# Some conditions for the existence of one $C^k$-invariant foliation for Lorenz-like map.

José Vidarte [1] PhD thesis supervised by Daniel Smania [2]
Departamento de Matemática, ICMC, USP. 1:vidarte@icmc.usp.br 2:smania@icmc.usp.br

## Introduction

In 1963, a meteorologist E. N. Lorenz in the paper titled "Deterministic nonperiodic flow" [N.L63] introduced the following differential equation so-called Lorenz System.

$$(\dot{x}, \dot{y}, \dot{z}) = (\sigma(y - x), \rho x - y - xz, -\beta z + xy), \quad (1)$$

Setting the parameters $(\sigma, \rho, \beta)$ to be at $(10, 28, \frac{8}{3})$. Lorenz numerically found a solution which remains bounded forever but behaves in a very complicated manner. This was one the first examples of what is now know as "chaos".

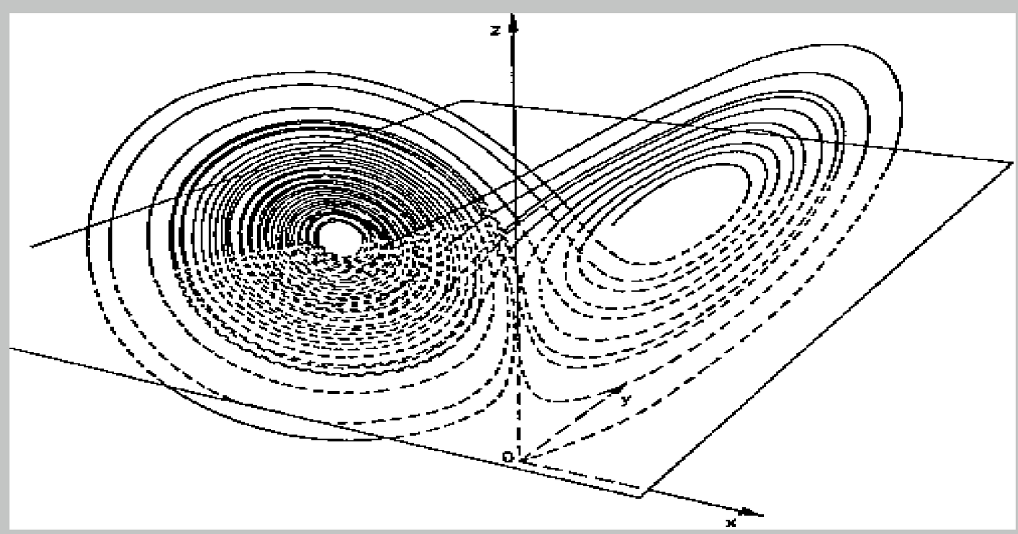

Figure : Numerical solution of the Lorenz equation (1):Graph obtained from the book of Guckenheimer [GH83]

Inspired in the numerically simulation of (1) Guckenheimer in the paper a strange, strange attractor, [J.G76] and Afraimovich, Bykov and Shil'nikov in the paper the origin and structure of the Lorenz attractor [ABS77] constructed a Geometrical Model as a abstract three-dimensional vector field which his corresponding Poincaré map satisfied some properties. Briefly with this model they have proven the existence of a well-defined attractor in the Geometrical Model. Although it was not until 1999 that the existence of this attractor for the Lorenz system itself (1) was rigorously proved by Warwick Tucker. In this poster, we will give the definition of the geometrical model due Afraimovich, Bykov and Shil'nikov [ABS77] so-called, **The Geometric Lorenz Flow**, as well as, we state some properties of this Flow, and our main theorem about the existence the of one $C^k-$ invariant stable foliation for the Lorenz-type map. We can with this foliation to associate an one-dimensional transformation $f$of class $C^k$ (defined on an interval). This allows us to study the dynamical properties for the original flow using techniques of one-dimensional dynamics.

## Geometric Lorenz Flow according to Afraimovich, Bykov and Shilnikov [ABS77]

A geometric Lorenz flow is a vector field $X$ of class $C^r - r \geq 1$, which satisfies the following properties:

L1) There is a equilibrium point of saddle type for $X$, in our case we assume that $0$ is the equilibrium point and suppose that the elements of $Spec(DX(0)) = \{\lambda_1, \lambda_2, \lambda_3\}$ satisfies the following relation:

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1 < -\lambda_2. \quad (2)$$

From the stable manifold Theorem for a Fixed Point and (2) we have that there are the stable manifold $W^s(0)$ and dimension two and the unstable manifold $W^u(0)$ of dimension one.

L2) There exist a transversal section $D := \{(x, y) \in \mathbb{R}^{n+1} :\| x \| < 1, |y| < 1\}$, and sets

$$D_{\pm} := \{(x, y) \in D : y(\pm)\}, \Gamma := \{(x, y) \in D \in D : y = 0\}$$

with following properties:

(1) $\Gamma = W^s(0) \cap D$, that is, for any trajectory begins on $D_0$ never comes back to beat $D$ for $t > 0$.

(2) There are a Poincaré map $P : D_+ \cup D_- \to D$ of $X$ given by the equation

$$P(x, y) = (F_i(x, y), G_i(x, y)) = (\overline{x}, \overline{y}),$$

$i = 1, 2.$

(3) $F_i$ and $G_i$ admit continuous extensions on $\Gamma$, and

$$\lim_{y\to 0} F_i(x, y) = x_i^*, \lim_{y\to 0} G_i(x, y) = y_i^* i = 1, 2.$$

(4) it is assumed that the images $P(D^{\pm})$ are depicted in Figure (2),

(5) Let us impose the following restrictions on $P$ called hyperbolicity conditions

(a) $||(f_x)|| < 1,$

(b) $1 - ||(g_y)^{-1}||||f_x|| > 2\sqrt{||(g_y)^{-1}||.||(g_y)^{-1}.f_y||},$

(c) $||(g_y)^{-1}|| < 1,$

(d) $||(g_y)^{-1}.f_y||.||g_x|| < (1 - ||f_x||)(1 - (g_y)^{-1}),$

Hereafter, $\| . \| = \sup_{(x,y)\in D_+\cup D_-} \| . \|$

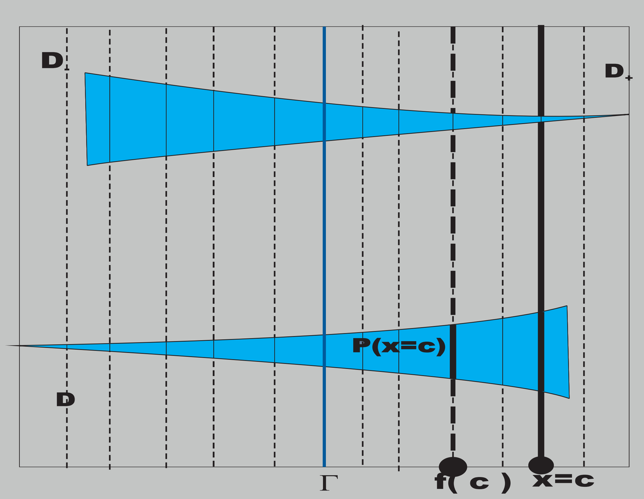


Figure : Graph of the images $P(D^{\pm})$

## Some remarks about of the geometric Lorenz flow

1. It follows from analysis of the behaviour of trajectories near $W^s(0)$ that in a small neighborhood of $\Gamma$ the following representation is valid [ABS82]:

$$F_i = x_i^* + \varphi_i(x, y)|y|^{\alpha}, G_i = y_i^* + \psi_i(x, y)|y|^{\alpha}, i = 1, 2 \quad (3)$$

where the functions $\varphi_i, \psi_i$ are $C^r-$ smooth in $x$; moreover,

$$|(\partial_y\varphi_i)y^{1-\beta_i}| < K_i, |(\partial_y\psi_i)y^{1-\alpha_i}| < K_i,$$

where $K_i$, are constants and $\beta_i, \alpha_i$ are positive numbers less that 1.

2. Afraimovich-Bykov-Shilnikov [ABS82] with the about property $L2)$ proved the existence of an invariant continuous stable foliation $\mathcal{F}_D$ in the region $D$ invariant to $P$. Each leaf $\widetilde{h} \in \mathcal{F}_D$ is the graph of a function $y = h(x)$, where $h$ is Lypschitz-continuous: $|h(x_1) - h(x_2)| \leq M|x_1 - x_2|$.

3. We note that all the conditions $L1) - L2)$ hold for a systems in a small neighborhood $U$ od system $X$.

4. Define $A_i(X) = \lim_{y\to 0} \psi_i(X)(x, y)$, the values are called the separatrix values and we also assumed that $A_i$ do not vanish. Therefore, it is natural to distinguish the following cases [ABS82]:
   A. (Orientable):$A_1(X) > 0, A_2(X) > 0;$
   B. (Semiorientable):$A_1(X) > 0, A_2(X) < 0;$
   C. (Noorientable):$A_1(X) < 0, A_2(X) < 0.$

5. Let $B = \bigcup_{k=0}^{\infty} F^{-k}\Gamma$, then $F^i$ is well-defined on the set $D \setminus B$ for any $i \in \mathbb{N} \cup \{0\}$, so we can defined the set $A := \bigcap_{i=0}^{\infty} F^i(D \setminus B)$ so-called a geometric Lorenz attractor [VA03].

5. Guckeheimer [J.G76] who introduce the notions of Geometric Lorenz flow, in fact considered, the case where the function $G_i(x, y) := G_i(y)$ are independent of $x$, and the conditions $(a) - (d)$ of hyperbolicity became simpler:namely:$g_i'(y) > \sqrt{2}, i = 1, 2$ It was show in [ABS82] that all the main resulted related to the simplified situation still hold for the more general case described above.

## The cross section $D$ for the geometric model of the Lorenz equation

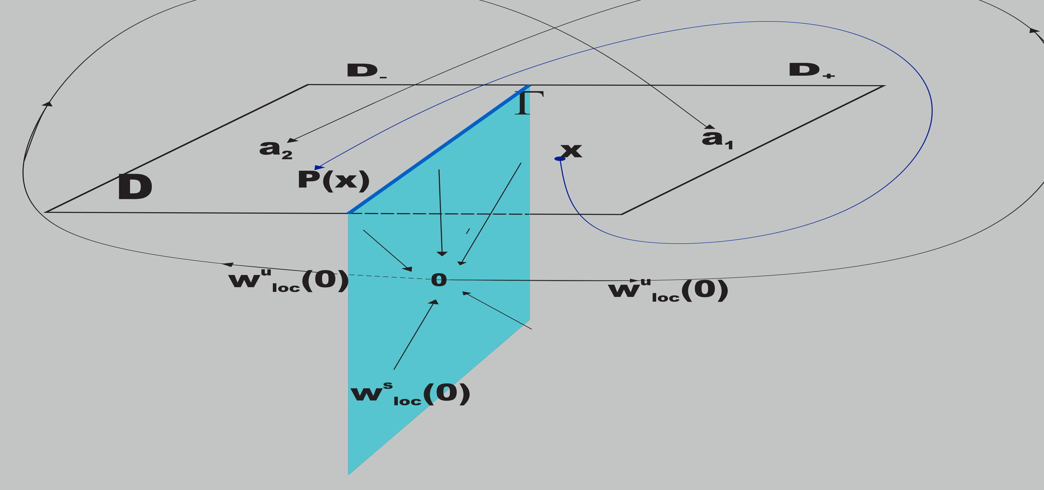


Figure : Behaviour of Poincaré map on $D$.

## Definitions and statement of result [SS94]

Consider the following sets: $D := \{(x, y) \in \mathbb{R}^{n+1} :\| x \| < 1, |y| < 1\}$,

$$D_{\pm} := \{(x, y) \in D : y(\pm)\}, \Gamma := \{(x, y) \in D \in D : y = 0\}$$

Consider the map $\widetilde{T} : D_+ \cup D_- \to D$ given by the equation

$$\widetilde{T}(x, y) = (F(x, y), G(x, y)) = (\overline{x}, \overline{y}),$$

such that the vector function $F$ and scalar function $G$ are differentiables in $D_+ \cup D_-$, and $\partial_y G(x, y) \neq 0$ for all $(x, y) \in D_+ \cup D_-$.

Setting $a(x, y) := (\partial_y G(x, y))^{-1}$. We define $A(x, y) := a(x, y).\partial_x F(x, y)$, $B(x, y) := a(x, y).\partial_y F(x, y)$, $C(x, y) := a(x, y).\partial_x G(x, y)$.

We assume the following conditions on $\widetilde{T}$ :

$(L_1)$ The function $F$ and $G$ have the form

$$F(x, y) = \begin{cases} x_+^* + |y|^{\alpha}[B_+^* + \varphi_+(x, y)], & \text{if } y > 0 \\ x_-^* + |y|^{\alpha}[B_-^* + \varphi_-(x, y)] & \text{if } y < 0 \end{cases},$$

$$G(x, y) = \begin{cases} y_+^* + |y|^{\alpha}[A_+^* + \psi_+(x, y)], & \text{if } y > 0 \\ y_-^* + |y|^{\alpha}[A_-^* + \psi_-(x, y)] & \text{if } y < 0 \end{cases}$$

where the functions $\varphi_{\pm} \in C^{k+1}(D_+ \cup D_-)$ and $\psi_{\pm} \in C^{k+1}(D_+ \cup D_-)$. The derivatives of $\varphi_{\pm}$ and $\psi_{\pm}$ are uniformly bounded with respect to $x$ and satisfies the estimates.

$$\left\| \frac{\partial^{l+m}\varphi_{\pm}(x, y)}{\partial x^l \partial y^m} \right\| \leq K|y|^{\gamma - m}, \left\| \frac{\partial^{l+m}\varphi_{\pm}(x, y)}{\partial x^l \partial y^m} \right\| \leq K|y|^{\gamma - m}, \quad (4)$$

where $k < \gamma$, $K$ is a positive constant, $l = 0, 1, 2$, $m = 0, 1, 2$, and $l + m \leq k + 1$.

$(L_2)$

$$1 - \|A\|_D \geq 2\sqrt{\|B\|_D \|C\|_D}. \quad (5)$$

$(L_3)$ Letting $L_{\pm} := (1 + \|A\| + \sqrt{(1 - \|A\|)^2 \pm 4\|B\|\|C\|})$

$$(\|\partial_x F\| + \|\partial_x G\|\|B\|) < \frac{2^{k-2}(L_-)^2}{k!(L_+)^k}. \quad (6)$$

## Main theorem:$C^k$-invariant foliation

For each map $\widetilde{T}$ that satisfies the conditions $(L_1) - (L_2) - (L_3)$ there is a foliation $\mathcal{F}_D$ of class $C^k$ with $C^{k+1}$ leaves, such that

(1) Each leaf $\mathcal{F}_{(x_0,y_0)} \in \mathcal{F}_D$ is the graph of a function $y = h(x)$;

(2) the hyperplane $\Gamma$ is a leaf of $\mathcal{F}_D$;

(3) the foliation $\mathcal{F}_D$ is $\widetilde{T}$ invariant, that is, for each leaf $\mathcal{F}_{(x_0,y_0)} \in \mathcal{F}_D$, $\mathcal{F}_{(x_0,y_0)} \neq \Gamma$, there is $\mathcal{F}_{\widetilde{T}(x_0,y_0)} \in \mathcal{F}$ such that $\widetilde{T}(\mathcal{F}_{(x_0,y_0)}) \subset \mathcal{F}_{\widetilde{T}(x_0,y_0)}$.

This theorem was proved using the some technics of the paper of Shashkov-Shilnikov [SS94] where is shown the theorem for the case $k = 1$.

## Factorization of the region $D$ and of the map $\widetilde{T}$.

1. [VA03] The existence of the foliation of last theorem allows us have a quotient space $D/\mathcal{F}_D$, where the equivalents class are the leaves of the foliation $\mathcal{F}_D$. Some authors call this quotient space factorization of the region $D$, note also one may identify quotient space $D/\mathcal{F}_D$ with $J := \{(0, \eta) : \eta \in I\}$ of the following way . Any leaf $\mathcal{F}_{(x,y)} \in D/\mathcal{F}_D$ can be identify with $\mathcal{F}_{(x,y)} \cap \{x = 0\}$, and if we identify $\eta$ with $(0, \eta)$ we can see $J$ as a interval thus $D/\mathcal{F}_D$.
2. [VA03] With the existence of the invariance foliation we may defined a factorization of map $\widetilde{T}$, $\varphi := \widetilde{T}/\mathcal{F}_D$. Then map $\varphi : J \to J$ looks like a unidimensional map. Indeed, take $(0, \eta) \in J$, so there is leaf $\mathcal{F}_{(x,y)} \in D/\mathcal{F}_D$. such that $(0, \eta) = \mathcal{F}_{(x,y)} \bigcap \{x = 0\}$, from the invariance a foliation we have $\widetilde{T}(\mathcal{F}_{(x,y)}) \subset \mathcal{F}_{\widetilde{T}(x,y)}$, we set $(0, \overline{\eta}) := \widetilde{T}(x, y) \bigcap \{x = 0\}$, and $\varphi((0, \eta)) = (0, \widetilde{\eta})$. Thus, we can write $y = h_{\eta}(x)$, $\eta := h_{\eta}(1)$. In other words, we may introduce new coordinate $\{(x, y)\}$ in $D$ such that the map $\widetilde{T}$ has the form
$$\overline{x} = \widehat{F}(x, \eta), \overline{\eta} = \widehat{G}(x, \eta) := \varphi(\eta),$$
where $\widehat{F}(x, \eta) = \widehat{F}(x, h_{\eta}(x))$.
From the existence of the $C^k-$ invariant foliation we have $\widehat{F}$ and $\widehat{G}$ are of class $C^k$ and we may use the powerful tools of one-dimensional dimensional dynamics.

## Bibliography

- V. S. Afraĭmovič, V. V. Bykov, and L. P. Sil′nikov.
  The origin and structure of the Lorenz attractor.
  *Dokl. Akad. Nauk SSSR*, 234(2):336–339, 1977.
- V. S. Afraĭmovich, V. V. Bykov, and L. P. Shil′nikov.
  On attracting structurally unstable limit sets of Lorenz attractor type.
  *Trudy Moskov. Mat. Obshch.*, 44:150–212, 1982.
- John Guckenheimer and Philip Holmes.
  *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, volume 42 of *Applied Mathematical Sciences*.
  Springer-Verlag, New York, 1983.
- J.Guckenheimer.
  A strange, strange attractor, in the Hopf bifurcation and its applications.
  19:368–381, 1976.
- Edward N.Lorenz.
  Deterministic nonperiodic flow.
  20:130–141, 1963.
- M. V. Shashkov and L. P. Shil′nikov.
  On the existence of a smooth invariant foliation in Lorenz-type mappings.
  *Differentsial′nye Uravneniya*, 30(4):586–595, 732, 1994.
- Sze-Bi Hsu. Valentin Afraimovich.
  *Lectures on Chaotic Dynamical Systems.*
  AMS/IP Studies in Advanced Mathematics Volume 28. American Mathematical Society / International Press, 2003.

This project is supported by FAPESP processo 2009/17153-9