ULRICH IDEALS AND MODULES, II
(THE CASE OF 2 DIMENSIONAL RATIONAL SINGULARITIES)

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The concept of Ulrich ideals and modules is explained by Goto and I will explain classification of Ulrich ideals and modules on 2-dimensional rational singularities. As in the talk of Goto, this lecture is based on [GOTWY].

Our language is somewhat geometric and we use one-to-one correspondence of integrally closed ideals and anti-nef cycles on the resolution of singularities. Also, we use MacKay correspondence of the set of exceptional curves on the minimal resolution of 2-dimensional rational double point and the set of indecomposable maximal Cohen-Macaulay modules.

1. Preliminaries

In the following, assume that A is a two-dimensional rational singularity and $f : X \to \operatorname{Spec} A$ is a resolution of singularities with $E := f^{-1}(\mathfrak{m})$ the exceptional divisor on X. Let $E = \cup_{i=1}^{r} E_i$ be the decomposition into irreducible components of E.

An $\mathfrak{m}$-primary ideal I is said to be *represented on* X if the sheaf $I\mathcal{O}_X$ is invertible and $I = H^0(X, I\mathcal{O}_X)$. If an ideal I is represented on some resolution $X \to \operatorname{Spec} A$, then it is integrally closed. Conversely, any integrally closed ideal can be represented on some (may not be minimal) resolution X of the singularity. In fact, Giraud [Gi] showed the following

Theorem 1.1. *Assume that A is a two-dimensional rational singularity. Then there is a one-to-one correspondence between the set of integrally closed $\mathfrak{m}$-primary ideals I in A that are represented on X and the set of anti-nef cycles $Z = \sum_{i=1}^{r} a_i E_i$ on X* (i.e. $Z \neq 0$, $ZE_i \leq 0$ for all i). *The correspondence is given by $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ and $I = H^0(X, \mathcal{O}_X(-Z))$.*

Lemma 1.2 (Riemann-Roch formula). *Let A be a 2-dimensional rational singularity, $f : X \to \operatorname{Spec}(A)$ a resolution of A, M a reflexive A module and $I = H^0(X, \mathcal{O}_X(-Z))$ be an integrally closed ideal such that $I\mathcal{O}_X = \mathcal{O}_X(-Z))$. We put $\mathcal{M} = f^*(M)/(torsion)$. Then we have $\ell_A(A/I) = -\dfrac{Z^2 + K_X Z}{2}$ and*

$$\ell_A(M/IM) = \operatorname{rank}_A M \cdot \ell_A(A/I) + c(\mathcal{M})Z,$$

where $c(\mathcal{M})$ is the Chern class of M.

A resolution $f : X \to \operatorname{Spec} A$ is *minimal* if X contains no (-1) curve C (i.e. $C \cong \mathbb{P}^1$ and $C^2 = -1$). In dimensional two, such a minimal resolution is unique up to isomorphism. Further, it is known that for each (-1)-curve C on X the morphism $f : X \to \operatorname{Spec} A$ is decomposed as $f = \pi \circ g$ such that $\pi : X \to X'$ is a contraction of C and $g : X' \to \operatorname{Spec} A$ is a morphism of schemes with X' regular.

Suppose that $f : X \to \operatorname{Spec} A$ is a minimal resolution. In the set $\mathcal{C}$ of cycles supported on $E = f^{-1}(\mathfrak{m})$, we define a partial order $\leq$ as follows: for Z, $Z' \in \mathcal{C}$, $Z \leq Z'$ if $Z' - Z$ is effective (i.e. every coefficient of E_i is non-negative). Then by the *fundamental cycle* (denoted by Z_0) on X is meant the minimum element with respect to the order $\leq$ among all (non-zero) anti-nef cycles.

2. Ulrich ideals in 2-dimensional rational singularities

Theorem 2.1. *Let $(A, \mathfrak{m})$ be a two-dimensional rational singularity and let I be an Ulrich ideal in A, which is not a parameter ideal. Then*

(1) *I is integrally closed,*
(2) *$I\mathcal{O}_X = \mathcal{O}_X(-Z)$ is invefrtible, where X is the minimal resolution of A.*
(3) *The number of Ulrich ideals is finite and we can explicitly write them down in terms of anti-nef cycle Z on the minimal resolution so that $I = H^0(X, I\mathcal{O}_X(-Z))$.*

Example 2.2. *Let A be a cyclic quotient singularity.*

(1) *If A is Gorenstein of type (A_n), then the number of Ulrich ideals in A is $[(n+1)/2]$.*
(2) *If A is not Gorenstein, then $\mathfrak{m}$ is the only Ulrich ideal of A.*

Example 2.3. *Let A be the localization of $k[T, xT^a, x^{-1}T^b, (x+1)^{-1}T^c]$ at the graded maximal ideal with $(2 \leq a \leq b \leq c)$. Then the number of Ulrich ideals in A is a.*

The following lemma plays the key role.

Lemma 2.4. *Let $(A, \mathfrak{m})$ be a two-dimensional local ring and let I be an $\mathfrak{m}$ primary ideal, which is not a parameter ideal and minimally generated by $\mu(I)$ elements. Then we have*

$$e(I) \leq (\mu(I) - 1)\ell_A(A/I)$$

and equality holds if and only if I is an Ulrich ideal.

If there is some time left, I will talk about Ulrich ideals in simple elliptic singularities.

References

[Gi] J. Giraud, *Improvement of Grauert-Riemenschneider's Theorem for a normal surface*, Ann. Inst. Fourier, Grebnoble **32** (1982), 13–23.

[GOTWY] S. Goto, K. Ozeki, R. Takahashi, K.-i. Watanabe and K. Yoshida, *Ulrich ideals and modules,* preprint.

[Li] J. Lipman, *Rational singularities with applications to algebraic surfaces and unique factorization*, Inst. Hautes Études Sci. Publ. Math. **36**, (1969), 195–279.

[WY] K.-i.Watanabe and K.Yoshida, *Hilbert-Kunz multiplicity, McKay correspondence and good ideals in two-dimensional rational singularities*, manuscripta math. 104 (2001), 275–294.