UNCERTAINTY PRINCIPLES AND SCHRÖDINGER FLOWS

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This talk is concerned with unique continuation properties of solutions of Schrödinger equations of the form

$$\partial_t u = i(\triangle u + V(x,t)u), \qquad (x,t) \in \mathbb{R}^n \times [0,T], \qquad T > 0. \tag{1}$$

The aim are : (I) to obtain sufficient conditions on the behavior of the solution u at two different times and on the potential V which guarantee that $u \equiv 0$ in $\mathbb{R}^n \times [0,T]$, and (II) under appropriate assumptions extend this result to the difference $v = u_1 - u_2$ of two solutions $u_1,\ u_2$ of semi-linear Schrödinger equation

$$\partial_t u = i(\triangle u + F(u,\overline{u})), \tag{2}$$

to conclude that $u_1 \equiv u_2$.

If potential $V \equiv 0$ the identity

$$\begin{aligned} &e^{it\Delta}u_0(x) = u(x,t) \\ &= (4\pi it)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{i|x-y|^2}{4t}} u_0(y)\,dy = (2\pi it)^{-\frac{n}{2}}\, e^{\frac{i|x|^2}{4t}}\, \widehat{e^{\frac{i|\,\cdot\,|^2}{4t}} u_0}\left(\frac{x}{2t}\right), \end{aligned}$$

shows that this kind of problem (the decay of the Schrödinger equation at two different times) for the free solution of the Schrödinger equation with data u_0 is intrinsically related to "uncertainty principles" concerning the decay of a function u_0 and its Fourier transform, $\widehat{u_0}$.

We shall show how to "extend" some uncertainty principles including those due to G. H. Hardy, G. W. Morgan and (specially) Paley-Wiener to solutions of the equations (1) and (2).

This lecture is based in works in collaboration with L. Escauriaza, C. E. Kenig, and L. Vega.