INTEGRABLE COMBINATORICS

A) INTRODUCTION

Integrable World : physics of integrable systems

(1) CLASSICAL

- continuous : time evolution governed by Hamiltonian, Poisson Structure $\{\ \}$

$$\frac{\partial f}{\partial t} = \{H, f\}$$

conservation laws $\frac{\partial C}{\partial t} = 0 = \{H, C\}$

- discrete : time $\in \mathbb{Z}$, evolution = explicit recursion relation. Integrability = existence of conserved quantities

Solvability = one can actually

solve for the general solution as a function of initial data.

In discrete setting $\equiv$ reducible to linear recursion relations (then easily solved).

TOY EXAMPLE : THE A_1 Q-SYSTEM

$$\boxed{R_{n+1}\, R_{n-1} = R_n^2 + 1 \qquad n \geq 0}$$

initial (Discrete Cauchy) data (R_0, R_1)

This is a discrete integrable equation, exactly solvable

Solution:
$$\begin{vmatrix} R_{n+1} & R_n \\ R_n & R_{n-1} \end{vmatrix} = 1$$

call this W_n (discrete Wronskian)

$$W_{n+1} - W_n = 0 = \begin{vmatrix} R_{n+2} + R_n & R_{n+1} \\ R_{n+1} + R_{n-1} & R_n \end{vmatrix}$$

$$\Rightarrow \exists\ c_n = \frac{R_{n+2} + R_n}{R_{n+1}} = \frac{R_{n+1} + R_{n-1}}{R_n}$$

c_n independent of n!

$c_n = c$ is a conserved quantity

Write $$c = \frac{R_{n+2} + R_n}{R_{n+1}} = \frac{\frac{R_{n+1}^2 + 1}{R_n} + R_n}{R_{n+1}}$$

$$\Rightarrow \boxed{\begin{aligned} c &= \frac{R_{n+1}}{R_n} + \frac{1}{R_n R_{n+1}} + \frac{R_n}{R_{n+1}} \\ &= \frac{R_1}{R_0} + \frac{1}{R_0 R_1} + \frac{R_0}{R_1} \end{aligned}}$$

= discrete "integral of motion"

we go from a 3 term recursion

$$R_{n+2} = \frac{R_{n+1}^2 + 1}{R_n}$$

to a 2 term recursion $c(R_n, R_{n+1}) = c$

Solvability:

We have a linear recursion relation: $\boxed{R_{n+1} - c R_n + R_{n-1} = 0}$

Use general theory (of linear differential eqns)

$$R_n = \lambda_+ (x_+)^n + \lambda_- (x_-)^n$$

where $x_\pm$ solutions of the characteristic equation:

$$x^2 - c x + 1 = 0$$

$$\underbrace{\lambda_\pm, \ x_\pm}_{\text{action-angle variables}} = \text{fctns of } (R_0, R_1)$$

more explicit?

yes: consider the generating function:

$$F(t) = \sum_{n=0}^{\infty} t^n \frac{R_n}{R_0} = \frac{1-\left(c-\frac{R_1}{R_0}\right)t}{1-ct+t^2}$$

set $\boxed{y_1 = \frac{R_1}{R_0}; \; y_2 = \frac{1}{R_0 R_1}; \; y_3 = \frac{R_0}{R_1}}$

(Laurent monomials of initial data)

then $$F(t) = \frac{1-t(y_2+y_3)}{1-t(y_1+y_2+y_3)+t^2 y_1 y_3}$$

$$F(t) = \frac{1}{1-\frac{t y_1(1-t y_3)}{1-t(y_2+y_3)}}$$

$$\boxed{F(t) = \frac{1}{1-\frac{t y_1}{1-\frac{t y_2}{1-t y_3}}}}$$

This gives an explicit solution in terms of initial data

(cluster algebra, positivity conjecture of Fomin–Zelevinsky).

Remark: The non-commutative A_1–Q-system is also integrable and exactly solvable:

$$\boxed{R_{n+1} R_n^{-1} R_{n-1} = R_n + R_n^{-1}}$$

R_n non-commuting. Solution (R_0, R_1)?

2 conserved quantities:

- $d_n = R_{n+1}^{-1} R_n R_{n+1} R_n^{-1} = d$
- $c_n = R_{n+1} R_n^{-1} + R_{n+1}^{-1} R_n^{-1} + R_{n+1}^{-1} R_n = c$

left linear recursion

$$R_{n+1} - c\,R_n + d\,R_{n-1} = 0$$

$c = y_1 + y_2 + y_3 \quad d = y_3 y_1$

$$\begin{cases} y_1 = R_1 R_0^{-1} \\ y_2 = R_1^{-1} R_0^{-1} \\ y_3 = R_1^{-1} R_0 \end{cases}$$

Path solution :

$$F(t) = \sum_{n=0}^{\infty} t^n R_n R_0^{-1}$$

$$= \left(1 - t\left(1 - t\left(1 - t\,y_3\right)^{-1} y_2\right)^{-1} y_1\right)^{-1}$$

= g.f. same paths w/ NC weights

$\Rightarrow$ reduces to weighted path combinatorics

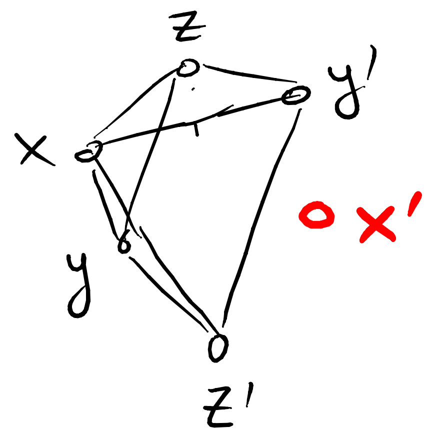
* * * *

Other examples :

- T-systems (for each Lie algebra) = identities for transfer matrices of generalized Heisenberg quantum spin chains. Ex : A case :

$$\boxed{T_{i,j,k+1}\,T_{i,j,k-1} = T_{i,j+1,k}\,T_{i,j-1,k} + T_{i+1,j,k}\,T_{i-1,j,k}}$$

also known as tetrahedron equation



$$xx' = yy' + zz'$$

$\rightarrow$ Combinatorics of weighted paths, Domino Tilings of the Aztec Diamond, Alternating Sign Matrices, Little-Wood-Richardson coefficients.

(Speyer, Knutson-Tao, Mills, Robbins Rumsey, etc...)

- Q-systems (for each Lie Algebra)

recursion relations for characters of special modules (A-case = rectangular Young diagram Schur functions).

→ general solution = weighted paths on target graphs
(Kedem + PDF)

Main example for these lectures = enumeration of planar maps with marked points at prescribed geodesic distances

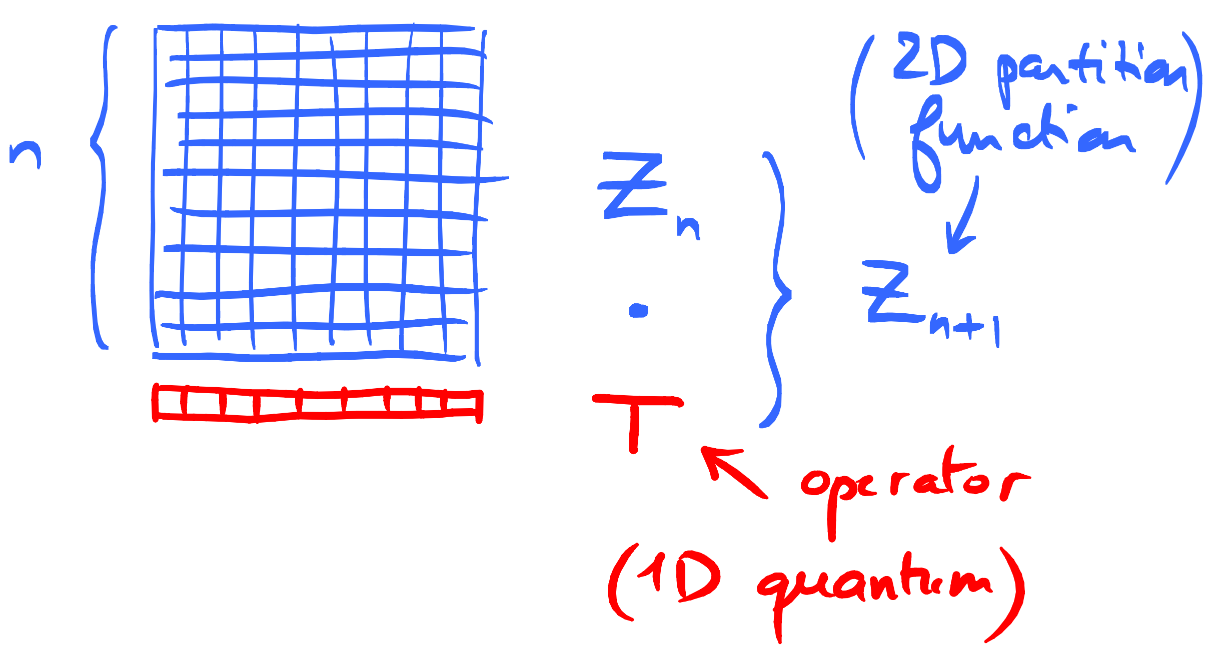
→ discrete integrable recursion relations, where "time" = geodesic distance

(2) QUANTUM

quantization : $\{,\} \to [,]$

conserved quantities commute with the Hamiltonian $\Leftrightarrow$ symmetries

Relation between 1D quantum statistical models ("spin chains") and 2D statistical models ("lattice models", vertex models, IRF = Interaction-round-a-face etc...)

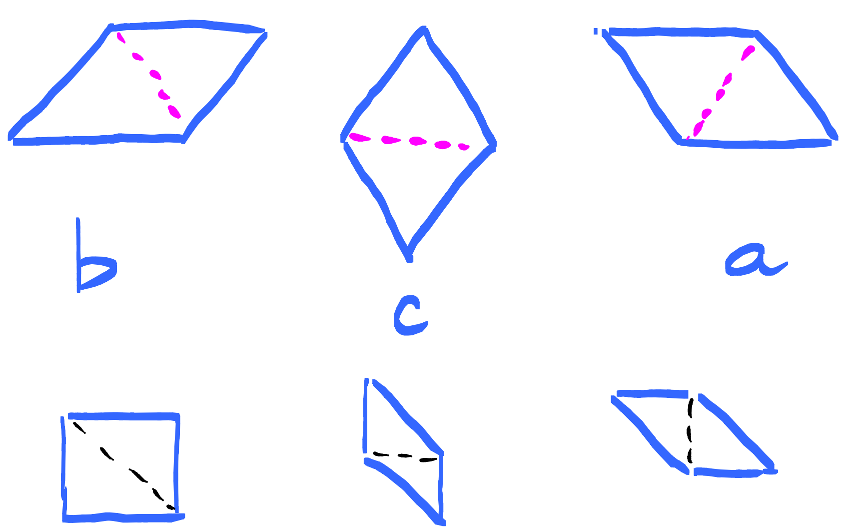


quantum integrability existence of a parametric family of operators $T(t)$ that commute with each-other

$$[T(t), T(t')] = 0 \quad \forall t, t'$$

Example = Rhombus Tilings (or dimer covering) of the triangular lattice

Pb: given a domain $\mathcal{D}$ of triangular lattice tile by means of 3 possible pairs of adjacent triangles



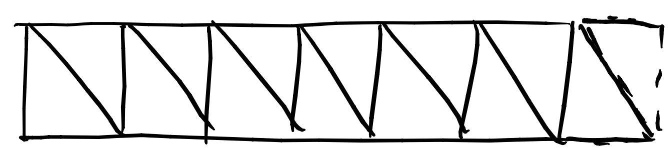
Compute the partition function

$$Z_{\mathcal{D}} = \sum_{\text{tilings}} a^{N_a}\, b^{N_b}\, c^{N_b}$$

tiles of each type

The system is integrable, and exactly solvable

Transfer matrix: cut the domain into consecutive slices (take cylinder e.g.)

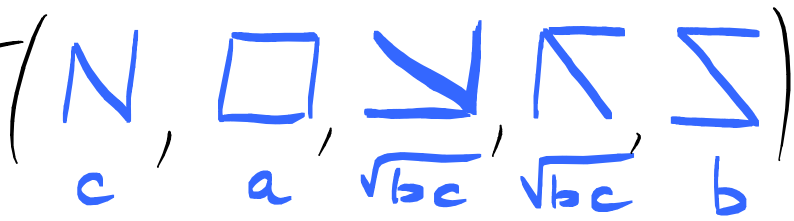


configuration of occupied horizontal edges = vector $e_{\{0,1\}} \otimes \cdots \otimes e_{\{0,1\}} \in V^{\otimes L}$

$\begin{cases} 0 \longrightarrow \text{empty} \\ 1 \longrightarrow \text{occupied} \end{cases}$ $\quad \langle e_0, e_1 \rangle = V$

Transfer matrix: $V^{\otimes L} \longrightarrow V^{\otimes L}$

matrix elements = from bottom $|B\rangle$ to top $|T\rangle$

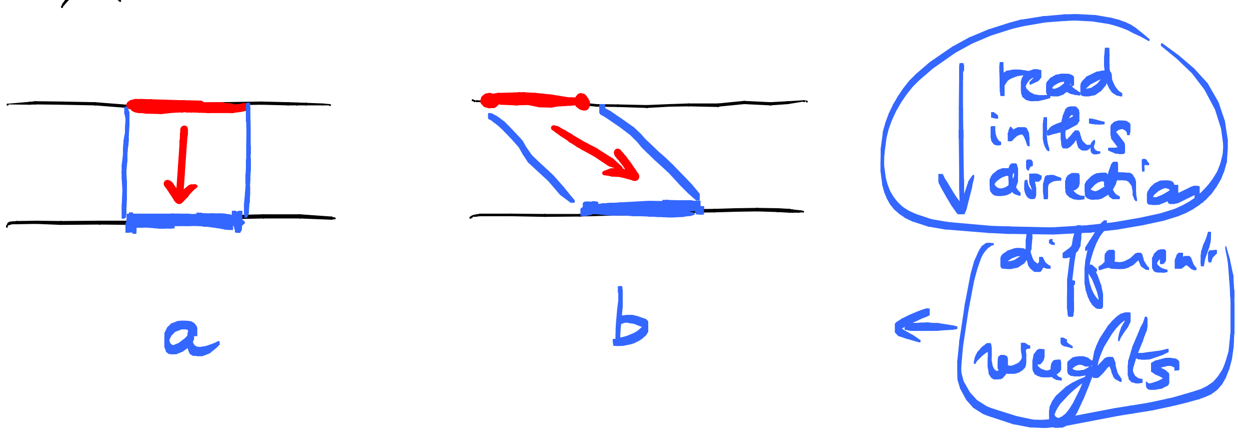
$T_{B,T} \sum \prod$ 

diagonal/vertical edge states

along the raw

Conservation Laws: horizontal

occupied edges propagate to the next line



$\Rightarrow$ Transfer matrix splits into $L+1$ blocks T_k $k=0,\dots,L$ where exactly k edges are occupied $\Rightarrow$

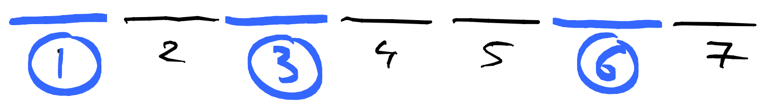
vector space $V_k = \sum\limits_{\substack{n_1+\dots+n_L=k \\ n_i\in\{0,1\}}} c_{n_1 \dots n_L}\, e_{n_1}\otimes\dots\otimes e_{n_L}$

represent such vectors

$$e_{n_1}\otimes\dots\otimes e_{n_L} = |x_1 x_2 \dots x_k\rangle$$

where x_i = position of the i-th occupied edge along the line.

Ex:



$$\Updownarrow$$

$$|1,3,6\rangle$$

In this notation, the transfer matrix is ultra-simple!

$$\boxed{T(a,b) = \prod_{i=1}^{k} (a + b\, s_i)}$$

s_i = shift operator in i-th variable

$$s_i \, |x_1 \cdots \underbrace{x_i \cdots x_k\rangle = |x_1 \cdots x_i+1}_{\text{position shifted by } +1} \cdots x_k\rangle$$

$$\boxed{\text{THM (trivial)} \quad [T(a,b), T(a',b')] = 0}$$

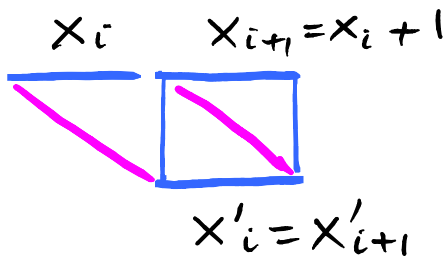
→ Solve by diagonalization of T!
(Bethe Ansatz).

Exactly solvable by diagonalization of T = "free fermions"

- <u>Good news</u>: S_i commute and simultaneously diagonalizable

$$S_i \cdot z^{x_i} = z \cdot z^{x_i} \quad \text{(plane wave)}$$

- <u>Bad news</u>: Problem with configurations that allow overlap:



Solution = consider vector space

$$V_k = \{ \, |x_1 \cdots x_k\rangle \;|\; 0 \leq x_1 \leq \cdots \leq x_k \leq L-1$$
$$\text{and such that } |x_1 \cdots x_i x_{i+1} \cdots x_k\rangle = 0$$
$$\text{whenever } x_i = x_{i+1} \text{ for some } i \, \}$$

Then <u>eigenvectors</u> = Slater determinants (Schur fctns)

$$S_\lambda = \det_{1 \leq i,j \leq k} \left(z_i^{x_j} \right)$$

<u>Bethe eqns</u>

$$z_i^L = (-1)^{k-1} \quad \text{(Periodicity)}$$

Eigenvalues

sol: 1. $z_j = e^{\frac{2i\pi}{L}\left(\frac{k-1}{2} + \ell_j\right)}$ integers mod L

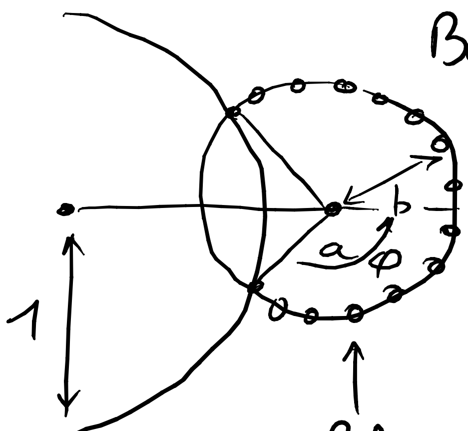
$\ell_1 < \ell_2 \cdots < \ell_k \mod L$

$\left(\dim V_k = \binom{L}{k}\right.$, OK, complete)

2. Eigenvalue reads

$$\Lambda(\ell_1 \ldots \ell_k) = \prod_{j=1}^{k} (a + b z_j)$$

Maximum? $\rightarrow$ largest number of Bethe roots z_j, such that $|a + b z_j| > 1$



$$\boxed{\frac{1}{L} \operatorname{Log} \Lambda_{max} \xrightarrow[L\to\infty]{} \frac{1}{2\pi} \int_{-\varphi}^{\varphi} \operatorname{Log}(a + b e^{i\theta})\, d\theta}$$

$Z \underset{L\to\infty}{\sim} \Lambda_{max}{}^{L}$ dominates

SUMMARY

In the following we will consider integrable lattice models for compact loops in various geometries, and solve them to get combinatorial/probabilistic results.

Plan

1 - Enumeration of Planar maps

- matrix models
- Tree bijections
- integrability/exact solvability

2 - Loop models, ASM and TSSCPP

- Integrable lattice models:
  6 Vertex - O(n) loops
- The ASM - TSSCPP theorem