Fire sales forensics: measuring endogenous risk

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May 4, 2012

Abstract

We propose a tractable framework for quantifying the impact of fire sales on the volatility and correlations of asset returns in a multi-asset setting. Our results enable to quantify the impact of fire sales on the covariance structure of asset returns and provide a quantitative explanation for spikes in volatility and correlations observed during liquidation of large portfolios. These results allow to estimate the impact and magnitude of fire sales from observation of market prices: we give conditions for the identifiability of model parameters from time series of asset prices, propose an estimator for the magnitude of fire sales in each asset class and study the consistency and large sample properties of the estimator. We illustrate our estimation methodology with two empirical examples: the hedge fund losses of August 2007 and the Great Deleveraging following the Lehman default.

Contents

1 Introduction

Fire sales or, more generally, the sudden deleveraging of large financial portfolios, have been recognized as a destabilizing factor in recent (and not-so-recent) financial crises, contributing to unexpected spikes in volatility and correlations of asset returns and resulting in spirals of losses for investors (Carlson, 2006; Khandani and Lo, 2007; Brunnermeier, 2008). In particular, unexpected spikes in correlations across asset classes have frequently occurred during market downturns (Cont and Wagalath, 2012; Bailey et al., 2012), leading to a loss of diversification benefits for investors, precisely when such effects were desirable.

For instance, during the first week of August 2007, when a large fund manager deleveraged its positions in long-short market neutral equity strategies, other long-short market neutral equity funds experienced huge losses, while in the meantime, index funds were left unaffected (Khandani and Lo, 2007). On a larger scale, the Great Deleveraging of financial institutions portfolios subsequent to the default of Lehman Brothers in fall 2008 led to an unprecedented peak in correlations across asset returns (Fratzscher, 2011).

The importance of fire sales as a factor of market instability is recognized in the economic literature. Shleifer and Vishny (1992, 2011) characterize an asset fire sale by a financial institution as a forced sale in which potential high valuation buyers are affected by the same shocks as the financial institution, resulting in a sale of the asset at a discounted price to non specialist buyers. They underline the fact that in the presence of fire sales, losses by financial institutions with overlapping holdings become self-reinforcing, leading to downward spirals for asset prices and, ultimately, to systemic risk. Pedersen (2009) describes qualitatively the effects of investors running for the exit and the spirals of losses and spillover effects they generate. Shin (2010) proposes a two period equilibrium model which takes into account the supply and demand generated by investors reacting to a price move and shows how feedback effects contribute to the amplification of market moves and systemic risk. Brunnermeier (2008) describes the channel through which losses in mortgage backed securities during the recent financial crisis led to huge losses in equity markets, although those two assets classes had been historically uncorrelated.

The empirical link between fire sales and correlation between asset returns has been documented in several recent studies. Coval and Stafford (2007) give empirical evidence for fire sales by open end mutual funds by studying the transactions caused by capital flows. They show that funds in distress experience outflows of capital by investors which result in fire sales in existing positions, creating a price pressure in the securities held in common by distressed funds. Jotikasthira et al. (2011) lead an empirical investigation on the effects of fund flows from developed countries to emerging markets. They show that such investment flows generate forced trading by fund managers, affecting asset prices and correlations between emerging markets and creating a new channel through which shocks are transmitted from developed markets to emerging markets. Anton and Polk (2008) find empirically that common active mutual fund ownership predicts cross-sectional variation in return realized covariance.

However, although the empirical examples cited above are related to liquidation of large *portfolios*, most theoretical studies focus for simplicity on fire sales in a single asset market and thus are not able to investigate the effect of fire sales on asset return correlations and the resulting limits to diversification alluded to above.

Greenwood and Thesmar (2011) propose a simple framework for modeling price dynamics which takes into account the ownership structure of financial assets, considered as given exogenously. Cont and Wagalath (2012) model the systematic supply and demand generated by investors exiting a large distressed fund and quantify its impact on asset returns.

We propose here a tractable framework for modeling and estimating the impact of fire sales in multiple funds on the volatility and correlations of asset returns in a multi-asset setting. We explore the mathematical properties of the model in the continuous-time limit and derive analytical results relating the realized covariance of asset returns to the parameters describing the volume of fire sales. In particular, we show that, starting from homoscedastic inputs, such feedback effects naturally generate heteroscedasticity in the covariance structure of asset returns, thus providing an economic interpretation for various multivariate models of heteroscedasticity in the recent literature (Engle, 2002; Da Fonseca et al., 2008; Gouriéroux et al., 2009; Stelzer, 2010). Our results allow for a structural explanation for the variability observed in measures of cross sectional dependence in asset returns (Bailey et al., 2012), by linking such increases in cross-sectional correlation to the deleveraging of large portfolios.

The analytically tractable nature of these results allows to explore in detail the problem of *estimating* these parameters from empirical observations of price series; we explore the corresponding identification problem and propose a method for estimating the magnitude of distressed selling in each asset class, and study the consistency and large sample properties of the proposed estimator. These results provide a quantitative framework for the 'forensic analysis' of the impact of fire sales and distressed selling, which we illustrate with two empirical examples: the August 2007 hedge fund losses and the Great Deleveraging of bank portfolios following the default of Lehman Brothers in September 2008.

Our framework allows to explain large shifts in the realized covariance structure of asset returns in terms of supply and demand patterns across asset classes, which makes such events easier to analyze and understand. This estimation procedure may be useful for regulators in view of investigating unusual market events in a systematic way, moving a step in the direction proposed by Fielding et al. (2011), who underlined the importance of systematically investigating all 'systemic risk' events in financial markets, as done by the National Transportation Safety Board for major civil transportation accidents.

Outline This paper is organized as follows. Section 2 presents a simple framework for modeling the impact of fire sales in various funds on asset returns. Section 3 resolves the question of the identification and estimation of the model parameters, characterizing the fire sales. Section 4 displays the results of our estimation procedure on liquidations occurring after the collapse of Lehman Brothers while Section 5 is focused on the study of the positions liquidated during the first week of August 2007.

2 Fire sales and endogenous risk

2.1 Impact of fire sales on price dynamics: a multiperiod model

Consider a financial market where n assets are traded at discrete dates $t_k = k\tau$, multiples of a time step τ (taken to be a trading day in the empirical examples). The value of asset i at date t_k is denoted S^i_k. We consider J institutional investors trading in these assets: fund j initially holds α^j_i units of asset i. The value of this (benchmark) portfolio at date t_k is denoted $V^j_k = \sum_{1\le i\le n} \alpha^j_i S^i_k$. Note that α^j_i may be negative and the funds can have short positions.

The impact of (exogenous) economic factors ('fundamentals') on prices is modeled through an IID sequence $(\xi_k)_{k\geq 1}$ of $\mathbb{R}^n$-valued centered random variables, such that in the absence of fire sales, the return of asset i during period $[t_k, t_{k+1}]$ is given by $\tau m_i + \sqrt{\tau}\xi^i_{k+1}$. Here m_i represents the expected return of asset i in the absence of fire sales and the 'fundamental' covariance matrix Σ, defined by

$$\Sigma_{i,j} = \mathrm{cov}(\xi^i_k, \xi^j_k)$$

represents the covariance structure of returns in the absence of large systematic trades by institutional investors.

Due to such exogenous shocks, the value of the benchmark portfolio j changes, during $[t_k, t_{k+1}]$, from V^j_k to

$$(V^j_{k+1})^* = \sum_{1\le i\le n} \alpha^j_i S^i_k (1 + \tau m_i + \sqrt{\tau}\xi^i_{k+1}).$$

Typically, over short time horizons of a few days, institutional investors do not alter their allocations and hold on to their positions. However, the occurrence of large losses typically leads the fund to sell off part of its assets (Coval and Stafford, 2007; Jotikasthira et al., 2011; Shleifer and Vishny, 2011). Such *distressed selling* may be due to

- investors redeeming (or expanding) their positions depending on the performance of the funds, causing inflows and outflows of capital. This mechanism is described by Coval and Stafford (2007), who show empirically that funds in distress experience outflows of capital by investors and explain that, as the ability of borrowing is reduced for distressed funds and regulation and self-imposed constraints prevent them from short-selling other securities, such outflows of capital result in fire sales in existing positions.

- capital requirements which lead fund managers to deleverage their portfolios when faced with trading losses (Danielsson et al., 2004),

- rule based strategies –such as portfolio insurance– which result in selling when a fund underperforms (Genotte and Leland, 1990),
- sale of assets held as collateral by creditors of distressed funds (Shleifer and Vishny, 2011).

The impact of fire sales may also be exacerbated by short selling and predatory trading: Brunnermeier and Pedersen (2005) show that, in the presence of fire sales in a distressed fund, the mean-variance optimal strategy for other investors is to short sell the assets held by the distressed fund and buy them back after the period of distress. A common feature of these mechanisms is that they react to a (negative) change in fund value.

Here we do not model each of these mechanisms in detail but focus instead on their aggregate effect. This aggregate effect may be modeled in a parsimonious manner by introducing a *deleveraging schedule*, represented by a function f_j which measures the systematic supply/demand generated by the fund j as a function of the fund's return: when the value of the portfolio j drops from V_k^j to $(V_{k+1}^j)^*$ due to market shocks, a portion

$$f_j\left(\frac{V_k^j}{V_0^j}\right) - f_j\left(\frac{(V_{k+1}^j)^*}{V_0^j}\right)$$

of the fund is liquidated between t_k and t_{k+1}.

As shown by Jotikasthira et al. (2011), negative returns for a fund lead to outflows of capital from this fund: this implies that f_j is an increasing function. Furthermore, we choose f_j to be concave, capturing the fact that fire sales accelerate as the fund exhibits larger losses. Figure 1 displays an example of such a deleveraging schedule f_j.

When the trades are sizable with respect to the average trading volume, the supply / demand generated by this deleveraging strategy impacts asset prices. We introduce, for each asset i, a *price impact* function $\phi_i(.)$ which captures this effect: the impact of buying v shares (where $v < 0$ represents a sale) on the return of asset i is $\phi_i(v)$. We assume that $\phi_i : \mathbb{R} \mapsto \mathbb{R}$ is increasing and $\phi_i(0) = 0$.

The impact of fire sales on the return of asset i is then equal to

$$\phi_i\left(\sum_{1\leq j\leq J} \alpha_i^j \left(f_j\left(\frac{(V_{k+1}^j)^*}{V_0^j}\right) - f_j\left(\frac{V_k^j}{V_0^j}\right)\right)\right) \tag{1}$$

At each period, the return of asset i is the sum of a fundamental term $\tau m_i + \sqrt{\tau}\xi_{k+1}^i$ plus an endogenous term, which is due to the price impact of fire sales.

$$S_{k+1}^i = S_k^i\left(1 + \tau m_i + \sqrt{\tau}\xi_{k+1}^i + \phi_i\left(\sum_{1\leq j\leq J} \alpha_i^j \left(f_j\left(\frac{(V_{k+1}^j)^*}{V_0^j}\right) - f_j\left(\frac{V_k^j}{V_0^j}\right)\right)\right)\right) \tag{2}$$

where

$$V_k^j = \sum_{1\leq i\leq n} \alpha_i^j S_k^i \tag{3}$$

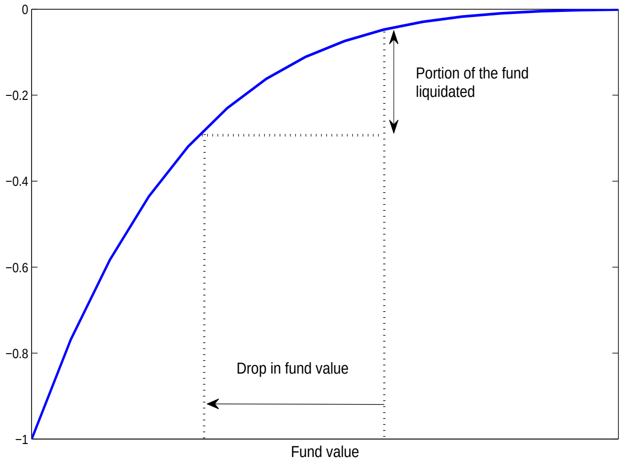

Figure 1: Example of a deleveraging schedule f_j

and

$$(V_{k+1}^j)^* = \sum_{1\leq i\leq n} \alpha_i^j S_k^i (1 + \tau m_i + \sqrt{\tau}\xi_{k+1}^i) \tag{4}$$

$$\boxed{S_k, (V_k^j)_{1\leq j\leq J}} \xrightarrow[\text{factors } (\xi_{k+1})]{\text{exogenous}} \boxed{S_{k+1}^*, ((V_{k+1}^j)^*)_{1\leq j\leq J}} \xrightarrow{\text{fire sales}} \boxed{S_{k+1}, (V_{k+1}^j)_{1\leq j\leq J}}$$

Equations (2), (3) and (4) show that S_{k+1} depends only on its value at t_k and on ξ_{k+1}, which is independent of events previous to t_k. The price vector S is thus a Markov chain.

This multiperiod model exhibits interesting properties: in particular, as shown in Cont and Wagalath (2012), the presence of distressed selling induces an endogenous, heteroscedastic component in the covariance structure of returns, which leads to state-dependent correlations, even in the absence of any heteroscedasticity in the fundamentals.

Figure 2 shows an example of such endogenous correlations: we simulated 10^6 price trajectories of this multiperiod model with the parameters used in Section 3.4 and for each trajectory, we computed the realized correlation between all pairs of assets. We find that even in the case where the exogenous shocks driving the asset values are independent (i.e. the 'fundamental' covariance matrix Σ is diagonal), the presence of distressed selling leads to significant realized correlations, thereby increasing the volatility experienced by investors holding the fund during episodes of fire sales. This phenomenon may substantially decrease the benefits of diversification.

Our goal is to explore such effects systematically and propose a method for estimating their impact on price dynamics.

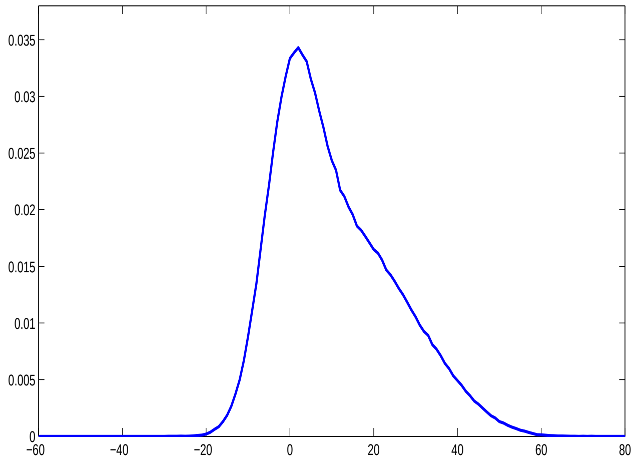

Figure 2: Distribution of realized correlation between two securities in the presence of distressed selling (case of zero fundamental correlation)

2.2 Continuous time limit

The multiperiod model above is rather cumbersome to study directly; in the sequel we focus its continuous time limit, which is analytically tractable and more easily related to commonly used diffusion models for price dynamics. This will allow us to compute realized covariances between asset returns in the presence of feedback effects from distressed selling.

For two n-dimensional vectors x and y, we denote $x.y = \sum_{1 \le i \le n} x_i y_i$ the scalar product between vectors x and y. For $M \in \mathcal{M}_n(\mathbb{R})$, M^t is the transpose of matrix M. $\mathcal{S}_n(\mathbb{R})$ (resp. $\mathcal{S}_n^+(\mathbb{R})$) denotes the set of real valued symmetric matrices (resp. real valued symmetric positive semi-definite matrices). For a sequence $X^{(\tau)}$ of random variables, we denote the fact that $X^{(\tau)}$ converges in law (resp. in probability) to X when τ goes to zero by $X^{(\tau)} \underset{\tau \to 0}{\Rightarrow} X$ (resp. $X^{(\tau)} \underset{\tau \to 0}{\overset{\mathbb{P}}{\to}} X$).

We make the following assumption on the deleveraging schedules f_j:

Assumption 2.1 *For all* $1 \le j \le J$, f_j *is increasing and*

$$f_j \in \mathcal{C}_0^3(\mathbb{R})$$

where $\mathcal{C}_0^p(\mathbb{R})$ *denotes the set of real-valued, p-times continuously differentiable maps whose derivatives of order* $1 \le l \le p$ *have compact support.*

In particular f_j is constant for large values and very small values of its argument, and monotone in between. This assumption has a natural interpretation in our context: fire sales occur when funds underperform, i.e. when the value of the fund relative to a benchmark falls below a threshold and cease when the fund defaults.

Theorem 2.2 *Under Assumption 2.1, if $\phi_i \in \mathcal{C}^3$ for all $1 \le i \le n$ and $\mathbb{E}(\|\xi\|^4) < \infty$, the process $(S_{\lfloor \frac{t}{\tau} \rfloor})_{t \ge 0}$ converges weakly, as the time step τ goes to 0, to a diffusion process $\left(P_t = (P_t^1, ...P_t^n)\right)_{t\ge 0}$ solution of the stochastic differential equation*

$$\frac{dP_t^i}{P_t^i} = \mu_i(P_t)dt + (\sigma(P_t)dW_t)_i \qquad 1 \le i \le n$$

where μ (resp., σ) is a $\mathbb{R}^n$-valued (resp. matrix-valued) mapping defined by

$$\mu_i(P_t) = m_i + \phi_i'(0) \sum_{1 \le j \le J} \alpha_i^j \left(f_j' \left(\frac{V_t^j}{V_0^j} \right) \frac{\pi_t^j.m}{V_0^j} + \frac{1}{2} f_j'' \left(\frac{V_t^j}{V_0^j} \right) \frac{\pi_t^j.\Sigma\pi_t^j}{\left(V_0^j\right)^2} \right) \tag{5}$$

$$+ \frac{\phi_i''(0)}{2} \sum_{1 \le j,l \le J} \left(\alpha_i^j \alpha_i^l f_j' \left(\frac{V_t^j}{V_0^j} \right) f_l' \left(\frac{V_t^l}{V_0^l} \right) \frac{\pi_t^j.\Sigma\pi_t^l}{V_0^j V_0^l} \right)$$

$$\sigma_{i,k}(P_t) = A_{i,k} + \phi_i'(0) \sum_{1 \le j \le J} \alpha_i^j f_j' (\frac{V_t^j}{V_0^j}) \frac{\left(A\pi_t^j\right)_k}{V_0^j} \tag{6}$$

Here W_t is an n-dimensional Brownian motion, $\pi_t^j = \begin{pmatrix} \alpha_1^j P_t^1 \\ \vdots \\ \alpha_n^j P_t^n \end{pmatrix}$ is the (dollar) allocation of fund j, $V_t^j = \sum_{1 \le k \le n} \alpha_k^j P_t^k$ the value of fund j and A is a square-root of the fundamental covariance matrix: $AA^t = \Sigma$.

The proof of this Theorem is given in Appendix A.

Remark 2.3 *The limit price process that we exhibit in Theorem 2.2 depends on the price impact functions only through their first and second derivatives in 0, $\phi_i'(0)$ and $\phi_i''(0)$. In particular, the expression of σ in (6) shows that realized volatilities and realized correlations of asset returns depend only on the slope $\phi_i'(0)$ of the price impact function. This remark shows that, under our assumptions, a linear price impact function would lead to the same realized covariance structure for asset returns in the continuous time limit.*

As a consequence, in the remainder of this paper, we use the assumption of linear price impact: $\lambda_i = \frac{1}{\phi_i'(0)}$ then corresponds to the market depth for asset i and is interpreted as the number of shares an investor has to buy in order to increase the price of asset i by 1%.

Corollary 2.4 (Case of linear price impact) *Under Assumption 2.1, if* $\mathbb{E}(\|\xi\|^4) < \infty$*, the process* $(S_{\lfloor \frac{t}{\tau} \rfloor})_{t\geq 0}$ *converges weakly, as the time step* τ *goes to* 0*, to a diffusion process* $\left(P_t = (P_t^1, ...P_t^n)\right)_{t\geq 0}$ *solution of the stochastic differential equation*

$$\frac{dP_t^i}{P_t^i} = \mu_i(P_t)dt + (\sigma(P_t)dW_t)_i \qquad 1 \leq i \leq n$$

where μ *(resp.,* σ*) is a* $\mathbb{R}^n$*-valued (resp. matrix-valued) mapping defined by*

$$\mu_i(P_t) = m_i + \frac{1}{\lambda_i} \sum_{1\leq j\leq J} \alpha_i^j \left(f_j^{'} \left(\frac{V_t^j}{V_0^j} \right) \frac{\pi_t^j.m}{V_0^j} + \frac{1}{2} f_j^{''} \left(\frac{V_t^j}{V_0^j} \right) \frac{\pi_t^j.\Sigma\pi_t^j}{\left(V_0^j\right)^2} \right) \tag{7}$$

$$\sigma_{i,k}(P_t) = A_{i,k} + \frac{1}{\lambda_i} \sum_{1\leq j\leq J} \alpha_i^j f_j^{'}(\frac{V_t^j}{V_0^j}) \frac{\left(A\pi_t^j\right)_k}{V_0^j} \tag{8}$$

where W_t*,* π_t^j*,* V_t^j *and* A *are defined in Theorem 2.2.*

When market depths are infinite, the price dynamics follows a multivariate exponential Brownian motion. In the presence of fire sales by distressed sellers, the fundamental dynamics of the assets is modified.

Using Theorem 2.4 and Ito's formula, we deduce that the log price $X_t^i = \ln(P_t^i)$ verifies the following stochastic differential equation:

$$dX_t^i = \left(\mu_i(\exp X_t) - \frac{1}{2}(\sigma(\exp X_t)\sigma(\exp X_t)^t)_{i,i} \right) dt + (\sigma(\exp X_t)dW_t)_i \tag{9}$$

where σ, μ and W are defined in Theorem 2.4 and $\exp X_t$ is a n dimensional column vector with i-th term equal to $\exp X_t^i$.

2.3 Realized covariance in the presence of fire sales

The *realized covariance* (Andersen et al., 2003; Barndorff-Nielsen and Shephard, 2004) between dates t_1 and t_2 computed at time resolution τ is defined as

$$\widehat{C}_{[t_1,t_2]}^{(\tau)} = \frac{1}{t_2 - t_1}([X, X]_{t_2}^{(\tau)} - [X, X]_{t_1}^{(\tau)}) \tag{10}$$

where $[X, X]_t^{(\tau)} = \left([X^i, X^k]_t^{(\tau)}\right)_{1\leq i,k\leq n}$ with

$$[X^i, X^k]_t^{(\tau)} = \sum_{1\leq l\leq \lfloor t/\tau \rfloor} (X_{l\tau}^i - X_{(l-1)\tau}^i)(X_{l\tau}^k - X_{(l-1)\tau}^k) \tag{11}$$

As the time step τ goes to zero, $\left([X, X]^{(\tau)}\right)_{t\geq 0}$ converges in probability to an increasing, $\mathcal{S}_n^+(\mathbb{R})$-valued process $([X, X])_{t\geq 0}$, the quadratic covariation of X Jacod and Protter

(2012). We define the $\mathcal{S}_n^+(\mathbb{R})$-valued process $c = (c_t)_{t\geq 0}$, which corresponds intuitively to the 'instantaneous covariance' of returns, as the derivative of the quadratic covariation process:

$$[X,X]_t = \int_0^t c_s ds \tag{12}$$

Theorem 2.4 allows to compute the realized covariance matrix for the n assets.

Proposition 2.5 *The instantaneous covariance matrix of returns, c_t, defined in* (12)*, is given by:*

$$c_t = \Sigma + \sum_{1\leq j\leq J}\left[\frac{1}{V_0^j} f_j'(\frac{V_t^j}{V_0^j})\left(\Lambda_j(\pi_t^j)^t\Sigma + \Sigma\pi_t^j\Lambda_j^t\right)\right] + \sum_{1\leq j,k\leq J}\frac{\pi_t^j.\Sigma\pi_t^k}{V_0^jV_0^k} f_j'(\frac{V_t^j}{V_0^j}) f_k'(\frac{V_t^k}{V_0^k})\Lambda_j\Lambda_k^t$$

where $\pi_t^j = \begin{pmatrix} \alpha_1^j P_t^1 \\ \vdots \\ \alpha_n^j P_t^n \end{pmatrix}$ *denotes the (dollar) holdings of fund* j *and* $\Lambda_j = \begin{pmatrix} \frac{\alpha_1^j}{\lambda_1} \\ \vdots \\ \frac{\alpha_n^j}{\lambda_n} \end{pmatrix}$ *represents the positions of fund* j *in each market as a fraction of the respective market depth.*

The realized covariance matrix between t_1 *and* t_2 *is then defined as*

$$C_{[t_1,t_2]} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} c_t \, \mathrm{d}t$$

Denote

$$\Lambda = (\Lambda_1, ..., \Lambda_J) \in \mathcal{M}_{n\times J}(\mathbb{R}) \tag{13}$$

where Λ_j is defined in Proposition 2.5. We observe that the excess realized covariance terms due to fire sales contain a term of order one in $\|\Lambda\|$ plus higher order terms:

$$c_t = \Sigma + \sum_{1\leq j\leq n}\left[\frac{1}{V_0^j} f_j'(\frac{V_t^j}{V_0^j})\left(\Lambda_j(\pi_t^j)^t\Sigma + \Sigma\pi_t^j\Lambda_j^t\right)\right] + O(\|\Lambda\|^2)$$

where $\frac{O(\|\Lambda\|^2)}{\|\Lambda\|^2}$ is bounded when $\|\Lambda\| \to 0$. This result is due to the fact that under Assumption 2.1, the second order terms $\frac{\pi_t^j.\Sigma\pi_t^k}{V_0^jV_0^k} f_j'(\frac{V_t^j}{V_0^j}) f_k'(\frac{V_t^k}{V_0^k})$ in the expression of c_t in Proposition 2.5 are bounded. In practice, this first order approximation is precise enough Cont and Wagalath (2012) and we will focus on this approximation in the numerical examples.

Fire sales impact realized covariances between assets. In the presence of fire sales, realized covariance is the sum of the fundamental covariance matrix Σ and an excess realized covariance which is liquidity dependent and path dependent. The magnitude of this endogenous impact is measured by the vectors Λ_j, which represent the position of each fund as a fraction of market depth. The volume generated by fire sales in fund

j on each asset i is equal to $\alpha_i^j \times f_j'$ and its impact on the return of asset i is equal to $\frac{\alpha_i^j}{\lambda_i} \times f_j'$. This impact can be significant even if the asset is very liquid, when the positions liquidated are large enough compared to the asset's market depth. *Thus, fire sales endogenously lead to empirically plausible patterns of heteroscedasticity in the covariance structure of asset returns.*

In the remainder of this paper, we assume that the following assumption holds:

Assumption 2.6 *There are no fire sales between* 0 *and* T *and each fund* j *liquidates between* T *and* $T + \tau_{liq}$ *at a constant rate* γ_j*.*

The following Corollary is a direct consequence of Proposition 2.5.

Corollary 2.7 *The realized covariance matrices between 0 and* T *and between* T *and* $T + \tau_{liq}$ *are respectively equal to:*

$$C_{[0,T]} = \frac{1}{T} \int_0^T c_t \, \mathrm{d}t = \Sigma$$

and

$$C_{[T,T+\tau_{liq}]} = \frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} c_t \, \mathrm{d}t = \Sigma + LM_0\Pi\Sigma + \Sigma\Pi M_0 L + O(\|\Lambda\|^2) \tag{14}$$

with

$$M_0 = \sum_{1 \leq j \leq J} \frac{\gamma_j}{V_0^j} \times \alpha^j (\alpha^j)^t \tag{15}$$

where $\alpha^j = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ *is the vector of positions of fund* j *and* L *and* Π *are diagonal matrices with i-th diagonal term equal respectively to* $\frac{1}{\lambda_i}$ *and* $\frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} P_t^i \, \mathrm{d}t$*.*

In the absence of distressed selling between 0 and T, the realized covariances between asset returns during this period are equal to their fundamental value. Between T and $T+\tau_{liq}$, fire sales affect the realized covariance between asset returns. The excess realized covariance is characterized by a matrix M_0, defined in (15), which reflects the magnitude of the fire sales. Note that we do not assume that all the funds are liquidating between T and $T + \tau_{liq}$. A fund j which is not subject to fire sales during this period of time has a rate of liquidation γ_j equal to zero.

In (15), $\alpha^j(\alpha^j)^t$ is a $n \times n$ symmetric matrix representing an orthogonal projection on fund j's positions and hence M is a sum of projectors. The symmetric matrix M captures the direction and intensity of liquidations in the J funds.

2.4 Spillover effects

Consider now the situation where a distressed fund with positions $(\alpha_1, ..., \alpha_n)$ is liquidating its assets over a period $[t_1, t_2]$. As argued above, this leads to endogenous volatility and correlations in asset prices, which then modifies the volatility experienced by any other fund holding the same assets.

Proposition 2.5 allows to compute the magnitude of this *volatility spillover* effect (Cont and Wagalath, 2012). The following result shows that the realized variance of a (small) fund with positions $(\mu_t^i, i = 1..n)$ is the sum of its the realized volatility in absence of distressed selling and an endogenous term which represents the impact of distressed selling in the

Corollary 2.8 (Spillover effects) *In the presence of fire sales in a reference fund with positions α, the realized variance for a target fund with positions $(\mu_t^i)_{1\le i\le n}$ between t_1 and t_2 is equal to $\frac{1}{t_2-t_1}\int_{t_1}^{t_2} \gamma_s \quad ds$ where*

$$\gamma_s M_s^2 = \pi_s^\mu.\Sigma\pi_s^\mu + \frac{2f'(\frac{V_s}{V_0})}{V_0}(\pi_s^\mu.\Sigma\pi_s^\alpha)(\Lambda.\pi_s^\mu) + \frac{f'(\frac{V_s}{V_0})^2}{V_0^2}(\pi_s^\alpha.\Sigma\pi_s^\alpha)(\Lambda.\pi_s^\mu)^2 \qquad (16)$$

where $\pi_s^\alpha = \begin{pmatrix} \alpha_1 P_s^1 \\ \vdots \\ \alpha_n P_s^n \end{pmatrix}$ *and* $\pi_s^\mu = \begin{pmatrix} \mu_t^1 P_s^1 \\ \vdots \\ \mu_t^n P_s^n \end{pmatrix}$ *denote respectively the (dollar) holdings of the reference fund and the target fund,* $M_s = \sum_{1\le i\le n} \mu_s^i P_s^i$ *is the target fund's value, and* $\Lambda = (\frac{\alpha_1}{\lambda_1}, ..., \frac{\alpha_n}{\lambda_n})$ *represents the positions of the reference fund in each market as a fraction of the respective market depth.*

The second term in (16), which represents the price-mediated contagion of endogenous risk from the distressed fund to other funds holding the same assets, is maximal for funds whose positions are colinear to those of the distressed fund. On the other hand, these endogenous terms are zero if the two portfolio verify an 'orthogonality condition'

$$\Lambda.\pi_t^\mu = \sum_{i=1}^n \frac{\alpha_i}{\lambda_i}\mu_t^i P_t^i = 0, \qquad (17)$$

in which case the fund with positions μ_t is not affected by the fire sales of assets by the distressed fund.

3 Identification and estimation

3.1 Inverse problem and identifiability

Corollary 2.7 gives us the modification of the realized covariance matrix due to fire sales in J funds, where each fund j has holdings $(\alpha_1^j, ..., \alpha_n^j)$ and liquidates at a constant rate γ_j between dates T and $T + \tau_{liq}$. We now consider the inverse problem of explaining 'abnormal' patterns in realized covariance and volatility in the presence of fire sales and estimating the parameters of the liquidated portfolio from empirical observations. Mathematically, this boils down to answering the following question: knowing that liquidations occurred at a constant rate between two dates T and $T+\tau_{liq}$ and observing the modification of realized covariance matrices that such distressed selling generates, is it possible, given Σ, $C_{[T,T+\tau_{liq}]}$, L and Π, to find M such that

$$C_{[T,T+\tau_{liq}]} = \Sigma + LM\Pi\Sigma + \Sigma\Pi ML \tag{18}$$

The following proposition gives conditions under which this inverse problem is well-posed i.e. the parameter M is identifiable:

Proposition 3.1 (Identifiability) *Let L and Π be diagonal matrices with*

$$L_{ii} = \frac{1}{\lambda_i} \qquad \Pi_{ii} = \frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} P_t^i \,\mathrm{d}t$$

If $\Pi\Sigma L^{-1}$ is diagonalizable with strictly positive eigenvalues i.e. there exists an invertible matrix Ω and $\phi_1 > 0$,...,$\phi_n > 0$ such that

$$\Omega^{-1}\Pi\Sigma L^{-1}\Omega = \begin{pmatrix} \phi_1 & & 0 \\ & \ddots & \\ 0 & & \phi_n \end{pmatrix}$$

then there exists a unique symmetric $n \times n$ matrix M verifying (18) which is given by

$$M = \Phi(\Sigma, C_{[T,T+\tau_{liq}]}) \tag{19}$$

where $\Phi(\Sigma, C)$ is a matrix defined by

$$\left[\Omega^t \Phi(\Sigma, C)\Omega\right]_{p,q} = \frac{1}{\phi_p + \phi_q} \times \left[\Omega^t L^{-1}(C-\Sigma)L^{-1}\Omega\right]_{p,q} \tag{20}$$

In this case, the unique solution M of (18) verifies

$$M = M_0 + O(\|\Lambda\|^2) \tag{21}$$

where M_0 is defined in (15).

The proof of this proposition is given in Appendix B. Thanks to (21), we deduce the following corollary:

Corollary 3.2 *The knowledge of M allows to estimate, up to an error term of order one in $\|\Lambda\|$, the volume of fire sales in asset class i between T and $T+\tau_{liq}$:*

$$\sum_{1\le j\le J} \frac{\alpha_i^j P_T^i}{V_T^j} \times \gamma_j \times \left(\frac{V_T^j - V_{T+\tau_{liq}}^j}{V_0^j}\right) \times V_T^j$$

$$= (0,...,0,P_T^i,0,...,0)M(P_T - P_{T+\tau_{liq}}) + O(\|\Lambda\|^2)$$

Note that the knowledge of M does not allow in general to reconstitute the detail of fire sales in each fund. Indeed, the decomposition of M given in (15) is not always unique. Nevertheless, when different funds engage in similar patterns of fire sales, the common component of these patterns may be recovered from the principal eigenvector of M. In the empirical examples, we find that M has one large eigenvalue, meaning that liquidations were concentrated in one direction.

3.2 Consistency and large sample properties

In the remainder of the paper, we make the following assumptions, which guarantees that the identification problem is well-posed in the sense of Proposition 3.1

Assumption 3.3 $\Pi\Sigma L^{-1}$ *is diagonalisable with distinct strictly positive eigenvalues.*

As a consequence, (19) (20) (21) hold. We require that the eigenvalues of $\Pi\Sigma L^{-1}$ are distinct so that the set of matrices Σ verifying Assumption 3.3 is an open subset of $\mathcal{S}_n(\mathbb{R})$ which allows for the study of the differentiability of Φ defined in (20).

Proposition 3.1 states that if we know $L = diag(\frac{1}{\lambda_i})$, $\Pi = diag(\frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} P_t^i \, \mathrm{d}t)$, the realized covariance matrix between 0 and T, Σ, and the realized covariance matrix between T and $T+\tau_{liq}$, $C_{[T,T+\tau_{liq}]}$, we can reconstitute M and hence the characteristics of the aggregate liquidations between T and $T+\tau_{liq}$, according to Corollary 3.2.

The market depth matrix L can be estimated precisely using intraday data. Obizhaeva Obizhaeva (2011) calculates the market depths of US stocks. We will discuss about the methodology to estimate L in Section 4. Π may be computed from time series of prices.

However, in practice, we only have estimators of Σ and $C_{[T,T+\tau_{liq}]}$, denoted respectively $\widehat{\Sigma}^{(\tau)}$ and $\widehat{C}^{(\tau)}$, which converge in probability (see for example (Jacod and Protter, 2012, Theorem 3.3.1, Ch. 5)) to Σ and $C_{[T,T+\tau_{liq}]}$ respectively.

$$\widehat{\Sigma}^{(\tau)} = \frac{1}{T}[X,X]_T^{(\tau)} \underset{\tau\to 0}{\overset{\mathbb{P}}{\to}} \Sigma \tag{22}$$

and

$$\widehat{C}^{(\tau)} = \frac{1}{\tau_{liq}}\left([X,X]_{T+\tau_{liq}}^{(\tau)} - [X,X]_T^{(\tau)}\right) \underset{\tau\to 0}{\overset{\mathbb{P}}{\to}} C_{[T,T+\tau_{liq}]} \tag{23}$$

where the process $[X,X]^{(\tau)}$ is defined in (11). $\widehat{\Sigma}^{(\tau)}$ and $\widehat{C}^{(\tau)}$ are the realized covariance with resolution τ between 0 and T and T and $T+\tau_{liq}$ respectively.

We can hence define an estimator $\widehat{M}^{(\tau)}$ of M by:

$$\widehat{M}^{(\tau)} = \Phi(\widehat{\Sigma}^{(\tau)}, \widehat{C}^{(\tau)}) \tag{24}$$

where Φ is defined in (20).

Proposition 3.4 (Consistency) *The estimator* $\widehat{M}^{(\tau)}$ *defined in* (20)-(22)-(23)-(24) *is consistent:*

$$\widehat{M}^{(\tau)} = \Phi(\widehat{\Sigma}^{(\tau)}, \widehat{C}^{(\tau)}) \underset{\tau\to 0}{\overset{\mathbb{P}}{\to}} M$$

The proof of this proposition is given in Appendix B.

To study the asymptotic joint distribution of the estimators $\widehat{\Sigma}^{(\tau)}$ and $\widehat{C}^{(\tau)}$, we need to extend $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ to a larger probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$, which supports a Brownian motion $\tilde{W}$ describing the estimation errors in (24).

Lemma 3.5 *There exists a $n \times n$ dimensional process $\overline{Z}$, defined on a very good filtered extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ verifying*

$$\overline{Z}_t^{ij} = \frac{1}{\sqrt{2}} \sum_{1\leq k,l\leq n} \int_0^t \left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl}\right) \mathrm{d}\tilde{W}_s^{kl} \tag{25}$$

where $\tilde{W}$ is a n^2-dimensional Brownian motion adapted to $\tilde{\mathcal{F}}_t$ and $\tilde{V}$ is a $\mathcal{M}_{n^2\times n^2}(\mathbb{R})$-valued process verifying

$$(\tilde{V}_t\tilde{V}_t^t)^{ij,kl} = [\sigma\sigma^t(P_t)]_{i,k}[\sigma\sigma^t(P_t)]_{j,l} \tag{26}$$

such that, when τ goes to zero:

$$\frac{1}{\sqrt{\tau}}\left([X,X]^{(\tau)} - [X,X]\right) \underset{\tau\to 0}{\Rightarrow} \overline{Z}$$

where σ is defined in (8) *and $[X,X]^{(\tau)}$ and $[X,X]$ in* (11)*.*

This implies that the estimator $(\widehat{\Sigma}^{(\tau)}, \widehat{C}^{(\tau)})$ defined in (22) *and* (23) *verifies the following central limit theorem:*

$$\frac{1}{\sqrt{\tau}}\left[\begin{pmatrix} \widehat{\Sigma}^{(\tau)} \\ \widehat{C}^{(\tau)} \end{pmatrix} - \begin{pmatrix} \Sigma \\ C_{[T,T+\tau_{liq}]} \end{pmatrix}\right] \underset{\tau\to 0}{\Rightarrow} \begin{pmatrix} \frac{1}{T}\overline{Z}_T \\ \frac{1}{\tau_{liq}}(\overline{Z}_{T+\tau_{liq}} - \overline{Z}_T) \end{pmatrix} \tag{27}$$

Proof Recall that the log price $X_t^i = \ln(P_t^i)$ verifies the stochastic differential equation given in (9). This implies that X is continuous and that its drift and volatility, which are bounded, verify $\forall t \geq 0$:

$$\int_0^t \left(\sum_{1\leq i\leq n}\left(\mu_i(P_t) - \frac{1}{2}(\sigma(P_t)\sigma_t(P_t)^t)_{i,i}\right)^2 + \|\sigma\sigma^t(P_t)\|^2\right)\mathrm{d}s < \infty$$

By (Jacod and Protter, 2012, Theorem 5.4.2, Ch.5) we find that Lemma 3.5 holds.

The following proposition gives us the rate of convergence of the estimator $\widehat{M}^{(\tau)}$ and its asymptotic distribution.

Proposition 3.6 (Asymptotic distribution of estimator)

$$\frac{1}{\sqrt{\tau}}\left(\widehat{M}^{(\tau)} - M\right) \underset{\tau\to 0}{\Rightarrow} \nabla\Phi\left(\Sigma, C_{[T,T+\tau_{liq}]}\right)\cdot\begin{pmatrix} \frac{1}{T}\overline{Z}_T \\ \frac{1}{\tau_{liq}}(\overline{Z}_{T+\tau_{liq}} - \overline{Z}_T) \end{pmatrix} \tag{28}$$

where $\overline{Z}$ is defined in (25) *and $\nabla\Phi$ is the derivative of Φ.*

The proof of this proposition is given in Appendix B. Proposition 3.6 allows to compute confidence intervals, following the approach outlined in Jacod and Protter (2012).

3.3 Testing for the presence of fire sales

Proposition 3.6 allows to test whether $M \neq 0$ i.e. if significant fire sales occurred between T and $T + \tau_{liq}$. Consider the null hypothesis

$$M = 0 \qquad (H_0)$$

Under assumption (H_0), there are no fire sales in the J funds between T and $T + \tau_{liq}$. The central limit theorem given in Proposition 3.6 can be simplified as follows:

Proposition 3.7 *Under the null hypothesis (H_0), the estimator $\widehat{M}^{(\tau)}$ verifies the following central limit theorem:*

$$\frac{1}{\sqrt{\tau}}\widehat{M}^{(\tau)} \underset{\tau\to 0}{\Rightarrow} \Phi\left(\Sigma, \Sigma + \frac{1}{\tau_{liq}}(\overline{Z}_{T+\tau_{liq}} - \overline{Z}_T) - \frac{1}{T}\overline{Z}_T\right)$$

where $\overline{Z}$ is a n^2-dimensional Brownian motion with covariance

$$cov(\overline{Z}^{i,j}, \overline{Z}^{k,l}) = \Sigma_{i,k}\Sigma_{j,l} + \Sigma_{i,l}\Sigma_{j,k}$$

and Φ is defined in (20).

The proof of this proposition is given in Appendix C.

This result allows to test whether the variability in the realized covariance of asset returns during $[T, T + \tau_{liq}]$ may be explained by the superposition of homoscedastic fundamental covariance structure and feedback effects from fire sales. To do this, we estimate the matrix M and test the nullity of the liquidation volumes derived in Corollary (3.2):

Corollary 3.8 *Under the null hypothesis (H_0),*

$$\frac{1}{\sqrt{\tau}}\left(P_T^t\widehat{M}^{(\tau)}(P_T - P_{T+\tau_{liq}})\right) \underset{\tau\to 0}{\Rightarrow} \mathcal{N}\left(0, \left(\frac{1}{T} + \frac{1}{\tau_{liq}}\right)\sum_{1\leq i,j,k,l\leq n} m_{ij}m_{kl}\left(\Sigma_{ik}\Sigma_{jl} + \Sigma_{jk}\Sigma_{il}\right)\right)$$

with $m_{ij} = \sum_{1\leq p,q\leq n} \frac{[\Omega^{-1}P_T]_p[\Omega^{-1}(P_T - P_{T+\tau_{liq}})]_q}{\phi_p + \phi_q}\Omega_{ip}\Omega_{jq}\lambda_i\lambda_j$ *where Ω and $(\phi_i)_{1\leq i\leq n}$ are defined in Proposition 3.1, P_t is the vector of prices at date t and $(\lambda_i)_{1\leq i\leq n}$ are the asset market depths.*

The proof of this corollary is given in Appendix D.

Corollary 3.8 gives the asymptotic law of $\left(P_T^t\widehat{M}^{(\tau)}(P_T - P_{T+\tau_{liq}})\right)$, the estimated volume of liquidations. We can then define a level l such that

$$\mathbb{P}\left(\left|P_T^t\widehat{M}^{(\tau)}(P_T - P_{T+\tau_{liq}})\right| > l\right) \leq 1 - p_l$$

where p_l is typically equal to 95% or 99%. If we find that $\left|P_T^t\widehat{M}^{(\tau)}(P_T - P_{T+\tau_{liq}})\right| > l$, then the null hypothesis of no fire sales may be rejected at confidence level p_l.

3.4 Numerical experiments

To assess the accuracy of these estimators in samples of realistic size, we first apply this test to a simulated discrete-time market. We consider the case of one fund investing in $n = 20$ assets, with fundamental volatility 30% and zero fundamental correlation. Furthermore, we assume that all assets have the same market depth λ and that the fund is initially equally weighted across these assets: $\frac{\alpha_i P_0^i}{V_0} = \frac{1}{n}$. The size of the fund can be captured by the vector Λ, defined in Proposition 2.5, which represents the size of the fund's position in each asset as a fraction of the asset's market depth. In our simulations, we choose this ratio equal to 20%.

We examine the results of our estimation method in the two following cases:

- the fund is not subject to distressed selling
- the fund is subject to distressed selling: when the fund value drops below $\beta_0 = 95\%$ of its initial value, the manager deleverages the fund portfolio.

Figure 3 displays a trajectory for the fund's value, where the fund was subject to distressed selling between T=116 days and $T + \tau_{liq} = 127$ days.

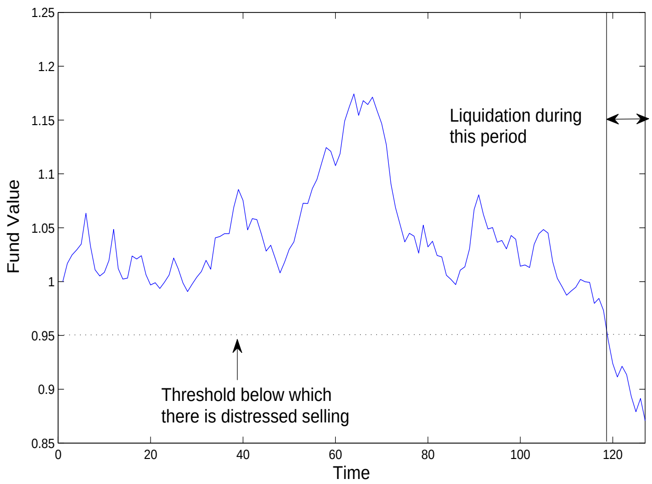

Figure 3: Fund value

We consider a market where trading is possible every hour of each trading day ($\tau = \frac{1}{6\times 250}$ if we consider that a trading day is 6 hour long). We calculate $\widehat{\Sigma}^{(\tau)}$ and $\widehat{C}^{(\tau)}$ and we apply our estimation procedure and calculate in each case (no liquidation and liquidation cases) an estimate for the volume of liquidations. Using 3.8, we can determine, at confidence level 95%, for example, whether there has been a liquidation or not.

Under the assumption (H_0) that $M = 0$ and using Lemma 3.8 we find that

$$\mathbb{P}\left(\left|P_T^t \widehat{M}^{(\tau)}(P_T - P_{T+\tau_{liq}})\right| > 3.2 \times 10^3\right) \leq 5\%$$

We find that

- when there are no fire sales, $\left(P_T^t \widehat{M}^{(\tau)}(P_T - P_{T+\tau_{liq}})\right) = 203 < 3.2 \times 10^3$ and we cannot reject assumption (H_0)
- when fire sales occur, $\left(P_T^t \widehat{M}^{(\tau)}(P_T - P_{T+\tau_{liq}})\right) = 7 \times 10^3 > 3.2 \times 10^3$ and, with a confidence level of 95%, we reject assumption (H_0).

Let us now focus on the results of our estimation procedure in the case where there were liquidations and check whether it allows for a proper reconstitution of the liquidated portfolio. We find that the estimates for the proportions liquidated $\frac{\alpha_i P_0^i}{V_0}$ are all positive and ranging from 2% to 10%, around the true value which is $\frac{1}{20} = 5\%$. As expected by the central limit theorem, the error between the true value and the estimated value is of the order of $\frac{1}{\sqrt{6 \times 250}} = 2.5\%$.

4 The Great Deleveraging of Fall 2008

Lehman Brothers was the fourth largest investment bank in the USA. During the year 2008, it experienced severe losses, caused mainly by the subprime mortgage crisis, and on September, 15^{th}, 2008, it filed for chapter 11 bankruptcy protection, citing bank debt of \$613 billion, \$155 billion in bond debt, and assets worth \$639 billion, becoming the largest bankruptcy filing in the US history.

The failure of Lehman Brothers generated liquidations and deleveraging in all asset classes all over the world. The collapse of this huge institution was such a shock to financial markets - major equity indices all lost around 10% on that day - that it triggered stop loss and deleveraging strategies among a remarkable number of financial institutions worldwide. Risk measures of portfolios, for example the value at risk, increased sharply, obliging financial institutions to hold more cash, which they got by deleveraging their portfolios, rather than by issuing debt which would have been very costly at such distressed times.

This massive deleveraging has been documented in several empirical studies. Fratzscher (2011) studies the effect of key events, such as the collapse of Lehman Brothers, on capital flows. He uses a dataset on portfolio capital flows and performance at the fund level, from EPFR, and containing daily, weekly and monthly flows for more than 16000 equity funds and 8000 bond funds, domiciled in 50 countries. He aggregates the net capital flows (ie net of valuation changes) for each country and finds that they are negative for all the countries of the study. This means that fund managers of such funds deleveraged their positions after the collapse of Lehman Brothers, sometimes in dramatic proportions: in some cases, the ouflows can represent up to 30% of the assets under management by the funds.

Our method allows to estimate the aggregate portfolio of liquidations during this period. We report below the result of the estimation method described in Section 3 SPDRs and components of the Eurostoxx 50 index. Figure 4 shows that the increase of average correlation in these two equity baskets lasted for around three months after September, 15^{th}, 2008. As a consequence, we examine liquidations that occured between September, 15^{th}, 2008 and December, 31^{st}, 2008.

We calculate the realized covariance matrices respectively between 02/01/2008 and $T = 09/15/2008$ and between $T = 09/15/2008$ and $T + \tau_{liq} = 12/31/2008$ and apply the estimation procedure described in Section 3. We use a linear price impact model

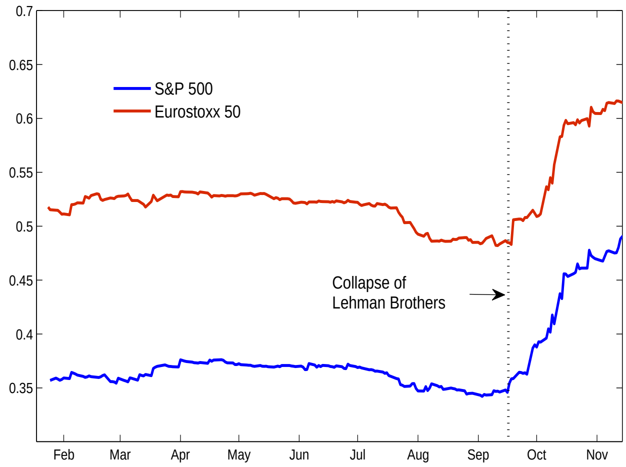

Figure 4: One-year average correlation among SPDRs and Eurostoxx 50

Obizhaeva (2011); Cont et al. (2010). To calibrate the market depth parameters λ_i, we follow the approach proposed by Obizhaeva Obizhaeva (2011): denoting by σ_i the average daily volatility of asset i and ADV_i the average daily trading volume, it was shown in Obizhaeva (2011) for a large panel of US stocks that the ratio $\frac{1}{\lambda}\frac{ADV}{\sigma_r}$ does not vary significantly from one asset to another and

$$\frac{1}{\lambda}\frac{ADV}{\sigma_r} \approx 0.33 \tag{29}$$

Obizhaeva (2011) also argues empirical evidence that the difference in price impact of buy-originated trades and sell-originated trades is not statistically significant. In order to lead our empirical study, we use average daily volumes and average daily volatility to estimate the market depth of each asset, using (29). Alternatively one could use intraday data, following the methodology proposed in Cont et al. (2010).

4.1 Sector ETFs

We first study fire sales among sector SPDRs, which are sector sub indices of the S&P 500. There exist nine sector SPDRs: Financials (XLF), Consumer Discretionary (XLY), Consumer Staples (XLP), Energy (XLE), Health Care (XLV), Industrials (XLI), Materials (XLB), Technology (XLK) and Utilities (XLU) and our goal is to determine

Sector SPDR	Estimated Market Depth $\times 10^8$ shares
Financials	34.8
Consumer Discretionary	4.4
Consumer Staples	6.2
Energy	8.8
Health Care	6.4
Industrials	8.1
Materials	7.0
Technology	7.9
Utilities	7.1

Table 1: Estimated market depth for SPDRs.

how economic actors investing in those SPDRs liquidated their portfolios following the collapse of Lehman Brothers.

In order to compute our estimation procedure, we need to know the market depth of each SPDR, which we can estimate as described in the previous section. Market depths are given in Table 1. We find that financials have the highest market depth and that other SPDRs have similar market depths.

We can then apply the estimation method described in Section 3 and find the magnitude of fire sales in each SPDR between September, 15^{th}, 2008 and December, 31^{st}, 2008.

Our method yields an estimate of 86 billion dollars for fire sales afffecting SPDRs between September, 15^{th}, 2008 and December, 31^{st}, 2008. Using Corollary 3.8, we can state that with a confidence interval of 99%, the hypothesis that the liquidation matrix M is equal to zero can be rejected and hence there were significant liquidations during this period, on this universe of assets. The liquidation volume that we find is equivalent to a daily liquidation volume of 1.2 billion dollars per day. In comparison, the average volume on SPDRs before Lehman Brother's collapse was 5.1 billion dollars per day. This shows how massive the liquidations were after this market shock.

Corollary 3.2 allows us to determine the composition of the aggregate portfolio liquidated between September 15^{th} 2008 and December, 31^{st}, 2008. The daily liquidated volumes and the proportions of each SPDR are given in Table 2. This shows that the aggregate portfolio liquidated after Lehman Brother's collapse was a long portfolio. This is consistent with the observation that many financial institutions liquidated equity holdings in order to meet capital requirements during this period, due to the increase of the risk associated with Lehman Brother's collapse. The highest volume of liquidations are associated with financial stocks, followed by the energy sector. Those two sectors represent 60% of the liquidations and more that 50 billion dollars liquidated before December, 31^{st}, 2008. All other sectors were liquidated in equivalent proportions.

As discussed in Section 3.1, the principal eigenvector of M reflects the common

Sector SPDR	Daily amount liquidated $\times 10^6\$$	Weight
Financials	320	28%
Consumer Discretionary	55	5%
Consumer Staples	38	3.5%
Energy	300	26%
Health Care	63	5.5%
Industrials	90	8%
Materials	110	9.5%
Technology	65	5.5%
Utilities	100	9%

Table 2: Daily volume and proportions of fire sales for SPDR between September 15^{th}, 2008 and Dec 31,2008.

Sector SPDR	Weight
Financials	78%
Consumer Discretionary	0%
Consumer Staples	2.5%
Energy	4%
Health Care	0%
Industrials	0%
Materials	2.5%
Technology	10%
Utilities	3%

Table 3: Proportions of fire sales between September 15^{th}, 2008 and December, 31^{st}, 2008 associated to the principal eigenvector of M

patterns of fire sales. Table 3 gives the proportions of fire sales associated to the principal eigenvector of M. We see that this portfolio is essentially made of financials, which have a weight of 78%. The large weight of XLF, the financial sector index, may be explained in terms of the loss of investor confidence in banks in the aftermath of the Lehman's collapse.

4.2 Eurostoxx 50

We now conduct our analysis on stocks belonging to the Eurostoxx 50 in order to determine the average composition of portfolios diversified among the components of the Eurostoxx 50 and that were liquidated after Lehman Brother's filing for bankruptcy. The Eurostoxx 50 is an equity index regrouping the 50 largest capitalizations of the Euro zone. It is the most actively traded index in Europe and is used as a benchmark to measure the financial health of the euro zone.

We use the same methodology as in the previous section (choice of dates, estimation of Σ and market depths). Note that we restricted our study to 45 stocks of the index, for which we had clean data. The 5 stocks left correspond to the lowest capitalizations among the index components, with very low liquidity.

We find that 350 billion euros were liquidated on stocks belonging to the Eurostoxx 50 between September, 15^{th}, 2008 and December, 31^{st}, 2008. Our statistical test described in Corollary 3.8 allows us to reject the hypothesis of no liquidation with a confidence interval of 99%. Our estimate for the liquidated volume is equivalent to a daily liquidation of 5 billion euros, which is equal to one third of the average daily volume of the index components before September, 15^{th}, 2008.

Figure 5, where each bar represents the weight of a stock in the aggregate liquidated portfolio, shows that most of the liquidations following Lehman Brother's collapse involved (long) *selling* of stocks, i.e. funds selling their holdings.

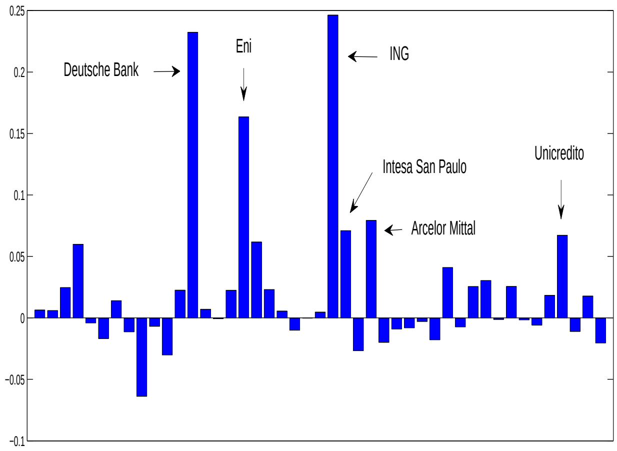

Figure 5: Fire sales in Eurostoxx 50 stocks in Fall 2008: each bar represents the weight of one stock in the aggregate liquidated portfolio

Figure 5 shows that fire sales are more intense for some stocks than others. Table 4 gives the detail of those stocks. As suggested by the previous section, we see that the fire sales in the Eurostoxx 50 index were concentrated in the financial and energy sectors. ING and Deutsche Bank account for almost half of the volume liquidated on the whole index.

5 The hedge fund losses of August 2007

From August 6^{th} to August 9^{th} 2007, long-short market-neutral equity funds experienced large losses: many funds lost around 10% per day and experienced a rebound of around

Stock	Amount liquidated $\times 10^6$ €	Weight
ING	1100	25%
Deutsche Bank	1000	23%
Eni	750	16%
Arcelor Mittal	350	8%
Intesa San Paolo	320	7%
Unicredito	300	6.5%

Table 4: Most liquidated stocks in the Eurostoxx 50 during the three months following September, 15^{th}, 2008

15% on August 10^{th}, 2007. During this week, as documented by Khandani and Lo (2007), market-neutral equity funds whose returns previously had a low historical volatility exhibited negative returns exceeding 20 standard deviations, while no major move was observed in equity market indices.

Khandani and Lo (2007) suggested that this event was due to a large market-neutral fund deleveraging its positions. They simulate a contrarian long-short equity market neutral strategy implemented on all stocks in the CRSP Database and were able to reconstitute qualitatively the empirically observed profile of returns of quantitative hedge funds : low volatility before August 6^{th}, huge losses during three days and a rebound on August 10^{th}. We reconstituted empirically the returns for Khandani and Lo's equity market neutral strategy on the S&P500 for the first three quarters of 2007. Figure 6 shows that this strategy underperforms significantly during the second week of August 2007, while no major move occured in the S&P 500. Such empirical results tend to confirm the hypothesis of the unwind of a large portfolio, which generated through price impact large losses across similar portfolios, as predicted by our model.

Using historical data on returns of 487 stocks from the S&P500 index, we have reconstituted the composition of the fund that deleveraged its positions during the second week of August 2007 using the estimation procedure described in Section 3 for the periods $[0, T] = [08/03/2006, 08/03/2007]$ and $[T, T + \tau_{liq}] = [08/06/2007, 08/09/2007]$.

Figure 7 displays the composition of the aggregate portfolio liquidated on the S&P500 during this period and found by our estimation method. The first and striking difference with the case of the deleveraging after Lehman Brother's collapse is that, during this quant event, the liquidated portfolio was a long-short portfolio. We clearly see in Figure 7 that for some stocks the liquidated position is significantly negative, meaning that a short position is being exited. More precisely, 250 stocks have positive weights in the liquidated portfolio, whereas 237 have negative weights. Furthermore, we find that the liquidated portfolio was highly leveraged: for each dollar of capital, 15 dollars are invested in long positions and 14 dollars are invested in short positions.

Importantly, the estimated portfolio is market-neutral in the sense of Equation (30):

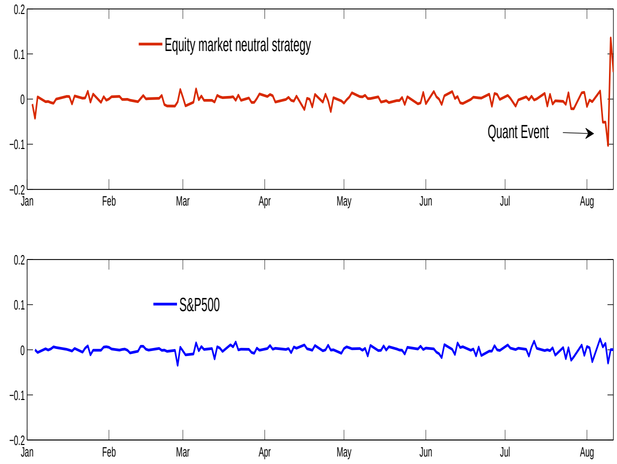

Figure 6: Returns of an market-neutral equity portfolio in 2007, compared with S&P500 returns.

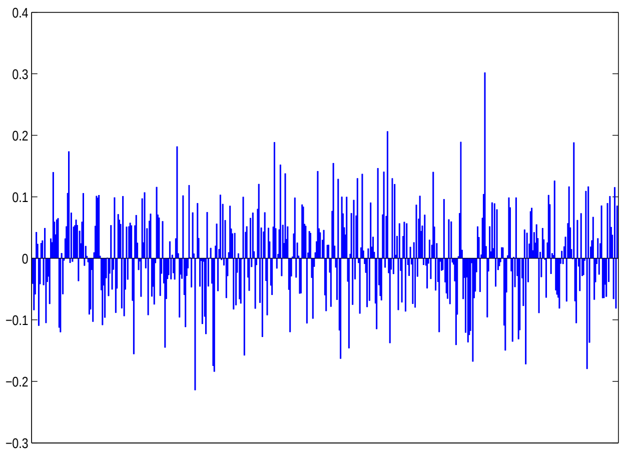

Figure 7: Composition of the aggregate equity portfolio liquidated during the 2^{nd} week of August

using the notations of Section 2.4 we find

$$\frac{\hat{\Lambda}.\pi_t^{\hat{\mu}}}{\|\hat{\Lambda}\|\|\pi_t^{\hat{\mu}}\|} = \frac{\sum_{i=1}^{n} \frac{\alpha_i}{\lambda_i} \mu_t^i P_t^i}{\|\hat{\Lambda}\|\|\pi_t^{\hat{\mu}}\|} = 0.0958 \tag{30}$$

which corresponds to an angle of 0.47π between the vectors $\hat{\Lambda}$ and $\pi_t^{\hat{\mu}}$, i.e. very close to orthogonality. This provides a quantitative explanation for the fact that, although massive liquidations occurred in the equity markets, index funds were not affected by this event. Note that, unlike other explanations proposed at the time, this explanation does not involve any assumption of liquidity drying up during the period of hedge fund turbulence.

Appendices

A Proof of Theorem 2.2

Assume that for all $1 \leq j \leq J$, $f_j \in \mathcal{C}_0^3$, for all $1 \leq i \leq n$, $\phi_i \in \mathcal{C}^3$ and $\mathbb{E}(\|\xi\|^4) < \infty$. Let $r > 0$.

As for all $1 \leq j \leq J$, $f_j \in \mathcal{C}_0^3$, f_j is bounded. As a consequence, there exists $R > 0$ such that, when $\|S_k\| \leq r$, for all $1 \leq i \leq n$

$$\sum_{1 \leq j \leq J} \alpha_i^j \left(f_j \left(\frac{(V_{k+1}^j)^*}{V_0^j} \right) - f_j \left(\frac{V_k^j}{V_0^j} \right) \right) \in [-R, R]$$

where V_k^j and $(V_{k+1}^j)^*$ are defined respectively in (3) and (4). As ϕ_i is $\mathcal{C}^3$, its third derivative is bounded on $[-R, R]$. As a consequence, using a Taylor expansion of ϕ_i and the fact that $\phi_i(0) = 0$, there exists $M > 0$ such that for all $1 \leq i \leq n$ and for all $x \in [-R, R]$:

$$|\phi(x) - x\phi'(0) - \frac{x^2}{2}\phi''(0)| \leq M|x|^3 \tag{31}$$

Fix $\|S\| \leq r$ and $1 \leq i \leq n$. Given (2) and (31), we have

$$\left|S_{k+1}^i - S_k^i\right| \leq |S_k^i| \left(\sqrt{\tau}|\xi_{k+1}^i| + \tau|m_i| + M' \left(\sum_{1 \leq j \leq J} |\alpha_i^j| \|f_j'\|_\infty |(V_{k+1}^j)^* - V_k^j| \right) \right)$$

As ξ has third-order moments, we find for $\|S\| \leq r$:

$$\mathbb{E}\left(|S_{k+1}^i - S_k^i|^3 | S_k = S\right) \leq C\tau^{\frac{3}{2}}$$

As a consequence, for $\epsilon > 0$:

$$\mathbb{P}(|S_{k+1}^i - S_k^i| \geq \epsilon | S_k = S) \leq \frac{1}{\epsilon^3} C\tau^{\frac{3}{2}}$$

and we find that for all $\epsilon > 0$ and $r > 0$:

$$\lim_{\tau \to 0} \sup_{\|S\| \leq r} \frac{1}{\tau} \mathbb{P}(|S_{k+1}^i - S_k^i| \geq \epsilon | S_k = S) = 0 \tag{32}$$

Fix $\|S\| \le r$ and let us now calculate for $1 \le j \le J$:

$$\mathbb{E}\left(f_j\left(\frac{(V^j_{k+1})^*}{V^j_0}\right) - f_j\left(\frac{V^j_k}{V^j_0}\right)|S_k = S\right)$$

$$= \frac{1}{V^j_0} f'_j\left(\frac{V^j_k}{V^j_0}\right)\mathbb{E}\left((V^j_{k+1})^* - V^j_k|S_k = S\right) + \frac{1}{2}\frac{1}{(V^j_0)^2} f''_j\left(\frac{V^j_k}{V^j_0}\right)\mathbb{E}\left(\left((V^j_{k+1})^* - V^j_k\right)^2|S_k = S\right) + o(\tau)$$

where, as the third derivative of f_j is bounded and ξ has third order moments, $o(\tau) = \tau h(\tau)$ with $h(\tau) \to 0$ when $\tau \to 0$, uniformly for $\|S\| \le r$.

As a consequence, given (3) and (4), we find that for all $1 \le j \le J$

$$\mathbb{E}\left(f_j\left(\frac{(V^j_{k+1})^*}{V^j_0}\right) - f_j\left(\frac{V^j_k}{V^j_0}\right)|S_k = S\right) = f'_j\left(\frac{V^j_k}{V^j_0}\right)\frac{\pi^j.m}{V^j_0} + \frac{1}{2}f''_j\left(\frac{V^j_k}{V^j_0}\right)\frac{\pi^j.\Sigma\pi^j}{(V^j_0)^2} + o(\tau) \tag{33}$$

and, using the fact that ξ has fourth order moments, we find that for all $1 \le j, l \le J$

$$\mathbb{E}\left(\left(f_j\left(\frac{(V^j_{k+1})^*}{V^j_0}\right) - f_j\left(\frac{V^j_k}{V^j_0}\right)\right)\left(f_l\left(\frac{(V^l_{k+1})^*}{V^l_0}\right) - f_l\left(\frac{V^l_k}{V^l_0}\right)\right)|S_k = S\right) = \tag{34}$$

$$f'_j\left(\frac{V^j_k}{V^j_0}\right) f'_l\left(\frac{V^l_k}{V^l_0}\right)\frac{\pi^j.\Sigma\pi^l}{V^j_0 V^l_0} + o(\tau)$$

where $\pi^j = (\alpha^j_1 S^1, ..., \alpha^j_n S^n)^t$.

Denote $a : \mathbb{R}^n \mapsto \mathcal{S}_n(\mathbb{R})$ and $b : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$a_{i,p}(S) = S^i S^p (\sigma(S)\sigma^t(S))_{i,p}$$

and

$$b_i(S) = S^i \mu_i(S)$$

where μ and σ are given in Equations 5 and 6. Thanks to (31), (33) and (34) we have for $\|S\| \le r$

$$\mathbb{E}\left(\phi_i\left(\sum_{1\le j\le J}\alpha^j_i\left(f_j\left(\frac{(V^j_{k+1})^*}{V^j_0}\right) - f_j\left(\frac{V^j_k}{V^j_0}\right)\right)\right)|S_k = S\right)$$

$$= \tau\phi'_i(0)\sum_{1\le j\le J}\alpha^j_i\left(f'_j\left(\frac{V^j_k}{V^j_0}\right)\frac{\pi^j.m}{V^j_0} + \frac{1}{2}f''_j\left(\frac{V^j_k}{V^j_0}\right)\frac{\pi^j.\Sigma\pi^j}{(V^j_0)^2}\right)$$

$$+\frac{\tau}{2}\phi''_i(0)\sum_{1\le j,l\le J}\alpha^j_i\alpha^l_i f'_j\left(\frac{V^j_k}{V^j_0}\right) f'_l\left(\frac{V^l_k}{V^l_0}\right)\frac{\pi^j.\Sigma\pi^l}{V^j_0 V^l_0} + o(\tau)$$

As a consequence

$$\mathbb{E}(S^i_{k+1} - S^i_k | S_k = S) = \tau b_i(S) + o(\tau)$$

which implies that for all $1 \leq i \leq n$ and $r > 0$:

$$\lim_{\tau \to 0} \sup_{\|S\| \leq r} \left| \frac{1}{\tau} \mathbb{E}(S^i_{k+1} - S^i_k | S_k = S) - b_i(S) \right| = 0 \tag{35}$$

Furthermore, for $1 \leq i, p \leq n$ and $\|S\| \leq r$, we have

$$\mathbb{E}\left(\sqrt{\tau} \xi^i_{k+1} \phi_p \left(\sum_{1 \leq j \leq J} \alpha^j_p \left(f_j \left(\frac{(V^j_{k+1})^*}{V^j_0} \right) - f_j \left(\frac{V^j_k}{V^j_0} \right) \right) \right) | S_k = S \right)$$

$$= \tau \phi'_p(0) \sum_{1 \leq j \leq J} \alpha^j_p f'_j \left(\frac{V^j_k}{V^j_0} \right) \frac{(\Sigma \pi^j)_i}{V^j_0} + o(\tau)$$

and

$$\mathbb{E}\left(\phi_i \left(\sum_{1 \leq j \leq J} \alpha^j_i \left(f_j \left(\frac{(V^j_{k+1})^*}{V^j_0} \right) - f_j \left(\frac{V^j_k}{V^j_0} \right) \right) \right) \phi_p \left(\sum_{1 \leq j \leq J} \alpha^j_p \left(f_j \left(\frac{(V^j_{k+1})^*}{V^j_0} \right) - f_j \left(\frac{V^j_k}{V^j_0} \right) \right) \right) | S_k = S \right)$$

$$= \tau \phi'_i(0) \phi'_p(0) \sum_{1 \leq j, l \leq J} \alpha^j_i \alpha^l_p f'_j \left(\frac{V^j_k}{V^j_0} \right) f'_l \left(\frac{V^l_k}{V^l_0} \right) \frac{\pi^j . \Sigma \pi^l}{V^j_0 V^l_0} + o(\tau)$$

As a consequence, for all $1 \leq i, p \leq n$ and $r > 0$:

$$\lim_{\tau \to 0} \sup_{\|S\| \leq r} \left| \frac{1}{\tau} \mathbb{E}[(S^i_{k+1} - S^i_k)(S^p_{k+1} - S^p_k) | S_k = S] - a_{i,p}(S) \right| = 0 \tag{36}$$

a and b are continuous functions and for all S, $a(S)$ is positive and (32), (35) and (36) hold. Furthermore, as for all $1 \leq j \leq J$, $f_j \in C^3_0$, a and b are Lipschitz. Define the differential operator $G : C^\infty_0(\mathbb{R}^n) \mapsto C^1_0(\mathbb{R}^n)$ by

$$Gh(x) = \frac{1}{2} \sum_{1 \leq i, j \leq n} a_{i,j}(x) \partial_i \partial_j h + \sum_{1 \leq i \leq n} b_i(x) \partial_i h$$

associated to the stochastic differential equation

$$dP_t = b(P_t)dt + \sqrt{a(P_t)} dW_t$$

which is the same equation as the stochastic differential equation given in Theorem 2.4 and which, as a and b are Lipschitz, has a unique strong solution $(P_t)_{t \geq 0}$.

By (Ethier and Kurtz, 1986, Theorem 4.2, Ch.7), the process $(S_{\lfloor \frac{t}{\tau} \rfloor})_{t \geq 0}$ converges in distribution to the solution $(\mathbb{P}, (P_t)_{t \geq 0})$ of the martingale problem for (G, δ_{S_0}) when $\tau \to 0$. $(P_t)_{t \geq 0}$ is thus the unique solution of the stochastic differential equation given in Theorem 2.4.

B Proofs of Proposition 3.1, 3.4 and 3.6

Let us invert (18) under the assumptions of Proposition 3.1. Denote $\Omega^{(i)} = \begin{pmatrix} \Omega_{1,i} \\ \vdots \\ \Omega_{n,i} \end{pmatrix}$ the i-th column of the matrix Ω. By definition, we know that $\Pi\Sigma L^{-1}\Omega^{(p)} = \phi_p \Omega^{(p)}$ which is equivalent to $(\Omega^{(p)})^t L^{-1}\Sigma\Pi = \phi_p (\Omega^{(p)})^t$.

As (18) is equivalent to $M\Pi\Sigma L^{-1} + L^{-1}\Sigma\Pi M = L^{-1}(C_{[T,T+\tau_{liq}]} - \Sigma)L^{-1}$ and multiplying this equality on the left by $(\Omega^{(p)})^t$ and on the right by $\Omega^{(q)}$, we find that

$$(\phi_p + \phi_q)[\Omega^t M \Omega]_{p,q} = [\Omega^t L^{-1}(C_{[T,T+\tau_{liq}]} - \Sigma)L^{-1}\Omega]$$

which gives the matrix $\Omega^t M \Omega$ as a function of Σ and $C_{[T,T+\tau_{liq}]}$. As Ω is invertible, this characterizes the matrix M, as a function, denoted Φ of Σ and $C_{[T,T+\tau_{liq}]}$, proving (19) and (20) of Proposition 3.1.

Furthermore, notice that $M_0 = \Phi\left(\Sigma, C_{[T,T+\tau_{liq}]} + O(\|\Lambda\|^2)\right)$. Given the expression for Φ in (20), (21) follows directly. This concludes the proof of Proposition 3.1.

Assume now that Assumption 3.3 holds. This implies that (19), (20) and (21) hold.

Lemma B.1 *The mapping Φ defined in* (20) *is C^∞ in a neighborhood of* (Σ, C).

Proof Consider the following mapping:

$$F : \mathcal{S}_n^3(\mathbb{R}) \mapsto \mathcal{S}_\mathrm{n}(\mathbb{R}), (\mathrm{S}, \mathrm{C}, \mathrm{N}) \mapsto \mathrm{LN\Pi S} + \mathrm{S\Pi NL} + \Sigma - \mathrm{C} \tag{37}$$

F is infinitely differentiable. The first derivative of F with respect to N, is equal to

$$\frac{\partial F}{\partial N}(S, C, N).H_3 = LH_3\Pi S + S\Pi H_3 L$$

As Σ verifies Assumption 3.3, we showed that $\frac{\partial F}{\partial N}(\Sigma, C, N)$ is invertible for all C. As $\Phi(\Sigma, C)$ is defined as the only matrix verifying $F(\Sigma, C, \Phi(\Sigma, C)) = 0$, the implicit function theorem states that Φ is C^∞ in a neighborhood of (Σ, C).

As convergence in probability implies that a sub-sequence converges almost surely, we assume from now on that the estimators defined in (22) and (23) converge almost surely. As a consequence, for τ sufficiently small, $\widehat{\Sigma}^{(\tau)}$ also verifies Assumption 3.3 and we can define $\widehat{M}^{(\tau)}$ as in (24).

Lemma B.1 implies in particular that Φ is continuous which implies that $\Phi(\widehat{\Sigma}^{(\tau)}, \widehat{C}^{(\tau)})$ is a consistent estimator of $\Phi(\Sigma, C_{[T,T+\tau_{liq}]})$, meaning that $\widehat{M}^{(\tau)}$ is a consistent estimator of M. This shows Proposition 3.4.

Furthermore $\Phi \in C^1$ and the central limit theorem for $(\widehat{\Sigma}^{(\tau)}, \widehat{C}^{(\tau)})$ given in Proposition 3.6 is a direct consequence of the delta method for estimators.

C Proof of Proposition 3.7

Under the null hypothesis (H_0),

$$\frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} c_t \,\mathrm{d}t = \Sigma$$

and hence

$$\Phi\left(\Sigma, \frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} c_t \,\mathrm{d}t\right) = \Phi(\Sigma, \Sigma) = 0$$

Let us calculate now the first derivative of Φ on (Σ, Σ). Recall that $\Phi(\Sigma, C)$ is defined as the only element of $\mathcal{S}_n(\mathbb{R})$ such that $F(\Sigma, C, \Phi(\Sigma, C)) = 0$, where F is defined in (37). F is affine in each component and as a consequence is C^∞ and we can define its derivatives on (Σ, C, M), $\frac{\partial F}{\partial \Sigma}(\Sigma, C, M)$, $\frac{\partial F}{\partial C}(\Sigma, C, M)$ and $\frac{\partial F}{\partial M}(\Sigma, C, M)$ which are linear mappings from $\mathcal{S}_n(\mathbb{R})$ to $\mathcal{S}_n(\mathbb{R})$ defined by:

$$\frac{\partial F}{\partial \Sigma}(\Sigma, C, M).H_1 = LM\Pi H_1 + H_1 \Pi M L + H_1$$

$$\frac{\partial F}{\partial C}(\Sigma, C, M).H_2 = -H_2$$

$$\frac{\partial F}{\partial M}(\Sigma, C, M).H_3 = LH_3\Pi\Sigma + \Sigma\Pi H_3 L$$

As a consequence, we have

$$dF(\Sigma, C, M).(H_1, H_2, H_3) = LM\Pi H_1 + H_1\Pi M L + H_1 - H_2 + LH_3\Pi\Sigma + \Sigma\Pi H_3 L$$

In the proof of Lemma B.1, we showed that $\frac{\partial F}{\partial \Sigma}(\Sigma, C, M)$ is invertible. As a consequence we can apply the implicit function theorem in order to compute the derivative of Φ. As $F(\Sigma, C, \Phi(\Sigma, C)) = 0$ and $\Phi(\Sigma, \Sigma) = 0$, we find the derivative of Φ on (Σ, Σ):

$$\nabla\Phi(\Sigma, \Sigma).(H_1, H_2) = \left(\frac{\partial F}{\partial M}(\Sigma, \Sigma, 0)\right)^{-1}(H_2 - H_1)$$

which is equivalent to

$$\nabla\Phi(\Sigma, \Sigma).(H_1, H_2) = \Phi(\Sigma, \Sigma + H_2 - H_1)$$

Using Proposition 3.6, we find that

$$\frac{1}{\sqrt{\tau}} \widehat{M}^{(\tau)} \overset{\mathcal{L}}{\Rightarrow} \Phi\left(\Sigma, \Sigma + \frac{1}{\tau_{liq}}(\overline{Z}_{T+\tau_{liq}} - \overline{Z}_T) - \frac{1}{T}\overline{Z}_T\right)$$

D Proof of Corollary 3.8

Under (H_0), we have $\sigma\sigma^t = \Sigma$ and the expression for the process $\tilde{V}_t$ defined in (26) is simplified as

$$(\tilde{V}_t\tilde{V}_t^t)^{ij,kl} = \Sigma_{i,k}\Sigma_{j,l} \tag{38}$$

which implies that the process $\overline{Z}$ defined in (25) is gaussian.

Furthermore, given Proposition 3.7, under (H_0), $\frac{1}{\sqrt{\tau}}\left(P_T^t\widehat{M}^{(\tau)}(P_T - P_{T+\tau_{liq}})\right)$ converges in law when τ goes to zero to the random variable

$$P_T^t\Phi\left(\Sigma, \Sigma + \frac{1}{\tau_{liq}}(\overline{Z}_{T+\tau_{liq}} - \overline{Z}_T) - \frac{1}{T}\overline{Z}_T\right)(P_T - P_{T+\tau_{liq}})$$

Given the explicit expression for Φ given in (20), we can find the expression for:

$$P_T^t\Phi(\Sigma, C)(P_T - P_{T+\tau_{liq}})$$

$$= \sum_{1\leq p,q\leq n}(\Omega^{-1}P_T)_p\frac{\left[\Omega^tL^{-1}(C-\Sigma)L^{-1}\Omega\right]_{p,q}}{\phi_p + \phi_q}(\Omega^{-1}(P_T - P_{T+\tau_{liq}}))_p$$

Given the fact that $L^{-1} = diag(\lambda_i)$, we have $(\Omega^{-1}L^{-1})_{p,i} = \Omega_{p,i}\lambda_i$ and $(L^{-1}\Omega)_{j,q} = \Omega_{j,q}\lambda_j$. As a consequence, denoting

$$m_{i,j} = \sum_{1\leq p,q\leq n}\frac{[\Omega^{-1}P_T]_p[\Omega^{-1}(P_T - P_{T+\tau_{liq}})]_q}{\phi_p + \phi_q}\Omega_{ip}\Omega_{jq}\lambda_i\lambda_j$$

we can write $P_T^t\Phi(\Sigma, C)(P_T - P_{T+\tau_{liq}})$ as $\sum_{1\leq i,j\leq n} m_{ij}(C_{i,j} - \Sigma_{i,j})$. Hence the limit of $\frac{1}{\sqrt{\tau}}\left(P_T^t\widehat{M}^{(\tau)}(P_T - P_{T+\tau_{liq}})\right)$ is equal to

$$\sum_{1\leq i,j\leq n} m_{ij}\left(\frac{1}{\tau_{liq}}(\overline{Z}_{T+\tau_{liq}} - \overline{Z}_T) - \frac{1}{T}\overline{Z}_T\right)_{i,j}$$

Under assumption (H_0), (38) holds and the limit process is gaussian. We now calculate its variance.

First, let us calculate the variance of $\sum_{1\leq i,j\leq n} m_{ij}\overline{Z}_t^{i,j}$ which, given the expression of $\overline{Z}$ in (25), can be written as

$$\sum_{1\leq k,l\leq n}\int_0^t \frac{1}{\sqrt{2}}\sum_{1\leq i,j\leq n} m_{i,j}\left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl}\right)\mathrm{d}\tilde{W}_s^{kl}$$

which has a variance equal to

$$\sum_{1\leq k,l\leq n}\int_0^t\left(\sum_{1\leq i,j\leq n}\frac{1}{\sqrt{2}}m_{i,j}\left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl}\right)\right)^2\mathrm{d}s$$

$$= \frac{t}{2} \sum_{1 \le k,l \le n} \left(\sum_{1 \le i,j,p,q \le n} m_{i,j} m_{p,q} \left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl} \right) \left(\tilde{V}_s^{pq,kl} + \tilde{V}_s^{qp,kl} \right) \right)$$

$$= \frac{t}{2} \sum_{1 \le i,j,p,q \le n} m_{i,j} m_{p,q} \left(\sum_{1 \le k,l \le n} \left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl} \right) \left(\tilde{V}_s^{pq,kl} + \tilde{V}_s^{qp,kl} \right) \right)$$

$$= t \sum_{1 \le i,j,p,q \le n} m_{i,j} m_{p,q} \left(\Sigma_{i,p} \Sigma_{j,q} + \Sigma_{i,q} \Sigma_{j,p} \right)$$

using the fact that $\sum_{1 \le k,l \le n} \tilde{V}_s^{ij,kl} \tilde{V}_s^{pq,kl} = \Sigma_{i,p} \Sigma_{j,q}$ as $\tilde{V}$ verifies (38).

As a consequence, we find that the variance of the gaussian limit is equal to

$$\left(\frac{1}{T} + \frac{1}{\tau_{liq}} \right) \sum_{1 \le i,j,k,l \le n} m_{ij} m_{kl} \left(\Sigma_{ik} \Sigma_{jl} + \Sigma_{jk} \Sigma_{il} \right)$$

which concludes the proof of Corollary 3.8.

Bibliography

Andersen, T., Bollerslev, T., Diebold, F., and Labys, P. (2003). Modeling and forecasting realized volatility. *Econometrica*, 71(2):579–625.

Anton, M. and Polk, C. (2008). Connected stocks. *Working paper, London Sch. Econ.*

Bailey, N., Kapetanios, G., and Pesaran, M. (2012). Exponent of cross-sectional dependence: Estimation and inference. *working paper*.

Barndorff-Nielsen, O. E. and Shephard, N. (2004). Econometric analysis of realised covariation: high frequency covariance, regression and correlation in financial economics. *Econometrica*, 72(2):885–925.

Brunnermeier, M. (2008). Deciphering the liquidity crunch 2007-2008. *Journal of Economic Perspectives*, 23:77–100.

Brunnermeier, M. and Pedersen, L. (2005). Predatory trading. *Journal of Finance*, 60(4):1825–1863.

Carlson, M. (2006). A brief history of the 1987 stock market crash with a discussion of the federal reserve response. *FEDS working paper*, 13.

Cont, R., Kukanov, A., and Stoikov, S. (2010). The price impact of order book events. Working paper.

Cont, R. and Wagalath, L. (2012). Running for the exit: distressed selling and endogenous correlation in financial markets. *Mathematical Finance*, in press.

Coval, J. and Stafford, E. (2007). Asset fire sales (and purchases) in equity markets. *Journal of Financial Economics*, 86(2):479–512.

Da Fonseca, J., Grasselli, M., and Tebaldi, C. (2008). A multifactor volatility Heston model. *Quant. Finance*, 8(6):591–604.

Danielsson, J., Shin, H. S., and Zigrand, J.-P. (2004). The impact of risk regulation on price dynamics. *Journal of Banking and Finance*, 28(5):1069 – 1087.

Engle, R. (2002). Dynamic conditional correlation: A simple class of multivariate garch models. *Journal of Business and Economic Statistics*, 20:339–350.

Ethier, S. and Kurtz, T. (1986). *Markov Processes: Characterization and Convergence.* Wiley.

Fielding, E., Lo, A. W., and Yang, J. H. (2011). The National Transportation Safety Board: A model for systemic risk management. *Journal Of Investment Management.*, 9(1).

Fratzscher, M. (2011). Capital flows, push versus pull factors and the global financial crisis. *European Central Bank working paper.*

Genotte, G. and Leland, H. (1990). Market liquidity, hedging and crashes. *American Economic Review*, 80:999–1021.

Gouriéroux, C., Jasiak, J., and Sufana, R. (2009). The Wishart autoregressive process of multivariate stochastic volatility. *Journal of Econometrics*, 150(2):167–181.

Greenwood, R. and Thesmar, D. (2011). Stock price fragility. *Journal of Financial Economics*, 102(3):471–490.

Jacod, J. and Protter, P. (2012). *Discretization of processes.* Springer.

Jotikasthira, P., Lundblad, C., and Ramadorai, T. (2011). Asset fire sales and purchases and the international transmission of funding shocks. *Journal of finance (forthcoming).*

Khandani, A. and Lo, A. (2007). What happened to the quants in August 2007? *Journal of investment management*, 5:5–54.

Obizhaeva, A. (2011). Selection bias in liquidity estimates. *Working Paper.*

Pedersen, L. (2009). When everyone runs for the exit. *International journal of central banking*, 5(4):177–199.

Shin, H. S. (2010). *Risk and Liquidity.* Oxford University Press.

Shleifer, A. and Vishny, R. (1992). Liquidation values and debt capacity: A market equilibrium approach. *Journal of Finance*, 47(4):1343–1366.

Shleifer, A. and Vishny, R. (2011). Fire sales in finance and macroeconomics. *Journal of Economic Perspectives*, 25(1):29–48.

Stelzer, R. (2010). Multivariate COGARCH$(1,1)$ processes. *Bernoulli*, 16(1):80–115.