Degrees of Noether-Lefschetz loci

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Noether-Lefschetz theorem tells us that very general surfaces of degree ≥ 4 are not supposed to contain curves besides complete intersections with other surfaces.

Asking the surface to contain say, a line, or a conic, a twisted cubic, etc, yields subvarieties of the appropriate $\mathbb{P}^N = |\mathcal{O}_{\mathbb{P}^3}(d)|$. There are *polynomial* formulas for their degrees.

For example, while a general surface f_d of degree $d \geq 4$ contains no line, the locus consisting of f_d's containing some (varying) line is a subvariety of codimension $d-3$ and degree

$$\binom{d+1}{4}(3d^4+6d^3+17d^2+22d+24)/4! \qquad \text{in} \quad P^N.$$

We show how to obtain similar formulas for the locus $NL(\mathbb{W}, d)$ consisting of f_d's containing some member of a given family of curves $\mathbb{W}$.

In all cases we've met so far, namely

- 1, 2 or 3 lines;
- plane curves;
- twisted cubics
- elliptic quartics,

the polynomial formula obtained is of degree twice the dimension of the family of curves, though the potential degree bounded by GRR is $\leq 3 \dim \mathbb{W}$.

Perhaps Kontsevich's spaces of stable maps could be used to handle the loci of surfaces of sufficiently high degree containing a rational curve of fixed degree.