INTEGRABLE COMBINATORICS

P. DI FRANCESCO / P Zinn-Justin
JB Zuber

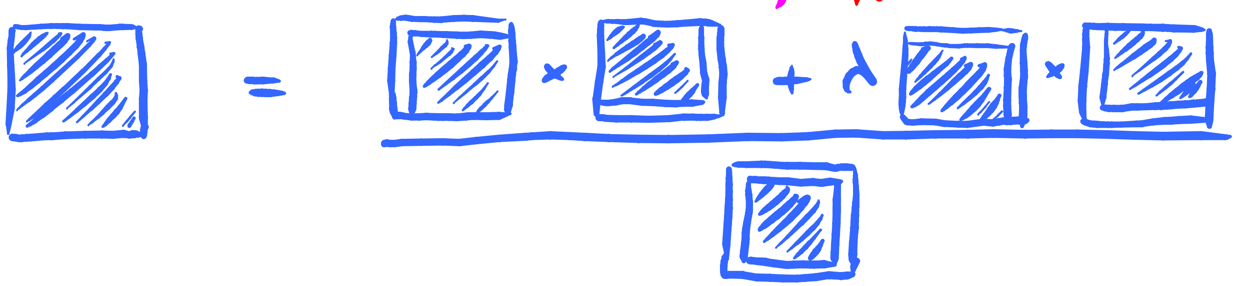
- Alternating Sign Matrices **ASM**
- Ice model (Integrable lattice model) **6V**
- Totally Symmetric Self-complementary Plane Partitions **TSSCPP**
- Loop models **DPL**, **FPL**
- Quantum Knizhnik-Zamolodchikov equation **QKZ**
- Degree / Multidegree of algebraic varieties **$M^2=0$**

ASM (Robbins Rumsey) (+ Mills)

generalized, "λ-determinant"

$M = (m_{ij})$

$$|M|_\lambda = \frac{|M^1_1|_\lambda \, |M^n_n|_\lambda + \lambda \, |M^1_n|_\lambda \, |M^n_1|_\lambda}{|M^{1,n}_{1,n}|_\lambda}$$

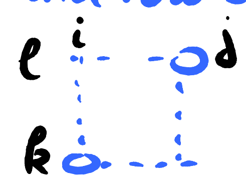


NB: $\lambda = -1$, ordinary determinant (Dodgson)

THM:

$$|M|_\lambda = \sum_{A \in ASM_n} \lambda^{I(A)} (1+\lambda^{-1})^{N(A)} \prod_{i,j} m_{ij}^{A_{ij}}$$

- $ASM_n = \{$ $n \times n$ matrices $a_{ij} \in \{0, \pm 1\}$ st ± 1 alternate along each row and column, and row sum = col sums = 1 $\}$
- $I(A) = \sum_{\substack{i<j \\ k>\ell}} A_{ki} A_{\ell j}$



- $N(A) = \#\{(-1)\text{'s in } A\} = \frac{1}{2}\left[\sum_{i,j} |A_{ij}| - n\right]$

Ex:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} \begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

THM:

$$A_n = |ASM_n| = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

(Zeilberger / Kuperberg)

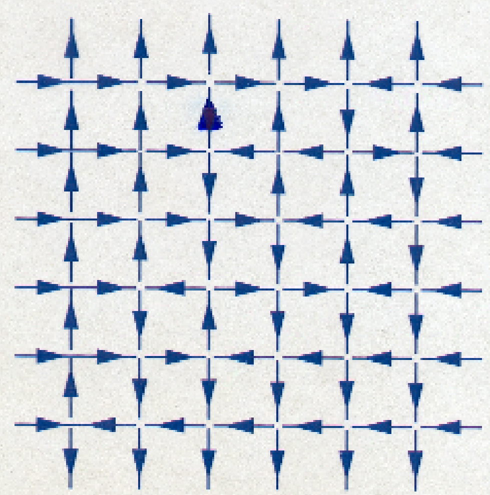
Both proofs relate ASM to other objects

→ TSSCPP "A=B"

→ ICE computes

ASM

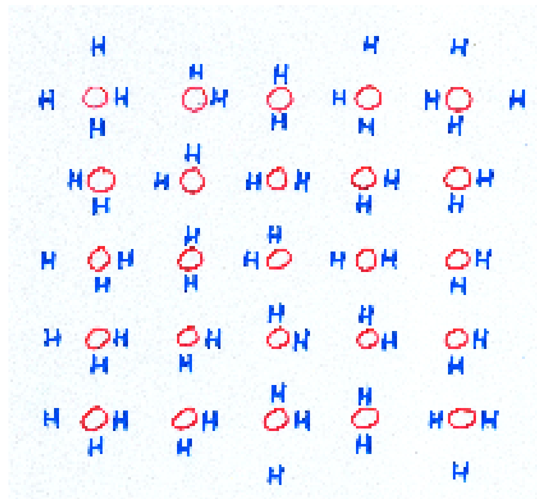
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



6vertex model

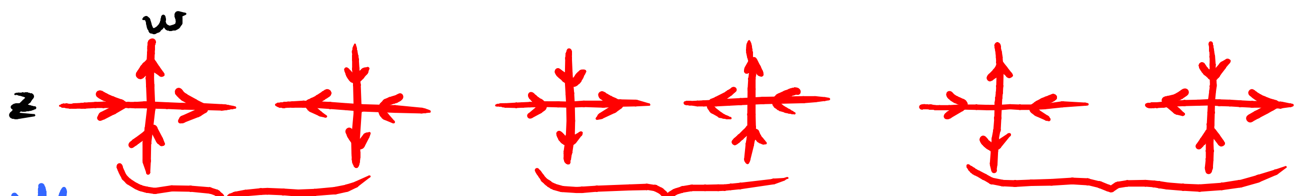
bijection between

- $ASM_{n\times n}$ and configs of the
- 6vertex model with Domain Wall Boundaries



ICE Model (Lieb)

6V + DWBC



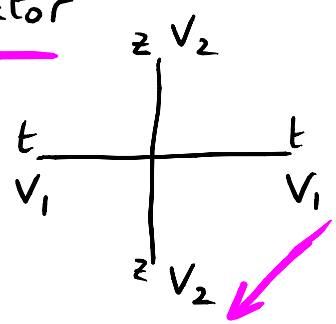
Weights

$a = qz - q^{-1}w$ $\qquad$ $b = z - w$ $\qquad$ $c = (q - q^{-1})\sqrt{zw}$

This model is integrable : Space $V \cong \mathbb{C}^2 = \langle \uparrow, \downarrow \rangle$

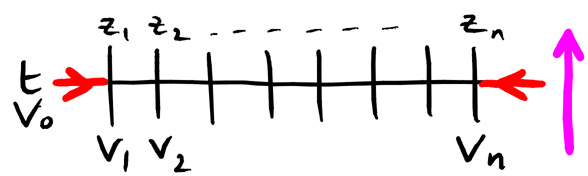
R-operator

$R_{12}(t,z) =$



acts $V_1 \otimes V_2 \to V_2 \otimes V_1$

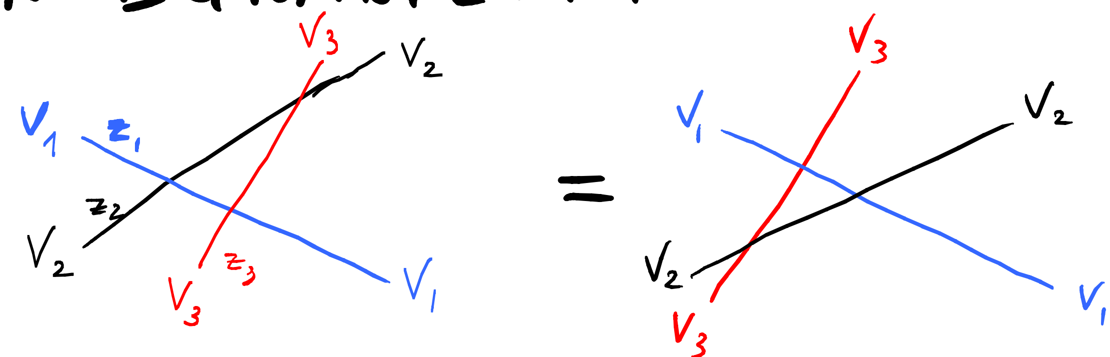
Transfer Operator



acts $V_1 \otimes V_2 \cdots \otimes V_n$ ↺

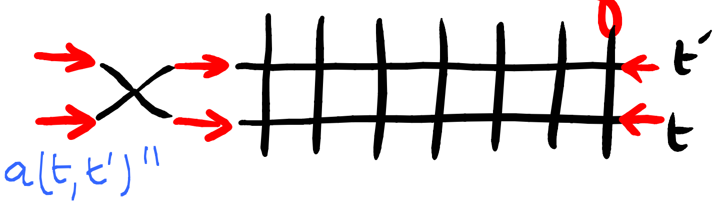
$T(t;\vec{z}) = \left[R_{01}(t,z_1)\, R_{02}(t,z_2) \cdots R_{0n}(t,z_n)\right]_{\rightarrow,\leftarrow}$

INTEGRABILITY

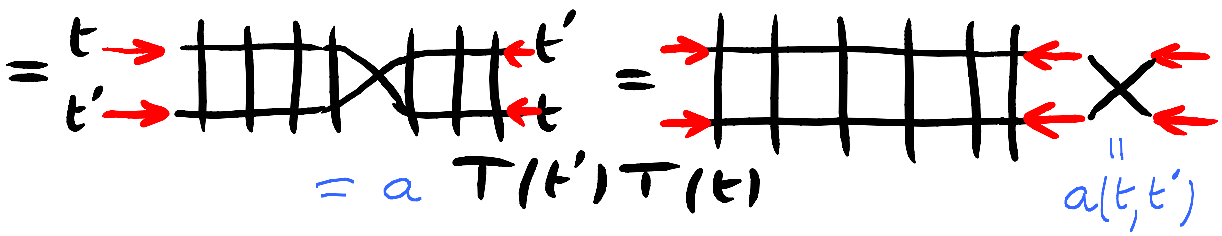


$$\boxed{R_{12}\,R_{13}\,R_{23} = R_{23}\,R_{13}\,R_{12}}$$ (Yang-Baxter eqn)

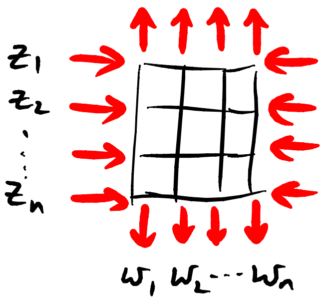
$\Rightarrow$ Commutation of Transfer matrices



$= a\,T(t)\,T(t')$



$= a\,T(t')\,T(t)$



Partition function of 6V + DWBC

$$Z_n = \frac{\displaystyle\sum_{\substack{\text{configs}\\ \text{on grid}}} \ \prod_{\substack{\text{vertices}\\ (ij)}} \text{weights}(z_i, w_j)}{\displaystyle\prod_i c(z_i, w_i)}$$

THM

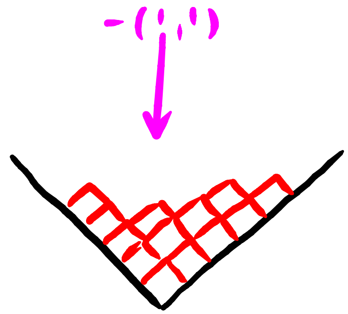
$$\boxed{Z_n = \frac{\displaystyle\prod_{i,j} a(z_i,w_j)\, b(z_i,w_j)}{\displaystyle\prod_{i<j} (z_i - z_j)(w_i - w_j)} \det\left\{ \frac{1}{a(z_i,w_j)\, b(z_i,w_j)} \right\}_{1\le i,j\le n}}$$

(Izergin - Korepin)

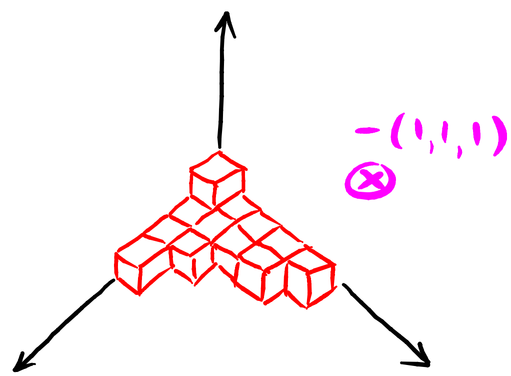
recursion relation + symmetries (from commutation of Transfer matrices).

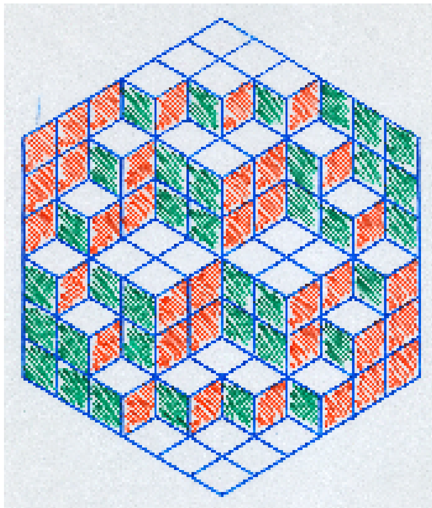
TSSCPP

Partitions
(2D)

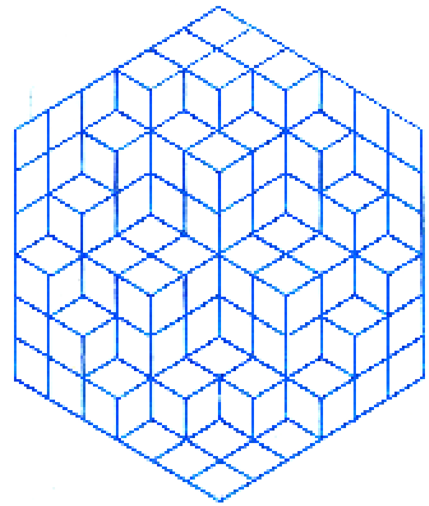


Plane Partitions
(3D)



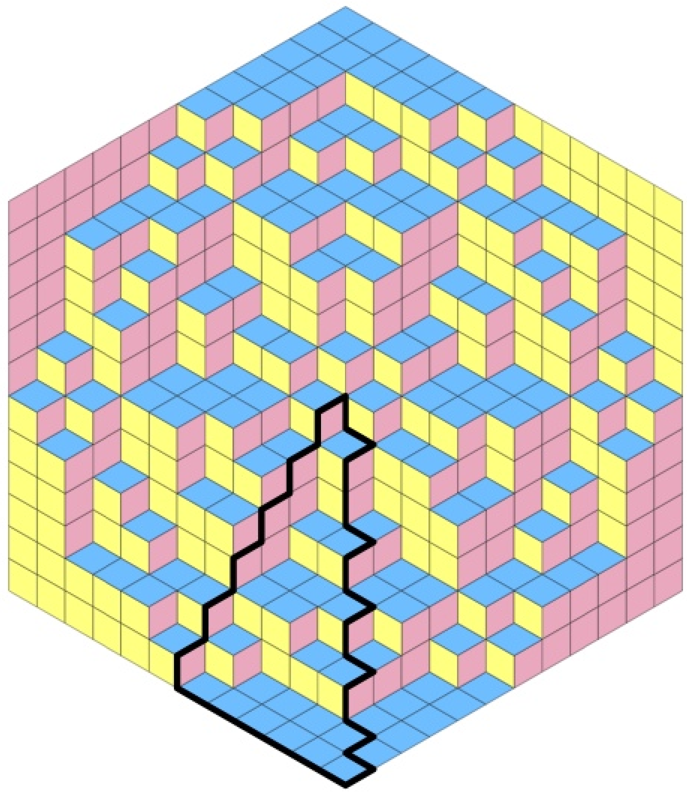


Plane Partition



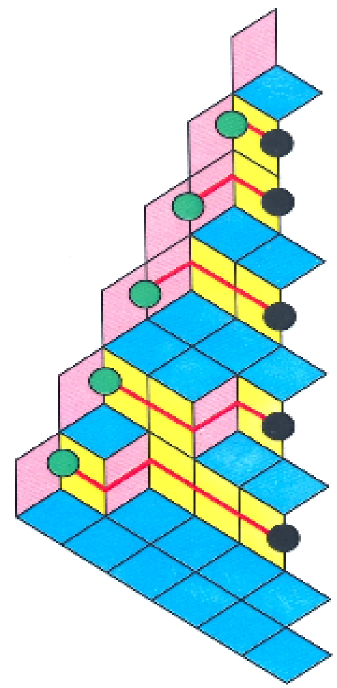
Rhombus Tiling of a hexagon

+ symmetries



TSSCPP

Totally Symmetric Self-Complementary Plane Partition



NILP

Non-intersecting Lattice Paths

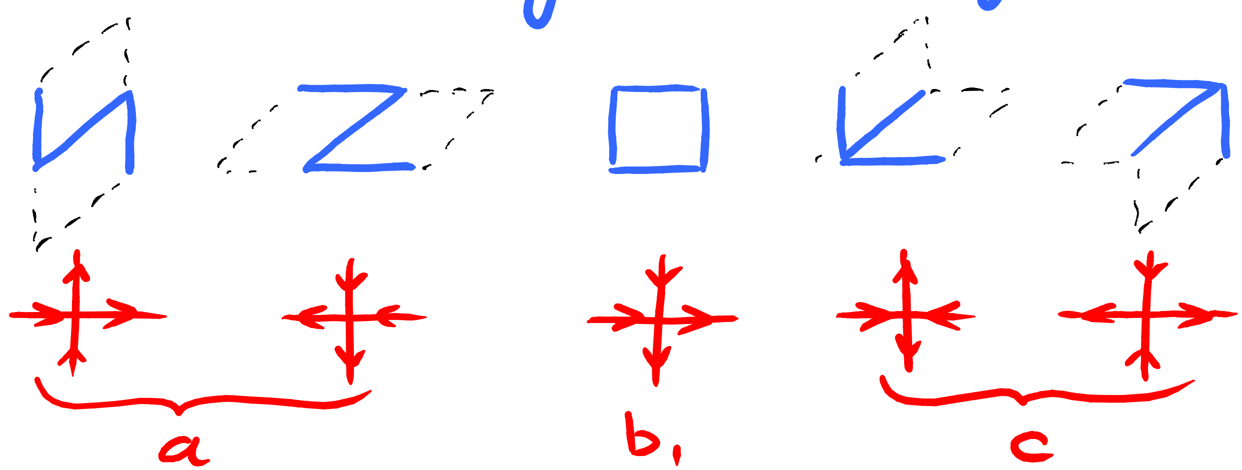
THM: The total number of TSSCPP in a $2n \times 2n \times 2n$ hexagon is:

$$\mathrm{TSSCPP}(n) = A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

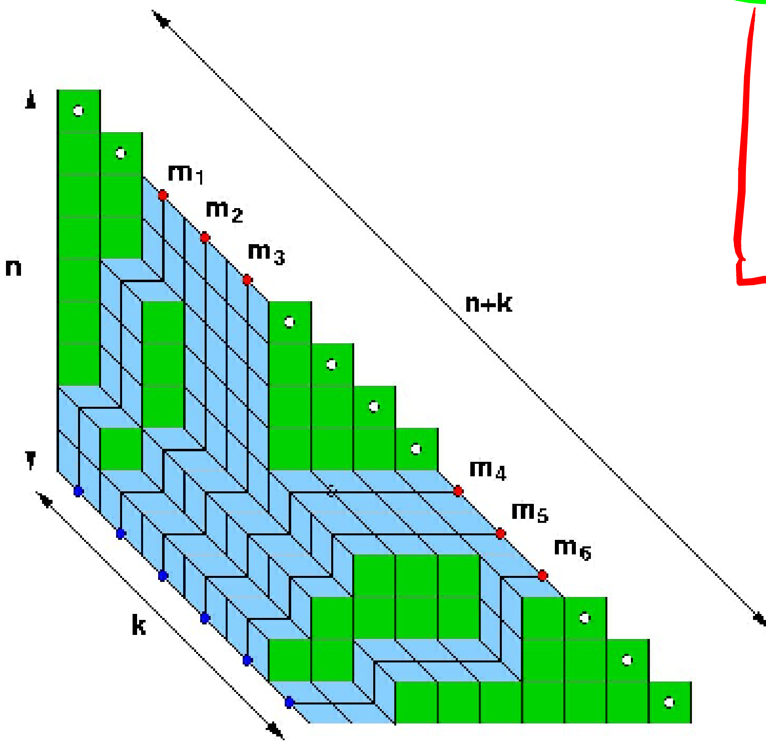
(Andrews) (Krattenthaler)

Recall:

The rhombus tiling model is integrable = 5V model



REMARK : Distribution of endpoints



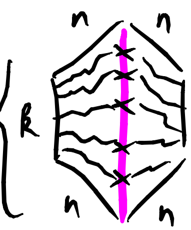
(PDF + Reshetikhin)

THM: # { NILP with ends $m_1, m_2 \cdots m_k$ }

$$= \det_{1 \le i,j \le k} \binom{n}{m_j - i + 1}$$

$$= \prod_{i>j} (m_i - m_j) \prod \frac{(n+k-i)!}{m_i!\,(n+k-m_i-1)!}$$

$$\Downarrow$$



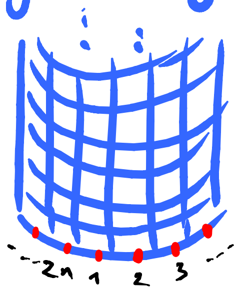
$$\# \{ \cdots \} = \Delta(m)^2 e^{V(m)}$$

matrix models !!!

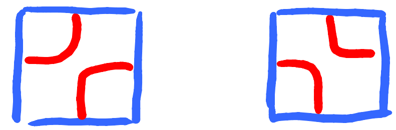
DPL (densely-Packed Loop model)

(Nienhuis)

geometry = semi-infinite cylinder of square lattice



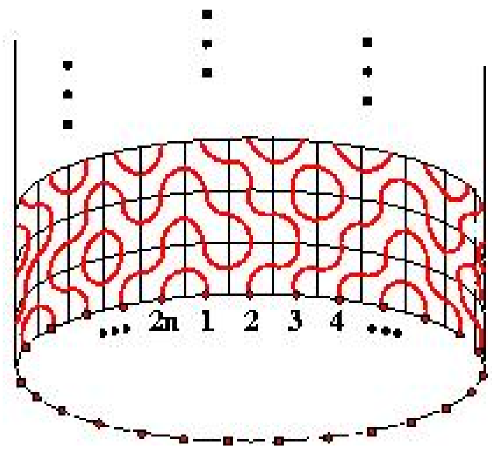
draw on each face either of

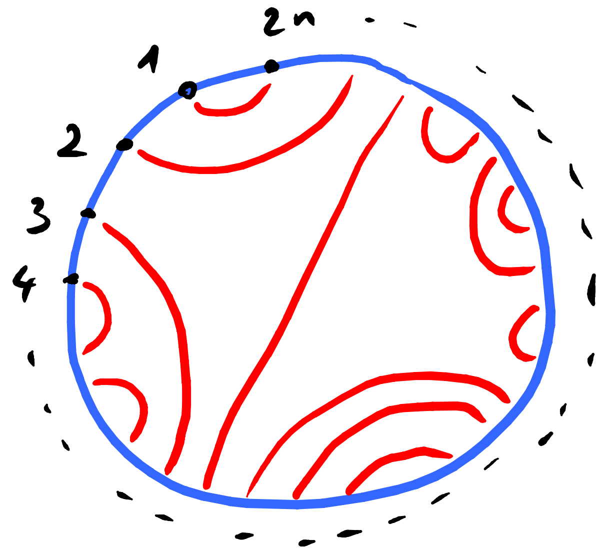


With Proba : → p $\qquad$ $1-p$ $\qquad$ homogeneous

→ p_i $\qquad$ $1-p_i$ $\qquad$ inhomogeneous

Question = connection of the boundary points?





Link Pattern π

(= connectivity diagram, planar, non-crossing chord diagram)

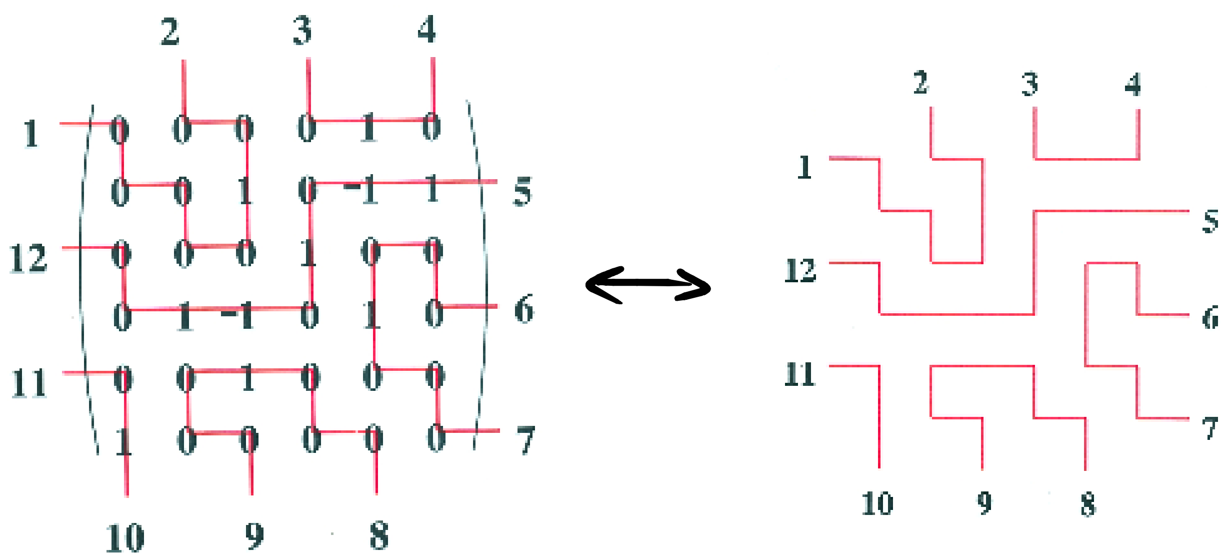
set = LP_n

Compute

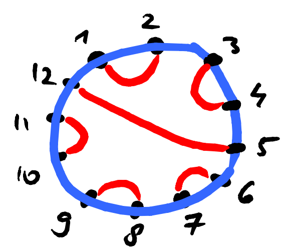
$\mathrm{Prob}(\pi)$?

ASM as Loop (FPL) configurations

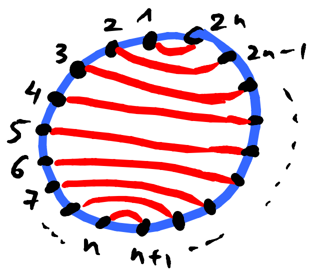
yet another bijection ASM $\Leftrightarrow$ FPL



$\mathrm{FPL}(\pi) = \#\left\{\begin{array}{l}\text{FPL configurations} \\ \text{that connect external} \\ \text{edges according to } \pi\end{array}\right\}$



π

let $\pi_0 :=$ 

Homogeneous case $(p_i = p)$

~~CON~~J: THM (Cantini - Sportiello) $\boxed{2010}$

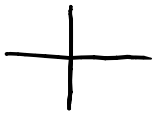
$$\Psi_\pi = \frac{\mathrm{Prob}(\pi)}{\mathrm{Prob}(\pi_0)} = FPL(\pi)$$

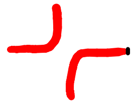
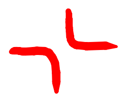
(Razumov - Stroganov) $\boxed{2001}$

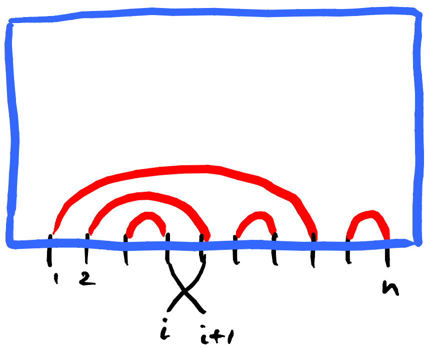
from integ. (1) $\frac{\mathrm{Prob}(\pi)}{\mathrm{Prob}(\pi_0)} \in \mathbb{N}$ (PDF + Zinn-Justin)

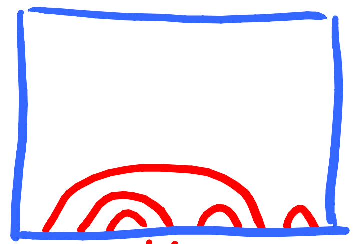
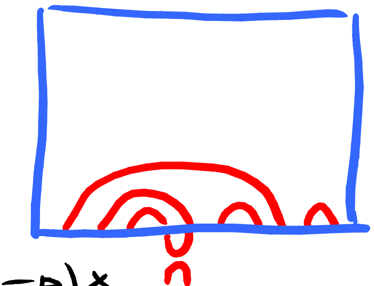
(2) sum rule $\sum_\pi \frac{\mathrm{Prob}(\pi)}{\mathrm{Prob}(\pi_0)} = A_n$ (conj. Batchelor De Gier - Nienhuis)

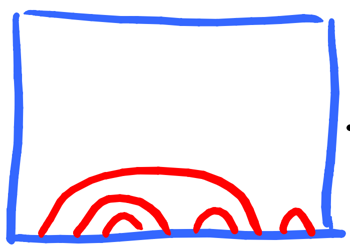
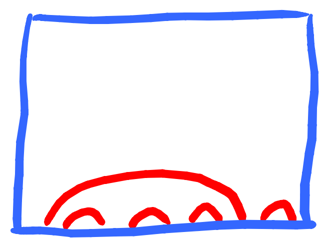
The DPL is integrable

$R_i =$ 

$= p_i$  $+(1-p_i)$  acts on $\pi \in LP_n$

π 

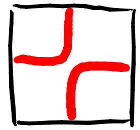
$\rightarrow$  $p_i \times$ $(\mathbb{1})$ $+(1-p_i)\times$  (e_i)

$R_{i,i+1} \cdot \pi = p_i$  $+(1-p_i)$ 

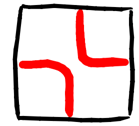
satisfies the Yang-Baxter eqn for some p_i

e_i generate Temperley-Lieb algebra $e_i e_{i\pm1} e_i = e_i$; $e_i^2 = e_i$.

$$R_{i,i+1}(t,z_i) = \underbrace{\frac{qt-q^{-1}z_i}{qz_i-q^{-1}t}}_{p_i}$$



$$+ \underbrace{\frac{t-z_i}{qz_i-q^{-1}t}}_{1-p_i}$$

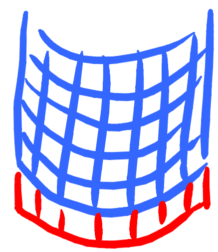


$(q^3=1)$

- Form the vector $\boxed{\sum_{\pi} \frac{\mathrm{Prob}(\pi)}{\mathrm{Prob}(\pi_0)} \cdot \pi = \Psi} = (\Psi_\pi)_{\pi \in LP_n}$

- Then $\boxed{T(t;\, z_1, \dots, z_n)\, \Psi = \Psi}$

Transfer operator describing the addition of a raw to the $\frac{1}{2}\infty$ cylinder



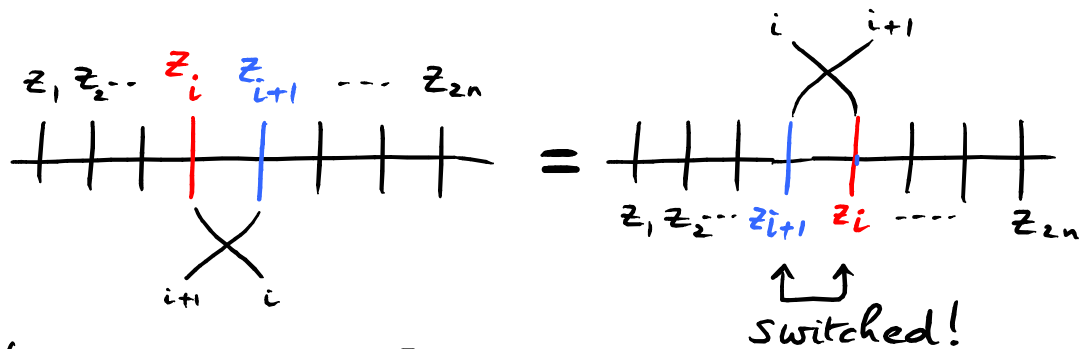
$\overset{\Psi}{\underset{T}{\cdot}} = \Psi$

Determines Ψ up to overall factor.

choose Ψ = polynomial of the z_i (indep. of t!)

qKZ (quantum Knizhnik-Zamolodchikov eqn)

Lemma



$$T(t; z_1 \cdots z_{i-1} z_i z_{i+1} z_{i+2} \cdots z_{2n})\, R_{i+1,i}(z_{i+1}, z_i)$$

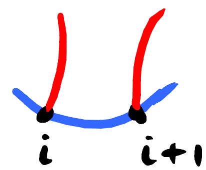
$$= R_{i+1,i}(z_{i+1}, z_i)\, T(t; z_1 \cdots z_{i-1}, z_{i+1}, z_i, z_{i+2} \cdots z_{2n})$$

$$\Rightarrow \boxed{\begin{aligned}&\Psi(z_1 \cdots z_{i-1}, z_{i+1} z_i\, z_{i+2} \cdots z_{2n}) \\ &\qquad = R_{i+1,i}(z_{i+1}, z_i)\, \Psi(z_1 \cdots z_{2n})\end{aligned}}$$

qKZ (Frenkel–Reshetikhin Smirnov)
Jimbo – Miwa)

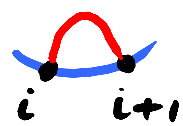
turns into:

(1) $\pi \notin \mathrm{Im}(e_i)$



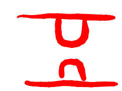
$$\boxed{(q z_i - q^{-1} z_{i+1}) \mid \Psi_\pi}$$

(2) $\pi \in \mathrm{Im}(e_i)$



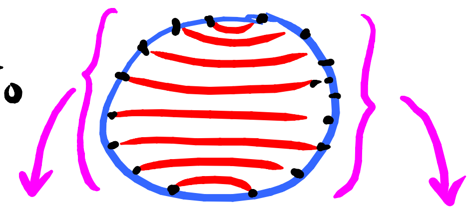
$$\boxed{(q z_{i+1} - q^{-1} z_i)\, \frac{(i,i+1) - 1}{z_{i+1} - z_i}\, \Psi_\pi = \sum_{\substack{\pi' \neq \pi \\ e_i \pi' = \pi}} \Psi_{\pi'}}$$

e_i = Temperley-Lieb algebra generators



$e_i^2 = -(q+q^{-1})\, e_i$; $e_i e_{i\pm1} e_i = e_i$

Solution :

π_0 

(A) $\Psi_{\pi_0} = \prod_{1\le i<j\le n} (q z_i - q^{-1} z_j) \prod_{n+1\le i<j\le 2n} (q z_i - q^{-1} z_j)$

by (1)

(B) Ψ_{π} determined triangularly by (2)

THM: at $q^3=1$

$$\sum_{\pi \in LP_n} \Psi_\pi(z_1 \cdots z_{2n}) = Z_n(z_1 \cdots z_n, z_{n+1} \cdots z_{2n}, q)$$

6V + DWBC

(PDF + Zinn Justin)

Proof:

- recursion from qKZ
- symmetry from YB

where are the TSSCPP?

Keep q generic (relax $q^3=1$)

$$\boxed{\begin{array}{l} e_i: \; e_i e_{i\pm1} e_i = e_i \\ \quad e_i^2 = -(q+q^{-1}) e_i \end{array}}$$

Then qKZ:

$$\boxed{\begin{array}{ll} (1) & (i,i+1)\,\Psi = R_{i+1,i}\,\Psi \qquad 1 \le i \le 2n-1 \\ (2) & \Psi(z_2 \ldots z_{2n}, s z_1)\sigma = c\,\Psi(z_1 \ldots z_{2n}) \end{array}}$$

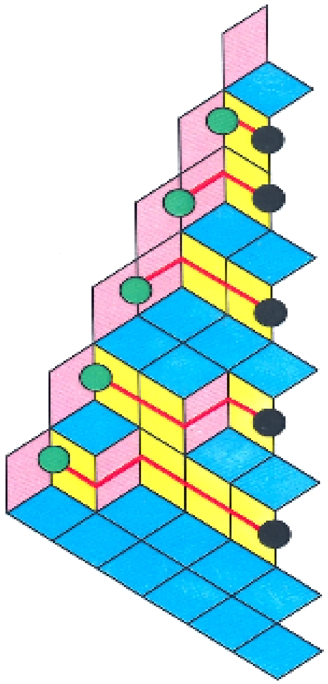
$s = q^6$

rotation of LP by 1 unit

has the same solution (q generic)

but

$$\sum_{\pi} \Psi_\pi(\vec{z}, q) \neq Z_n(\vec{z}, q) \;\; !$$



Refined enumeration of TSSCPP:

weight τ / pink rhombus

$$\boxed{\mathrm{TSSCPP}(n,\tau) := \sum_{\substack{\text{TSSCPP} \\ \text{of size } n}} \tau^{\#(\text{pink rhombus})}}$$

then ...

THM When $z_1 = \cdots z_{2n} = 1$, $\tau = -(q + \frac{1}{q})$,
we have $\displaystyle\sum_{\pi \in LP_n} \Psi_\pi(\vec{\tau}, q) = \mathrm{TSSCPP}(n, \tau)$

(PDF + Zinn-Justin) (+help Zeilberger)

Proof:
- multiresidue integral sol'n to qKZ
- generating fctns for NILP
- Zeilberger's lemma

L = 6 qKZ solution

```
Clear[a, b, z, g]
a[i_, j_] := q z[i] - q^-1 z[j]
b[i_, j_] := q^-2 z[i] - q^2 z[j]
```

```
(* (16) (25) (34) *)
g[1] := (a[1, 2] a[1, 3] a[2, 3]) * (a[4, 5] a[4, 6] a[5, 6])
```

```
(* (16) (23) (45) *)
g[2] := a[1, 2] a[3, 4] a[5, 6] * (a[1, 3] a[4, 5] b[6, 2] + a[2, 4] a[3, 6] b[5, 1])
```

```
(* (12) (36) (45) *)
g[3] := (a[2, 3] a[2, 4] a[3, 4]) * (a[5, 6] b[5, 1] b[6, 1])
```

```
(* (14) (23) (56) *)
g[4] := (a[3, 4] a[3, 5] a[4, 5]) * (b[6, 1] b[6, 2] a[1, 2])
```

```
(* (12) (34) (56) *)
g[5] := a[2, 3] a[4, 5] b[6, 1] * (a[2, 4] a[5, 6] a[1, 3] + a[3, 5] b[4, 1] b[6, 2])
```

```
Table[Expand[Simplify[g[i]/g[1] /. {z[i_] → 1, q → 1/2 (-τ + √(-4 + τ^2))}]], {i, 5}]
Expand[Total[%]]
({"tau=", τ → %% /. {{τ → 1}, {τ → 2}}}) // TableForm
```

$\{1, 2\tau, \tau^2, \tau^2, \tau + \tau^3\}$

$\boxed{1 + 3\tau + 2\tau^2 + \tau^3 = \mathrm{TSSCPP}(6, \tau)}$

tau=

$1 \to \{1, 2, 1, 1, 2\}$ $\quad 2 \to \{1, 4, 4, 4, 10\}$

so what ?

$M^2=0$ degree and multidegree

Set $\underset{(\tau=2)}{q=-1}$ in the previous solution,

actually do $\lim_{\varepsilon\to 0}$ with $\begin{cases} q=-e^{-\varepsilon a/2} \\ z_i=e^{-\varepsilon u_i} \end{cases}$

$\Rightarrow \Psi_\pi(a,\vec{u})$

Then:

(1) if $u_i=0\ \forall i$, Ψ_π are integers

again: ex: $n=3$

$$\boxed{\Psi=(1,4,4,4,10) \quad \Sigma=23}$$

$\boxed{\begin{array}{l} M^2 = 0 \\ M \text{ complex,} \\ \text{upper-triang.} \\ 2n \times 2n \end{array}}$

degree!

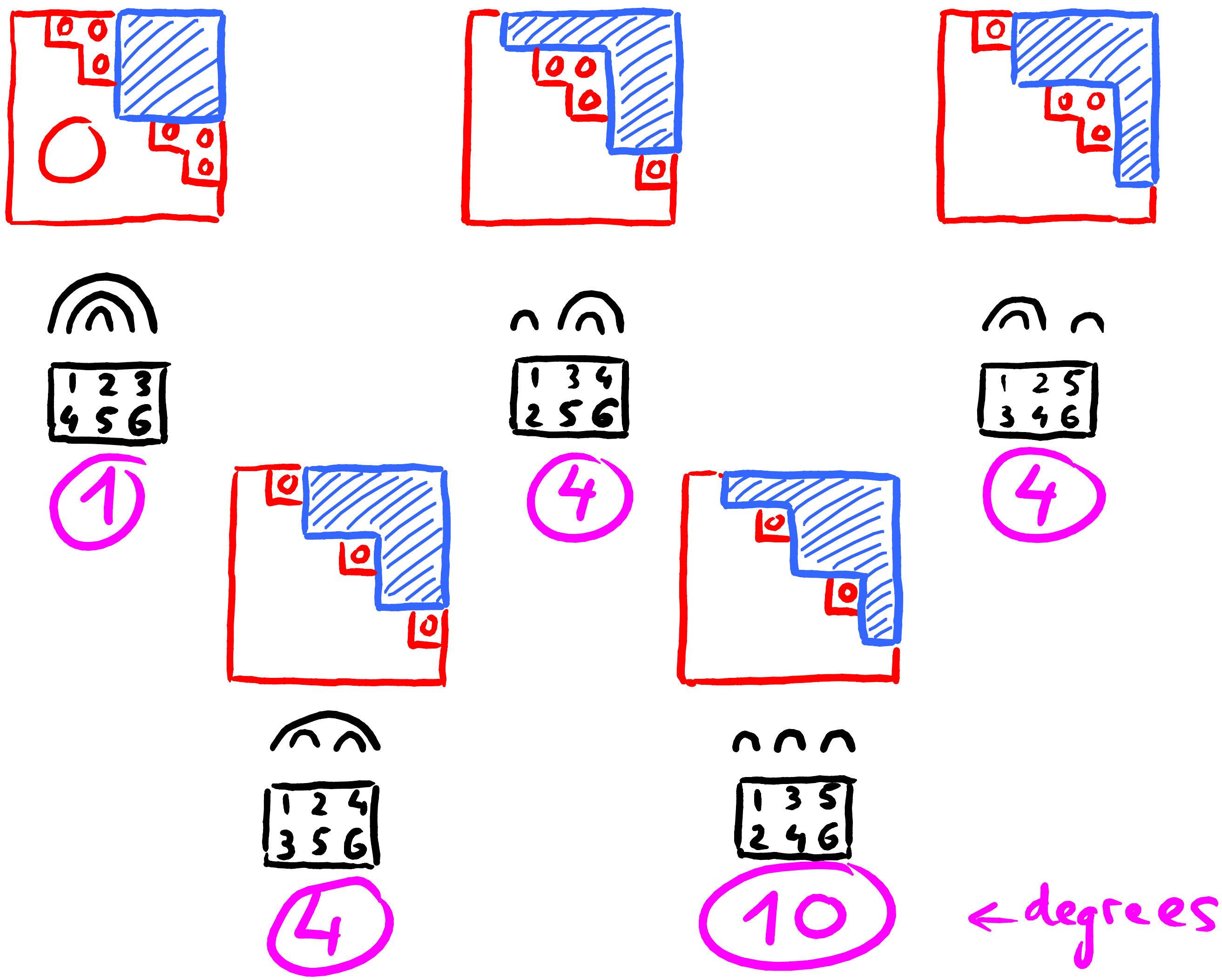
```
Clear[f, f0, g, g0, i, j, l, L, num, m]
n = 6;
p[i_, j_] := Block[{val}, val = 0; If[i < j, val = m[i, j]]; val]
M[n_] := Array[p, {n, n}]
M[n] // MatrixForm
f0 = Flatten[Table[Table[M[n][[i, j]], {j, i + 1, n}], {i, 1, n - 1}]];
f = Flatten[Table[Table[(M[n].M[n])[[i, j]], {j, i + 2, n}], {i, 1, n - 2}]];
f // TableForm
L = Length[f0];
num = Floor[n/2] Floor[(n + 1)/2];
For[i = 1, i ≤ num, i++, l[i] = Random[Complex];
 For[j = 1, j ≤ L, j++, l[i] = l[i] + Random[Complex] * f0[[j]]]]
Clear[i, j]
NSolve[Join[f, Table[l[i], {i, 1, num}]] == 0, f0];
Length[%]
```

$$\begin{pmatrix} 0 & m[1,2] & m[1,3] & m[1,4] & m[1,5] & m[1,6] \\ 0 & 0 & m[2,3] & m[2,4] & m[2,5] & m[2,6] \\ 0 & 0 & 0 & m[3,4] & m[3,5] & m[3,6] \\ 0 & 0 & 0 & 0 & m[4,5] & m[4,6] \\ 0 & 0 & 0 & 0 & 0 & m[5,6] \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
m[1, 2] m[2, 3]
m[1, 2] m[2, 4] + m[1, 3] m[3, 4]
m[1, 2] m[2, 5] + m[1, 3] m[3, 5] + m[1, 4] m[4, 5]
m[1, 2] m[2, 6] + m[1, 3] m[3, 6] + m[1, 4] m[4, 6] + m[1, 5] m[5, 6]
m[2, 3] m[3, 4]
m[2, 3] m[3, 5] + m[2, 4] m[4, 5]
m[2, 3] m[3, 6] + m[2, 4] m[4, 6] + m[2, 5] m[5, 6]
m[3, 4] m[4, 5]
m[3, 4] m[4, 6] + m[3, 5] m[5, 6]
m[4, 5] m[5, 6]
```

23

IRREDUCIBLE COMPONENTS



$$V = \{ M \in \mathcal{M}_{2n}(\mathbb{C}) \text{, st. upper triang, } M^2 = 0 \} \quad \text{variety}$$

$$= \bigcup_{\pi \in LP_n} V_\pi \quad \text{irred. components}$$

THM

(1) $\boxed{\deg V_\pi = \Psi_\pi}$ (qKZ sol at $a=0$, $u_i=0$)

(2) $\boxed{\mathrm{mdeg}\, V_\pi = \Psi_\pi(a, \vec{u})}$

PDF + Zinn Justin

multidegree (equivariant cohom of V_π, w/ torus action on M's by conjugation w/ diagonal (u_i) and multiplication by scalar (a) / $([M_{ij}] = a + u_i - u_j)$)

Corollary:

$$\boxed{\deg(\{M^2=0,\ M \text{ u.tri } 2n\times 2n\}) = TSSCPP(n,2)}$$

Conjecture:

TSSCPP form a natural degeneration of $\{M^2=0\}$ into intersections of hyperplanes (deg 1) and quadrics (deg 2).

CONCLUSION

- use of integrable models to obtain new connections

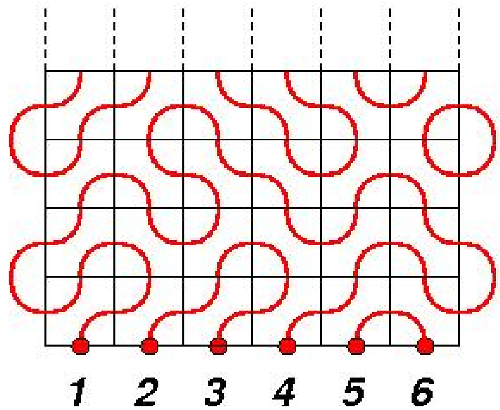
$$\begin{tikzcd}[column sep=small] \mathrm{ASM} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \mathrm{TSSCPP} \arrow[dl, leftrightarrow] \\ & M^2=0 & \end{tikzcd}$$

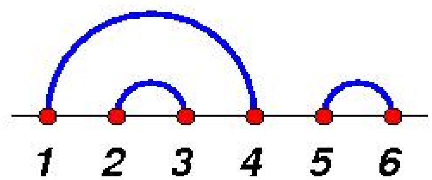
via qKZ

- generalizations
    - other Boundary Conditions (qKZ for root systems)
    - other groups (qKZ for higher rank groups)

OTHER BOUNDARY CONDITIONS

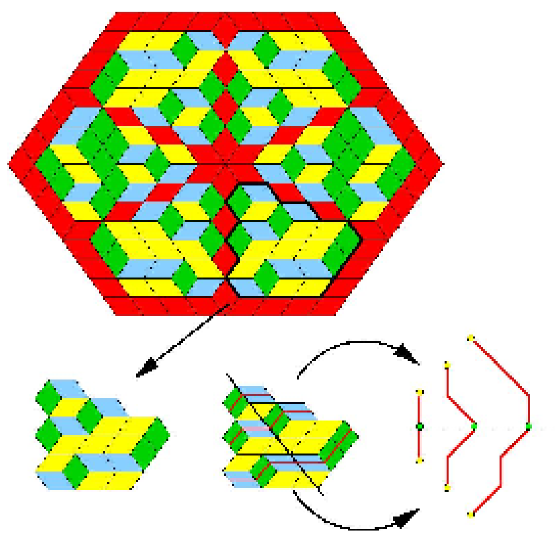
B-qKZ

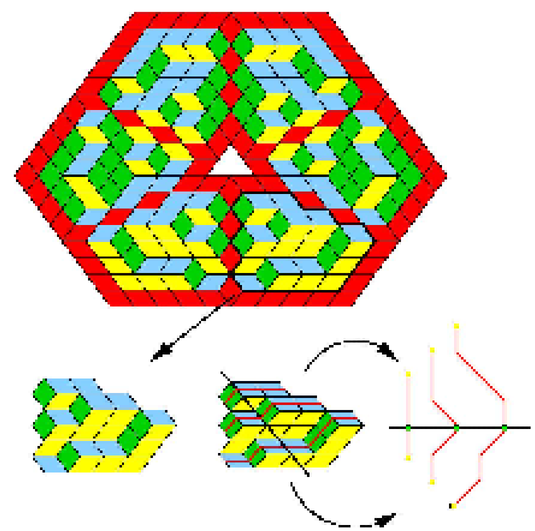




$M^2 = 0$ in B_n

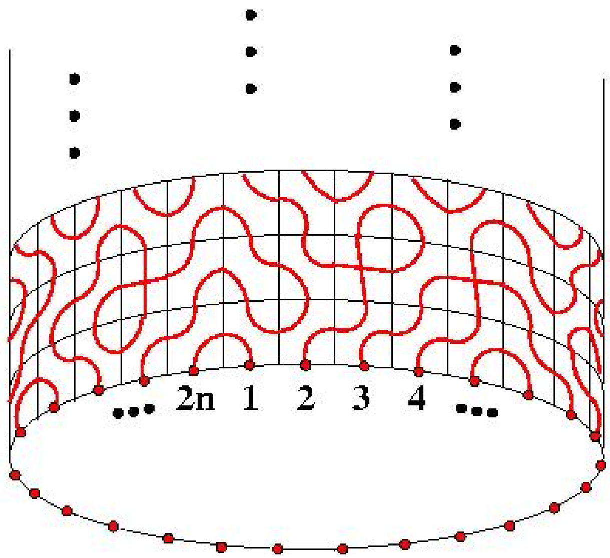
CSTC PP

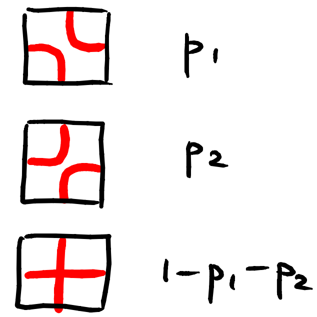




OTHER GROUPS

Brauer model





Temperley-Lieb → Brauer algebra

multidegree of <u>commuting variety</u> and more

(PDF, Zinn Justin)(Knutson, Zinn Justin)

question : generalize ASM ? TSSCPP ?

HIGHER RANK

SL_k - qKZ ↔ multidegree of variety

$\{M \in \mathcal{M}_{kn}(\mathbb{C}), M^k = 0\}$

u.triang.

Combinatorial sum rules

generalized ASM numbers

Temperley-Lieb → higher rk quotients of Hecke Algebra

PDF + Zinn-Justin