Quenched invariance principle for a balanced, non-elliptic, random walk in balanced random environment

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Resumo/Abstract:
Let M^d be all probability measures on $\{\pm e_i\}_{i=1}^d$ An **environment** is a point $\omega \in \Omega = \left(M^d\right)^{\mathbb{Z}^d}$

$$\omega = \{\omega(x, \pm e_i), i = 1, ..., d\}_{x \in \mathbb{Z}^d}$$

The law of environment P is an i.i.d. measure, i.e.

$$P = \mu^{\mathbb{Z}^d}$$

for some distribution μ on M^d.

For an environment $\omega \in \Omega$, the *Random Walk* on ω is a time-homogenous Markov chain with transition kernel

$$P_\omega\left(X_{n+1} = z + e \middle| X_n = z\right) = \omega(z, e).$$

The **quenched law** P_ω^z is defined to be the law on $\left(\mathbb{Z}^d\right)^{\mathbb{N}}$ induced by the kernel P_ω and $P_\omega^z(X_0 = z) = 1$. An environment ω is said to be *balanced* if for every $z \in \mathbb{Z}^d$ and neighbor e of the origin, $\omega(z, e) = \omega(z, -e)$.

An environment ω is said to be *genuinely d-dimensional* if for every neighbor e of the origin, there exists $z \in \mathbb{Z}^d$ such that $\omega(z, e) > 0$.

Throughout this paper we make the following assumption. P-almost surely, ω is balanced and genuinely d-dimensional.

Set

$$X_t^N = \frac{1}{\sqrt{N}} X_{[tN]} + \frac{tN - [tN]}{\sqrt{N}} \left(X_{[tN]+1} - X_{[tN]}\right), \quad t \geq 0.$$

The **quenched invariance principle holds** if for P a.a. ω the law of $\{X_t^N\}_{t \geq 0}$ under P_ω^0 converges weakly to a Brownian motion with deterministic non-degenerate matrix.

Theorem Let $d \geq 2$ and assume that the environment is i.i.d., genuinely d-dimensional and balanced, then the quenched invariance principle holds with non-degenerate limiting covariance matrix.

Lawler showed in [L] the quenched invariance principle for ergodic uniformly elliptic environments: that is, if there exits $\epsilon_0 > 0$ with

$$P(\forall i = 1, ..., d,\ \omega(z, e_i) > \epsilon_0) = 1.$$

Guo and Zeitouni showed in [GZ] the quenched invariance principle for i.i.d elliptic environments

$$P(\forall i = 1, ..., d,\ \omega(z, e_i) > 0) = 1.$$

and for ergodic environments under the moment condition

$$E[\big(\prod_{i=1}^{d} \omega(x, e_i)\big)^{-p/d}] < \infty \quad \text{for some} \quad p > d$$

One can find an example of ergodic elliptic balanced environment, where the invariance principle fails.

Note that, due to the balanced environment, $\{X_n\}$ is a martingale.

Let $\{\bar{\omega}_n\}_{n\in\mathbb{N}}$ be the environment viewed from the point of view of the particle:

$$\bar{\omega}_n = \tau_{X_n}\omega,$$

where τ is the shift on Ω This is a Markov chain on Ω under P with transition kernel

$$M(\omega', d\omega) = \sum_{i=1}^{d} \left[\omega'(0, e_i)\delta_{\tau_{e_i}\omega'} + \omega'(0, -e_i)\delta_{\tau_{-e_i}\omega'}\right]$$

The quenched invariant principle follows once we can find a probability measure $Q \ll P$ which is an invariant ergodic measure for $\{\bar{\omega}_n\}$ and such that P-almost surely, after some finite time the shifted environment is in the support of Q.

Note that in the elliptic case it follows immediately when $Q \ll P$ that $P \ll Q$, but in our case it is possible to have $Q \ll P$ but $P \not\ll Q$. Thus we need to be more careful.

Our proof is based on analytical methods, in particular on the maximum principle which we have to adapt to the non-elliptic setting. The estimates are based on the rescaled random walk, obtained from the original walk stopped after each coordinate has been upgraded. The maximum inequality allows us to control for $p > 1$ the L^p- norm of the density of the invariant measure of the walk on the reflected-periodized cube of size N, uniformly in large N. From this we get the existence of an invariant measure $Q \ll P$, however due to the non-ellipticity of the walk, the proof of the ergodicity of Q, which is related to the uniqueness of a maximal strongly connected component, is more delicate. In the 2 dimensional case a simple coupling argument is applicable, while in higher dimensions we need to adapt the Burton-Keane argument [BK], to our setting, where we only have a weak version of the finite energy condition. We compensate for the weaker finite energy condition by using density bounds on the support of the invariant measure.

References

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