Buzios course on *Self-Avoiding Walks* by Gordon Slade, University of British Columbia

http://www.math.ubc.ca/~slade/buzios.html

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This is material for a course in Buzios, Brazil, Summer 2010, as part of:
Clay Mathematics Institute Summer School July 11 to August 7, 2010, and XIV Escola Brasileira de Probabilidade August 2-7, 2010.

The course will consist of a 90-minute lecture each morning August 2-7 and a tutorial each afternoon August 2-6.
Tutorials will be led by Roland Bauerschmidt and Jesse Goodman, University of British Columbia.
There will be a **guest lecture by Hugo Duminil-Copin** on the afternoon of August 3rd.

Self-Avoiding Walks. The course will focus on rigorous results for self-avoiding walks on the d-dimensional integer lattice. The model is defined by assigning equal probability to all random walk paths starting from the origin and without self-intersections. These probability measures are not consistent as the path length is varied, and thus do not define a stochastic process; the model is combinatorial in nature. Despite its simple definition, the self-avoiding walk is difficult to study in a mathematically rigorous manner, and many of the important problems remain unsolved; they encompass many of the features and challenges of critical phenomena.

General references:
N. Madras and G. Slade, *The Self-Avoiding Walk*, Birkhäuser, Boston, (1993).
G. Slade. *The Lace Expansion and its Applications*, Lecture Notes in Mathematics #1879. Springer, Berlin, (2006). PDF file
B.D. Hughes, *Random walks and random environments*, Volume 1, Oxford University Press, Oxford (1995).

Lecture 1. Introduction to the self-avoiding walk model, connective constant, critical exponents, dependence of critical behaviour on dimension, overview of what is known and of open problems.
References:
G. Slade, The Self-Avoiding Walk: A Brief Survey. To appear in *Surveys in Stochastic Processes*, Proceedings of the 33rd SPA Conference in Berlin, 2009, to be published in the EMS Series of Congress Reports, eds. J. Blath, P. Imkeller, S. Roelly. PDF file
N. Madras and G. Slade, *The Self-Avoiding Walk*, Chapter 1. You can read it at google books.

Tutorial 1. Let (a_n) be a real sequence with a_{n+m} &le $a_n + a_m$ for all n,m; prove that lim $n^{-1} a_n$ exists (possibly minus infinity) and equals inf $n^{-1} a_n$.
Prove that the connective constant in dimension 2 is strictly between 2 and 3.
Exercises on random walks and self-avoiding walks from *The Lace Expansion and its Applications*: Exercises 1.6, 1.7, 2.1, 2.2, 7.7. PDF file

Lecture 2. The Hammersley-Welsh upper bound on the number of n-step self-avoiding walks. Although very far from the predicted power law behaviour, this 1962 bound has not been improved (for dimension d=2) in almost 50 years. Its elegant proof employs the useful notion of *bridges*. These apply to the study of self-avoiding polygons.
References:
1) N. Madras and G. Slade, *The Self-Avoiding Walk*. Sections 3.1-3.2. You can read it at google books.
2) J.M. Hammersley and D.J.A. Welsh, Further results on the rate of convergence to the connective constant of the hypercubical lattice. *Quart. J. Math. Oxford*, (2), **13**, 108-110, (1962).

Tutorial 2: Guest lecture by Hugo Duminil-Copin on the following paper:
H. Duminil-Copin and S. Smirnov, The connective constant of the honeycomb lattice equals $\sqrt{2+\sqrt{2}}$. Preprint (2010). PDF file

Lectures 3 and 4. Derivation and application of the lace expansion for self-avoiding walks. The lace expansion has been used to prove that self-avoiding walks in dimensions 5 and higher behave like simple random walks. The mean-square displacement is linear in the number of steps and the scaling limit is Brownian motion.
Reference:
G. Slade. *The Lace Expansion and its Applications*, Lecture Notes in Mathematics #1879. Springer, Berlin, (2006). Section 2.2 and Chapters 3,4,5. PDF file

Tutorials 3 and 4. Exercises on the lace expansion from *The Lace Expansion and its Applications*: Exercises 3.6, 3.7, 3.8 (a,b,d), 5.17, 5.18, 5.19. PDF file

Lecture 5. Integral representations for the self-avoiding walk. The two-point function for strictly self-avoiding walk or weakly self-avoiding walk can be written as a Gaussian integral involving anti-commuting variables. Such representations have been useful as the point of departure for renormalisation group analysis.
Reference:
D.C. Brydges, J.Z. Imbrie, G. Slade. Functional integral representations for self-avoiding walk. *Probability Surveys*, **6**, 34-61, (2009). PDF file

Tutorial 5. Exercises on the Simon-Lieb inequality and its application to self-avoiding walks. PDF file

Lecture 6. Renormalisation group analysis. Using the integral representation, renormalisation group methods have been developed and used to prove inverse square decay for the critical two-point function of the continuous-time weakly self-avoiding walk in dimension 4.

Reference:
D. Brydges and G. Slade. Renormalisation group analysis of weakly self-avoiding walk in dimensions four and higher. To appear in the *Proceedings of the International Congress of Mathematicians*, Hyderabad, 2010. PDF file