A Long-Memory Model for the Term-Structure of Commodity Futures

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Abstract. We present a long-memory affine term-structure model for the prices of commodity futures. Our model specifies that the dynamics for spot commodity prices and the market price of risk follow an ARFIMA process, which can potentially capture mean-reverting behaviour and long-range dependencies. Under the observation that the ARFIMA process has a state-space representation we are able to derive expressions for futures prices that are exponential-affine in the state variable. Parameter estimation is possible using maximum likelihood and the Kalman filter.

Key words: Commodity-futures prices, Term-structure models, ARFIMA processes.

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I. INTRODUCTION

Golinski & Zaffaroni (2011) develop an affine model of the interest-rate term structure where observed yields follow processes in the ARFIMA class. In this paper, we are able to follow their methodology to develop a similar model for commodity futures prices.

II. THE MODEL

Throughout this paper $\mathbb{E}[-]$ denotes the expectation operator under the physical measure $\mathbb{P}$. The operator $\mathbb{E}[-|\mathcal{F}_t] = \mathbb{E}_t[-]$ denotes the conditional expectation under $\mathbb{P}$ with respect to the observed filtration $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$. The operation v^{T} on a vector v indicates vector transpose. The operator L is the lag operator, i.e.,

$$LY_t = Y_{t-1}. \tag{1}$$

We define an ARFIMA(p, d, q) process $\{Y_t\}_{t\in\mathbb{N}}$, with i.i.d. Gaussian error terms $\{\eta_t\}_{t\in\mathbb{N}}$, by the expression

$$\Phi(L)(1-L)^d Y_t = \Theta(L)\eta_t. \tag{2}$$

The autoregressive behaviour is captured by the autoregression operator:

$$\Phi(L) = \left(1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p\right), \tag{3}$$

and the moving average behaviour is captured by the moving average operator

$$\Theta(L) = \left(1 - \theta_1 L - \theta_2 L^2 - \ldots - \theta_q L^q\right). \tag{4}$$

The fractional difference operator $(1-L)^d$, which captures long-range dependency, can be written as

$$(1-L)^d = \sum_{k=0}^{\infty} \pi_k L^j, \tag{5}$$

where $d < |1/2|$. The coefficients of the fractional difference operator are given by

$$\pi_k = \frac{d\Gamma(k-d)}{\Gamma(1-d)\Gamma(k+1)}, \tag{6}$$

where $\Gamma(-)$ is the gamma function (cf. Chan & Palma 1998).

A causal ARFIMA(p, d, q) process is one such that $\Phi(z) \neq 0$ for all $|z| < 1$ and

$$Y_t = \sum_{j=0}^{\infty} \varphi_j \eta_{t-j}. \tag{7}$$

The coefficients φ_j are the coefficients of

$$\frac{\Phi(z)}{\Theta(z)}(1-z)^{-d} = \sum_{j=0}^{\infty} \varphi_j z^j. \tag{8}$$

Through expression (7) we may derive the following state space representation of the ARFIMA process (see Chan & Palma 1998, and references cited therein)

$$\begin{aligned} Y_t &= GX_t \\ X_{t+1} &= FX_t + H\eta_t, \end{aligned} \tag{9}$$

where

$$\begin{aligned} G &= (1\ 0\ 0\ 0\ \ldots), \\ F &= \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}, \\ H &= (\varphi_0\ \varphi_1\ \varphi_2\ \ldots)^{\mathrm{T}}, \end{aligned} \tag{10}$$

and

$$X_t = \begin{pmatrix} Y(t|t) \\ Y(t+1|t) \\ Y(t+2|t) \\ \vdots \end{pmatrix}, \tag{11}$$

with

$$Y(i|j) = \mathbb{E}\left[Y_i | \{Y_s\}_{s\in[0,j]}\right]. \tag{12}$$

III. AFFINE TERM-STRUCTURE MODEL

Let $\{F_t(n)\}_{t\in\mathbb{N}}$ denote the process followed by the n-maturity futures price on a commodity with price process $\{S_t\}_{t\in\mathbb{N}}$. By the definition of the futures price we observe the following relationship to the underlying spot price:

$$F_t(n) = \mathbb{E}_t^*[S_{t+n}], \tag{13}$$

where $\mathbb{E}_t^*[-]$ denotes the expectation operator under the equivalent-martingale measure $\mathbb{Q}$.

We assume that $\{Y_t\}$ genetates the market filtration,

$$\mathcal{F}_t = \sigma\left(\{Y_s\}_{0\leq s\leq t}\right). \tag{14}$$

We also assume that the spot commodity price is an exponential-affine function of the ARFIMA(p, d, q) process $\{Y_t\}$, i.e.,

$$S_t = \mathrm{e}^{\delta_0+\delta_1 Y_t} \tag{15}$$

where δ_0 and δ_1 are constants. We then postulate that, for each n, the futures price is an exponential-affine function of the state-variable process $\{X_t\}$. In particular, given the deterministic scalar function $A(-)$ and the vector-valued function $B(-)$ we write

$$F_t(n) = \mathrm{e}^{A(n)+B(n)X_t}. \tag{16}$$

It is left to derive expressions for $A(-)$ and $B(-)$. We shall assume that $\eta_t \sim N(0, \sigma^2)$ for all t, and Thus, that $\{X_t\}$ is $\{\mathcal{F}_t\}$-measurablethere. We shall also assume that there exists a market price of risk process $\{\lambda_t\}_{t\in\mathbb{N}}$ and that it is an affine function of the ARFIMA process, so that

$$\lambda_t = \lambda_0 + \lambda_1 Y_t, \tag{17}$$

where λ_0 and λ_1 are constant. We may then obtain

$$\begin{aligned} F_t(n+1) &= \mathbb{E}_t^*\left[F_{t+1}(n)\right] \\ &= \mathbb{E}_t\left[\mathrm{e}^{-\lambda_t\eta_{t+1}-\frac{1}{2}\lambda_t^2\sigma^2}\mathrm{e}^{A(n)+B(n)X_{t+1}}\right] \\ &= \mathrm{e}A(n) - \tfrac{1}{2}\lambda_t^2\sigma^2 + -\tfrac{1}{2}\left(B(n)H - \lambda_t\right)^2\sigma^2 + B(n)FX_t. \end{aligned} \tag{18}$$

And we are able to dervie the following recursive expressions for $A(-)$ and $B(-)$:

$$A(n+1) = A(n) - \lambda_0\sigma^2(B(n)H) + \tfrac{1}{2}\sigma^2(B(n)H)^2, \tag{19}$$

and

$$B(n+1) = B(n)F + \lambda_1\sigma^2(B(n)H)G, \tag{20}$$

with $A(0) = \delta_0$ and $B(0) = \delta_1 G$.

When we consider τ seperate maturities in the term structure of futures prices we shall write

$$F_t = \mathrm{e}^{A+BX_t}, \tag{21}$$

where F_t is a $(\tau \times 1)$ vector, A is a $(\tau \times 1)$ vector, and B is a $(\tau \times \infty)$ vector whose n^{th} rows are denoted by $F_t(n)$, $A(n)$, and $B(n)$ respectively.

IV. KALMAN FILTER

Chan & Palma (1998) show that it is possible for the parameters of the ARFIMA(p, d, q) model to be estimated using maximum likelihood and the Kalman filter. They show that for a time series of length N the likelihood function depends only on the first N elements of the vector-valued process $\{X_t\}$. Thus, our model does not suffer from the use of infinite dimensional vectors.

For convenience we shall study the logarithm of futures prices as the observation variable and we shall assume that a measurement error is observed for each maturity. We assume that measurement errors are i.i.d. and are independent across maturities. Specifically, for the n-maturity futures price we assume that we observe the process $\{Z_t\}_{t\in\mathbb{N}}$ where

$$Z_t = \log F_t + \nu_t, \tag{22}$$

and $\{\nu_t\}_{t\in\mathbb{N}}$ is a $(\tau \times 1)$ vector of measurement errors with distribution $\nu_t(n) \sim N(0, \nu_n^2)$ for all t and each maturity n.

To derive the relevant Kalman recursive equations we must adjust the time-indices slightly. In particular, we let $X_t \to X_{t+1}$ and

$$X_{t+1} = FX_t + H\eta_t. \tag{23}$$

Our state-space model is then given by

$$\begin{aligned} Z_t &= A + DX_t + V_t \\ X_{t+1} &= FX_t + W_t, \end{aligned} \tag{24}$$

where $D = BF$, $V_t = BH\eta_t + \nu_t$, and $W_t = H\eta_t$. The error terms $\{V_t\}_{t\in\mathbb{N}}$ and $\{W_t\}_{t\in\mathbb{N}}$ are Guassian and correlated such that

$$\mathbb{E}\left[\begin{pmatrix} V_t \\ W_t \end{pmatrix} \begin{pmatrix} V_t^{\mathrm{T}} & W_t^{\mathrm{T}} \end{pmatrix}\right] = \begin{pmatrix} R_t & S_t \\ S_t^{\mathrm{T}} & Q_t \end{pmatrix}. \tag{25}$$

Here, $\{R_t\}_{t\in\mathbb{N}}$, $\{S_t\}_{t\in\mathbb{N}}$ and $\{Q_t\}_{t\in\mathbb{N}}$ are matrix-valued processes of dimension $(\tau\times\tau)$, $(\infty\times\infty)$ and $(\tau \times \infty)$ respectively.

We shall study the ARFIMA$(1, d, 1)$ model with $\phi_1 = \phi$ and $\theta_1 = \theta$. The parameter set, which we denote bu $\underline{\Theta}$, is

$$\underline{\Theta} = \{\phi, d, \theta, \sigma, \delta_0, \delta_1, \lambda_0, \lambda_1\}. \tag{26}$$

For a series of observations $(x_1, x_2, \ldots, x_N)$ the log-likelihood function is written as

$$LL(\underline{\Theta}) = \sum_{t=1}^{N} \log f(x_t|\underline{\Theta}). \tag{27}$$

Our goal is to find the parameters that maximise $LL(\underline{\Theta})$ where, excluding a constant, the probability density function $f(-|\underline{\Theta})$ is given by

$$\log f(x_t|\underline{\Theta}) = -\tfrac{1}{2}\log\left(\det\Sigma_t\right) - \tfrac{1}{2}\left(x_t - \hat{Z}_t\right)^{\mathrm{T}} \Sigma_t^{-1} \left(x_t - \hat{Z}_t\right). \tag{28}$$

Here we have defined

$$\Sigma_t = \mathbb{E}\left[\left(Z_t - \hat{Z}_t\right)\left(Z_t - \hat{Z}_t\right)^{\mathrm{T}}\right] \tag{29}$$

and

$$\hat{Z}_t = \mathbb{E}\left[Z_t|\mathcal{F}_{t-1}\right] \tag{30}$$

as the mean and variance terms that can be computed using the Kalman filter.

Let us define

$$\Omega_t = \mathbb{E}\left[\left(Z_t - \hat{Z}_t\right)\left(Z_t - \hat{Z}_t\right)^{\mathrm{T}}\right], \tag{31}$$

and

$$I_t = Z_t - \hat{Z}_t. \tag{32}$$

We observe that

$$\hat{Z}_t = A + D\hat{X}_t, \tag{33}$$

where

$$\hat{X}_t = \mathbb{E}_{t-1}[X_t]. \tag{34}$$

By deriving the Kalman recursive equations we are able to derive expressions for $\hat{X}_t$, Ω_t, and Σ_t. In particular we obtain

$$\hat{X}_{t+1} = F\hat{X}_t + K_t I_t, \tag{35}$$

where

$$K_t = \left(F\Omega_t D^{\mathrm{T}} + S_t\right)\Sigma_t^{-1}, \tag{36}$$

$$\Omega_{t+1} = \left(F - K_t D\right)\Omega_t\left(F - K_t D\right)^{\mathrm{T}} + Q_t + K_t R_t K_t^{\mathrm{T}} - S_t K_t^{\mathrm{T}} - K_t S_t^{\mathrm{T}}, \tag{37}$$

and

$$\Sigma_t = D\Omega_t D^{\mathrm{T}} + R_t. \tag{38}$$

In a set of observations of length N we use only the first $N - \tau_N$ observations where τ_N is the value of the largest maturity in the data set. This is the nature of using the Kalman filter with the ARFIMA process. The log-likelihood function calculated using the τ_N^{th} to the N^{th} elements of X_t will then be equal to the exact likelihood function (see Chan & Palma 1998 for further details).

V. REFERENCES

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