Orthogonal Polynomial Technique

1. $$\frac{Z(N,V)}{Z(N,V_0)} = \frac{\int dM\, e^{-N\,\mathrm{Tr}\,V(M)}}{\int dM\, e^{-N\,\mathrm{Tr}\frac{M^2}{2}}}$$

$V_0 = \frac{\lambda^2}{2}$

$$= \frac{\int d^N\lambda\ \Delta(\lambda)^2\ e^{-N\,\mathrm{Tr}\,V(\lambda)}}{\int d^N\lambda\ \Delta(\lambda)^2\ e^{-N\,\mathrm{Tr}\frac{\lambda^2}{2}}}$$

where $\lambda = N\times N$ diagonal matrix

$\Delta(\lambda) =$ Vandermonde det $\prod_{i<j}(\lambda_i - \lambda_j)$

(Proof: by diagonalization $M = U\lambda U^+$

$\lambda \in \mathbb{R}^N$, $U \in U(N)/U(1)^N$, $\Delta^2 =$ Jacobian).

2. $$\Delta(\lambda) = \det_{1\le i,j\le N} p_{i-1}(\lambda_j)$$ for any

sequence of monic polynomials $p_{i-1}(\lambda) = \lambda^{i-1} + O(\lambda^{i-2})$.

3. Pick for p_i the polynomials $\perp$

wrt $$\boxed{(f,g) = \int_{\mathbb{R}} d\lambda\ e^{-NV(\lambda)} f(\lambda)g(\lambda)}$$

$\exists!$ set of monic polynomials, $\perp$ wrt $(\,,\,)$.

then $Z(N,V) = \sum_{\sigma,\tau\in \mathfrak{S}_n} (-1)^{\sigma\circ\tau} \times$

$$\times \prod_{i=1}^{N} \big(p_{\sigma(i)-1}(\lambda_i), p_{\tau(i)-1}(\lambda_i)\big)$$

$\sigma = \tau$ by $\perp$

$$\boxed{Z(N,V) = N! \prod_{i=0}^{N-1} (p_i, p_i)}$$

4. Compute the norms $(p_i, p_i) = h_i$

$Q = \lambda \times$ on basis $(p_i(\lambda))_{i\in\mathbb{N}}$

$P = \frac{d}{d\lambda} \times$ ---- -----

(a) $Q p_n = p_{n+1} + R_n p_{n-1}$ $\quad R_n = \dfrac{h_n}{h_{n-1}}$

(b) $P p_n = n p_{n-1} + \text{lower}$

$$\Rightarrow \frac{(P p_n, p_{n-1})}{(p_{n-1}, p_{n-1})} = n \underset{\text{by parts}}{=} \frac{N(V'(Q) p_n, p_{n-1})}{(p_{n-1}, p_{n-1})}$$

$$\int d\lambda \, -\frac{d}{d\lambda}\left(p_{n-1} e^{-NV(\lambda)}\right) p_n$$

$$= N\big(V'(Q) p_{n-1}, p_n\big)$$

$$\Rightarrow \quad \boxed{\frac{n}{N} = \frac{(V'(Q)\,p_n, p_{n-1})}{(p_{n-1}, p_{n-1})}}$$

$$V'(Q) = Q - \sum_{k \geq 2} g_{2k}\, Q^{2k-1}$$

path interpretation

Q^{2k-1} = paths of length $2k-1$, with

steps up from height $n \longrightarrow n+1$

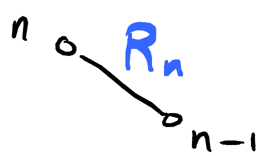
weight = 1

down " " $n \longrightarrow n-1$

weight = R_n

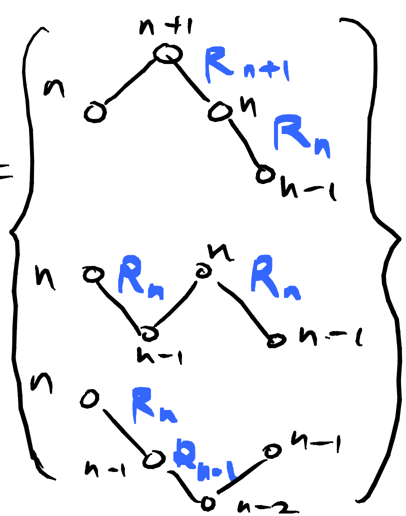
Ex:

$k=1$ $\quad \frac{(Q p_n, p_{n-1})}{(p_{n-1}, p_{n-1})} =$



$= \boxed{R_n}$

$k=2$ $\quad \frac{(Q^3 p_n, p_{n-1})}{(p_{n-1}, p_{n-1})} =$



$$= \boxed{\begin{matrix} R_{n+1} R_n \\ + \\ R_n^2 \\ + \\ R_n R_{n-1} \end{matrix}}$$

$k=k$ $\qquad \binom{2k-1}{k-1}$ paths $(n) \longrightarrow (n-1)$ with

$2k-1$ steps $\begin{cases} k-1 & \nearrow \\ k & \searrow \end{cases}$

General equation:

$$\frac{n}{N} = \mathrm{Pol}(R_n, R_{n\pm 1}, \dots, R_{n\pm m})$$

<u>Ex</u>: tetravalent maps:

$$\frac{n}{N} = ((Q - gQ^3)p_n, p_{n-1}) \,/\, (p_{n-1}, p_{n-1})$$

$$\boxed{\frac{n}{N} = R_n - g R_n (R_{n+1} + R_n + R_{n-1})}$$

$\Rightarrow$ recursion relation + initial

data $\begin{cases} R_0 = 0 \quad (\lambda \times p_0 = p_1) \\ R_1 = \text{integral involving } V. \end{cases}$

and $\text{Log}\, Z = N \sum_{n=0}^{N} \left(1 - \frac{n}{N}\right) \text{Log} \frac{R_n(V)}{R_n(V_0)}$

assume when N large

$$R_n \to r\left(x = \frac{n}{N}\right)$$

then: $\begin{cases} x = r - \sum \binom{2k-1}{k} g_{2k} r^k & (V) \\ x = r & (V_0) \end{cases}$

$$\lim_{N\to\infty} \frac{\text{Log}\, Z}{N^2} = f = \int_0^1 (1-x) \,\text{Log}\left(\frac{r}{x}\right) dx$$

change variables to r, via.

$$\begin{cases} x = r - \sum_{k\geq 2} \binom{2k-1}{k} g_{2k} r^k \\ 1 = R - \sum_{k\geq 2} \binom{2k-1}{k} g_{2k} R^k \qquad (x=1,\ n=N) \end{cases}$$

$$R_N \xrightarrow[N\to\infty]{} R$$

5. $\boxed{R = \langle (\mathrm{Tr} M)^2 \rangle_{N\to\infty}}$

$$\langle (\mathrm{Tr} M)^2 \rangle = \frac{1}{Z_0}\int d^N\lambda \sum_{\sigma,\tau\in S_N}{}' (-1)^{\sigma\circ\tau} \prod_i p_{\sigma(i)}(\lambda_i)\, p_{\tau(i)}(\lambda_i)$$

$$\times \prod e^{-N W(\lambda_i)} \times \left(\sum \lambda_i\right)^2$$

$\tau = \sigma \circ (\ell m) \Rightarrow$ sign

$$\langle (\mathrm{Tr} M)^2 \rangle = N \langle \lambda_1^2 \rangle - N(N-1)\langle \lambda_1 \lambda_2 \rangle$$

$$\lambda^2 p_n = \lambda(p_{n+1} + R_n p_{n-1}) = p_{n+2} + (R_{n+1} + R_n) p_n + R_n R_{n-1} p_{n-2}$$

$$\langle (\mathrm{Tr} M)^2 \rangle = \sum_{n=0}^{N} \underbrace{(R_n + R_{n+1})}_{= \frac{(\lambda^2 p_n, p_n)}{(p_n, p_n)}} - 2 \sum_{n=0}^{N-1} \underbrace{\frac{(\lambda p_n, p_{n+1})(\lambda p_{n+1}, p_n)}{(p_n, p_n)(p_{n+1}, p_{n+1})}}_{R_{n+1}}$$

$$\langle (\mathrm{Tr} M)^2 \rangle = R_{N+1} + \underbrace{\cancel{R_0}}_{=0} \xrightarrow[N\to\infty]{} R$$

qed

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