Deforming submanifolds while preserving the Gauss map

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IMPA

THE CHRISTOFFEL-SAMUEL PROBLEM:

TO WHAT EXTENT IS A SUBMANIFOLD OF EUCLIDEAN SPACE DETERMINED BY ITS METRIC OR CONFORMAL STRUCTURE AND ITS GAUSS MAP?

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- To solve the problem one needs to find “all exceptions”, that is, describe all submanifolds of Euclidean space that admit isometric or conformal deformations preserving the Gauss map.
- Basically, this is a problem of local nature.

The Christoffel-Samuel Problem

The problem for surfaces in $\mathbb{R}^3$ was solved by Christoffel who found all local exceptions.

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The problem for submanifolds of any dimension and codimension was studied by the eminent algebraic geometer Pierre Samuel, a second generation member of the Bourbaki group.

P. Samuel,
On conformal correspondence of Surfaces and Manifolds.
American J. Math. 1947.

The Christoffel problem

Correspondance conforme de deux surfaces a plans tangents paralleles. Ann. Univ. Lyon, 1942 [The author's name was omitted in the journal.]

Let S and S_0 be parallel surfaces; that is, S and S_0 are surfaces such that the normals to S and S_0 at corresponding points are parallel. The author studies directly the classical problem of determining conditions under which the parallel correspondence is also conformal. Cases of direct conformality and inverse conformality are treated separately.

In the first case the author indicates that the only solutions are minimal surfaces, and shows that any two minimal surfaces can be put in the desired representation.

In the second case, it is shown that the class of solutions is the class of isothermic surfaces; that is, the class of surfaces for which the lines of curvature form an isothermic system. When the first surface is given, the second is determined to within a homothetic transformation.

Reviewed by E. F. Beckenbach

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- Examples of isothermic surfaces are cylinders, surfaces of revolution, cones, quadrics, surfaces of constant mean curvature and the images of such surfaces by any Moebius transformation of $\mathbb{R}^3$.
- The notion of isothermic surface extends to surfaces in $\mathbb{R}^N$ with flat normal bundle.

Definitions

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The Gauss map $p \in M^n \mapsto G_{N,n}$ of f into the Grassmann manifold $G_{N,n}$ of unoriented n-planes in $\mathbb{R}^N$ assigns to $p \in M^n$ the tangent space $f_* T_p M$.

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Definition

That an immersion $g\colon M^n \to \mathbb{R}^N$ has the same Gauss map as f means

$$f_* T_pM = g_* T_pM \;\text{ for all }\; p \in M^n$$

up to a rigid motion of $\mathbb{R}^N$.

Samuel's problem

Question

Under which conditions two immersions $f, g\colon M^n \to \mathbb{R}^N$ that induced conformal metrics, i.e., there exists $\varphi \in C^\infty(M)$ such that the induced metrics satisfy that

$$\langle\ ,\ \rangle_g = e^{2\varphi}\langle\ ,\ \rangle_f,$$

have the same Gauss map?

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Samuel's problem

Classify all pairs of immersions $f, g\colon M^n \to \mathbb{R}^N$ that have the same Gauss map and induce conformal metrics on M^n.

Of course, the classification is up to homothety and rigid motion.

A long history

- Surfaces in $\mathbb{R}^3$ – E. Christoffel – 1867.
- Surfaces in $\mathbb{R}^3$ – P. Samuel – 1942.
- The general case – P. Samuel – 1947.
- Surfaces in $\mathbb{R}^N$ – D. Hoffman and R. Osserman –1982.
- Surfaces in $\mathbb{R}^N$ – B. Palmer – 1988.
- Surfaces in $\mathbb{R}^N$ – E. Vergasta – 1992.
- Surfaces in $\mathbb{R}^4$ – V. Gor'kavyi – 2003.
- Isometric case in $\mathbb{R}^N$– M. Dajczer and D. Gromoll – 1985.
- Conformal hypersurfaces in $\mathbb{R}^N$– M. Dajczer and E. Vergasta – 1995.
- Conformal case in $\mathbb{R}^N$– M. Dajczer and R. Tojeiro.

Real Kaehler submanifolds

By a Real Kaehler submanifold we mean an isometric immersion

$$f\colon M^{2n} \to \mathbb{R}^N$$

of a Kaehler manifold (M^{2n}, J) where $2n$ stands for the real dimension.

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We have that

$$f \text{ is minimal} \iff f \text{ is pseudo-holomorphic},$$

i.e., the second fundamental form α_f satisfies

$$\alpha_f(X, JY) = \alpha_f(JX, Y).$$

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Equivalently, the restriction of f to any holomorphic curve in M^{2n} is a minimal surface in $\mathbb{R}^N$. Thus, it is also called pluriminimal.

Minimal real Kaehler submanifolds

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If M^{2n} is simply connected, there is an isometric immersion

$$f_\theta \colon M^{2n} \to \mathbb{R}^N$$

such that

$$f_{\theta *} = f_* \circ J_\theta,$$

and hence f_θ has the same Gauss map as f.

Minimal real Kaehler submanifolds

The set $\{f_\theta : \theta \in [0, 2\pi)\}$ is called the associated family of f where

$$f_\theta(x) = \int_{x_0}^{x} f_* \circ J_\theta$$

and $x_0 \in M^{2n}$.

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The second fundamental form α_θ of f_θ is given by

$$\alpha_\theta(X, Y) = \alpha_f(J_\theta X, Y)$$

and thus the f_θ's are also minimal.

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There is a holomorphic representative $F\colon M^{2n} \to \mathbb{C}^N$ given by

$$F = \frac{1}{\sqrt{2}}\big(f \oplus f_{\pi/2}\big).$$

Minimal real Kaehler submanifolds

For minimal real Kaehler submanifold there is a Weierstrass type representation.

C. Arezzo, G Pirola and M. Solci,
The Weierstrass representation for pluriminimal submanifolds.
Hokkaido Math. J. 2004.

In the following result a subspace $V \subset \mathbb{C}^N$ is *isotropic* if

$$u.v = 0 \text{ for all } u, v \in V$$

where "." denotes the standard symmetric inner product in $\mathbb{C}^N$.

Minimal real Kaehler submanifolds

Proposition. Let $f\colon M^{2n} \to \mathbb{R}^N$ be a minimal real Kaehler submanifold. Given a simply connected coordinate chart U of M^{2n} with $z_j = x_j + iy_j$, define the maps $\varphi_j\colon U \to \mathbb{C}^N$, by

$$\varphi_j = \sqrt{2}\, f_{z_j} = \frac{1}{\sqrt{2}}\left(f_{x_j} - i f_{y_j}\right), \ \ 1 \le j \le n.$$

Then the φ_j satisfy the following conditions:

(i) The vectors $(\varphi_1, \ldots, \varphi_n)$ are linearly independent at any point in U,

(ii) The functions φ_j are holomorphic,

(iii) The subspace $\operatorname{span}\{\varphi_1, \ldots, \varphi_n\}$ of $\mathbb{C}^N$ is isotropic,

(iv) The integrability conditions $\partial\varphi_j/\partial z_k = \partial\varphi_k/\partial z_j$, $1 \le j, k \le n$.

Furthermore, if $F\colon U \to \mathbb{C}^N$ is the holomorphic representative of f, then

$$\varphi_j = F_{z_j}, \ \ 1 \le j \le n. \qquad (*)$$

Conversely, let U be a simply connected open subset of $\mathbb{C}^N$ and

$$\varphi_1, \ldots, \varphi_n \colon U \to \mathbb{C}^N$$

be maps that satisfy conditions (i) through (iv). Then, there is a holomorphic map $F \colon U \to \mathbb{C}^N$ such that $(*)$ is satisfied, and if

$$f \colon M^{2n} \to \mathbb{R}^N$$

is defined by

$$f = \sqrt{2} Re\,(F),$$

then $M^{2n} = (U, f^*\langle\,,\,\rangle)$ is a Kaehler manifold and f is a minimal real Kaehler submanifold whose holomorphic representative is F.

The isometric case

THEOREM

Minimal real Kaehler submanifolds are the only irreducible ones that admit local isometric deformations preserving the Gauss map.

M. Dajczer and D. Gromoll,
Real Kaehler submanifolds and uniqueness of the Gauss map.
J. Diff. Geom. 1985.

Hypersurfaces

THEOREM

Hypersurfaces $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, admitting conformal non-isometric deformations are the hypersurfaces of rotation over either plane curves or minimal surfaces in $\mathbb{R}^3$.

M. Dajczer and E. Vergasta,
Conformal Hypersurfaces with the Same Gauss Map.
Trans. Amer. Math. Soc. 1995

Real Kaehler cones

A submanifold is a cone if it admits a foliation by straight lines through a common point of $\mathbb{R}^N$.

Proposition. A simply connected minimal real Kaehler submanifold $f\colon M^{2n} \to \mathbb{R}^N$ is a cone iff it is the real part of a holomorphic isometric immersion $F\colon M^{2n} \to \mathbb{C}^N$ obtained lifting a holomorphic immersion

$$\bar{F}\colon M^{2n-1} \to \mathbb{CP}^{N-1}$$

by the Hopf projection $\pi\colon \mathbb{C}^N \to \mathbb{CP}^{N-1}$.

- There is a complete classification for $n = 2$.

Example 1. Let $f\colon\ M^n \to \mathbb{R}^N$ be a minimal real Kaehler cone and let f_θ be a member of its associated family. Consider an inversion $\mathcal{I}$ with respect to a sphere centered at the vertex of f_θ, and set

$$g = \mathcal{I} \circ f_\theta.$$

Then g is conformal to f with the same Gauss map.

An inversion with respect to the origin $\mathcal{I}\colon \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$\mathcal{I}(X) = \frac{X}{\|X\|^2}.$$

Warped product of two immersions

Consider $(M, \langle\ ,\ \rangle_M)$, $(N, \langle\ ,\ \rangle_N)$ and $\rho \in C^\infty(M)$ with $\rho > 0$.

Warped product

$$M \times_\rho N = (M \times N, \langle\ ,\ \rangle) \text{ where } \langle\ ,\ \rangle = \langle\ ,\ \rangle_M + \rho^2 \langle\ ,\ \rangle_N.$$

Consider the map

$$\psi_1 \colon R^m_+ \times_{x_m} \mathbb{S}^{N-m} \to \mathbb{R}^N$$

given by

$$\psi_1(x, y) = (x_1, \ldots, x_{m-1}, x_m y)$$

and immersions $f_1 \colon M_1 \to \mathbb{R}^m_+$ and $f_2 \colon M_2 \to \mathbb{S}^{N-m}$.

Warped product of two immersions

$$f \colon M_1 \times M_2 \to \mathbb{R}^N \text{ given by } f = \psi_1(f_1, f_2).$$

Example of warped product

Example 2. Let $\beta, \gamma\colon I \subset \mathbb{R} \to \mathbb{R}^m_+$ be two regular curves and let $h\colon L^{n-1} \to \mathbb{S}^{N-m}$ be an isometric immersion. Let

$$f, g\colon M^n = I \times L^{n-1} \to \mathbb{R}^N$$

be given by

$$f = \psi_1(\beta, h) \;\text{ and }\; g = \psi_1(\gamma, h).$$

Then f, g are conformal with the same Gauss map if and only if

$$\gamma = C \int \frac{\beta'(\tau)}{\beta_m^2(\tau)} d\tau$$

for a constant $C < 0$.

Example 3. Let $f_0, g_0 \colon N^2 \to \mathbb{R}^m_+$ be two minimal surfaces and let $h \colon L^{n-2} \to \mathbb{S}^{N-m}$ be an isometric immersions. Let

$$f, g \colon M^n = N^2 \times L^{n-2} \to \mathbb{R}^N$$

be given by

$$f = \psi_1(f_0, h) \text{ and } g = \psi_1(g_0, h).$$

Then f, g are conformal with the same Gauss map iff

$$f_0 = (a_1, \ldots, a_{m-1}, a)$$

is parametrized by an isothermal coordinate z and

$$g_0 = -\int \frac{1}{A^2}(f_0)_z dz$$

where $A = a + i\tilde{a}$ is holomorphic.

Warped product of three immersions

Consider the map

$$\psi_2 \colon R^m_{++} \times_{x_{m-1}} \mathbb{S}^{m_1} \times_{x_m} \mathbb{S}^{m_2} \to \mathbb{R}^N, \quad N = m + m_1 + m_2$$

given by

$$\psi_2(x, y_1, y_2) = (x_1, \ldots, x_{m-2}, x_{m-1}y_1, x_m y_2)$$

and immersions $f_1 \colon M_1 \to \mathbb{R}^m_+$, $f_2 \colon M_2 \to \mathbb{S}^{m_1}$ and $f_3 \colon M_3 \to \mathbb{S}^{m_2}$

Warped product of three immersions

$$f \colon M_1 \times M_2 \times M_3 \to \mathbb{R}^N \text{ given by } f = \psi_2(f_1, f_2, f_3).$$

Example 4. Let $f_0, g_0 \colon N^2 \to \mathbb{R}^m_+$ be two minimal surfaces and let $h_j \colon L^{s_j} \to \mathbb{S}^{s_j}$, $j = 1, 2$, be isometric immersions. Let

$$f, g \colon M^n = N^2 \times L^{s_1} \times L^{s_2} \to \mathbb{R}^N$$

be given by

$$f = \psi_1(f_0, h_1, h_2) \text{ and } g = \psi_1(g_0, h_1, h_2).$$

Then f, g are conformal with the same Gauss map iff f_0, g_0 satisfy:

(1) If $f_0 = (a_1, \ldots, a_{m-2}, a, \tilde{a})$, $a, \tilde{a} > 0$ in isothermal coordinates with $A = a + i\tilde{a}$ holomorphic and if $g_0 = (b_1, \ldots, b_{m-2}, b, \tilde{b})$, $b, \tilde{b} > 0$ in isothermal coordinates with $B = b + i\tilde{b}$ holomorphic

(2) If $\mathcal{R}$ is a reflexion with respect to the hyperplane orthogonal to e_m then $\mathcal{R} \circ g_0 = (b_1, \ldots, b_{m-2}, b, -\tilde{b})$ is such that

$$\mathcal{R} \circ g_0 = \int \frac{1}{A^2}(f_0)_z dz.$$

The general case

THEOREM

Any nontrivial pair $f, g\colon M^n \to \mathbb{R}^N$, $n \geq 3$, of conformal immersions with the same Gauss map is as in one of the examples already given.

M. Dajczer and R. Tojeiro,
A complete solution of P. Samuel's problem.
Preprint.

Same Gauss map

That an immersion $g\colon M^n \to \mathbb{R}^N$ has the same Gauss map as f is equivalent to the existence of a tensor $\Phi \in C^\infty(T^*M \otimes TM)$ so that

$$g_* = f_* \circ \Phi.$$

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Proposition. The following holds:

(i) Φ is a Codazzi tensor, i.e.,

$$(\nabla_X \Phi)Y = (\nabla_Y \Phi)X \quad \text{for all } \; X, Y \in TM.$$

(ii) The second fundamental form α_f of f commutes with Φ, i.e.,

$$\alpha_f(X, \Phi Y) = \alpha_f(\Phi X, Y) \quad \text{for all } \; X, Y \in TM.$$

Conversely...

Same Gauss map

Assume, in addition, that f and g are conformal. Then,

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Proposition. The pair (T, φ) satisfies the differential equation

$$(\nabla_X T)Y = \langle Y, \nabla\varphi\rangle TX - \langle X, Y\rangle T\nabla\varphi \ \text{ for all } X, Y \in TM.$$

Moreover,

$$\alpha_f(TX, Y) = \alpha_f(X, TY) \ \text{ for all } X, Y \in TM.$$

Conversely, for a given isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ of a simply connected Riemannian manifold, any pair (T, φ) satisfying the above equations gives rise to a conformal immersion $g\colon M^n \to \mathbb{R}^{n+p}$ with the same Gauss map.

The proof

- Complexify the tangent bundle TM.
- Decompose TM as the orthogonal sum of proper subspaces

$$TM \otimes \mathbb{C} = L_+ \oplus L_- \oplus L_{\mathbb{C}}.$$

- Analyze all possible cases.
- Use the Theorems of Hiepko (1979) and Nölker (1996).

Hiepko's Theorem

Theorem. Let M^n be a Riemannian manifold and let

$$TM = L \oplus S_1 \oplus \cdots \oplus S_k$$

be an orthogonal decomposition into nontrivial vector subbundles such that $S_1, \ldots, S_k$ are spherical and $S_1^\perp, \ldots, S_k^\perp$ totally geodesic. Then, there is locally a decomposition of M^n into a Riemannian warped product

$$M^n = N_0 \times_{\varrho_1} N_1 \times \cdots \times_{\varrho_k} N_k$$

such that $L = TN_0$ and $S_i = TN_i$ for $1 \leq i \leq k$.

The proof

Nölker's Theorem

Theorem. Let $f\colon\ M^n \to \mathbb{R}^N$ be an isometric immersion of a warped product manifold $M^n = M_0 \times_{\varrho_1} M_1 \times \cdots \times_{\varrho_k} M_k$ whose sff satisfies

$$\alpha(X_i, X_j) = 0 \quad \text{for all} \;\; X_i \in TM_i,\; X_j \in TMj, \;\; i \neq j.$$

Given $\bar{p} = (\bar{p}_0, \ldots, \bar{p}_k) \in M^n$, set $f_i = f \circ \tau_i^{\bar{p}}\colon\ M_i \to \mathbb{R}^N$ for $\tau_i^{\bar{p}}(p_i) = (\bar{p}_0, \ldots, p_i, \ldots, \bar{p}_k)$, and let S_i be the spherical hull of f_i, $1 \leq i \leq k$. Then f_0 is an isometric immersion, f_i is a homothetical immersion with homothety factor $\rho_i(\bar{p}_0)$ and $(f(\bar{p}); S_1, \ldots, S_k)$ determines a warped product representation

$$\Phi\colon\ S_0 \times_{\sigma_1} S_1 \times \cdots \times_{\sigma_k} S_k \to \mathbb{R}^N$$

such that $f_0(M_0) \subset S_0$, $\rho_i = \rho_i(\bar{p}_0)(\sigma_i \circ f_0)$ and

$$f = \Phi \circ (f_0 \times \cdots \times f_k),$$

where f_i is regarded as a map into S_i for $1 \leq i \leq k$.