PCA Explanatory Power in Stochastic Interest Rate Models

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Abstract

For affine interest rate models, it is provided a proposition relating the number of risk factors and the number of principal components required to explain the whole variabilty given by the covariance matrix of the term structure along its paths. Moreover, using a Monte Carlo simulation, the analysis is extended to Dothan and Exponencial Vasicek non-affine models. In the both cases, the results are similar.

Keywords: Stochastic Interest Rate Models, Monte Carlo Method, Principal Component Analysis

Basic Relationships and PCA

Let's consider the following two equations:

$$P(t.T) = E[e^{-\int_t^T r(x)\mathrm{d}x}]$$

$$-lnP(t,T) = f(t,T)$$

where is $r(t)$ is the short rate, $P(t,T)$ is the price of a zero coupon bond price (with maturity at T) and $f(t,T)$ is the spot rate (to maturity T).

The term structure at t is defined as the function $f(t,\cdot) : \Re^+ \to \Re^+; T \mapsto f(t,T), \forall T > t$.

The initial goal is to decompose the evolution of the term structure in a few principal components. This analysis is important in order to hedge the movements of term structure. To achieve this, we need to write the term structure in a discrete form so it is possible to compute the decomposition.

$$\begin{bmatrix} f(t,T_1) & f(t,T_2) & \dots & f(t,T_m) \\ f(t+1,T_1+1) & f(t+1,T_2+1) & \dots & f(t+1,T_m+1) \\ \vdots & \vdots & \ddots & \vdots \\ f(t+n,T_1+n) & f(t+n,T_2+n) & \dots & f(t+n,T_m+n) \end{bmatrix}$$

Hence we can estimate covariance matrix Σ from a particular path of realizations of the term structure. Since this matrix is symmetric, we apply the spectral theorem to write it as:

$$\Sigma = V\Psi V'$$

$$VV' = I$$

where Ψ is the diagonal matrix of eigenvalues $\psi_1, ..., \psi_m$ of Σ. We define the Principal Component vector:

$$F = \Psi^{-\frac{1}{2}} V' \bar{X}.$$

$\bar{X}$ is term structure yields for some observation. The explanatory power of each principal component is $\frac{\psi_i}{\sum_{j=1}^n \psi_j}$

For further reading, see [4]

Interest Rate Models

The interest rate models used in this work may be written as:

$$r(t) = \textstyle\sum_{i=1}^k x_i;$$

$$\mathrm{d}x_i(t) = f(x_i(t),t)\mathrm{d}t + g(x_i(t),t)\mathrm{d}W_i,$$

where $W = (W_1, \cdots, W_k)$ is a k-dimensional Wiener process and k is the number of factors.

These models can be divided into affine processes and non-affine. An affine model admit writing the bond price as

$$P(t,T) = \prod_{i=1}^k A(p_i,t,T)e^{-B(p_i,t,T)x_i(t)},$$

where k is the number of factors and p_i is the parameter set associated to factor i.

Table I presents the models that are evaluated in our work.

Model	$g(x(t),t)$	$h(x(t),t)$	Affine?	Factors (n)
Dothan	$\lambda x(t)$	$\sigma x(t)$	No	1
EV	$x_t[\theta - k\ln x_t]$	σx_t	No	1
G2	$-kx(t)$	σ	Yes	2
CIR2	$k(\theta - x(t))$	$\sigma\sqrt{x(t)}$	Yes	2

Table 1

For affine models we apply the following proposition:

Proposition 1 *For any term structure of a k-dimensional affine model, the number of principal components necessary to explain the variability is equal or lesser than k.*

Simulations made in Matlab 7.12.0

Number of simulations per parameter sets 10.000

Algorithms from: [1] (Dothan), [2] (G2 and CIR2) and [3] (EV)

J.Chevitarese acknowledges the financial support from PIBIC-CNPq.

G2 and CIR2

G2 is the simplest multifactor model. It has analytic solutions for bond prices. Its main critic is the absence of mean revertion.

Its short rate is given by the equation

$$r(t) = \sum_{i=1}^2 x_i(s)e^{-k_i(T-t)} + \sigma_i \int_t^T e^{-k_i(T-s)}\mathrm{d}W_i(s)$$

and its transition equation by

$$r(t+h) = \sum_{i=1}^2 x_i(t)e^{-k_i(h)} + \sigma_i\sqrt{\frac{1}{2k_i}[1-e^{-2k_i(h)}]}Z_i(t+h)$$

where Z_1 and Z_2 are realizations drawn from a multivariate normal with correlation ρ. By that, we write $A_1(t,T)$ and $A_2(t,T)$ as only one $A(t,T)$. $A(t,T)$ and $B_i(t,T)$ are:

$$A(t,T) = e^{\frac{1}{2}V(t,T)}$$

$$B_i(t,T) = \tfrac{1}{k_i}[1-e^{-k_i(T-t)}]$$

and

$$V(t,T) = \frac{\sigma_1^2}{k_1^2}[T-t+\frac{2}{k_1}e^{-k_1(T-t)} - \frac{1}{2k_1}e^{-2k_1(T-t)} - \frac{3}{2k_1}]$$

$$+\frac{\sigma_2^2}{k_2^2}[T-t+\frac{2}{k_2}e^{-k_2(T-t)} - \frac{1}{2k_2}e^{-2k_2(T-t)} - \frac{3}{2k_2}]$$

$$+2\rho\frac{\sigma_1\sigma_2}{k_1k_2}[T-t-B_1(t,T)-B_2(t,T)+\frac{1-e^{-(k_1+k_2)(T-t)}}{k_1+k_2}]$$

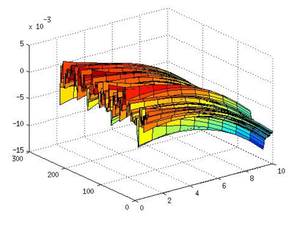

Figure 1: default

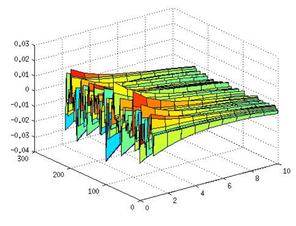

Figure 2: default

Figure 3, 4, 5 and 6 show results of parameter variations (k_1 and k_2) at PCA.

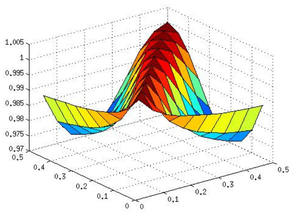

Figure 3: $\sigma_1 = 0.01$, $\sigma_2 = 0.01$ and $\rho = -0.9$

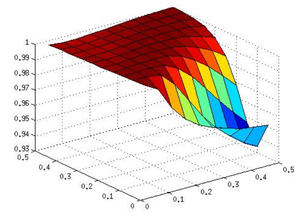

Figure 4: $\sigma_1 = 0.01$, $\sigma_2 = 0.05$ and $\rho = -0.9$

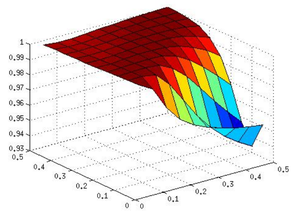

Figure 5: $\sigma_1 = 0.02$, $\sigma_2 = 0.01$ and $\rho = -0.9$

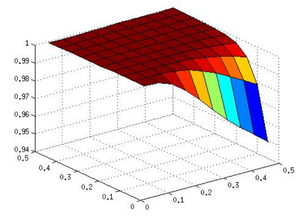

Figure 6: $\sigma_1 = 0.02$, $\sigma_2 = 0.005$ and $\rho = -0.9$

The x-axis and y-axis are k_1 and k_2 and z-axis is the value of mean first eigenvalue. It is important to notice that Figure 4 and 5 are almost equal.

The table 2 gives the value of ρ, $E(PC_1)$ and $SD(PC_1)$.

ρ	$E(PC_1)$	$SD(PC_1)$
0.9	0.9983	0.0004
0.6	0.9977	0.0006
0.3	0.9971	0.0009
0	0.9966	0.0011
-0.3	0.9963	0.0012
-0.6	0.9961	0.0013
-0.9	0.9964	0.0011

Table 2

The two-factor Cox-Ingersoll-Ross model (CIR2) has two squared-root processes generating the short rate. Its transition is stated as:

$$x_i(t) = \frac{\sigma_i^2(1-e^{k_i(t-u)})}{4k_i}\chi'^2 d_i(\lambda_i)$$

λ_i is the non-centrality coefficient for a chi-square distribution with d_i degrees of freedom. d_i and λ_i are defined as:

$$d_i = \tfrac{4k_i\theta_i}{\sigma_i^2}$$

$$\lambda_i = \tfrac{4k_ie^{k_i(t-u)}}{\sigma_i^2(1-e^{k_i(t-u)})}x(u)$$

CIR2 is also affine and its A_i and B_i are defined as:

$$A_i(t,T) = \left[\tfrac{2h_i\exp((k_i+h_i)(T-t)/2)}{2h_i}\right]^{2k\theta_i/\sigma_i^2}$$

$$B_i(t,T) = \tfrac{2\exp\{(T-t)h_i-1\}}{2h_i+(k+h)\exp\{(T-t)h_i-1\}}$$

and

$$h_i = \sqrt{k_i^2+2\sigma_i^2}$$

Results for CIR2:

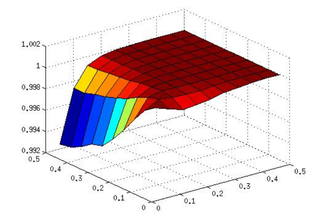

Figure 7: $\sigma_1 = 0.01$, $\sigma_2 = 0.01$ and $\rho = -0.9$

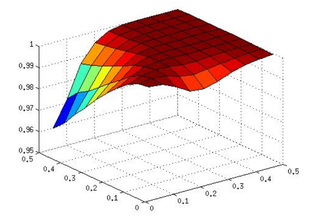

Figure 8: $\sigma_1 = 0.01$, $\sigma_2 = 0.05$ and $\rho = -0.9$

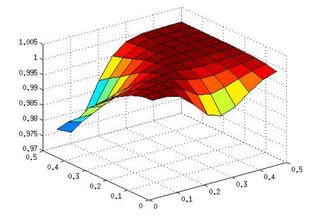

Figure 9: $\sigma_1 = 0.01$, $\sigma_2 = 0.05$ and $\rho = -0.9$

Dothan Model

Dothan Model is basically assuming the same diffusion of Black-Scholes. So:

$$\mathrm{d}r(t) = \sigma r(t)\mathrm{d}W^O(t)$$

This is valid under a objective probability measure Q_O, in a risk-neutral measure the process is:

$$\mathrm{d}r(t) = \lambda r(t)\mathrm{d}t + \sigma r(t)\mathrm{d}W(t)$$

It is important to say that λ is a constant market risk. The only model we simulated it is Dothan. We used Cox-Ross-Rubinstein algorithm to perform the simulation:

$$p = \frac{e^{a\frac{T}{N}} - d}{u-d}$$

T is the maturity and N is the number of steps. u and d are well-known in the literature for this model.

The simulations are expressed in this tables:

σ	$E(PC_1)$	$SD(PC_1)$
0.025	0.9898	0.0093
0.050	0.9912	0.0076
0.075	0.9912	0.0078
0.100	0.9914	0.0075
0.125	0.9913	0.0076
0.150	0.9915	0.0074
0.175	0.9914	0.0074
0.200	0.9913	0.0076

$\lambda = 0.0001$

σ	$E(PC_1)$	$SD(PC_1)$
0.025	0.9850	0.0133
0.050	0.9901	0.0090
0.075	0.9910	0.0079
0.100	0.9911	0.0078
0.125	0.9914	0.0075
0.150	0.9913	0.0077
0.175	0.9912	0.0077
0.200	0.9915	0.0075

$\lambda = 0.0002$

Exponential Vasicek Model

This model should be seen - as the name says - as the exponential Vasicek model. To simulate it we can create a binomial tree for the Ornstein-Uhlenbeck as Nelson-Ramaswamy (1991). The process we have to simulate is:

$$\mathrm{d}z(t) = -kz(t)\mathrm{d}t + \sigma\mathrm{d}W(t), z(0) = 0$$

Hence, we apply:

$$r(t) = \exp z(t) + (\log x_0 - \frac{\theta}{k})e^{-kt} + \frac{\theta}{k}$$

to discount the bonds in the tree.

Results of EV: TO BE COMPLETED

Conclusions

- **Affine models don't have more principal components than risk factors**
- **This is not necessarily true for non-affine models**
- **For all tested parameters the mean of first principal component is always above 98%**
- **For all tested models the only information to perform a good immunization is the first principal component**

References

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[3] D.B. Nelson and K. Ramaswamy. Simple binomial processes as diffusion approximations in financial models. *Review of Financial Studies*, 3(3):393–430, 1990.

[4] M. Valli and G. Varga. Movimentos da estrutura a termo da taxa de juros brasileira e imunização. *Revista de Estudos Avançados da USP*, 2001.