is non-decreasing in the market portfolio.
- Using **conditioning** by the market portfolio at T.
- If the conditional payoff is not non-decreasing, it can be improved a second time by cost-efficiency techniques.

Improving a buy and hold strategy by conditioning

It is interesting to see that the conditional expectation to the market portfolio can be explicitly calculated. Hence let us consider next the path-independent payoff $H^*(T) := E_{\mathbb{P}}[H(T) \mid X^{\pi^*}(T)]$. Using properties of multivariate normal distributions we find that

$$H^*(T) = W_0 \cdot \sum_{i=1}^{m} \alpha_i e^{\mathbb{M}_i + \frac{1}{2}\mathbb{V}_i} + W_0 \cdot \left(1 - \sum_{i=1}^{m} \alpha_i\right) e^{rT}$$

where

$$\begin{cases} \mathbb{M}_i &= E_{\mathbb{P}}[X^i(T)] + \frac{Cov_{\mathbb{P}}[X^i(T), X^{\pi^*}(T)]}{Var_{\mathbb{P}}[X^{\pi^*}(T)]} \left(X^{\pi^*}(T) - E_{\mathbb{P}}[X^{\pi^*}(T)]\right) \\ \mathbb{V}_i &= Var_{\mathbb{P}}[X^i(T)] - \frac{Cov^2_{\mathbb{P}}[X^i(T), X^{\pi^*}(T)]}{Var_{\mathbb{P}}[X^{\pi^*}(T)]}. \end{cases}$$

Hence, using the notation introduced before we find that the buy-and-hold payoff $H(T)$ can be dominated by $H^*(T)$:

$$H^*(T) = W_0 \sum_{i=1}^{m} \alpha_i e^{b_i} \left(e^{X^{\pi^*}(T)}\right)^{c_i} + W_0 \left(1 - \sum_{i=1}^{m} \alpha_i\right) e^{rT},$$

with constants b_i and c_i $(i = 1, 2, \ldots, m)$ given by

$$b_i = \left(\left(\mu_i - \frac{\sigma_i^2}{2}\right) - \rho_i \frac{\sigma_i}{\sigma_*}\left(\mu_* - \frac{\sigma_*^2}{2}\right) + \frac{1}{2}(1-\rho_i^2)\sigma_i^2\right) T,$$

and

$$c_i = \rho_i \frac{\sigma_i}{\sigma_*},$$

respectively where $\rho_i =$Corr$_{\mathbb{P}}[X^{i}(T), X^{\pi^*}(T)]$. Note that b_i can also be determined from

$$b_i = \left(r - \rho_i \frac{\sigma_i}{\sigma_*}\left(r - \frac{\sigma_*^2}{2}\right) - \frac{1}{2}\rho_i^2\sigma_i^2\right) T.$$

Hence the optimal payoff $H^*(T)$ only depends on the drift parameters μ_i $(i = 1, 2, ..., m)$ through the weights π_i^* of the market portfolio indeed.

By substituting the component $e^{X^{\pi^*}(T)}$ by $\frac{S^{\pi^*}(T)}{S^{\pi^*}(0)}$ we see that $H^*(T)$ can also be interpreted as a weighted sum of power options with strike 0 written on the market portfolio.

As a special case, it is in general not optimal to hold an equally weighted portfolio with $1/m$ of the wealth in each asset. This portfolio is indeed dominated by the above payoff were $\alpha_i = 1/m$.

Example with 2 assets and a risky portfolio

Let us take $T=1$, $m=2$, $W_0=1$ and $\alpha_1+\alpha_2=1$.

Assume drift parameters $\mu_1=0.06$ and $\mu_2=0.10$, volatilities $\sigma_1=0.10$ and $\sigma_2=0.20$, and as correlation coefficient between the two risky assets we take $\rho=0.5$.

The risk free rate r is equal to 0.03. The market portfolio is determined by $\pi^*=(\frac{5}{9},\frac{4}{9})$. Moreover, the logreturn $X^{\pi^*}(1)$ has a drift $\mu_*=7/90$ and a volatility $\sigma_*=\frac{1}{30}\sqrt{\frac{43}{3}}$. We also find that $b_1=\frac{14}{1075}, c_1=\frac{27}{43}$, $b_2=-\frac{5}{258}$ and $c_2=\frac{63}{43}$.

Hence, the buy-and-hold payoff $H(1)$ can be dominated by the following payoff:

$$E_{\mathbb{P}}[H(1)\mid X^{\pi^*}(1)]=\alpha_1(e^{\frac{14}{1075}})(e^{X^{\pi^*}(1)})^{\frac{27}{43}}+\alpha_2(e^{-\frac{5}{258}})(e^{X^{\pi^*}(1)})^{\frac{63}{43}}$$

Example of a portfolio constituted with one stock

As a special case, in a multidimensional Black-Scholes market investing in a portfolio with a single stock is not optimal.

$$H(T) = W_0 \left(\alpha e^{X^1(T)} + (1-\alpha) e^{rT} \right)$$

There are several ways to dominate this strategy with payoff $H(T)$.

In our example, there are m assets and the wealth is invested in the first risky asset and in the bond.

Then it is possible to calculate explicitly the cost-efficient payoff as well as the conditional expectation.

Improving using Cost-efficiency

The cheapest way to achieve the distribution F of $H(T)$ is the payoff

$$Z^*(T) = F^{-1}\left(F_{S^{\pi^*}(T)}\left(S^{\pi^*}(T)\right)\right)$$

Since $\ln(S^{\pi^*}(T)) = \ln(S^{\pi^*}(0)) + \left(\mu_* - \frac{\sigma_*^2}{2}\right)T + \sigma_* B^{\pi^*}(T)$, one has

$$\forall x > 0, \quad F_{S^{\pi^*}(T)}(x) = \Phi\left(\frac{\ln\left(x/S^{\pi^*}(0)\right) - \left(\mu_* - \frac{\sigma_*^2}{2}\right)T}{\sigma_*\sqrt{T}}\right). \quad (9)$$

Let us calculate $F(x) = P(H(T) \leqslant x)$ and its inverse. There are two cases ($\alpha > 0$ and $\alpha < 0$). We report the case $\alpha > 0$.

Improving using Cost-efficiency

When $\alpha > 0$, the cdf of $H(T)$ is equal to

$$F(x) = \begin{cases} 0 & \text{if } x < W_0(1-\alpha)e^{rT}, \\ \Phi\left(\dfrac{\ln\left(\frac{x}{\alpha W_0} - \frac{(1-\alpha)e^{rT}}{\alpha}\right) - \left(\mu_1 - \frac{\sigma_1^2}{2}\right)T}{\sigma_1\sqrt{T}}\right) & \text{otherwise.} \end{cases}$$

Then, for $y \in (0,1)$, $y > 0$,

$$F^{-1}(y) = (1-\alpha)e^{rT}W_0 + \alpha W_0 \exp\left(\Phi^{-1}(y)\,\sigma_1\sqrt{T} + \left(\mu_1 - \frac{\sigma_1^2}{2}\right)T\right)$$

Improving by Conditioning

Using the expressions of c_1 and b_1,

$$H^*(T) = W_0\,\alpha\,e^{b_1}\,\left(e^{X^{\pi^*}(T)}\right)^{c_1} + W_0(1-\alpha)e^{rT}$$

dominates $H(T)$ for risk averse investors and has the same cost as $H(T)$.
This expression is valid for all $\alpha \in \mathbb{R}$.

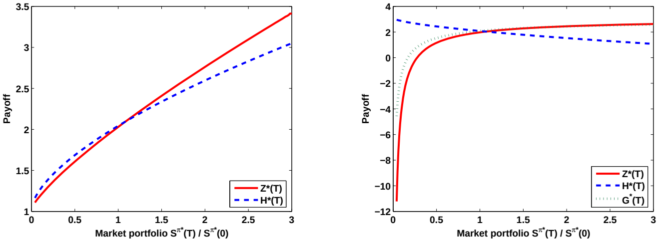
When $\alpha > 0$, since $c_1 > 0$ it is cost-efficient. In the case when $\alpha < 0$ it can be improved by the following payoff

$$G^*(T) = \alpha W_0 e^{b_1} \left(\frac{S^{\pi^*}(T)}{S^{\pi^*}(0)} \right)^{-c_1} + W_0 e^{rT}(1-\alpha) \tag{10}$$

which dominates in the sense of first stochastic dominance the payoff $H^*(T)$.

Comparison of the payoffs $Z^*(T)$, $G^*(T)$ and $H^*(T)$ for the buy-and-hold strategy with $\alpha = 0.5$ and $\alpha = -0.5$.

Parameters are $r = 3\%$, $\mu_1 = 0.06$, $\sigma_1 = 0.1$, $\mu_2 = 0.1$, $\sigma_2 = 0.2$, $\rho = 0.5$, $T = 1$ year, $W_0 = 2$. $\sigma_* = \frac{1}{30}\sqrt{\frac{43}{3}}$ and $\mu_* = \frac{7}{90}$. When $\alpha = 0.5$, we represent the two cost-efficient payoffs $H^*(T)$ and $Z^*(T)$ on the left panel. When $\alpha = -0.5$, we represent the cost-efficient payoff $G^*(T)$ and the inefficient payoff $H^*(T)$ on the right panel.



Lévy model with the Esscher transform (Vanduffel et al. (2008))

Any path-dependent financial derivative is inefficient. Indeed

$$\xi_t = e^{-rt}\frac{e^{h\frac{S_t}{S_0}}}{m_t(h)}$$

where $h \in \mathbb{R}$ is the unique real number such that $\xi_t S_t$ is a martingale under the physical measure.

$m_t(h)$ is a normalization factor such that $f_t^{(h)}$ defined by $f_t^{(h)}(x) = \frac{e^{hx} f_t(x)}{m_t(h)}$ is a density where f_t denotes the density of S_t under the physical measure.

To be cost-efficient, the payoff has to be written as:

$$X_T^* = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right) = F^{-1}\left(F_S\left(S_T\right)\right)$$

It is a European derivative written on the stock S_T (**and the payoff is increasing with** S_T when $h < 0$).

Extension to State-Dependent Strategies

Summary of Part I:
Optimal portfolio selection for law-invariant investors

Characterization of optimal investment strategies for an investor with **law-invariant preferences** and a **fixed investment horizon**

- Optimal strategies are "cost-efficient".
- **Cost-efficiency** $\Leftrightarrow$ Minimum correlation with the state-price process $\Leftrightarrow$ Anti-monotonicity
- Explicit representations of the **cheapest** and **most expensive** strategies to achieve a given distribution.
- In the **Black-Scholes** setting,
  - *Optimality* of strategies increasing in S_T.
  - *Suboptimality* of path-dependent contracts.

Explaining the Demand for Inefficient Payoffs

1. **Other sources of uncertainty:** Stochastic interest rates or stochastic volatility
2. **Transaction costs, frictions**
3. **Intermediary consumption.**
4. Often we are looking at an **isolated contract**: the theory applies to the complete portfolio.
5. **State-dependent needs**
   - **Background risk:**
     - Hedging a long position in the market index S_T (background risk) by purchasing a put option,
     - the background risk can be path-dependent.
   - **Stochastic benchmark or other constraints:** If the investor wants to outperform a given (stochastic) benchmark Γ such that:

$$P\left\{\omega \in \Omega \,/\, W_T(\omega) > \Gamma(\omega)\right\} \geqslant \alpha.$$

Part 2: Investment with State-Dependent Constraints

Problem considered so far

$$\min_{\{X_T \mid X_T \sim F\}} \mathbb{E}\left[\xi_T X_T\right].$$

A payoff that solves this problem is **cost-efficient**.

New Problem

$$\min_{\{Y_T \mid Y_T \sim F,\, \mathbb{S}\}} \mathbb{E}\left[\xi_T Y_T\right].$$

where $\mathbb{S}$ denotes a set of constraints. A payoff that solves this problem is called a $\mathbb{S}-$**constrained cost-efficient payoff**.

How to formulate "state-dependent constraints"?

Y_T and S_T have given distributions.

- The investor wants to ensure a **minimum** when the market falls
$$\mathbb{P}(Y_T > 100 \mid S_T < 95) = 0.8.$$
This provides some additional information on the joint distribution between Y_T and S_T $\Rightarrow$ information on the joint distribution of (ξ_T, Y_T) in the Black-Scholes framework.
- Y_T is **decreasing** in S_T when the stock S_T falls below some level (to justify the demand of a put option).
- Y_T is **independent** of S_T when S_T falls below some level.

All these constraints impose the strategy Y_T to pay out in given states of the world.

Formally

Goal: Find the **cheapest** possible payoff Y_T with the distribution F and which **satisfies additional constraints** of the form

$$\mathbb{P}(\xi_T \leqslant x, Y_T \leqslant y) = Q(F_{\xi_T}(x), F(y)),$$

with $x > 0, y \in \mathbb{R}$ and Q a given feasible function (for example a copula).

Each constraint gives information on the dependence between the state-price ξ_T and Y_T and is, for a given function Q, determined by the pair $(F_{\xi_T}(x), F(y))$.

Denote the finite or infinite set of all such constraints by $\mathbb{S}$.

Theorem (Case of one constraint)

Assume that there is only one constraint (a, b) *in* $\mathbb{S}$ *and let* $\vartheta := Q(a, b)$. *The* $\mathbb{S}-$*constrained cost-efficient payoff* $Y_T^\star$ *exists and is unique. It can be expressed as*

$$Y_T^\star = F^{-1}\left(G(F_{\xi_T}(\xi_T))\right), \tag{11}$$

where $G : [0,1] \to [0,1]$ *is defined as* $G(u) = \ell_u^{-1}(1)$ *and can be written as*

$$G(u) = \begin{cases} 1-u & \text{if } 0 \leqslant u \leqslant a-\vartheta, \\ a+b-\vartheta-u & \text{if } a-\vartheta < u \leqslant a, \\ 1+\vartheta-u & \text{if } a < u \leqslant 1+\vartheta-b, \\ 1-u & \text{if } 1+\vartheta-b < u \leqslant 1. \end{cases} \tag{12}$$

Example 1: $\mathbb{S}$ contains 1 constraint

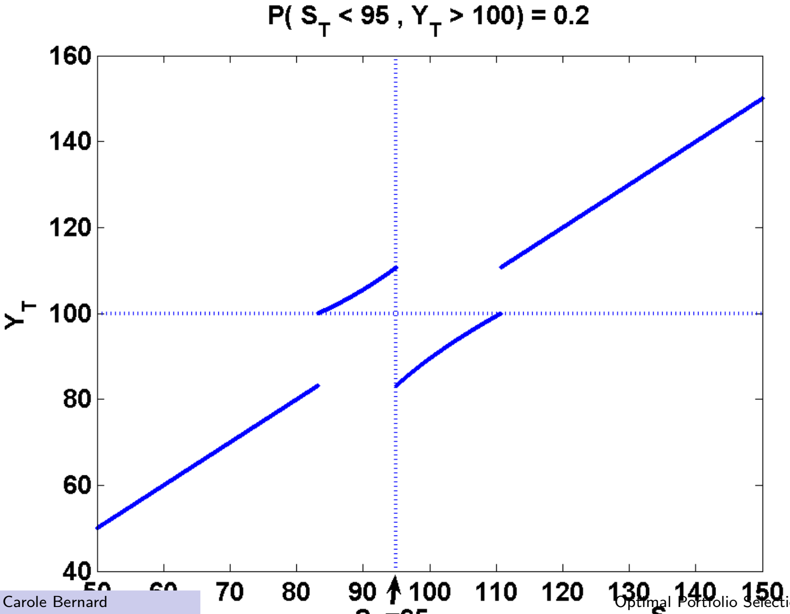
Assume a Black-Scholes market. We suppose that the investor is looking for the payoff Y_T such that $Y_T \sim F$ (where F is the cdf of S_T) and satisfies the following constraint

$$\mathbb{P}(S_T < 95,\ Y_T > 100) = 0.2.$$

The optimal strategy, where $a = 1 - F_{S_T}(95)$, $b = F_{S_T}(100)$ and $\vartheta = 0.2 - F_{S_T}(95) + F_{S_T}(100)$ is given by the previous theorem.

Its price is 100.2

Example: Illustration



Example 2: $\mathbb{S}$ is infinite

A cost-efficient strategy with the same distribution F as S_T but such that it is decreasing in S_T when $S_T \leqslant \ell$ is unique a.s. Its payoff is equal to

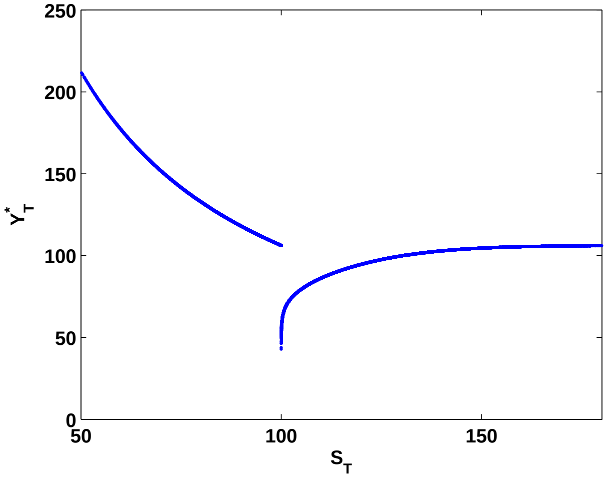
$$Y_T^\star = F^{-1}\left[G(F(S_T))\right],$$

where $G : [0,1] \to [0,1]$ is given by

$$G(u) = \begin{cases} 1-u & \text{if } \ 0 \leqslant u \leqslant F(\ell), \\ u - F(\ell) & \text{if } \ F(\ell) < u \leqslant 1. \end{cases}$$

The **constrained cost-efficient payoff** can be written as

$$Y_T^\star := F^{-1}\left[(1-F(S_T))\mathbb{1}_{S_T<\ell} + (F(S_T)-F(\ell))\,\mathbb{1}_{S_T\geqslant\ell}\right].$$



$Y_T^\star$ as a function of S_T. Parameters: $\ell = 100$, $S_0 = 100$, $\mu = 0.05$, $\sigma = 0.2$, $T = 1$ and $r = 0.03$. The price is 103.4.

"Tail Diversification" of Cost-Efficient Strategies

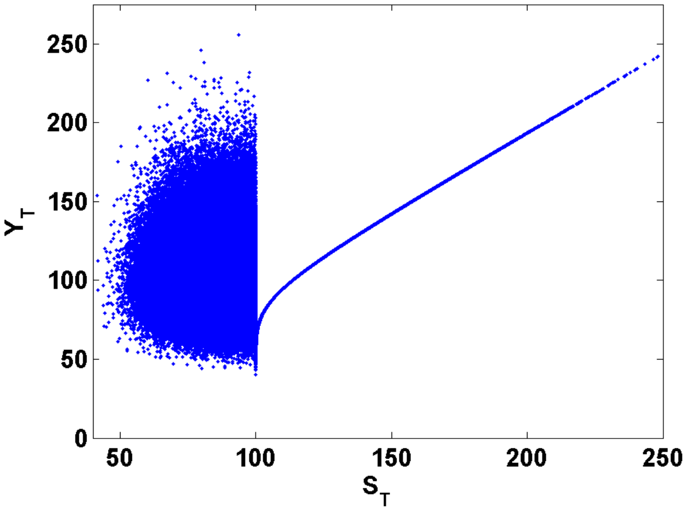
Theorem (Constraints on the tail)

In a one-dimensional Black-Scholes market, the cheapest path-dependent strategy with a cumulative distribution F but such that it is independent of $S_1(T)$ when $S_1(T) \leqslant q_\alpha$ can be constructed as

$$\begin{cases} F^{-1}\left(\frac{F_{S_1(T)}(S_1(T)) - F_{S_1(T)}(q_\alpha)}{1 - F_{S_1(T)}(q_\alpha)}\right) & \text{when} \quad S_1(T) > q_\alpha \\ F^{-1}\left(\Phi\left(\frac{\ln\left(\frac{S_1(t)}{(S_1(T))^{t/T}}\right) - \left(1 - \frac{t}{T}\right)\ln(S_1(0))}{\sigma_1\sqrt{t - \frac{t^2}{T}}}\right)\right) & \text{when} \quad S_1(T) \leqslant q_\alpha \end{cases}$$

where $t \in (0, T)$ can be chosen freely.

(**No uniqueness and path-dependent optimum**).



10,000 realizations of $Y_T^\star$ as a function of S_T where $\ell = 100$, $S_0 = 100$, $\mu = 0.05$, $\sigma = 0.2$, $T = 1$, $r = 0.03$ and $t = T/2$. Its price is 101.1

Conclusion

- Characterization of cost-efficient strategies.
- **Path-dependent strategies are never optimal in the Black and Scholes model** for investors with law-invariant preferences.
- Optimal investment choice under state-dependent constraints. **In the presence of state-dependent constraints, optimal strategies**
  - **are not always non-decreasing with the stock price S_T.**
  - **are not anymore unique and could be path-dependent.**

Further Research Directions / Work in Progress (1/2)

- Extension to the presence of **stochastic interest rates** (work in progress with Jit Seng Chen and Phelim Boyle).
- Application to executive compensation (work in progress with Jit Seng Chen and Phelim Boyle).
- Robustness of the optimum. Characterization of cost-efficient strategies in the presence of uncertainty (joint work with Steven Vanduffel)
  1. Uncertainty on the state-price process (incompleteness of the market).
  2. Uncertainty on the cdf F.

Further Research Directions / Work in Progress (2/2)

- Using cost-efficiency to derive **bounds for insurance prices** derived from indifference utility pricing (working paper on "Bounds for Insurance Prices" with Steven Vanduffel) and more generally application to utility indifference pricing in incomplete market.
- Further extend the work on state-dependent constraints:
  1. Solve with **expectations constraints** between ξ_T and X_T.
  $$\mathbb{E}[g_i(\xi_T, X_T)] \in I_i$$
  where I_i is an interval, possibly reduced to a single value.
  2. Solve with the probability constraint of outperforming a benchmark
  $$\mathbb{P}(X_T > h(S_T)) \geqslant \varepsilon$$
  3. Extend the literature on optimal portfolio selection in specific models under state-dependent constraints.

Do not hesitate to contact me to get updated working papers!

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