Deforming submanifolds while preserving the Gauss map

Marcos Dajczer

IMPA
The Christoffel–Samuel problem

THE CHRISTOFFEL-SAMUEL PROBLEM:

TO WHAT EXTENT IS A SUBMANIFOLD OF EUCLIDEAN SPACE DETERMINED BY ITS METRIC OR CONFORMAL STRUCTURE AND ITS GAUSS MAP?
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- To solve the problem one needs to find “all exceptions”, that is, describe all submanifolds of Euclidean space that admit isometric or conformal deformations preserving the Gauss map.

- Basically, this is a problem of local nature.
The problem for surfaces in $\mathbb{R}^3$ was solved by Christoffel who found all local exceptions.

E. Christoffel, 
_Ueber einige allgemeine Eigenschaften der Minimumsflächen._ Crelle’s J. 1867.
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The problem for submanifolds of any dimension and codimension was studied by the eminent algebraic geometer Pierre Samuel, a second generation member of the Bourbaki group.

P. Samuel, 
*On conformal correspondence of Surfaces and Manifolds.* 
Correspondance conforme de deux surfaces a plans tangents paralleles.
Ann. Univ. Lyon, 1942 [The author’s name was omitted in the journal.]

Let $S$ and $S_0$ be parallel surfaces; that is, $S$ and $S_0$ are surfaces such that the normals to $S$ and $S_0$ at corresponding points are parallel. The author studies directly the classical problem of determining conditions under which the parallel correspondence is also conformal. Cases of direct conformality and inverse conformality are treated separately.

In the first case the author indicates that the only solutions are minimal surfaces, and shows that any two minimal surfaces can be put in the desired representation.

In the second case, it is shown that the class of solutions is the class of isothermic surfaces; that is, the class of surfaces for which the lines of curvature form an isothermic system. When the first surface is given, the second is determined to within a homothetic transformation.

Reviewed by E. F. Beckenbach
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Christoffel’s problem

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- Any **pair of minimal surfaces** is (locally) a solution of the problem because the Gauss map of a minimal surface is conformal.
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- Examples of isothermic surfaces are cylinders, surfaces of revolution, cones, quadrics, surfaces of constant mean curvature and the images of such surfaces by any Moebius transformation of $\mathbb{R}^3$. 
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The notion of isothermic surface extends to surfaces in $\mathbb{R}^N$ with flat normal bundle.
The map $f : M^n \to \mathbb{R}^N$ denotes an immersion of an $n$-dimensional connected differentiable manifold into Euclidean space.

**Notation**

The Gauss map $p \in M^n \mapsto G^{N,n}$ of $f$ into the Grassmann manifold $G^{N,n}$ of unoriented $n$-planes in $\mathbb{R}^N$ assigns to $p \in M^n$ the tangent space $f^* T_p M^n$.

That an immersion $g : M^n \to \mathbb{R}^N$ has the same Gauss map as $f$ means $f^* T_p M^n = g^* T_p M^n$ for all $p \in M^n$ up to a rigid motion of $\mathbb{R}^N$. 

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Definitions

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The map $f : M^n \rightarrow \mathbb{R}^N$ denotes an immersion of an $n$-dimensional connected differentiable manifold into Euclidean space.

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The Gauss map $p \in M^n \mapsto G_{N,n}$ of $f$ into the Grassmann manifold $G_{N,n}$ of unoriented $n$-planes in $\mathbb{R}^N$ assigns to $p \in M^n$ the tangent space $f_* T_p M$. 
Definitions

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**Definition**

That an immersion \( g : M^n \to \mathbb{R}^N \) has the same Gauss map as \( f \) means

\[
f_* T_p M = g_* T_p M \quad \text{for all} \quad p \in M^n
\]

up to a rigid motion of \( \mathbb{R}^N \).
**Question**

Under which conditions two immersions $f, g : M^n \to \mathbb{R}^N$ that induced conformal metrics, i.e., there exists $\varphi \in C^\infty(M)$ such that the induced metrics satisfy that

$$\langle \cdot, \cdot \rangle_g = e^{2\varphi} \langle \cdot, \cdot \rangle_f,$$

have the same Gauss map?

The function $e^\varphi$ is the conformal factor of $\langle \cdot, \cdot \rangle_g$ with respect to $\langle \cdot, \cdot \rangle_f$. 

**Samuel’s problem**

Classify all pairs of immersions $f, g : M^n \to \mathbb{R}^N$ that have the same Gauss map and induce conformal metrics on $M^n$. Of course, the classification is up to homothety and rigid motion.
Samuel’s problem

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The Christoffel-Samuel problem
Surfaces in $\mathbb{R}^3$ – E. Christoffel – 1867.

Surfaces in $\mathbb{R}^3$ – P. Samuel – 1942.


Surfaces in $\mathbb{R}^4$ – V. Gor’kavyi – 2003.


Conformal case in $\mathbb{R}^N$ – M. Dajczer and R. Tojeiro.
By a **Real Kaehler submanifold** we mean an isometric immersion

\[ f : M^{2n} \rightarrow \mathbb{R}^N \]

of a Kaehler manifold \((M^{2n}, J)\) where \(2n\) stands for the real dimension.
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We have that

\[ f \text{ is minimal } \iff f \text{ is pseudo-holomorphic,} \]

i.e., the second fundamental form \(\alpha_f\) satisfies

\[ \alpha_f(X, JY) = \alpha_f(JX, Y). \]
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Equivalently, the restriction of \(f\) to any holomorphic curve in \(M^{2n}\) is a minimal surface in \(\mathbb{R}^N\). Thus, it is also called **pluriminimal**.
For each $\theta \in [0, 2\pi)$ the tensor

$$J_\theta = \cos \theta I + \sin \theta J$$

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Minimal real Kaehler submanifolds

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is also parallel.

If $M^{2n}$ is simply connected, there is an isometric immersion

$$f_\theta : M^{2n} \rightarrow \mathbb{R}^N$$

such that

$$f_{\theta *} = f_* \circ J_\theta,$$

and hence $f_\theta$ has the same Gauss map as $f$. 

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The Christoffel-Samuel problem
The set \( \{ f_\theta : \theta \in [0, 2\pi) \} \) is called the associated family of \( f \) where

\[
f_\theta(x) = \int_{x_0}^{x} f_* \circ J_\theta \]

and \( x_0 \in M^{2n} \).
The set \( \{ f_\theta : \theta \in [0, 2\pi) \} \) is called the associated family of \( f \) where
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and \( x_0 \in M^{2n} \).

The second fundamental form \( \alpha_\theta \) of \( f_\theta \) is given by
\[
\alpha_\theta(X, Y) = \alpha_f(J_\theta X, Y)
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and thus the \( f_\theta \)'s are also minimal.
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There is a **holomorphic representative** \( F : M^{2n} \to \mathbb{C}^N \) given by

\[
F = \frac{1}{\sqrt{2}} (f \oplus f_{\pi/2}).
\]
For minimal real Kaehler submanifold there is a Weierstrass type representation.

C. Arezzo, G Pirola and M. Solci,
*The Weierstrass representation for pluriminimal submanifolds.*

In the following result a subspace $V \subset \mathbb{C}^N$ is *isotropic* if

$$u \cdot v = 0 \text{ for all } u, v \in V$$

where “.” denotes the standard symmetric inner product in $\mathbb{C}^N$. 
Proposition. Let \( f : M^{2n} \to \mathbb{R}^N \) be a minimal real Kaehler submanifold. Given a simply connected coordinate chart \( U \) of \( M^{2n} \) with \( z_j = x_j + iy_j \), define the maps \( \varphi_j : U \to \mathbb{C}^N \), by

\[
\varphi_j = \sqrt{2} f_{z_j} = \frac{1}{\sqrt{2}} (f_{x_j} - if_{y_j}), \quad 1 \leq j \leq n.
\]

Then the \( \varphi_j \) satisfy the following conditions:

(i) The vectors \((\varphi_1, \ldots, \varphi_n)\) are linearly independent at any point in \( U \),

(ii) The functions \( \varphi_j \) are holomorphic,

(iii) The subspace span\(\{\varphi_1, \ldots, \varphi_n\}\) of \( \mathbb{C}^N \) is isotropic,

(iv) The integrability conditions \( \partial \varphi_j / \partial z_k = \partial \varphi_k / \partial z_j, \quad 1 \leq j, k \leq n \).

Furthermore, if \( F : U \to \mathbb{C}^N \) is the holomorphic representative of \( f \), then

\[
\varphi_j = F_{z_j}, \quad 1 \leq j \leq n. \tag{\ast}
\]
Conversely, let $U$ be a simply connected open subset of $\mathbb{C}^N$ and

$$\varphi_1, \ldots, \varphi_n : U \to \mathbb{C}^N$$

be maps that satisfy conditions (i) through (iv). Then, there is a holomorphic map $F : U \to \mathbb{C}^N$ such that $(\ast)$ is satisfied, and if

$$f : M^{2n} \to \mathbb{R}^N$$

is defined by

$$f = \sqrt{2} \text{Re} (F),$$

then $M^{2n} = (U, f^* \langle , \rangle)$ is a Kaehler manifold and $f$ is a minimal real Kaehler submanifold whose holomorphic representative is $F$. 
Minimal real Kaehler submanifolds are the only irreducible ones that admit local isometric deformations preserving the Gauss map.

M. Dajczer and D. Gromoll, 
Real Kaehler submanifolds and uniqueness of the Gauss map. 
Hypersurfaces $f : M^n \rightarrow \mathbb{R}^{n+1}$, $n \geq 3$, admitting conformal non-isometric deformations are the hypersurfaces of rotation over either plane curves or minimal surfaces in $\mathbb{R}^3$.

M. Dajczer and E. Vergasta,
*Conformal Hypersurfaces with the Same Gauss Map.*
A submanifold is a cone if it admits a foliation by straight lines through a common point of $\mathbb{R}^N$.

**Proposition.** A simply connected minimal real Kaehler submanifold $f : M^{2n} \to \mathbb{R}^N$ is a cone iff it is the real part of a holomorphic isometric immersion $F : M^{2n} \to \mathbb{C}^N$ obtained lifting a holomorphic immersion

$$\bar{F} : M^{2n-1} \to \mathbb{C}\mathbb{P}^{N-1}$$

by the Hopf projection $\pi : \mathbb{C}^N \to \mathbb{C}\mathbb{P}^{N-1}$.

- There is a complete classification for $n = 2$. 
**Example 1.** Let \( f : \mathcal{M}^n \rightarrow \mathbb{R}^N \) be a minimal real Kaehler cone and let \( f_\theta \) be a member of its associated family. Consider an inversion \( \mathcal{I} \) with respect to a sphere centered at the vertex of \( f_\theta \), and set

\[
g = \mathcal{I} \circ f_\theta.
\]

Then \( g \) is conformal to \( f \) with the same Gauss map.

An inversion with respect to the origin \( \mathcal{I} : \mathbb{R}^N \rightarrow \mathbb{R}^N \) is given by

\[
\mathcal{I}(X) = \frac{X}{\|X\|^2}.
\]
Consider \((M, \langle , \rangle_M), (N, \langle , \rangle_N)\) and \(\rho \in C^\infty(M)\) with \(\rho > 0\).

**Warped product**

\[ M \times_\rho N = (M \times N, \langle , \rangle) \text{ where } \langle , \rangle = \langle , \rangle_M + \rho^2\langle , \rangle_N. \]

Consider the map

\[ \psi_1 : R^m_+ \times_{x_m} S^{N-m} \to R^N \]

given by

\[ \psi_1(x, y) = (x_1, \ldots, x_{m-1}, x_my) \]

and immersions \(f_1 : M_1 \to R^m_+\) and \(f_2 : M_2 \to S^{N-m}\).

**Warped product of two immersions**

\[ f : M_1 \times M_2 \to R^N \text{ given by } f = \psi_1(f_1, f_2). \]
Example 2. Let $\beta, \gamma : I \subset \mathbb{R} \to \mathbb{R}_+^m$ be two regular curves and let $h : L^{n-1} \to S^{N-m}$ be an isometric immersion. Let

$$f, g : M^n = I \times L^{n-1} \to \mathbb{R}^N$$

be given by

$$f = \psi_1(\beta, h) \quad \text{and} \quad g = \psi_1(\gamma, h).$$

Then $f, g$ are conformal with the same Gauss map if and only if

$$\gamma = C \int \frac{\beta'(\tau)}{\beta_2^m(\tau)} d\tau$$

for a constant $C < 0$. 
Example 3. Let $f_0, g_0 : N^2 \to \mathbb{R}_+^m$ be two minimal surfaces and let $h : L^{n-2} \to S_+^{N-m}$ be an isometric immersions. Let

$$f, g : M^n = N^2 \times L^{n-2} \to \mathbb{R}^N$$

be given by

$$f = \psi_1(f_0, h) \quad \text{and} \quad g = \psi_1(g_0, h).$$

Then $f, g$ are conformal with the same Gauss map iff

$$f_0 = (a_1, \ldots, a_{m-1}, a)$$

is parametrized by an isothermal coordinate $z$ and

$$g_0 = -\int \frac{1}{A^2} (f_0)_z dz$$

where $A = a + i\tilde{a}$ is holomorphic.
Consider the map

\[ \psi_2 : \mathbb{R}^m_+ \times_{x_{m-1}} S^{m_1} \times_{x_m} S^{m_2} \rightarrow \mathbb{R}^N, \quad N = m + m_1 + m_2 \]

given by

\[ \psi_2(x, y_1, y_2) = (x_1, \ldots, x_{m-2}, x_{m-1}y_1, x_m y_2) \]

and immersions \( f_1 : M_1 \rightarrow \mathbb{R}^m_+ \), \( f_2 : M_2 \rightarrow S^{m_1} \) and \( f_3 : M_3 \rightarrow S^{m_2} \)

Warped product of three immersions

\[ f : M_1 \times M_2 \times M_3 \rightarrow \mathbb{R}^N \quad \text{given by} \quad f = \psi_2(f_1, f_2, f_3). \]
Example 4. Let $f_0, g_0 : N^2 \rightarrow \mathbb{R}^m_+$ be two minimal surfaces and let $h_j : L^{s_j} \rightarrow S^{s_j}, j = 1, 2$, be isometric immersions. Let

$$f, g : M^n = N^2 \times L^{s_1} \times L^{s_2} \rightarrow \mathbb{R}^N$$

be given by

$$f = \psi_1(f_0, h_1, h_2) \quad \text{and} \quad g = \psi_1(g_0, h_1, h_2).$$

Then $f, g$ are conformal with the same Gauss map iff $f_0, g_0$ satisfy:

1. If $f_0 = (a_1, \ldots, a_{m-2}, a, \tilde{a})$, $a, \tilde{a} > 0$ in isothermal coordinates with $A = a + i\tilde{a}$ holomorphic and if $g_0 = (b_1, \ldots, b_{m-2}, b, \tilde{b})$, $b, \tilde{b} > 0$ in isothermal coordinates with $B = b + i\tilde{b}$ holomorphic

2. If $R$ is a reflection with respect to the hyperplane orthogonal to $e_m$ then $R \circ g_0 = (b_1, \ldots, b_{m-2}, b, -\tilde{b})$ is such that

$$R \circ g_0 = \int \frac{1}{A^2} (f_0)_z dz.$$
The general case

**THEOREM**

Any nontrivial pair $f, g : M^n \to \mathbb{R}^N$, $n \geq 3$, of conformal immersions with the same Gauss map is as in one of the examples already given.

M. Dajczer and R. Tojeiro,
*A complete solution of P. Samuel’s problem.*
Preprint.
That an immersion $g: M^n \to \mathbb{R}^N$ has the same Gauss map as $f$ is equivalent to the existence of a tensor $\Phi \in C^\infty(T^*M \otimes TM)$ so that

$$g_* = f_* \circ \Phi.$$
That an immersion \( g : M^n \to \mathbb{R}^N \) has the same Gauss map as \( f \) is equivalent to the existence of a tensor \( \Phi \in C^\infty(T^*M \otimes TM) \) so that

\[
g_* = f_* \circ \Phi.
\]

**Proposition.** The following holds:

(i) \( \Phi \) is a Codazzi tensor, i.e.,

\[
(\nabla_X \Phi)Y = (\nabla_Y \Phi)X \quad \text{for all } X, Y \in TM.
\]

(ii) The second fundamental form \( \alpha_f \) of \( f \) commutes with \( \Phi \), i.e.,

\[
\alpha_f(X, \Phi Y) = \alpha_f(\Phi X, Y) \quad \text{for all } X, Y \in TM.
\]

Conversely...
Assume, in addition, that \( f \) and \( g \) are conformal. Then,

\[ T = e^{-\varphi} \Phi \]

is an orthogonal tensor with respect to \( \langle \ , \ \rangle_f \).
Assume, in addition, that $f$ and $g$ are conformal. Then,

$$T = e^{-\varphi} \Phi$$

is an orthogonal tensor with respect to $\langle \cdot, \cdot \rangle_f$.

**Proposition.** The pair $(T, \varphi)$ satisfies the differential equation

$$(\nabla_X T) Y = \langle Y, \nabla \varphi \rangle T X - \langle X, Y \rangle T \nabla \varphi$$

for all $X, Y \in TM$.

Moreover,

$$\alpha_f(TX, Y) = \alpha_f(X, TY)$$

for all $X, Y \in TM$.

Conversely, for a given isometric immersion $f : M^n \to \mathbb{R}^{n+p}$ of a simply connected Riemannian manifold, any pair $(T, \varphi)$ satisfying the above equations gives rise to a conformal immersion $g : M^n \to \mathbb{R}^{n+p}$ with the same Gauss map.
Complexify the tangent bundle $TM$.

Decompose $TM$ as the orthogonal sum of proper subspaces

$$TM \otimes \mathbb{C} = L_+ \oplus L_- \oplus L_{\mathbb{C}}.$$ 

Analyze all possible cases.

Use the Theorems of Hiepko (1979) and Nölker (1996).
Hiepko’s Theorem

**Theorem.** Let $M^n$ be a Riemannian manifold and let

$$TM = L \oplus S_1 \oplus \cdots \oplus S_k$$

be an orthogonal decomposition into nontrivial vector subbundles such that $S_1, \ldots, S_k$ are spherical and $S_1^\perp, \ldots, S_k^\perp$ totally geodesic. Then, there is locally a decomposition of $M^n$ into a Riemannian warped product

$$M^n = N_0 \times_{\varrho_1} N_1 \times \cdots \times_{\varrho_k} N_k$$

such that $L = TN_0$ and $S_i = TN_i$ for $1 \leq i \leq k$. 
Nölker’s Theorem

**Theorem.** Let $f: M^n \to \mathbb{R}^N$ be an isometric immersion of a warped product manifold $M^n = M_0 \times_{\varrho_1} M_1 \times \cdots \times_{\varrho_k} M_k$ whose sff satisfies

$$\alpha(X_i, X_j) = 0 \text{ for all } X_i \in TM_i, X_j \in TM_j, \ i \neq j.$$ 

Given $\bar{p} = (\bar{p}_0, \ldots, \bar{p}_k) \in M^n$, set $f_i = f \circ \tau_i^{\bar{p}}: M_i \to \mathbb{R}^N$ for $\tau_i^{\bar{p}}(p_i) = (\bar{p}_0, \ldots, p_i, \ldots, \bar{p}_k)$, and let $S_i$ be the spherical hull of $f_i$, $1 \leq i \leq k$. Then $f_0$ is an isometric immersion, $f_i$ is a homothetical immersion with homothety factor $\rho_i(\bar{p}_0)$ and $(f(\bar{p}); S_1, \ldots, S_k)$ determines a warped product representation

$$\Phi: S_0 \times_{\sigma_1} S_1 \times \cdots \times_{\sigma_k} S_k \to \mathbb{R}^N$$

such that $f_0(M_0) \subset S_0$, $\rho_i = \rho_i(\bar{p}_0)(\sigma_i \circ f_0)$ and

$$f = \Phi \circ (f_0 \times \cdots \times f_k),$$

where $f_i$ is regarded as a map into $S_i$ for $1 \leq i \leq k$. 

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