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Critical site percolation on the triangular lattice

In the whole session, we consider site percolation on the triangular lattice with parameter $p = 1/2$ (the measure is denoted $\mathbb{P}$). When handling percolation, you want to express your artistic qualities: draw some pictures!

Background (FKG inequality) An event A is *increasing* if it is stable by addition of black sites. The FKG inequality yields

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B)$$

for any increasing events A and B.

Exercise 1: Russo-Seymour-Welsh theorem for site percolation

A *horizontal crossing* of a rectangle is a sequence of neighboring black sites going from the left side to the right side.

Question 1 What is the probability that there exists an horizontal crossing of $[0,n]^2$?

Question 2 Assume $[0,n]^2$ is crossed from left to right and set Γ to be the lowest horizontal crossing. Let γ be a deterministic path from left to right, prove that $\{\Gamma = \gamma\}$ is measurable with respect to the σ-algebra spanned by the sites **below** γ and the sites of γ. When conditioning on $\{\Gamma = \gamma\}$, what can be said about the law of sites above γ?

Question 3 Consider the rectangle $[0,2n]\times[0,n]$ and assume that $[0,n]^2$ is crossed horizontally. Can you bound from below the probability that the lowest crossing Γ is connected to $\big([n,2n]\times\{n\}\big)\cup\big(\{2n\}\times[0,n]\big)$ by an open path? *Hint: condition on $\{\Gamma = \gamma\}$ and consider the reflected path $\sigma(\gamma)$ with respect to the line $y = n + \frac{1}{2}$.*

Question 4 Deduce that the probability of crossing the rectangle $[0,2n]\times[0,n]$ horizontally is bounded away from 0 when n goes to infinity.

Question 5 a) Let $\rho > 1$. Deduce that the probability to cross the rectangle $[0,\rho n]\times[0,n]$ horizontally is uniformly (in n) bounded away from 0.

b) Prove that the probability of a black circuit surrounding the origin in the annulus $[-2n,2n]^2 \setminus [-n,n]^2$ remains bounded away from 0 when n goes to infinity.

c) Show that almost surely there is no infinite cluster at $p = \frac{1}{2}$.

d) What can be said about $\mathbb{P}(0 \leftrightarrow \partial\Lambda_n)$?

e)** Explain a strategy to prove that $p_c = \frac{1}{2}$.

Exercise 2: universal exponents

Let σ be a finite sequence of colors (B for black, W for white). We associate to $n > 0$ and $\sigma = \{\sigma_1, .., \sigma_k\}$ the event $A_\sigma(n)$ that there exist paths γ_1, ..,γ_k such that the path γ_i has color σ_i, γ_i connects the origin to the boundary of $[-n,n]^2$ (When $k > 2$, we require only that the paths connect $[-k,k]^2$ to the boundary of $[-n,n]^2$) and γ_1,...,γ_k can be found in counter-clockwise order.

We define the same event in the upper half-plane (which we denote by $A_\sigma^{\mathbb{H}}(n)$). In this case, the paths must be found in counter-clockwise order, starting from the right.

Question 1 a) Prove that $\mathbb{P}(A^{\mathbb{H}}_{BW}(n)) \geq \frac{c}{n}$ for some universal constant c. *Hint. Use the RSW theorem to construct a point in* $\{0\} \times [-n/2, n/2]$ *which is connected to the boundary of the box by two arms of distinct colors.*

b) Assume $A^{\mathbb{H}}_{BW}(n)$ holds. We require that the site on the left of 0 is white and that it is the start of the white path, and the site on the right is black and is the start of the black path. Show that one can explore the interface between the black and the white paths without exploring any other site.

c) Let $B(n)$ be the event that there exist a white path connected to the left side of $[-n,n] \times [0,n]$ and a black path connected to the right side. Show that there exists a universal $c_1 > 0$ such that

$$\mathbb{P}(A^{\mathbb{H}}_{BW}(n)) \leq c_1 \mathbb{P}(B(n)).$$

d) Deduce that there exists $c_2 > 0$ such that

$$\mathbb{P}(A^{\mathbb{H}}_{BW}(n)) \leq \frac{c_2}{n}.$$

What did we proved?

Question 2* Prove that the exponent for $BWBWW$ in the plane is 2.

Question 3** Prove that the exponent for $BWBW$ in the plane is smaller than 2.

Rudiments of complex analysis

Exercise 1: Riemann mapping theorem

Theorem 1 (Riemann mapping theorem) *Let D and D' be two simply connected domains included in $\mathbb{C}$ and different from $\mathbb{C}$, there exists a conformal map (*i.e. *a bijection derivable in the complex variable) between D and D'.*

Question 1 Find a conformal map between the following domains:

- from $\mathbb{R}\times]0,\pi[$ to $\mathbb{H} = \{z, Im(z) > 0\}$;
- from the disk $\mathbb{D} = \{z, |z| < 1\}$ to $\mathbb{H}$;
- from $\mathbb{H} \setminus [0, ir]$ to $\mathbb{H}$;
- * from $\mathbb{D}$ to $\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$;
- from $S_\epsilon = \big(\mathbb{R}\times]0,2[\big) \setminus \big((i-\infty, i-\varepsilon] \cup [i+\varepsilon, i+\infty)\big)$ to $\mathbb{H}$
- ** from $\mathbb{H}$ to an equilateral triangle.

Question 2 a) Show that there is no conformal map from $D(0,1)$ to $\mathbb{C}$. *It confirms that the assumption $D \neq \mathbb{C}$ is necessary.*

b) Let D be a simply connected domain and f be a conformal map, why is $f(D)$ simply connected?

Question 3 Which are the conformal maps from $D(0,1)$ into $D(0,1)$? *Hint. One can guess what they are and make sure none is left behind using Schwarz's Lemma.* Deduce that there are three (real) degrees of freedom in the choice of a conformal map between two domains in the following sense:

- one can fix one point on the boundary and one point inside the domain;
- one can fix one point inside the domain and the direction of the derivative;
- one can fix three points on the boundary (keeping the order).

Exercise 2: Estimates for conformal maps

Question 1 (Lemme de Schwarz): Let f be a continuous map from $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$ such that $f(0) = 0$ and f is holomorphic inside $\mathbb{D}$. Show that $|f(z)| \le |z|$. *Hint. Think about the maximum principle.* Study the case where $|f'(0)| = 1$.

Question 2 * (Koebe 1/4-theorem) Let

$$\mathcal{S} := \{f : \mathbb{D} \to \mathbb{C}, \text{analytic, one-to-one with } f(0) = 0 \text{ and } f'(0) = 1\}$$

a) (Area theorem) Let $f \in \mathcal{S}$ and $K = \mathbb{C} \setminus \{1/z, z \in f(\mathbb{D})\}$, prove that

$$\text{area}(K) = \pi \left[1 - \sum_{n=1}^{\infty} n|b_n|^2\right]$$

where $1/f(1/z) = z + b_0 + \sum_{n\ge1} \frac{b_n}{z^n}$. Note that it implies $|b_1| \le 1$.

b)* Prove that if $f = z + a_2 z + \cdots$ is in $\mathcal{S}$, then $|a_2| \le 2$. *Hint: construct a function $h \in \mathcal{S}$ such that $h(z) = z + \frac{a_2}{2} z^3 + \cdots$ and conclude.*

c)* Deduce Koebe 1/4 theorem: if $f \in \mathcal{S}$, then $B(0, \frac{1}{4}) \subset f(\mathbb{D})$.

d) Suppose $f : D \to D'$ is a conformal transformation with $f(z) = z'$. Then

$$\frac{1}{4} \frac{d(z', \partial D')}{d(z, \partial D)} \le |f'(z)| \le 4 \frac{d(z', \partial D')}{d(z, \partial D)}.$$

Exercise 3 **

Prove the Riemann mapping theorem. *Hint. It is sufficient to handle the case of $D' = \mathbb{D}$. Fix ω in D. Consider the set of functions from D into $\mathbb{D}$ such that $f(\omega) = 0$ and $f'(\omega) > 0$. The function f maximizing $f'(\omega)$ will be your conformal map between D and $\mathbb{D}$.*

Itô's Formula

Let $(\mathcal{F}_t)$ be a filtration, *i.e.* an increasing family of σ-algebra. The process (M_t) is a *martingale* (with respect to $\mathcal{F}_t$) if for each $s < t$, $\mathbb{E}[|M_t|] < \infty$ and $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$. In this exercise sheet, B is a standard one-dimensional Brownian motion.

Exercise 1: integration with respect to Brownian motion

Question 1 We call H a *simple processes* if it is of the form

$$H_s = \sum_{j=1}^{n} C_j \mathbb{1}_{[t_{j-1}, t_j)}(s)$$

where (t_j) is increasing and C_j is $\mathcal{F}_{t_{j-1}}$-measurable.

a) For a (random) process $H = C\mathbb{1}_{[s,t)}$, where C is $\mathcal{F}_s$-measurable, find a natural candidate for the integral of H against the Brownian motion B, in other terms, what could $\int_0^\infty H_s dB_s$ be? How could the notion of integral be extended to any simple process?

b) We assume that the integral has been constructed as above. For any simple process H, check that

$$\mathbb{E}\left[\left(\int_0^\infty H_s dB_s\right)^2\right] = \int_0^\infty \mathbb{E}[H_s^2] ds.$$

Question 2 Let $\mathcal{L}^2$ the set of square integrable adapted processes (in the sense $\int_0^\infty \mathbb{E}[H_s^2] ds < \infty$). Explain how to extend the definition of integral to $\mathcal{L}^2$.

Question 3 For a bounded adapted process H, we define $\int_0^t H_s dB_s$ by $\int_0^\infty H_s \mathbb{1}_{[0,t)} dB_s$. Show that $M_t = \int_0^t H_s dB_s$ is a $\mathcal{F}_t$-martingale. *Hint. Check it in the case of simple processes first.* ** Show that it is a continuous process.

Remark 1 *Note that for any bounded adapted process a, $\int_0^t a_s ds$ is straightforward to define. It is also possible to check that $H_t = \int_0^t a_s dB_s + \int_0^t \sigma_s ds$ is a martingale if and only if $\sigma = 0$.*

Question 4 a) Let H_s be a bounded continuous adapted process and $t > 0$. Considering subdivisions $0 = t_1^n < .. < t_n^n = t$ with $\max(t_{i+1}^n - t_i^n) \to 0$, show that

$$\sum_{i=1}^{n-1} H_{t_i^n}(B_{t_{i+1}^n} - B_{t_i^n}) \xrightarrow{\mathcal{L}^2} \int_0^t H_s dB_s.$$

b) Let H_s be a bounded adapted process and $t > 0$. Considering subdivisions $0 = t_1^n < .. < t_n^n = t$ with $\max(t_{i+1}^n - t_i^n) \to 0$, show that

$$\sum_{i=1}^{n-1} H_{t_i^n}(B_{t_{i+1}^n} - B_{t_i^n})^2 \xrightarrow{\mathcal{L}^2} \int_0^t H_s ds.$$

Hint. Recall that $B_t^2 - t$ is a martingale.

c) Prove Itô formula.

Theorem 2 (Itô formula) *For any a, σ bounded adapted processes and $t > 0$, we set $Y_t = \int_0^t a_s dB_s + \int_0^t \sigma_s ds$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function twice continuously derivable, then*

$$\phi(Y_t) = \phi(Y_0) + \int_0^t \phi'(Y_s) a_s dB_s + \int_0^t \left[\phi'(Y_s)\sigma_s + \frac{1}{2}\phi''(Y_s) a_s^2\right] ds.$$

Remark 2 *In order to write the equality*

$$H_t = x + \int_0^t a_s dB_s + \int_0^t \sigma_s ds$$

in a concise way, we often write

$$H_0 = x \quad \textit{and} \quad dH_t = a_t dB_t + \sigma_t dt.$$

Exercise 2: A first application, Bessel processes

Let $d > 0$. We **admit** that there exists a unique process, denoted X_t^x, which is solution of the following stochastic differential equation

$$dX_t^x = dB_t + \frac{d-1}{2X_t^x} dt, \quad X_0^x = x$$

up to time $T_x := \inf\{t : X_t^x = 0\}$. This process is called a *d-dimensional Bessel process.* For integer d, this process is the norm of a d-dimensional vector with independent brownian entries. Let $0 < a < x < b < \infty$, τ the first exit time of the set $[a, b]$, and $\phi(x) = \mathbb{P}(X_\tau^x = a)$.

Question 1 Show that $\phi(X_{t\wedge\tau}^x)$ is a martingale with respect to $\mathcal{F}_{t\wedge\tau}$.

Question 2 a) Assume ϕ is twice continuously differentiable. Using Itô formula, deduce that

$$\frac{1}{2}\phi''(x) + \frac{d-1}{2x}\phi'(x) = 0, \quad a < x < b.$$

and compute ϕ when $d \geq 2$,

b) When $d > 0$, compute $\mathbb{P}(X_\tau^x = a)$. What can you deduce?

Question 3 ** (optional question) Using the Itô formula, show that $\psi(x, t) = \mathbb{P}_x(\tau > t)$ is the solution of a partial differential equation. Deduce an estimate for $\mathbb{P}_x(\tau > t)$ when t goes to infinity. *Hint. I might help you if I am in a good mood.*